Cosmology NAME:

Homework 3: The Friedmann Equation

1. "Let's play *Jeopardy*! For \$100, the answer is: It was derived from general relativity in 1922 with the assumptions of a homogeneous and isotropic universe and that all mass-energy in the universe could be modeled by a perfect fluid. A Newtonian derivation (which required extra natural hypotheses) was given in 1934.

   What is the ____________, Alex?

   a) Einstein equation b) Milne-McCrea equation c) Synge equation d) Bondi equation
   e) Friedmann equation

2. A Newtonian derivation of the Friedmann equation (with extra natural hypotheses) could easily have been done in the 19th century, but it wasn't. There were probably 3 reasons why 19th century astronomers did not think of such a derivation. First, many were still thinking of a universe that was static on average even though dynamic equilibrium seemed hard to arrange, even though the universe was obviously not in thermodynamic equilibrium and so why should be in dynamic equalibrium, and even though idea existed that the Milky was was held up by rotation around the center of mass located somewhere. Second, they did not know that other galaxies existed though some believed this and they had not observed the general redshifts of the objects they thought might be other galaxies. Third, they thought in terms of Newton's absolute space (i.e., a single fundamental inertial frame) and did not think of the alternative idea completely compatible with their data that all ____________ unrotating with respect to the observable universe were inertial frames (i.e., frames with respect to which Newtonian physics could be referenced to). Such frames could be sub frames of other frames of the same sort. There is whole hierarchy of them top out by the comoving frames of the expanding universe.

   What is the ____________, Alex?

   a) star frames b) planet frames c) free-fall frames d) thermodynamics frames
   e) gravity frames

3. "Let's play *Jeopardy*! For \$100, the answer is: This theorem (originally proven by Newton by primitive means) allows one to show as a corollary that spherically symmetric masses should interact gravitationally as though they are point masses as long as they are do not interpenetrate.

   What is the ____________, Alex?

   a) Newton theorem b) shell theorem c) point-mass theorem d) sphere theorem
   e) waste theorem

4. "Let's play *Jeopardy*! For \$100, the answer is: The theorem that states that the only attractive central forces that give closed orbits for all bound orbits are the inverse-square law force and the attractive linear force (AKA Hooke's law force or the radial harmonic oscillator force). All attractive central forces give closed **CIRCULAR** orbits, of course."

   What is ____________, Alex?

   a) the virial theorem b) Euler's theogonic proof c) the brachistochrone problem
   d) Schubert's unfinished symphony e) Bertrand's theorem

5. The solutions of the Friedmann equation have characteristic cosmological quantities some of which are called Hubble quantities since the Hubble constant is one of their ingredients. The table below displays some the cosmological quantities. Since the currently determined values of the quantities always fluctuate a bit depending on whose analysis is used, we have written the quantities as fiducial values with correction factors that are 1 to within a few percent: h_{70} is the Hubble constant divided by $70\,(\mathrm{km/s})/\mathrm{Mpc}$ (i.e., $H_0/(70\,(\mathrm{km/s})/\mathrm{Mpc})$), $\omega_{\mathrm{m},0} = \Omega_{\mathrm{m},0}/0.3$, and $\omega_\Lambda = \Omega_\Lambda/0.7$. The asymptotic Hubble quantities are those that will be the Hubble quantities as cosmic time goes to infinity if the Λ-CDM model is correct.

Table: Cosmological Quantities

Cosmic scale factor for the present cosmic time $a_0 = 1$ by convention
Hubble constant $H_0 = 70h_{70}$ (km/s)/Mpc
Hubble time $t_H = 1/H_0 = (13.968\ldots)/h_{70}$ Gyr
Hubble length $\ell_H = c/H_0 = (13.968\ldots)/h_{70}$ Gly $= (4.2827\ldots)/h_{70}$ Gpc
Critical density $\rho_{\rm critical} = [3H_0^2/(8\pi G)] = (9.2039 \times 10^{-27})h_{70}^2$ kg/m^3
$= (1.3599 \times 10^{11})h_{70}^2$ M$_\odot$/Mpc^3
AKA Hubble density (i.e., the density implied by the Hubble constant at cosmic present)
Cosmological constant matter density parameter $\Omega_{\rm m,0} = 0.3\omega_{\rm m,0}$
Cosmological constant Λ density parameter $\Omega_\Lambda = 0.7\omega_\Lambda$
Asymptotic Λ Hubble parameter $H_\Lambda = H_0\sqrt{\Omega_\Lambda} = \sqrt{\Lambda/3} = (58.566\ldots)h_{70}\sqrt{\omega_\Lambda}$ (km/s)/Mpc
Asymptotic Λ Hubble time $t_{H_\Lambda} = (16.6955\ldots)/(h_{70}\sqrt{\omega_\Lambda})$ Gyr

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Given that the Λ-CDM model is correct, to 1st order, the observable universe is already expanding like a cosmological-constant universe with $a = a_0 \exp(\Delta t/t_{H_\Lambda})$ (where $\Delta t = t - t_0$) and this formula becomes more correct as time advances. On what time scale Δt will the matter mass-energy density of the observable universe fall to of order 2 % of the total mass-energy? Note you have to solve for a/a_0 from

$$\Omega_{\rm m} = \Omega_{\rm m,0}\left(\frac{a_0}{a}\right)^3 \approx 0.02\Omega_\Lambda$$

and then solve for Δt.

a) t_{H_Λ}. b) $2t_{H_\Lambda}$ c) $3t_{H_\Lambda}$. d) $4t_{H_\Lambda}$. e) $5t_{H_\Lambda}$.

6. In this problem, we will derive the generic Gauss' law in its integral form and then specialize to the gravity and Coulomb force cases.

   NOTE: There are parts a,b,c,d. Some of the parts can be done independently, and so do not stop if you cannot do a part.

   a) Consider the generic inverse-square law central force

   $$\vec f = \frac{q}{r^2}\hat r ,$$

   where q is a generic charge for the force located at the origin. Now consider the differential surface area vector $d\vec A = dA\,\hat n$ for a **CLOSED** surface. The unit vector $\hat n$ is normal to the differential surface and points in outward direction. The differential solid angle subtended by the differential surface area is $d\Omega$. Prove

   $$\vec f \cdot d\vec A = q(\pm\, d\Omega)$$

   where the upper/lower cases are for the solid angle cone going outward/inward through the differential surface area. Note the charge could be inside or outside the closed surface. **Hint:** This is an easy question, but a few words of explanation are needed. But **NO** words are during exams.

   b) Consider a differentally small cone extending from the origin. It intersects the closed surface n times. Note that closed surface is finite, and so the cone must exit the closed surface for good at some point. We form the sum

   $$\sum_{i=1}^{n} \vec f \cdot d\vec A_i ,$$

   where sum is over all intersections. What is the sum equal to in terms of solid angle for all cases?

   c) Say you had multiple charges q_i with total charge Q and total charge $Q_{\rm enclosed}$ inside a closed surface. Evaluate

   $$\oint \vec f \cdot d\vec A .$$

   The result is the generic Gauss' law in its integral form. Specialize the result for the cases of gravity and the Coulomb force.

   d) What is the necessary condition for a force to obey Gauss' law?

7. Remarkably the linear force obeys analogues to Gauss's law and shell theorem for the inverse-square law force. Let the linear-force field (force per unit charge) for a point charge be

$$\vec{f} = kqr\hat{r} \ ,$$

where k is a constant which could be positive or negative, q is the charge (of some unspecified kind), and r is the distance from the point charge. We assume Newtonian physics, and so to maintain Newton's 3rd law, we require

$$\vec{F}_{1,2} = kq_1q_2r_{1,2}\hat{r}_{1,2} \ ,$$

where $\vec{F}_{1,2}$ is the force of point charge 1 on point charge 2.

There are parts a,b,c,d,f. Some of the parts can be done independently, and so do not stop if you cannot do a part.

NOTE: This question has **MULTIPLE PAGES** on an exam. Omit part (f) during exams.

a) Without words, for a close surface derive the linear-force Gauss' laww

$$\oint \vec{f} \cdot d\vec{A} = kQ \ ,$$

where $\vec{f}$ is the field due to the entire charge distribution, the integral is over the whole close surface, and Q is the total charge of the charge distribution wherever it is in space. **Hint:** Recall the divergence theorem (AKA Gauss' theorem)

$$\oint \vec{Y} \cdot d\vec{A} = \int \nabla \cdot \vec{Y} \, dV \ ,$$

where Y is a general vector field and the volume integral is over all volume V inclosed by the closed surface (Wikipedia: Divergence theorem). Recall also the divergence operator for spherically symmetric system in spherical coordinates obeys

$$\nabla \cdot \vec{Z} = \frac{1}{r^2}\frac{\partial(r^2 Z_r)}{\partial r} \ ,$$

where Z is spherically symmetric, but otherwise general, and Z_r is the radial component of $\vec{Z}$ (Arfken-104).

b) For what symmetries can the linear-force field be easily solved for directly from the linear-force Gauss' law?

c) Without words, solve for the linear-force field for a spherically symmetric charge distribution. What simple charge distribution would give an equivalent linear-force field for all radius r? What can this result be called? How is this equivalent linear-force field different from the analogue result with the inverse-square-law force?

d) Without words, show for a general charge distribution 1 and a spherical symmetric charge distribution 2 that the force of distribution 1 on distribution 2 is exactly the same as when distribution 2 is replaced point-charge 2. If charge distribution 1 were also spherically symmetric, what be the force between them be equal to and what would it be if their centers coincided exaclty?

e) Say you had a charge distribution that maintained spherically symmetry no matter what, that had its center of mass at its center, and the only external forces that acted on it were external linear forces. How would described its motion? Recall Newton's 2nd law:

$$\vec{F}_{\text{net external}} = m\vec{a}_{\text{cm}} \ ,$$

where $\vec{F}_{\text{net external}}$ is the net external force on a body of mass m and a_{cm} is the center of mass of the body. Given the result of part (d) Without words, show for two spherically symmetric distribution charges that the force of distribution 1 on distribution 2 is exactly the same **Hint:** Recall the part (d) answer.

f) Is the linear force for spherically symmetric mass distribution with mass as its charge consistent with linear force that occurs in the Newtonian derivation of the Friedmann equation:

$$\vec{F} = \frac{\Lambda}{3} m r \hat{r} ,$$

where m is a test particle mass. There is no right answer. This is a discussion question.

8. The Friedmann equation of general relativity (GR) cosmology in its most standard form (e.g., Wikipedia: Friedmann equations: Equations) is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} ,$$

where H is the Hubble parameter (which at current cosmic time is the Hubble constant H_0 and has fiducial value 70 (km/s)/Mpc), a is the cosmic scale factor, $\dot{a}$ is the time derivative of the cosmic scale factor with respect to cosmic time t, $G = 6.67430(15) \times 10^{-11}\,\mathrm{J\,m/kg^2}$ is the gravitational constant, ρ is the density of a uniform perfect fluid (in old-fashioned jargon AKA the cosmological substratum: Bo-75–76) which is used to model the universal mass distribution, k is called the curvature (Li-24,28) $k/(c^2 a^2)$ is called Gaussian curvature (CL-12,29), $c = 2.99792458 \times 10^8\,\mathrm{m/s}$ is the vacuum light speed as usual. and Λ is the cosmological constant which is the simplest form of the dark energy even though is only a from of energy in one interpretation. Note k is often defined with an unabsorbed c^2: i.e., the shown k is replaced by kc^2.

There are parts a,b,c. Some of the parts can be done independently, and so do not stop if you cannot do a part.

NOTE: This question has **MULTIPLE PAGES** on an exam. During exams do **ONLY** parts a,b,c,d.

a) Without words prove the Friedmann equation starting from the work-energy theorem

$$E_{\mathrm{mechanical}} = \frac{1}{2} m v^2 - \frac{GMm}{r} - \frac{1}{2}\frac{\Lambda}{3} m r^2 ,$$

where m is the mass of a test particle.

b) Without words prove the general Hubble law $v = Hr$, where v is recession velocity (i.e., the velocity between comoving frames) and r is proper distance (i.e., the distance measurable in with a ruler at one instant in cosmic time).

c) What is the asymptotic Hubble law (i.e., Hubble law valid in the limit $z \to 0$)?

9. The scaled Friedmann equation for power-law density components is

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_p \Omega_{p,0} x^{-p} ,$$

where 0 indicates the fiducial time which may be cosmic present, $h = H/H_0$ is the scaled Hubble parameter with H_0 being the Hubble constant, $x = a/a_0$ is the scaled cosmic scale factor, $\dot{x} = dx/d\tau$ is the rate of change of the scaled cosmic scale factor, $\tau = t/t_{H_0}$ is the scaled time with t_{H_0} being the Hubble time, the $\Omega_{p,0}$ are the density parameters for the density components at the fiducial time with their sum being 1, and p are the powers of the power-law density components.

NOTE: There are parts a,b,c,d,e. Part (e) can be done independently of the other parts. This question has **MULTIPLE PAGES** on an exam.

a) Without words, derive the general asymptotic solution $x(\tau)$ for the leading density component as $\tau \to 0$ (i.e., the density component with highest p). As a shorthand, this solution can be called the early universe solution. Assume $p > 0$. To avoid pointless generality, assume $x(\tau = 0) = 0$ (i.e., there is a point origin at time zero).

b) Without words, derive early universe formula for $\Omega_p(\tau)$ for $p > 0$.

c) Without words, derive the special case early universe solutions for $p = 1, 2, 3, 4$.

d) We now assume the universe has only one density component with power $p > 0$. Without words, derive the generic age of the universe formula for τ and t and give the fiducial value version for t with the Hubble time $t_{H_0} = (13.968\ldots\,\text{Gyr})/h_{70}$, where $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$. special case solutions for $p = 1, 2, 3, 4$.

e) We assume the universe has only one density component with power $p = 0$. Without words, derive $x(\tau)$ and $x(t)$ assuming $x(0) = 1$. Note this universe is the de Sitter universe and the Hubble constant $H_0 = \sqrt{\Lambda/3}$.

10. The differential equation (DE) for the perfect fluid of cosmology of the Friedmann equation is

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) ,$$

where ρ is mass-energy in the comoving frames of Friedmann cosmology and p is isotropic pressure in those frames (in some sense) (Liddle 26). The perfect fluid DE can be derived rigorously from general relativity (Carroll 333–334) or, perhaps fudgily, from classical thermodynamics. Remarkably, this equation does not guarantee conservation of energy in the ordinary sense of classical physics: it does embody the general relativity feature that the covariant derivative of the energy-momentum tensor is zero (Carroll 117,120): i.e., the energy-momentum conservation equation. General relativity may or may not in some sense conserve energy for cosmology, but certainly gravitating mass-energy is allowed to appear and disappear by the perfect fluid DE.

Multiple perfect fluids can exist and if they are assumed to act independently (which is the usual cosmological assumption), then they all obey there own perfect fluid DE: i.e., for perfect fluid i

$$\dot{\rho}_i = -3\frac{\dot{a}}{a}\left(\rho_i + \frac{p_i}{c^2}\right) .$$

In current standard cosmology (i.e., the ΛCDM model or simple variations thereof), it is assumed that the perfect fluid equation of state (EOS) is of the form

$$p = w\rho c^2 ,$$

where w is a constant parameter that seems to have no special name. Most standard/interesing values of w are given by

$$w = \begin{cases} 0 & \text{for nonrelativistic (NR) mass-energy (AKA “matter”} \\ & \qquad \text{or “dust”: Liddle-40);} \\ 1/3 & \text{for extreme relativistic (ER) mass-energy: most obviously photons,} \\ & \qquad \text{but also the ER neutrinos of the Big Bang era and early cosmic dark ages;} \\ -1 & \text{for cosmological constant (which name can also be used for constant dark energy;} \\ -1/3 & \text{for zero-acceleration (or constant } \dot{a}\text{) universes such as} \\ & \qquad \text{Fulvio Melia's } R_{\text{h}} = ct \text{ universe or a universe with cosmic scale} \\ & \qquad \text{determined only by negative curvature } k. \end{cases}$$

Solve for the formula for $\rho(a)$ for general w and the 4 special cases of w listed above. Assume a_0 and ρ_0 for cosmic present values.

11. The Friedmann equation and acceleration equation are, respectively,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} = H_0^2\frac{\rho}{\rho_{\text{C}}} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p_{\text{pressure}}}{c^2}\right) + \frac{\Lambda}{3} = -\frac{1}{2}H_0^2\frac{\rho}{\rho_{\text{C}}}(1+3w) + \frac{\Lambda}{3} ,$$

where following a usual convention c^2 has been absorbed into k and Λ (Li-55) and we have assumed a single equation of state for the second version of the acceleration equation. A standard set of solutions follows for perfect fluids with equation of state $p_{\text{pressure}} = w\rho c^2$ (with w constant) for the cases with $k = 0$ and $\Lambda = 0$ and density ρ obeying an inverse-power law of a. Following CL-36, we will call these solutions Einstein-de-Sitter universes (EdS universes) although originally only the $w = 0$ case was called an EdS universe. Note EdS universes do not the include the Einstein universe (which is a static, positive curvature universe), but do include the flat de Sitter universe with $k = 0$. The original de Sitter universe had positive curvature (O'Raifearty et al., 2017, p. 38). Explicitly, density as an inverse-power law of a is

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^p \qquad \text{with} \qquad p = 3(1+w) \qquad \text{and} \qquad \gamma = 2/p \ ,$$

where p is a power and not pressure p_{pressure}.

There are parts a,b.

NOTE: This question has **MULTIPLE PAGES** on an exam.

a) For the EdS universes, determine the general solutions for $a(t)$ (assuming a(0)=0, except for $w = -1$), t_0, q_0, and $\rho(t)$ in terms of a_0, t, H_0, w (or any convenient combination of w, p, and γ), and ρ_0 which equals

$$\rho_{\text{C}} = \frac{3H_0^2}{8\pi G}$$

for Euclidean universes (i.e., flat universes). Note the subscript 0 means present cosmic time and the $w = -1$ cases require special treatment. Recall the deceleration parameter formula

$$q = -\frac{\ddot{a}}{a}\frac{1}{H^2}$$

(Li-53).

b) Specialize the results of part (a) for w values 0 ("matter"), 1/3 ("radiation"), -1 (de Sitter universe: cosmological constant, constant dark energy, or steady-state universe), and $-1/3$ (zero acceleration universe). Organize the results in a table for easy understanding.

12. The Friedmann equation of general relativity (GR) cosmology in standard form (e.g., Wikipedia: Friedmann equations: Equations) is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ ,$$

where H is the Hubble parameter (which at current cosmic time is the Hubble constant H_0 and has fiducial value 70 (km/s)/Mpc), a is the cosmic scale factor, $\dot{a}$ is the time derivative of the cosmic scale factor with respect to cosmic time t, $G = 6.67430(15) \times 10^{-11}\,\text{J m/kg}^2$ is the gravitational constant, ρ is the density of a uniform perfect fluid (in old-fashioned jargon AKA the cosmological substratum: Bo-75–76) which is used to model the universal mass distribution, k is called the curvature (Li-24,28) $k/(c^2a^2)$ is called Gaussian curvature (CL-12,29), and $c = 2.99792458 \times 10^8\,\text{m/s}$ is the vacuum light speed as usual. Note k is often defined with an unabsorbed c^2: i.e., the shown k is replaced by kc^2.

The Friedmann equation is, as one can see, a 1st order nonlinear ordinary differential equation. The fact that is nonlinear means that linear combinations of solutions are not in general solutions though they may be in special cases or approximately. The Friedmann equation is also a homogeneous differential equation at least in the sense that it can be written $\dot{a} = g(a)$. The form $\dot{a} = g(a)$ implies that a must be strictly increasing or decreasing except possibly at $\pm\infty$ and possibly at points where the some order of derivative of g have infinities. Both exceptions do occur for some solutions of the Friedmann equation. For example, the latter exception occurs for the closed universe model (with only matter). The closed universe model solution is closely related to throwing a ball into the air: the maximum size of the closed universe model corresponds to the maximum height of the ball.

The Friedmann equation actually has an interesting nature in that its independent variable is cosmic time t, but the solution the cosmic scale factor $a(t)$ is the factor by which all distances scale with time in expanding universe models.

Let's derive the Friedmann equation from Newtonian physics with extra natural hypotheses as needed. A priori, it not clear that the Newtonian derivation must yield the Friedmann equation with the extra natural hypotheses. But it can be shown that it should (C.G. Wells 2014, ArXiv:1405.1656). Note that the Newtonian derivation can say nothing about the curvature of space and assumes any curvature does not affect the derivation. We will do a long preamble wherein, with any luck, the extra hyptheses are shown to be natural.

First, just as in the GR derivation, we assume for our universe model the cosmological principle which states that the universe has a homogeneous, isotropic mass-energy distribution when averaged on a sufficiently large scale. The cosmological principle is what allows us to approximate the observable universe in our model with a perfect fluid. Observationally, the cosmological principle has been verified to a degree, but some tension remains. The observational scale for the validity of the cosmological principle is 100 Mpc or maybe a factor of a few times that larger (Wikipedia: Cosmological principle: Observations). Note that well beyond the observable universe, the cosmological principle may well fail, but, just as in the GR derivation, we assume this has negligible effect for the observable universe.

As to the perfect fluid of our model, it has uniform rest-frame mass-energy density ρ (uniform in space, not in time). The mass-energy gravitating mass-energy, of course. The perfect fluid has no viscosity and has an isotropic pressure p in its own rest frame (Ca-34). The perfect fluid equation of state (EOS) is $p = p(\rho)$. Actually, the perfect fluid can have internal energy (i.e., thermal energy), but that is counted as part of ρ as follows from $E = mc^2$. Also note that we said "rest-frame mass-energy" which can be the energy of massless particles. In fact, a photon gas is a good realization of the perfect fluid. The actual cosmic background radiation since the recombination era approximates a perfect fluid to high accuracy. Its photons do pass through gravitational wells, scatter off free electrons, and sometime hit planets, etc., but to good approximation the photons act as if they never interacted with anything except gravitationally.

Next, we note a corollory of Birkhoff's theorem (a theorem in GR): a spherical cavity at the center of spherical symmetric mass-energy distribution (static or not, finite or infinite) is a flat Minkowski spacetime (CL-24; We-337–338, 474). The spherical symmetric mass distribution can be, in fact, an unbounded homogeneous, isotropic mass-energy distribution: it can be infinite or finite. Note that if the spherical symmetric mass distribution is finite, it must have positive curvature and be a closed universe model. We assume, just as in the GR derivation, that Birkhoff's theorem applies to good approximation even if the cosmological principle fails well beyond the observable universe. Inside the cavity, we can put mass-energy and it should behave exactly as superimposed on a universe of flat Minkowski spacetime (CL-24; We-337–338, 474) as long as it does not break spherical symmetry significantly, which would cause a significant perturbation of the spherical symmetry of the surroundings. The mass-energy we put in the cavity used for our derivation does not break spherical symmetry.

The situation for the Birkhoff-theorem cavity is analogous to a cavity in spherically symmetric mass distribution in Newtonian physics. Inside the Newtonian cavity, the gravitational field is zero: this is a corollory of the shell theorem first proven by Newton himself. However, what happens if the mass distribution is infinite is not defined by pure Newtonian physics. Analogous to the GR case, inside the cavity, we can put mass-energy and it should behave exactly as superimposed a region where there is no external gravitational field as long as it does not break spherical symmetry significantly which would cause a significant perturbation of the spherical symmetry of the surroundings.

Now consider general relativistic space infinite or finite and unbounded (which would be positive curvature space: Li-33). The space is filled with the aforementioned uniform perfect fluid. The fluid density ρ is a function of cosmic time t in general. The fluid's motions are determined only by gravity (i.e., the geometry of spacetime) and initial conditions, and so each element of the fluid moves along a geodesic in a GR interpretation and in free fall in the Newtonian physics interpretation. Since we demand homogeneity and isotropy, we can only have uniform expansion/contraction of the whole model. Note the fluid can have pressure (positive or negative), but uniformity means the pressure force cancels out everywhere locally. The fluid can also have a formal pressure that does not have to push/pull on anything. However, formal pressure does have a global effect as we will show below.

Now consider a Birkhoff-theorem cavity of radius r for our model which is also filled with the perfect fluid with density ρ. Everything inside the cavity behaves just as everything outside, and so the cosmological principle is maintained. The cavity fluid has total mass M. We assume that gravitational field due to the cavity fluid is asymmptotically Newtonian. This requires

$$\frac{R_{\rm Sch}}{r} = \frac{2GM/c^2}{r} = \frac{8\pi}{3}\frac{G\rho}{c^2}r^2 << 1 \ ,$$

where $R_{\rm Sch} = 2GM/c^2$ is the Schwarzschild radius (Wikipedia: Schwarzschild radius). So we just assume r is small enough. Note that Newtonian gravitational field is actually the classical limit of the left-hand side of the Einstein field equations (i.e., the spacetime geometry structure side: We-152), and so it does not itself contribute mass-energy (which comes from the right-hand side of the Einstein field equations and is described by the energy-momentum tensor). So we do not have to worry about the mass-energy contribution of the gravitational field to gravitating mass-energy since it does not contribute.

We also have to assume that r is small enough that the gravitational effects propagate with negligible time delay. Really, they propagate at the vacuum light speed relative to their local inertial frame.

We also have to assume that all relative velocities v of the fluid elements inside the cavity satisfy $v/c << 1$ so that we can employ Newtonian physics. This assumption is also asymptotically valid for small enough cavity radius r since the relative velocities between fluid elements are proportional to their separation distances as shown by Hubble's law which we derive nonrigorously below.

Recall all fluid elements in the perfect fluid are in free fall as aforesaid. This raises an interesting point. Special relativity gives the vacuum light speed c as the highest speed relative to inertial frames, but not between inertial frames. And the strong equivalence principle of GR shows that free-fall frames with uniform external gravity are exact inertial frames. The strong equivalence principle has been verified to very high accuracy (Archibald et al. 2018, arXiv:1807.02059). So the free-fall frames (which we will call comoving frames) of our model can grow apart at faster than c. In fact, Hubble's law shows that they must for large enough separation distances. Note that a light signal between comoving frames can only propagate at the vacuum speed light relative to the comoving frames it propagates through. So the fact that space can grow faster than the vacuum light speed does not imply there is faster-than-light signaling.

To summarize our assumptions for the Newtonian derivation, we require Birkhoff's theorem and that r be sufficiently small so that all relativistic and time-delay effects are small. If the aforesiad effects vanish in the differential limit as $r \rightarrow$, then the Newtonian derivation should be valid. Recall the Friedmann equation holds at every point in the universe model according to the GR derivation. Perhaps, there is some way that the Newtonian proof is still invalid, but it would have to be a very odd way.

Now we are ready to tear into the derivation of the Friedmann equation. We put a test particle of mass m at the surface of our cavity (i.e., at radius r). Given our setup, we have conservation of mechanical energy E:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv^2 - \frac{4\pi G}{3}\rho r^2 m \ ,$$

where the first term to the right of the equal signs is the kinetic energy of our test particle and the second is its gravitational potential energy which is also its gravitational field energy in Newtonian physics which as discussed above does not itself contribute to gravitating mass-energy. We now write

$$r = ar_0 \ ,$$

where a is the dimensionless cosmic scale factor and r_0 is a time-independent covoming distance. By usual convention the scale factor for the current cosmic time t_0 is defined to be 1: i.e., $a_0 = a(t_0) = 1$. This means that the r_0 are the proper distances for the current cosmic time: i.e., distances that you could measure with a ruler at current instant in cosmic time. Note $v = \dot{a}r_0$. Now defining the Hubble parameter $H = \dot{a}/a$, we get

$$v = Hr$$

which is the general-time Hubble's law. The current cosmic time Hubble's law (with the current Hubble parameter being Hubble's constant) is

$$v_0 = H_0 r_0 \ ,$$

The validity of this derivation of Hubble's law follows from the Friedmann equation itself, and so is valid insofar as our Newtonian derivation of the Friedmann equation is valid. A rigorous GR derivation is given by CL-13–14.

Re Hubble's law: it is an exact law for recession velocities (which are velocities between comoving frames: i.e., free-fall frames that are exact inertial frames) and proper distances (which are true physical distances that can be measured at one instant in cosmic time with a ruler). In fact, neither recession velocities nor proper distances are observables, except asymptotically as $r \rightarrow 0$. The exception allows Hubble's constant to be measured from cosmologically nearby galaxies.

We divide the conservation of mechanical energy equation by $-mr_0^2/2$ to get

$$-\frac{2E}{mr_0^2} = -\dot{a}^2 + \frac{8\pi G}{3}\rho a^2 \ .$$

The right-hand side of the second to last equation is independent of E, m, and r_0 and depends only on universal quantities of the universe model, and therefore the constant on the left-hand side must be a universal constant independent of the peculiarities of the test particle: i.e., E, m, and r_0. We use the symbol k for this universal constant: thus,

$$k = -\frac{2E}{mr_0^2} \ .$$

The constant k is called the curvature since GR tells us it describes the curvature of space which we cannot know from Newtonian physics (Li-24, CL-12–13). Note $k > 0$ gives positive curvature (hyperspherical geometry), $k < 0$ gives negative curvature (hyperbolical geometry), and $k = 0$ gives zero curvature (flat or Euclidean geometry): see Wikipedia: Shape of the universe. (As noted above, k is often defined with an unabsorbed c^2: i.e., $kc^2 = -2E/mr_0^2$.) Rearranging the second to last equation gives us the Friedmann equation itself:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} = H_0^2\left[\Omega + \Omega_k\left(\frac{a_0}{a}\right)^2\right] ,$$

(Li-24), where we have defined

$$\Omega = \frac{\rho}{\rho_{\rm c}} , \qquad \rho_{\rm c} = \frac{3H_0^2}{8\pi G} , \qquad \Omega_k = -\frac{k}{a_0^2 H_0^2} .$$

(Li-51,56). Note Ω is the density parameter (Li-51), $\rho_{\rm c} = 3H_0^2/(8\pi G)$ is the critical density (Li-51), and Ω_k is the curvature density parameter (Li-56). If $\Omega = 1$ at the current cosmic time (or any other cosmic time defined as current cosmic time), one has

$$H_0^2 = H_0^2(1 + \Omega_k)$$

implying $\Omega_k = 0$. So a universe model that is exactly flat at any cosmic time is exactly flat at all times.

There are several interesting points to be made about the Friedmann equation. First, we demanded r be small enough so that we could neglect relativistic and time travel effects. But we would derive the same Friedmann equation no matter what r we choose. So actually, all the effects we have neglected must cancel out for any r due to the conditions we imposed on the universe model: the cosmological principle and the perfect fluid.

A second interesting point is that Friedmann equation allows for mass-energy to appear or disappear as function of a. To explicate, mass-energy that is conserved (which called matter in cosmology jargon) has $\rho_{\rm m} \propto 1/a^3$. We show this below, but is in fact it is somewhat obvious: if the volume of a fluid element scales of up as a^3 and mass-energy is conserved, then density must decrease as $1/a^3$. But we allow other kinds of mass-energy dependence on a. For one example of mass-energy appearance/disappearance is that the cosmic background radiation and cosmic neutrino background (which in cosmology jargon is collectively called radiation) has $\rho_{\rm r} \propto 1/a^4$. The extra power of a is due to the cosmological redshift of extreme relativistic mass-energy which just causes radiation mass-energy to vanish from universe—it's just gone as gravitating mass-energy. Note general relativity cosmology does not have ordinary conservation of mass-energy: it just has the energy-momentum conservation equation $\nabla^\mu T_{\mu\mu} = 0$ (Carroll-120). Another point is that Noether's theorem that gives energy conservation when time invariance applies does not apply in an evolving universe model that does not have time invariance (Carroll-120). Another example of mass-energy appearance/disappearance is that constant dark energy (which is equivalent to the cosmological constant Λ in effect in the Friedmann equation if not otherwise) has ρ_Λ constant. The appearing/disappearing mass-energy contributes both gravitational field energy and, by the conservation of mechanical energy, the kinetic energy of the comoving frames which is sort of energy of expansion. (The disappearance of radiation also removes the kinetic energy of the comoving frames). To make more obvious the way mass-energy appearance/disappearance balances the gravitational field energy and the kinetic energy of the comoving frames , consider the Friedmann equation version

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k .$$

Holding a and k fixed, and increasing ρ (mass-energy) proportionally increases $\dot{a}^2$ (kinetic energy of comoving frames). This balanced contribution of gravitational field energy and kinetic energy for appearing/disappearing mass-energy arises only from starting our derivation from the conservation of mechanical energy equation. If we had started from Newton's 2nd law, we would have had no obvious path to include appearing/disappearing mass-energy.

You might ask what if k is a function of time or appearing/disappearing mass-energy is an explicit function of time not merely a function of a which is a function time. We have no guiding theory for these cases, and so far no observational or theoretical need for them.

We will now derive the fluid equation as it is called in cosmology jargon: i.e., the equation for $\dot{\rho}$. We assume that the perfect fluid obeys the 1st law of thermodynamices (which is actually implicit in the energy-momentum tensor for a perfect fluid: C.G. Wells 2014, ArXiv:1405.1656, p. 4). The 1st law is

$$dE = T\,dS - p\,dV + \mu\,dN \ ,$$

where here E is total mass-energy and not mechanical energy as above, T is temperature, S is entropy, p is pressure, V is volume, μ is chemical potential, and N is number of particles. The perfect fluid is adiabatic (i.e., $dS = 0$) and so the 1st law reduces to

$$dE = -p\,dV + \mu\,dN \ ,$$

For simplicity, we allow change in number of particles only to a species that is spontaneously created in such a way that N stays proportional to volume V. This means that $N = nV$ where n is the constant density of the spontaneously created particles. The spontaneously created particles are created at rest in the comoving frames, and so their chemical potential is just their rest-mass mass-energy. Given a volume $V \propto a^3$ for an amount of perfect fluid, we have

$$\begin{aligned}
&E = \rho c^2 V \\
&\dot{E} = (\dot{\rho}V + \rho\dot{V})c^2 = -p\dot{V} + \mu n\dot{V} \\
&\dot{\rho} = -\frac{\dot{V}}{V}\left(\rho + \frac{p}{c^2} - \frac{\mu n}{c^2}\right) \qquad \text{and using} \qquad \frac{\dot{V}}{V} = \frac{3a^2\dot{a}}{a^3} = 3\frac{\dot{a}}{a} \\
&\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2} - \frac{\mu n}{c^2}\right)
\end{aligned}$$

(Li-26). At the expense of clutter, we can explicitly allow for different species in the fluid equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}\sum_i\left(\rho_i + \frac{p_i}{c^2} - \frac{\mu_i n_i}{c^2}\right) \ ,$$

where $\mu_i = 0$ for those species which are not the spontaneously created particles we allowed for.

We note that in cosmology the equation of state is often parameterized thusly

$$p = \begin{cases} w\rho c^2 & \text{where } w \text{ is constant parameter just called w;} \\ 0 & \text{for matter where } w = 0; \\ \frac{1}{3}\rho c^2 & \text{for radiation where } w = 1/3; \\ -\rho c^2 & \text{for constant dark energy where } w = -1; \\ -\frac{1}{3}\rho c^2 & \text{for a non-accelerating universe where } w = -1/3. \end{cases}$$

One might well ask what the heck is the negative pressure of constant dark energy. Well for a hypothetical laboratory gas, its something with suction. So expanding it, requires adding internal energy. But the constant dark energy negative pressure may be just formal. There is no reason to require it to couple to anything except maybe itself, and so maybe nothing feels negative pressure, except maybe dark energy itself. In any case, the dark energy is uniform, and so there are no pressure gradients. Where does the mass-energy come from to keep dark energy constant as the universe expands? Well in simplest theory, it just appears as a fundamental fact. However, there are quantum field theory reasons for believing there could be dark energy, but quantum field theory in its simplest prediction gets the size of constant dark energy too big by more than 100 orders of magnitude. So maybe quantum field theory does not know what its talking about.

Why do we allow for constant dark energy? The universal expansion is positively accelerating and constant dark energy supplies a cause. Of course, constant dark energy insofar as it affects Friedmann equation (but perhaps not otherwise) can be replaced by Einstein's cosmological constant Λ with the appropriate positive value. The cosmological constant (if it exists) is a fundamental aspect of gravity and not mass-energy form at all.

The negative pressure for the non-accelerating universe is just a fix to get a non-accelerating universe which has been argued for by some (e.g., Melia 2015, arXiv:1411.5771). So it's just a formal pressure.

Why did we allow for spontaneously created particles? They represent an alternative idea to constant dark energy and the cosmological constant. In the Friedmann equation, they have the same effect as constant dark energy and the cosmological constant Λ with the appropriate positive value. What could such particles be? Very speculatively, dark matter particles, nonrelativistic neutrinos (which can exist even if we have never detected them), and/or baryonic matter (pairs of protons and electrons). All of these would have other effects than just giving a positively accelerating universe. They could clump eventually and affect large-scale structure evolution, and in the case of baryonic matter lead to new star formation. The particles, by the way, certainly have only positive pressure, but to first approximation that is negligible compared to their mass-energy contribution. The case of spontaneous creation of baryonic matter leads to the unlikely hypothesis that the observable universe started with a Big Bang, but is now evolving to the steady-state universe as hypothesized by Bondi, Gold, and Hoyle in 1948. Actually, Einstein anticipated the steady-state universe in unpublished work in 1931.

Now for some problems.

a) Derive the acceleration equation (AKA the 2nd Friedmann equation)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2} - \frac{3\mu n}{c^2}\right) .$$

Hint: Start by mulitpying the Friedmann equation through by a^2.

b) The deceleration parameter q is a dimensionless measure of the acceleration of the universal expansion. It is defined

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}$$

(Li-53), where the negative sign was included to get a positive value when people expected the acceleration to be negative. Some simple analytic solutions for $a(t)$ have only two unknown parameters and the observational determination of H_0 and q_0 determine those. This is why Allan Sandage (1926–2010) once, with admitted vast simplification, called the cosmology the search for two numbers: i.e., H_0 and q_0. Write q for general time in terms of general-time Ω, $\rho_{\rm c}$, p, and μn.

c) As discussed in the preamble, the cosmological constant is the alternative to constant dark energy insofar as the Friedmann equation alone is considered. One can derive it from the given standard form of the Friedmann equation by replacing ρ by $\rho + \rho_\Lambda$, where $\rho_\Lambda \equiv \Lambda/(8\pi G)$ (Li-56). Make this replacement in the Friedmann equation and then reverse engineer the derivation of the Friedmann equation to find the Newtonian potential energy U_Λ and the Newtonian force F_Λ implied by the cosmological constant.

d) What is peculiar about the Newtonian force F_Λ? **Hint:** The short answer is expected.

e) Write down the Friedmann equation and the acceleration equation with the explicit cosmological constant term. Set $\mu n = 0$ for simplicity. **Hint:** This is easy given $\rho_\Lambda \equiv \Lambda/(8\pi G)$, but you have to remember ρ_Λ has a formal pressure if it is attributed to constant dark energy as follows from the fluid equation for $\dot{\rho}_\Lambda = 0$.

f) The de Sitter solution of the Friedmann equation—which is grandly called the de Sitter universe—is obtained for the case where $\rho = 0$, $k = 0$, and $\Lambda > 0$. Find this solution in terms of current cosmic time t_0 and find the expressions for the Hubble parameter, the Hubble constant, and the deceleration parameter in general and for the current cosmic time. By the by, the de Sitter solution with the cosmological constant interpreted as constant density of ordinary matter is the steady-state universe.

13. Here we do the quick derivations of the Friedmann equation, the fluid equation, the Friedmann acceleration equation, and some other results.

NOTE: There are parts a,b,c,d,e,f. During exams do **ONLY** parts a,b,c,d. Some of the parts can be done independently, and so do not stop if you cannot do a part.

a) Without words, derive the Friedmann equation in standard form (with the cosmological constant force $F_\Lambda = (\Lambda/3)mr$ included) from classical physics with the hypotheses that all free-fall frames are fundamental inertial frames and that the shell theorem for a spherically symmetric mass distribution can be extended to infinite distance (which is validated by Birkhoff's theorem from general relativity). The derivation makes use of classical conservation of mechanical energy. You should end up with a $-k/a^2$ term among other things. You can a draw diagram if you like. **Hint:** Start with the conservation of mechanical energy of a test particle of mass m:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} - \left(\frac{1}{2}\right)\frac{\Lambda}{3}mr^2 .$$

b) Without words, derive the curvature equation:

$$\Omega_k = -\frac{k}{H^2 a^2} = 1 - \Omega_{\text{non-}k} \ ,$$

where $\Omega_{\text{non-}k} = \Omega_{\text{non-}k,\Lambda} + \Omega_\Lambda$ and Ω_k is the curvature density parameter (e.g., Steiner 2008, p. 5). Show explicitly the critical density formula: i.e., the formula for ρ_{c} that makes curvature k zero when

$$\rho_{\text{non-}k,\Lambda} + \rho_\Lambda = \rho_{\text{non-}k,\Lambda} + \frac{\Lambda}{8\pi G} = \rho_{\text{c}} \ .$$

c) Without words and starting from the 1st law of thermodynamics

$$dE = T\,dS - p\,dV + \mu\,dN \ ,$$

derive the cosmological fluid equation in standard form (which means with $dN = 0$) and in a form with $\dot{\rho}a/\dot{a}$ equal to something for use in part (e). Recall the rest-frame energy is $E = \rho c^2 V$.

d) Specialize the fluid equation to the special case where the equation of state is $p = w\rho c^2$ with w a constant. Determine the explicit solution $\rho(a)$ for the special case where $\rho_0 = \rho(a_0)$. **Hint:** You will have to eliminate the time derivative.

e) Without words, derive the acceleration equation (or Friedmann acceleration equation) in standard form using parts (a) and (c).

f) Without words, derive from the Friedmann equation the de Sitter universe solution which has $\rho = 0$ and $k = 0$, but $\Lambda \neq 0$.

14. Consider the following linear 1st order differential equation (DE):

$$x' = A - kx \ ,$$

where t is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant.
There are parts a,b,c,d. Parts (a) and (b) can be done independently at least.
NOTE: This question has **MULTIPLE PAGES** on an exam.

a) Solve for the constant solution x_A. **Hint:** This is easy.

b) We can now write the DE as

$$x' = k\,(x_A - x) \ .$$

Without solving for non-constant solution describe what it must look like as a function of t for arbitrary initial value $x_0 = x(t = 0)$. In particular, where are its stationary points if any? **Hint:** Consider the continuity of all orders of derivative of x.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$, $x'(t)$, and the 1st order in small t solution $x_{1\text{st}}(t)$. **Hint:** You can use an integrating factor, but there is a more straightforward way.

d) What is the e-folding time t_e of your solution and what does it signify? What is the $x(t_e)$? What is the $x_{1\text{st}}(t_e)$? What is remarkable about $x_{1\text{st}}(t_e)$?