Cosmology & Galaxies NAME:

Homework 3: The Friedmann Equation

1. "Let's play *Jeopardy*! For \$100, the answer is: It was derived from general relativity in 1922 with the assumptions of a homogeneous and isotropic universe and that all mass-energy in the universe could be modeled by a perfect fluid. A Newtonian derivation (which required extra natural hypotheses) was given in 1934.

   What is the ____________, Alex?

   a) Einstein equation b) Milne-McCrea equation c) Synge equation d) Bondi equation
   e) Friedmann equation

2. A Newtonian derivation of the Friedmann equation (with extra natural hypotheses) could easily have been done in the 19th century, but it wasn't. There were probably 3 reasons why 19th century astronomers did not think of such a derivation. First, many were still thinking of a universe that was static on average even though dynamic equilibrium seemed hard to arrange, even though the universe was obviously not in thermodynamic equilibrium (and so why should be in dynamic equalibrium), and even though idea existed that the Milky was held up by rotation around its center of mass located somewhere. Second, they did not know that other galaxies existed though some believed this and they had not observed the general redshifts of the objects they thought might be other galaxies. Third, they thought in terms of Newton's absolute space (i.e., a single fundamental inertial frame) and did not think of the alternative idea completely compatible with their data that all ____________ unrotating with respect to the observable universe were elementary inertial frames (i.e., frames with respect to which Newtonian physics and all other known physics could be referenced to). The elementary inertia frames could be incorporated into more general inertial frames (e.g., center-of-mass inertial frames) and the whole observable universe could organized into the more general inertial frames. Going beyond what 19th century astronomers probably could have thought of, there is whole hierarchy of general inertial frames that tops out with the comoving frames of the expanding universe.

   What is the ____________, Alex?

   a) star frames b) planet frames c) free-fall frames d) thermodynamics frames
   e) gravity frames

3. "Let's play *Jeopardy*! For \$100, the answer is: This theorem (originally proven by Newton by primitive means) allows one to show by means of a **COROLLARY** that spherically symmetric masses should interact gravitationally as though they are point masses as long as they are do not interpenetrate.

   What is the ____________, Alex?

   a) Newton theorem b) shell theorem c) point-mass theorem d) sphere theorem
   e) waste book theorem

4. "Let's play *Jeopardy*! For \$100, the answer is: The theorem that states that the only attractive central forces that give closed orbits for all bound orbits are the inverse-square law force and the attractive linear force (AKA Hooke's law force or the radial harmonic oscillator force). All attractive central forces give closed **CIRCULAR** orbits, of course."

   What is ____________, Alex?

   a) the virial theorem b) Euler's theogonic proof c) the brachistochrone problem
   d) Schubert's unfinished symphony e) Bertrand's theorem

5. The solutions of the Friedmann equation have characteristic cosmological quantities some of which are called Hubble quantities since the Hubble constant is one of their ingredients. The table below displays some the cosmological quantities. Since the currently determined values of the quantities always fluctuate a bit depending on whose analysis is used, we have written the quantities as fiducial values with correction factors that are 1 to within a few percent: h_{70} is the Hubble constant divided by $70\,(\text{km/s})/\text{Mpc}$ (i.e., $H_0/(70\,(\text{km/s})/\text{Mpc})$), $\omega_{\text{m},0} = \Omega_{\text{m},0}/0.3$, and $\omega_\Lambda = \Omega_\Lambda/0.7$. The asymptotic Hubble quantities are those that will be the Hubble quantities as cosmic time goes to infinity if the Λ-CDM model is correct.

Table: Cosmological Quantities

Cosmic scale factor for the present cosmic time $a_0 = 1$ by convention
Hubble constant $H_0 = 70h_{70}$ (km/s)/Mpc
Hubble time $t_H = 1/H_0 = (13.968\ldots)/h_{70}$ Gyr
Hubble length $\ell_H = c/H_0 = (13.968\ldots)/h_{70}$ Gly $= (4.2827\ldots)/h_{70}$ Gpc
Critical density $\rho_{\text{critical}} = [3H_0^2/(8\pi G)] = (9.2039 \times 10^{-27})h_{70}^2$ kg/m^3 $= (1.3599 \times 10^{11})h_{70}^2$ M$_\odot$/Mpc^3
AKA Hubble density (i.e., the density implied by the Hubble constant at cosmic present)
Cosmological constant matter density parameter $\Omega_{\text{m},0} = 0.3\omega_{\text{m},0}$
Cosmological constant Λ density parameter $\Omega_\Lambda = 0.7\omega_\Lambda$
Asymptotic Λ Hubble parameter $H_\Lambda = H_0\sqrt{\Omega_\Lambda} = \sqrt{\Lambda/3} = (58.566\ldots)h_{70}\sqrt{\omega_\Lambda}$ (km/s)/Mpc
Asymptotic Λ Hubble time $t_{H_\Lambda} = (16.6955\ldots)/(h_{70}\sqrt{\omega_\Lambda})$ Gyr

Given that the Λ-CDM model is correct, to 1st order, the observable universe is already expanding like a cosmological-constant universe with $a = a_0 \exp(\Delta t/t_{H_\Lambda})$ (where $\Delta t = t - t_0$) and this formula becomes more correct as time advances. On what time scale Δt will the matter mass-energy density of the observable universe fall to of order 2 % of the total mass-energy? Note, you have to solve for a/a_0 from

$$\Omega_{\text{m}} = \Omega_{\text{m},0}\left(\frac{a_0}{a}\right)^3 \approx 0.02\Omega_\Lambda$$

and then solve for Δt.

a) t_{H_Λ}. b) $2t_{H_\Lambda}$ c) $3t_{H_\Lambda}$. d) $4t_{H_\Lambda}$. e) $5t_{H_\Lambda}$.

6. That a function $x(\tau)$ satisfies the (scaled) Friedmann equation is necessary and sufficient for it being a cosmic scale factor solution since, among other things, it implies a constant curvature k. That the function satisfies the (Friedmann) acceleration equation is:

   a) also necessary and sufficient. b) only sufficient. c) only necessary.
   d) neither necessary nor sufficient. e) irrelevant to it being a cosmic scale factor.

7. In this problem, we consider the general Gausss law from which are derived the inverse-square-law Gauss' law (i.e., Gauss' law without qualification) and the linear-force Gauss' law.

   NOTE: There are parts a,b,c,d. On exams, omit parts d,e and use minimal words.

   a) Determine the most general function $f(r)$ satisfying

   $$\nabla \cdot \vec{f} = \frac{1}{r^2}\frac{\partial[r^2 f(r)]}{\partial r} = 3C \ ,$$

   where $\vec{f} = f(r)\hat{r}$, C is a constant, and the divergence is for a spherically symmetric vector field in spherical coordinates (Ar-104). What problem does this general function have for $r = 0$?

   b) To deal with the problem found in part (a), consider a general central force obeying a power law

   $$\vec{f} = f(r)\hat{r} = r^p\hat{r} \ ,$$

   where p is general: it can be positive, negative, and zero, and integer and noninteger. Imagine a small spherical Gaussian surface or radius R centered on the origin. For this surface determine in all cases

   $$\lim_{R\to 0}\oint \vec{f}\cdot d\vec{A} \ ,$$

   where the integral is over the whole surface and $d\vec{A}$ is a differential surface area vector. Given Gauss' theorem (AKA the divergence theorem)

   $$\oint \vec{g}\cdot d\vec{A} = \int \nabla\cdot\vec{g}\,dV$$

   (where $\vec{g}$ is a general field) and part (a), what must be

   $$\nabla\cdot(r^{-2}\hat{r}) \ ?$$

c) Given parts (a) and (b), what is the most general form of Gauss' law: i.e., the most general form of the equation

$$\oint \vec{f} \cdot d\vec{A} = \text{Constant ?}$$

Note, the constant is not completely independent of position. **HINT:** Consider two kinds of charge Q_1 and $Q_{(-2)}$. The point charges for these two kinds of charge each give rise to central forces.

d) So how many kinds of Gauss' law are there? Give their formulae.

e) Determine the differential equation form of the two kinds of Gauss' law. **HINT:** Make use of Gauss' theorem.

8. In this problem we prove the shell theorem for gravity and 4 corollaries.
NOTE: There are parts a,b,c,d,e. On exams, omit part e and use minimal words.

a) The shell theorem applies to a spherically symmetric mass distributions and is given by

$$\vec{g}(r) = -\frac{GM(r)}{r^2}\hat{r} ,$$

where $\vec{g}(r)$ is the gravitational field, r is the radius from the origin, and $M(r)$ is the mass enclosed by a sphere of radius r. Prove the theorem given Gauss' law for gravity

$$\oint \vec{g} \cdot d\vec{A} = -4\pi G M_{\text{enclosed}} ,$$

where the integral is over a Gaussian surface with differential surface vector $d\vec{A}$ and M_{enclosed} is the mass enclosed by the Gaussian surface.

b) First corollary: Determine the formula for the gravitational field outside of a spherically symmetric mass distribution with radius R and total mass M. What formula is this formula identical too? This formula was very hard for Isaac Newton (1643–1727) to prove with his primitive mathematical techniques, but the formula was vital for his pioneering work in celestial mechanics.

c) Second corollary: Determine the formula for the graviational field in a spherically symmetric cavity of radius R inside a spherically symmetric mass distribution.

d) Third corollary: First show that Newton's 3rd law holds explicily for two general gravitating mass systems of point masses. Then show that two non-overlapping spherically symmetric mass distributions gravitationally interact like point massess: i.e., show $\vec{F}_{1,2} = \vec{F}_{1_{\text{point}},2_{\text{point}}}$, where the numbers label the two systems and the numbers subscripted label their point mass replacements. This result is the third corollary itself and it may seem intuitively obvious, it still needs a proof.

e) The fourth corollary is actually more of interesting feature than a corollary. The feature is the local gravitational field contribution $\vec{g}_{\text{local}}$ to the total gravitational field of an infinitely thin (mass) shell (radius R and mass M) by an infinitesimal piece of the shell infinitesimally close to the shell. First, find the gravitational field of the shell for all radius r. Note, the gravitational field at $r = R$ is actually indeterminate without specifying a limiting process to obtain it. Second, consider an infinitesimal Gaussian pillbox (a cylindrical Gaussian surface) that straddles the shell with its symmetry axis aligned with radial direction. To the Gaussian pillbox, the shell surface is an infinite plane. Determine the gravitational field $\vec{g}_{\text{local}}$ on the top and bottom of the pillbox due to the mass contained in the pillbox using Gauss' law and then the gravitational field $\vec{g}_{\text{remote}}$ on the top and bottom of the pillbox due to the remote parts of the shell. Is $\vec{g}_{\text{remote}}$ continuous at $r = R$? How do the local and remote gravitational contributions just inside the shell compare in this ideal limit?

9. In this problem, we will derive the general Gauss' law (for the inverse-square law) in its integral form and then specialize to the gravity and Coulomb force cases.
NOTE: There are parts a,b,c,d. Some of the parts can be done independently, and so do not stop if you cannot do a part.

a) Consider the general inverse-square law central force

$$\vec{f} = \frac{q}{r^2}\hat{r} ,$$

where q is a general charge for the force located at the origin (which is the center of force), r is the distance to a point where the force is evaluated, and $\hat{r}$ is the direction to that point. Now consider a differential surface area vector $d\vec{A}$ for a Gaussian surface (i.e., a **CLOSED** surface) Note, $d\vec{A}$ is defined as pointing outward from the Gaussian surface. Let $d\vec{A}$ and $\hat{r}$ define a r-y plane that is picture as rotated about a z axis. We take the surface area defined by $d\vec{A}$ as a square of area $\Delta z \Delta y$ (where both Δz and Δy are defined as positive) since $d\vec{A}$ is differentially small. The area components of $d\vec{A}$ in the $\hat{r}$ and $\hat{y}$ directions are, respectively,

$$\hat{r} \cdot d\vec{A} = A_r = \Delta z \Delta y \cos(\theta) \qquad \text{and} \qquad \hat{y} \cdot d\vec{A} = A_y = \Delta z \Delta y \sin(\theta) \ ,$$

where θ is the angle of $d\vec{A}$ measured from the $\hat{r}$ direction. Note, A_r is positive for $\theta \in [-\pi/2, \pi/2]$ and negative for $\theta \in [\pi/2, 3\pi/2]$, and the area subtended from the origin is always $|A_r|$. Thus, we find

$$\hat{r} \cdot d\vec{A} = A_r = r^2(\pm\, d\Omega) \ ,$$

where $d\Omega$ is the differential solid angle subtended by $|A_r|$ and the upper/lower case is for positive/negative A_r (i.e., for the solid angle cone from the origin going outward/inward through the differential surface area $d\vec{A}$).

Prove

$$\vec{f} \cdot d\vec{A} = q(\pm\, d\Omega) \ ,$$

where the upper/lower cases are for the solid angle cone going outward/inward through the differential surface area. Note, the charge could be inside or outside the closed surface. **HINT:** This is easy.

b) Consider a differentally small cone extending from the origin. It intersects the Gaussian surface n times. Note, Gaussian surface is finite, and so the cone must exit Gaussian surface forever at some point. We form the sum

$$\sum_{i=1}^{n} \vec{f} \cdot d\vec{A}_i \ ,$$

where sum is over all intersections. What is the sum equal to in terms of solid angle for all cases? **HINT:** A few words of explanation and a diagram are needed.

c) Say you had multiple charges q_i with total charge Q and total charge Q_{enclosed} inside a closed surface. Evaluate

$$\oint \vec{f} \cdot d\vec{A} \ .$$

The result is the general Gauss' law in its integral form. Specialize the result for the cases of gravity and the Coulomb force.

d) What is the necessary condition for a force to obey Gauss' law?

10. Remarkably the linear force obeys analogues to Gauss's law and shell theorem for the inverse-square law force. Let the linear-force field (force per unit charge) for a point charge be

$$\vec{f} = kqr\hat{r} \ ,$$

where k is a constant which could be positive or negative, q is the charge (of some unspecified kind), and r is the distance from the point charge. We assume Newtonian physics, and so to maintain Newton's 3rd law, we require

$$\vec{F}_{1,2} = kq_1q_2r_{1,2}\hat{r}_{1,2} \ ,$$

where $\vec{F}_{1,2}$ is the force of point charge 1 on point charge 2.

There are parts a,b,c,d,f. Some of the parts can be done independently, and so do not stop if you cannot do a part. Omit part (f) during exams.

a) Without words, for a close surface derive the linear-force Gauss' law

$$\oint \vec{f} \cdot d\vec{A} = kQ \ ,$$

where $\vec{f}$ is the field due to the entire charge distribution, the integral is over the whole close surface, and Q is the total charge of the charge distribution wherever it is in space. **HINT:** Recall the divergence theorem (AKA Gauss' theorem)

$$\oint \vec{Y} \cdot d\vec{A} = \int \nabla \cdot \vec{Y}\, dV \ ,$$

where Y is a general vector field and the volume integral is over all volume V inclosed by the closed surface (Wikipedia: Divergence theorem). Recall also the divergence operator for spherically symmetric system in spherical coordinates obeys

$$\nabla \cdot \vec{Z} = \frac{1}{r^2}\frac{\partial (r^2 Z_r)}{\partial r} \ ,$$

where Z is spherically symmetric, but otherwise general, and Z_r is the radial component of $\vec{Z}$ (Arfken-104).

b) For what symmetries can the linear-force field be easily solved for directly from the linear-force Gauss' law?

c) Without words, solve for the linear-force field for a spherically symmetric charge distribution. What simple charge distribution would give an equivalent linear-force field for all radius r? What can this result be called? How is this equivalent linear-force field different from the analogue result with the inverse-square-law force?

d) Without words, show for a general charge distribution 1 and a spherical symmetric charge distribution 2 that the force of distribution 1 on distribution 2 is exactly the same as when distribution 2 is replaced point-charge 2. If charge distribution 1 were also spherically symmetric, what be the force between them be equal to and what would it be if their centers coincided exaclty?

e) Say you had a charge distribution that maintained spherically symmetry no matter what, that had its center of mass at its center, and the only external forces that acted on it were external linear forces. How would described its motion? Recall Newton's 2nd law:

$$\vec{F}_{\text{net external}} = m\vec{a}_{\text{cm}} \ ,$$

where $\vec{F}_{\text{net external}}$ is the net external force on a body of mass m and a_{cm} is the center of mass of the body. Given the result of part (d) Without words, show for two spherically symmetric distribution charges that the force of distribution 1 on distribution 2 is exactly the same **HINT:** Recall the part (d) answer.

f) Is the linear force for spherically symmetric mass distribution with mass as its charge consistent with linear force that occurs in the Newtonian derivation of the Friedmann equation:

$$\vec{F} = \frac{\Lambda}{3} m r \hat{r} \ ,$$

where m is a test particle mass. There is no right answer. This is a discussion question.

11. The (Newtonian) cosmological constant force **PER UNIT MASS** is given by

$$\vec{f} = \frac{\Lambda}{3}\vec{r} \ ,$$

where Λ is the cosmological constant, the $1/3$ factor is for consistency with cosmological constant as it appears in the Einstein field equations, and $\vec{r}$ is the displacement vector from any point in space. In an extra Newtonian hypothesis, one can hypothesize that Λ is set somehow by a universal force charge density that is constant in space and time and the Newtonian-3rd-law equal-and-opposite force caused by the cosmological constant force on a particular mass is exterted by the particular mass on on this charge throughout the universe. But this may be a useless hypothesis.

Consider a system of point masses m_i at displacements $\vec{r}_i$ relative to an external origin. The total mass of the system is $m = \sum_i m_i$. The center of mass of the system is $\vec{r}_{\text{cm}}$ and the relative displacements are $\Delta\vec{r}_i = \vec{r}_i - \vec{r}_{\text{cm}}$.

NOTE: There are parts a,b,c.

a) Write down the cosmological constant force $\vec{F}_i$ on point mass m_i relative to the **ORIGIN** both in terms of $\vec{r}_i$ and $\Delta\vec{r}_i$. **HINT:** This is easy.

b) Determine the net cosmological constant force $\vec{F} = \sum_i \vec{F}_i$ on the system and simplify as much as possible. **HINT:** Recall the definition of center of mass.

c) What simplifying conclusion can you draw from the part (b) answer?

12. The Friedmann equation of general relativity (GR) cosmology in its most standard form (e.g., Wikipedia: Friedmann equations: Equations) is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} ,$$

where H is the Hubble parameter (which at current cosmic time is the Hubble constant H_0 and has fiducial value 70 (km/s)/Mpc), a is the cosmic scale factor, $\dot{a}$ is the time derivative of the cosmic scale factor with respect to cosmic time t, $G = 6.67430(15) \times 10^{-11}\,\mathrm{J\,m/kg^2}$ is the gravitational constant, ρ is the density of a uniform perfect fluid (in old-fashioned jargon AKA the cosmological substratum: Bo-75–76) which is used to model the universal mass distribution, k is called the curvature (Li-24,28) $k/(c^2a^2)$ is called Gaussian curvature (CL-12,29), $c = 2.99792458 \times 10^8\,\mathrm{m/s}$ is the vacuum light speed as usual. and Λ is the cosmological constant which is the simplest form of the dark energy even though is only a from of energy in one interpretation. Note, k is often defined with an unabsorbed c^2: i.e., the shown k is replaced by kc^2.

There are parts a,b,c. Some of the parts can be done independently, and so do not stop if you cannot do a part. During exams do **ONLY** parts a,b,c,d.

a) Without words prove the Friedmann equation starting from the work-energy theorem

$$E_{\text{mechanical}} = \frac{1}{2}mv^2 - \frac{GMm}{r} - \frac{1}{2}\frac{\Lambda}{3}mr^2 ,$$

where m is the mass of a test particle.

b) Without words prove the general Hubble law $v = Hr$, where v is recession velocity (i.e., the velocity between comoving frames) and r is proper distance (i.e., the distance measurable in with a ruler at one instant in cosmic time).

c) What is the asymptotic Hubble law (i.e., Hubble law valid in the limit $z \to 0$)?

13. The scaled Friedmann equation for multi-component (power-law) density components is

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_p \Omega_{p,0}x^{-p} ,$$

where 0 indicates the fiducial time which may be cosmic present, $h = H/H_0$ is the scaled Hubble parameter with H_0 being the Hubble constant, $x = a/a_0$ is the scaled cosmic scale factor, $x_0 = 1$, $\dot{x} = dx/d\tau$ is the rate of change of the scaled cosmic scale factor, $\tau = H_0 t = t/t_{H_0}$ is the scaled time with t_{H_0} being the Hubble time, the $\Omega_{p,0}$ are the density parameters for the density components at the fiducial time with their sum being 1, and p are the powers of the power-law density components.
NOTE: There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c,d.

a) Without words, derive the general asymptotic solution $\tau(x)$ and its inverse $x(\tau)$ for the leading density component as $\tau \to 0$ (i.e., the density component with highest p). As a shorthand, this solution can be called the early universe solution. Assume $p > 0$. To avoid pointless generality, assume $x(\tau = 0) = 0$ (i.e., there is a point origin in time at time zero).

b) Without words, derive early universe formula for $\Omega_p(\tau) = \Omega_p[x(\tau)]$ for $p > 0$.

c) Without words, derive the special case early universe solutions for $p = 1, 2, 3, 4, 5$.

d) Without words, derive the Hubble parameter $h = \dot{x}/x$ and the deceleration parameter $q = -\ddot{x}x/(\dot{x})^2 = -\ddot{x}/(xh^2)$ for the general early universe with $p > 0$. Simplify the latter as much as possible. For what p values is the universe in positive/zero/negative acceleration?

e) We now assume the universe has only one density component with power $p > 0$. Without words, derive the general age of the universe formula (which we assume to the fiducial time where $x = 1$) for τ and t and give the fiducial value version for t with the Hubble time $t_{H_0} = (13.968\ldots\,\text{Gyr})/h_{70}$, where $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$.

f) We assume the universe has only one density component with power $p = 0$. Without words, derive $x(\tau)$ and $x(t)$ assuming $x(0) = 1$. Note, this universe is the de Sitter universe and the Hubble constant $H_0 = \sqrt{\Lambda/3}$.

g) Students are now welcome to view a table in the answer to this part that presents the single density component solutions plus relevant features for powers $p = 5, 4, 3, 2, 1, 0$. Note, if we assume that the dependence of the density components on the scale factor is due to a perfect fluid pressure obeying the equation of state $p_{\text{pressure}} = w\rho c^2$ where w is a constant parameter (with no special name), then power

$$p = 3(1 + w) \ .$$

The w values are included in the table.

14. The differential equation (DE) for the perfect fluid of Friedmann equation cosmology is

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ ,$$

where ρ is mass-energy in the comoving frames of Friedmann equation cosmology and p is isotropic pressure in those frames (in some sense) (Liddle 26). The perfect fluid DE can be derived rigorously from general relativity (Carroll 333–334) and, perhaps somewhat fudgily, from classical thermodynamics and special relativity. Remarkably, this equation does not guarantee conservation of energy in the ordinary sense of classical physics: it does embody the general relativity feature that the covariant derivative of the energy-momentum tensor is zero (Carroll 117,120): i.e., the energy-momentum conservation equation. General relativity may or may not in some sense conserve energy for cosmology, but certainly gravitating mass-energy is allowed to appear and disappear by the perfect fluid DE.

Multiple perfect fluids can exist and if they are assumed to act independently (which is the usual cosmological assumption), then they all obey there own perfect fluid DE: i.e., for perfect fluid i,

$$\dot{\rho}_i = -3\frac{\dot{a}}{a}\left(\rho_i + \frac{p_i}{c^2}\right) \ .$$

In current standard cosmology (i.e., the ΛCDM model or simple variations thereof), it is assumed that the perfect fluid equation of state (EOS) is of the form

$$p = w\rho c^2 \ ,$$

where w is a constant parameter that seems to have no special name. Most standard/interesing values of w are given by

$$w = \begin{cases} 2/3 & \text{for nonrelativistic kinetic energy of free-streaming nonrelativistic particles} \\ & \quad\text{(e.g., late-time cosmic neutrinos).} \\ 1/3 & \text{for extreme relativistic (ER) mass-energy (AKA ``radiation''): most obviously photons,} \\ & \quad\text{but also the ER neutrinos of the Big Bang era} \\ & \quad\text{and to some later not perfectly known cosmic time;} \\ 0 & \text{for nonrelativistic (NR) rest mass-energy (AKA ``matter''} \\ & \quad\text{or ``dust'': Liddle-40);} \\ -1/3 & \text{for zero-acceleration (or constant } \dot{a}\text{) universes such as} \\ & \quad\text{Fulvio Melia's } R_{\text{h}} = ct \text{ universe, cosmic strings or a universe with cosmic scale} \\ & \quad\text{determined only by negative curvature } k. \\ -2/3 & \text{for some kinds of quintessence.} \\ -1 & \text{for cosmological constant or equivalently (constant) dark energy;} \end{cases}$$

Solve for the formula for $\rho(a)$ for general w and the 6 special cases of w listed above. Assume a_0 and ρ_0 for cosmic present values.

15. Here we do the quick derivations of the Friedmann equation, the fluid equation, the Friedmann acceleration equation, and some other results.

NOTE: There are parts a,b,c,d,e. On exams, do **ONLY** parts a,b,c,d. Some of the parts can be done independently, and so do not stop if you cannot do a part.

a) Without words, derive the Friedmann equation in standard form (with the cosmological constant force $F_\Lambda = (\Lambda/3)mr$ included) from classical physics with the hypotheses that all free-fall frames are elementary inertial frames (as told to us by general relativity) and that the shell theorem for a spherically symmetric mass distribution can be extended to infinite distance (which is validated by Birkhoff's theorem from general relativity). The derivation makes use of the classical conservation of mechanical energy. You should end up with a $-k/a^2$ term among other things. You can a draw diagram if you like. **HINT:** Start with the conservation of mechanical energy of a test particle of mass m:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} - \left(\frac{1}{2}\right)\frac{\Lambda}{3}mr^2 \ .$$

b) Without words and starting from the 1st law of thermodynamics

$$dE = T\,dS - p\,dV + \mu\,dN \ ,$$

derive the cosmological fluid equation in standard form (which means with $dS = 0$ and $dN = 0$) and in a form with $\dot{\rho}a/\dot{a}$ equal to something for use in part (d). Recall the rest-frame energy is $E = \rho c^2 V$.

c) Specialize the fluid equation to the special case where the equation of state is $p = w\rho c^2$ where w is the equation-of-state constant (which seems to have no special name). Determine the explicit solution $\rho(a)$ for the special case where $\rho_0 = \rho(a_0)$. **HINT:** You will have to eliminate the time derivative.

d) Without words, derive the acceleration equation (or Friedmann acceleration equation) in standard form using parts (a) and (b). A subtle point is that you have to assume that the gravitational potential energy formula continues to be valid (though perhaps with a different meaning) for cases where mass is not conserved. There is an argument why it should, but that is beyond the scope of this question.

e) Without words, derive from the Friedmann equation the de Sitter universe solution which has $\rho = 0$ and $k = 0$, but $\Lambda \neq 0$.

16. The Friedmann equation in the most standard form is

$$H^2 = \left(\frac{dx/dt}{x}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{x^2} + \frac{\Lambda}{3} \ ,$$

where we use x rather than a as the cosmic scale factor since yours truly finds a intolerable as a variable.

NOTE: There are parts a,b,c,d,f,g. On exams, omit parts d,e,f,g.

a) Write the right-hand side of the Friedmann equation as H_0 times a bracketed factor. The quantity H_0 is the Hubble constant for a fiducial time t_0 which for expanding solutions with a point origin is the time since the point origin and is usually taken as cosmic present, but it could be any time. What is the bracketed factor equal to at fiducial time t_0?

b) Define scaled time by $d\tau = H_0\,dt$ and let $\dot{x} = dx/d\tau$. Now derive the scaled Friedmann equation

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \Omega_{\rm ME} + \Omega_k + \Omega_\Lambda \ ,$$

where $\Omega_{\rm ME}$ is the mass-energy density parameter, Ω_k is the curvature density parameter, Ω_Λ is the cosmological constant or Λ density parameter,. and the scalings are $x = a$, $d\tau = H_0\,dt$, $\dot{x} = dx/d\tau$, $h = H/H_0$, and $\rho_{\rm c} = 3H_0^2/(8\pi G)$. Note, the subscript 0 indicates fiducial time t_0 where $H = H_0$ and which is often chosen to be cosmic present. Note, the density parameters defined to sum to 1 at the fiducial time t_0. Give the explicit expressions for the critical density $\rho_{\rm c}$ defined by $\Omega_{\rm ME} = \rho/rho_{\rm c}$, $\rho_k = \rho_{\rm c}\Omega_k$, and $\rho_\Lambda = \rho_{\rm c}\Omega_\Lambda$. What is the curvature formula at the fiducial time t_0 or τ_0: i.e., the formula for $\Omega_{k,0}$? What does the value $\Omega_{k,0}$ imply as we know from general relativity?

c) We assume the pressure of a cosmological density component follows from a simple equation of state $P = w\rho c^2$ where w is a constant which seems to have no special name. Given the density evolution formula

$$\dot{\rho} = -3\frac{\dot{x}}{x}\left(\rho + \frac{P}{c^2}\right) ,$$

determine the w values for the commonly consider powers $p = 0, 1, 2, 3, 4, 5$ for (inverse) power-law density parameter evolution

$$\rho = \rho_0 x^{-p} ,$$

where set $x = 1$ at fiducial time t_0 (which is a usual, but not necessary assumption). Note, $p = 0$ for the cosmological constant (or constant dark energy) $p = 1$ for some kinds of quintessence $p = 2$ for curvature, maybe cosmic strings, and the $R_c = ht$ universe, $p = 3$ for matter (i.e., non-relativistic rest mass), $p = 4$ for radiation (i.e., photons or extreme relativistic particles), and $p = 5$ for nonrelativistic kinetic energy of free-streaming paricles (e.g., nonrelativistic cosmic neutrinos).

d) Write the scaled Friedmann equation for a general set of density components obeying (inverse) power laws including those for the cosmological constant and curvature. Assume $x = 1$ as the fiducial time t_0.

e) Given part (d), derive the formula for $\ddot{x}$.

f) Given parts (d) and (e), derive the formula for the deceleration parameter whose general formula is

$$q = -\frac{\ddot{x}x}{\dot{x}^2}$$

(Li-53), where the annoying minus sign is because people expected a decelerating universe once and wanted a positive diagnostic. The deceleration parameter is a sort of scaled accleration. Which terms tend to acceleration, coasting, deceleration?

g) Given the single density component elementary solution for power $p > 0$

$$\tau = \frac{1}{\sqrt{\Omega_{p,0}}}\left(\frac{2}{p}\right) x^{p/2} ,$$

determine the fiducial time t_0 in terms of p and H_0, in terms of p and Hubble time t_{H_0}, in terms of q and H_0, and q and Hubble time t_{H_0}. What is the condition for $t_0 = t_{H_0}$?

17. Consider the following linear 1st order autonomous differential equation (DE):

$$x' = A - kx ,$$

where t is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant. Note, the DE is autonomous because there is no explicit dependence on the independent variable t.

There are parts a,b,c,d. On exams, omit parts b,d.

a) Solve for the constant solution x_A. **HINT:** This is easy.

b) We can now write the DE as

$$x' = k(x_A - x) .$$

Without solving for non-constant solution describe what it must look like as a function of t for arbitrary initial value $x_0 = x(t = 0) \neq x_A$. Note, x_0 can be greater or less then x_A. In particular, where are the soltutions stationary points if there are any? **HINT:** Consider the continuity of all orders of derivative of x.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$ from the differential equation form $x' = k(x_A - x)$ and give its 1st order in small t solution $x_{\rm 1st}(t)$ and its asymptotic solution as $t \to \infty$. Verify that the $x(t)$ is monotonic with its only stationary point at $t = \infty$. **HINT:** You can use an integrating factor, but there is a more straightforward way.

d) What is the e-folding time t_e of your solution and what does it signify? What is remarkable about $x_{\rm 1st}(t_e)$?

18. First order autonomous ordinary differential equations (FAODEs), linear or nonlinear, only have solutions with stationary points at infinity (SPIs), (except for special cases which are not all that

rare) and constant solutions. Actually, each SPI corresponds to a constant solution which could also be viewed as a continuum of stationary points. Note, an autononous differential equation depends only on functions of the dependent variable, and so has no explicit dependence on the independent variable.

To investigate the SPI behavior of FAODEs consider the (somewhat general) FAODE

$$x^{(1)} = [f(x)]^{1/k} \ ,$$

where t (not necessarily time) is the independent variable, the superscript (1) means 1st derivative with respect to t, $f(x)$ is an infinitely differentiable function with zeros at set of values $\{x_i\}$, and $k > 0$. We limit k to being greater than zero to avoid uninteresting generality. Since $f(x)$ is infinitely differentiable at (general) x_i, we can expand $f(x)$ about x_i with some radius of convergence: i.e.,

$$f(\Delta x) = \sum_{j=\ell}^{\infty} \Delta x^j f_j = \Delta x^\ell f_\ell + \ldots \ ,$$

where $\Delta x = x - x_i$, the f_j are expansion constants, and $\ell > 0$ is the lowest (nonzero) order in the expansion. Note, $\ell \neq 0$ since we have assumed x_i is a zero of $f(x)$: i.e., $f(x_i) = 0$.

We will primarily be examining the lowest order solutions in Δx, and so we will be dealing with $\Delta x^{\ell/k} f_\ell^{1/k}$ and related expressions. Mathematically, if ℓ/k is not an integer, complex numbers can arise in these expressions. However, we are only interested FAODEs and their solutions corresponding to physical systems involving real numbers. In these systems, the solutions just never evolve into the complex number realm. So we are not going to concern ourselves with question what happens mathematically if some our expressions can give rise to complex numbers. They never give rise to complex numbers physically.

NOTE: There are parts a,b,c,d,e,f,g,h,i,j,k. On exams, do **ONLY** parts i,j.

a) What is the behavior of x as a function of t between the points in the set $\{x_i\}$.

b) In this question we are only interested in the SPI behavior and constant solution behavoir, and so we are only interested in the behavior of $x(t)$ when it is arbitrarily close to x_i where SPI and constant solutions occur. Therefore expand the FAODE about x_i with dependent variable Δx to lowest order in the exponent.

c) Determine the formula $p(n)$ for the exponent of Δx in the n derivative of Δx (for the lowest order of the FAODE) with respect to t. **HINT:** Drop all constants that turn up in the differentiations.

d) What is behavior of the t derivatives of Δx when $x = x_i$ for $\ell/k \geq 1$? What solutions $x(t)$ are implied by $\ell/k \geq 1$?

e) What is behavior of the t derivatives of Δx for $f(x_i)$ for $\ell/k < 1$ assuming the formula $p(n)$ never equals zero? What solution $x(t)$ behavior is implied by $\ell/k < 1$ in this case? Only a short answer is expected to the last question.

f) If $\ell/k < 1$ and the formula $p(n)$ goes to zero for a stopping $n_{\rm st}$, what is the formula for ℓ/k as a function of $n_{\rm st}$ and what are the values of ℓ/k for the set $n_{\rm st} = 1, 2, 3, \ldots, \infty$ and what do the $n_{\rm st} = 1$ and $n_{\rm st} = \infty$ cases mean? What is the formula $n_{\rm st}$ as a function of ℓ/k? What is this formula good for?

g) What is implied by a stopping $n_{\rm st} \in [2, \infty)$ (i.e., an actual integer $n_{\rm st}$ in this range)? Give the solution for small $\Delta x(t)$ with with initial condition $\Delta x(t = 0) = 0$. Describe the function behavior at $\Delta x(t = 0) = 0$: i.e., maximum or minimum stationary point or rising or falling inflection point.

h) What would you expect the two likeliest values for ℓ to be for physically relevant FAODEs? What would you expect the two likeliest value for $k \neq 1$ to be for physically relevant FAODEs?

i) Now we intuited for the case of $\ell/k \geq 1$ that the stationary point would be a stationary point at infinity (i.e., an SPI), but we did not prove this directly. To prove directly, we need to show that the small Δx (meaning small in absolute value) solutions of

$$\Delta x^{(1)} = \Delta x^{\ell/k} f_\ell^{1/k}$$

that go to zero only do so as $t \to \infty$. Solutions that go to zero are convergent solutions. This means that the constant solutions they correspond to are stable solutions: small perturbations from the constant solutions damp out. Those that do not go to zero are divergent solutions. This means that the constant solutions they correspond to are unstable solutions: small perturbations from the constant solutions cause non-stopping divergence from the constant solutions.

Here consider the $\ell/k = 1$ case and the solutions for $\Delta x(t)$ starting from $t = t_0$ and $\Delta x = \Delta x_0$ as initial conditions. Determine the solutions and under what conditions they are convergent/divergent. Does the convergent solution, in fact, have a SPI? **HINT:** Let $y = \pm\Delta x$ where the upper/lower case is for positive/negative Δx_0.

j) Repeat part (i) for the case of $\ell/k > 1$.

k) An optional continuation of the discussion of the part (h) answer.

19. In this problem, we will get some more insight into first order autonomous ordinary differential equations (FAODEs) with stationary points that are not stationary points at infinity (SPIs) by examining a solution beyond solution to lowest (nonzero) order around the stationary points. Consider the FAODE

$$x^{(1)} = f(x) ,$$

where $f(x_i) = 0$ (i.e., $x = x_i$ gives a stationary point of some kind) and the independent variable is t (not necessarily time). However,

$$x^{(2)} = \frac{df}{dx}x^{(1)} = \frac{df}{dx}f(x) \neq 0$$

for $x = x_i$. This means the stationary point is not a SPI.

NOTE: There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c.

a) Let

$$g(x) = \frac{df}{dx}f(x)$$

and determine a formal solution for $f(x)$.

b) Assume $x(t)$ has maximum and minimum at, respectively, x_i and $-x_i$. Now invent the simplest $f(x)$ you can starting from the part (a) answer, except it has a general constant coefficient so as to give a general scale to the derivative $x^{(1)}$.

c) Now solve for $x(t)$ given the part (b) answer. **HINT:** You could do this by integrating $x(t)$, but differentiating $x(t)$ lead to solution by inspection.

d) Say a FAODE is given by

$$x^{(1)} = [f(x)]^{1/k} ,$$

where t is the independent variable (not necessarily time), $k > 0$, $f(x)$ is infinitely differentiable, and $f(x) = \Delta x^\ell f_\ell + \ldots$ is the expansion of $f(x)$ around the stationary point x_i with $\Delta x = x - x_i$ starting with the lowest nonzero order. Then the lowest order FAODE is

$$\Delta x^{(1)} = x^{\ell/k} f_\ell^{1/k} ,$$

In order for a solution of the FAODE to have stationary point that is not a SPI, there must be a stopping (derivative order) $n_{\rm st}$ given the formula

$$n_{\rm st} = \frac{1}{1 - \ell/k}$$

where an actual stopping $n_{\rm st}$ must be an integer. If the formula gives a non-integer value, then there is a singularity in the behavior of some order of derivative of $x(t)$ at $x = x_i$ and that behavior takes some analysis to determine. An actual stopping $n_{\rm st}$ gives the only nonzero derivative order of $x(t)$ at $x = x_i$. What are the ℓ and k values for the FAODE used in the part (c) and are they consistent with a nonzero derivative order $n = 2$ which is what we imposed in the preamble?