Cosmology & Galaxies Name:

Homework 2: Miscellaneous Math Problems: All Questions

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002 qfull 00110 1 3 3 hard math: ancient Egyptians and unit fractions

1. The ancient Egyptian mathematicians thought there was something unfundamental about non-unit fractions (those not of the form $1/n$) though they made a bit of an exception for 2/3 (Boyer-13–14). So they thought it a good idea to expand non-unit fractions as sums of unit fractions (those of the form $1/n$).

   There are parts a,b,c.

   a) Show the general rational number m/n can be expanded into infinitely many possible unit fraction expansions. **HINT:** This is trivial.

   b) The ancient Egyptians apparently thought some kinds of unit-fraction expansions were good, but have not left us any definite rules (Boyer-14). Probably they never formulated any. However, we can formulate a rule/algorithm. Specify an rule/algorithm for expanding general m/n in unit fractions

$$m/n = \sum_{i=1}^{I} \frac{k_i}{n_i}$$

   where the denominators n_i are all divisors of n in increasing order of size, there are I divisors in total, and k_i are all zero or 1, except that k_I can be greater than 1. **HINT:** The proof just requires some subtraction using a recurrence relation.

   c) Use your rule/algorithm from the part (b) answer to expand 601/360 in unit fractions. You could do this by hand or write a small computer progam do to it. Note that 360 has 24 divisors which is probably one of the main reasons why the ancient Babylonian astronomers chose it for the division of the circle—they wanted easy division. The other main reason was probably to get angle unit nearly equal to the distance the Sun moved every day on the celestial sphere. **HINT:** If you write a computer code, make it find the divisors with the mod function for you then it will be general for any denominator n. Try your code out on 1170/360.

   d) Consider m/n and an expansion in the harmonic series with omissions:

$$\frac{m}{n} = \sum_{\ell=2}^{K} \frac{k_\ell}{\ell} ,$$

   where $k_i = 1$ or zero and K is in general ∞. Why is it always possible to make this expansion? Can the series truncate with K finite? I will give one buck to the first person who finds out by themselves or from some source whether or not the expansion truncates to finite K always.

SUGGESTED ANSWER:

a) Behold:

$$\frac{m}{n} = \sum_{m} \frac{1}{n} = \sum_{mk} \frac{1}{kn} ,$$

where k is general positive-definite integer. Since k is general, clearly there are infinitely many expansions. One, of course, can do all kinds of elaborate expansions.

b) Let $R_0 = m/n$ be the zeroth remainder. The algorithm has recurrence relation

$$R_i = R_{i-1} - \frac{k_i}{n_i} = \frac{m_i}{n} - \frac{p_i k_i}{n} = \frac{m_i - p_i k_i}{n} ,$$

where p_i is the complement divisor to n_i (i.e., $p_i = n/n_i$) and $R_i = m_i/n$ is the remainder after subtracting the ith unit/zero fraction: unit if $p_i k_i \leq m_i$, zero otherwise.

To prove that in general k_I is not 1 or zero, just consider any prime number n. Clearly, m/n cannot be expanded in any unit fractions in this case by our procedure. For general n, just construct an m' from the expansion

$$\frac{m'}{n} = \sum_{i=1}^{I} \frac{1}{n_i} .$$

Now add $1/n$ to get $m/n = (m'+1)/n$ which clearly cannot be expanded into all unit fractions by our rule.

c) For reference, the divisors of 360 are given in the following table.

Table: The 24 Divisors of 360

Count	Divisor	Divisor	count
1	1	360	24
2	2	180	23
3	3	120	22
4	4	90	21
5	5	72	20
6	6	60	19
7	8	45	18
8	9	40	17
9	10	36	16
10	12	30	15
11	15	24	14
12	18	20	13

Yours truly wrote code to do the expansion. See below. The required expansions are

$$\frac{601}{360} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{360}$$

and

$$\frac{1170}{360} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{18} + \frac{1}{20} + \frac{1}{24} + \frac{1}{30} + \frac{1}{36} + \frac{1}{40} + \frac{1}{45} + \frac{1}{60} + \frac{1}{72} + \frac{1}{90} + \frac{1}{120} + \frac{1}{180} + \frac{1}{360} \; .$$

As you can see, the expansion for 1170/360 uses all the divisors of 360.

d) It's always possible to have this expansion since the harmonic series diverges (Ar-279). Thus, by adding k_ℓ/ℓ terms it is always possible to keep getting the finite sum closer to general m/n.

Clearly, K can be finite in some expansions since the harmonic numbers themselves are defined by such expansion: i.e.,

$$H_K = \sum_{\ell=2}^{K} \frac{1}{\ell} \; ,$$

where K is finite.

My guess is that K cannot be finite in general because if that were true someone would have mentioned it like Boyer-13–14.

Fortran-95 Code

```
!23456789a123456789b123456789c123456789d123456789e123456789f123456789g12
! Code fragment:  Unit fraction expansion for divisors rule:
      print*
!      inum=1    ! 1/360 correct
!      inum=3    ! 1/120 correct
!      inum=10   ! 1/36 correct
!      inum=72   ! 1/5  correct
!      inum=601  ! 1/1, 1/2, 1/6, 1/360 correct
      inum=360+180+120+90+72+60+45+40+36+30+24+20+18+15+12+10           &
     &        +9+8+6+5+4+3+2+1   ! = 1170 correct
      iden=360
      print*,'Numerator,Denominator'
      print*,'inum,iden'
```

```
          print*,inum,iden
          irem=inum
          if(irem .le. 0) stop 'irem must be > 0'
          icount=0
          do i=1,iden
           if(mod(iden,i) .ne. 0) cycle ! cycle if not a divisor
           icount=icount+1
           icom=iden/i                  ! The complement divisor
           write(*,910) icount,i,icom,i*icom
           iremp=irem-icom
           if(iremp .ge. 0) then
               if(i .ne. iden  .or.  iremp .eq. 0) then
                   irem=iremp
                   jnum=1
                 else           ! You've used the last divisor and it's not given
zero remainder.
                   jnum=irem
               end if
               write(*,912) icount,jnum,'/',i
               if(iremp .eq. 0) exit
           end if
          end do
      910 format(4i5)
      912 format(i5,' Expansion term',i4,a1,i3)
!23456789a123456789b123456789c123456789d123456789e123456789f123456789g12
```

Redaction: Jeffery, 2018jan01

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002 qfull 00120 1 3 3 hard math: frustum volume AKA frustrum volume

2. A general cone is 3-dimensional shape formed from a planar base and continuum of line segments from the base's perimeter to a vertex (or apex) not in the plane of the base. The height of the cone is the length of the perpendicular from the base plane to the vertex. A general frustum—or, tripping off the tongue erroneously, frustrum—is a general cone with the top sliced off parallel to the base.

   The ancient Egyptian mathematicians were very interested in frustums because of the topless pyramid kind—they are were always designing and building things like that. They even knew the rule for the volume of square pyramidal frustum which in modern formula form is

$$V = \frac{\Delta h}{3}(a^2 + ab + b^2) ,$$

   where Δh is the height of the frustum (not the height of the pyramid it's cut from), a is the base square side length, and b is the top square side length. The ancient Egyptians probably deduced this rule by constructing a square pyramidal frustum from simpler parts (Boyer-21).

   There are parts a,b.

   a) By the power of pure guess, generalize the volume formula to that of a general frustum with base area A and top area B.

   b) Prove your generalization from the part (a) answer. **HINT:** Note the following factoids. Factoid 1: You can approximately replace any cone/frustum by a **SET** of equal-base-area square cones/frustums with their base-parallel slices slid appropriately: just picture it. Factoid 2: If you slide parallel slices of 3-dimensional shape, you don't change the volume of the shape (e.g., for parallelopiped obviously).

   c) Now derive the volume of a general cone with base area A and height h without using the equation in the preamble or the formula found in the parts (a) and (b) answers. **HINT:** The area of any base-parallel slice A_z is proportional to the square of the distance from the vertex to the slice z along the perpendicular from the base plane to the vertex. This is obvious if you envisage the slice as covered by a grid: each grid line obviously scales as z.

   d) Now what is the volume of a frustum with base of area A and height to the invisible vertex h, and top with area B and height to the invisible vertex h_B?

e) Given $\Delta h = h - h_B$, derive the formula found in the part (a) answer from the formula found in the part (d) answer. **HINT:** You will have to express h and h_B in terms of Δh, A, and B making use of the integrand used in the part (d) answer, and do some mildly tricky algebra which is accelerated by using the sum/difference of cubes formula:

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2) \ .$$

f) Who is responsible for

...
Come, every frustum longs to be a cone,
And every vector dreams of matrices.
Hark to the gentle gradient of the breeze:
It whispers of a more ergodic zone.
...
I see the eigenvalue in thine eye,
I hear the tender tensor in thy sigh.
Bernoulli would have been content to die,
Had he but known such $a^2 \cos(2\phi)$.

SUGGESTED ANSWER:

a) Behold:

$$V = \frac{\Delta h}{3}(A + \sqrt{AB} + B) \ .$$

Hero of Alexandria (c. 10–c. 70 CE) derived this formula first, but not in modern formula form, of course. The formula has as a factor the Heronian mean H of A and B:

$$H = \frac{1}{3}(A + \sqrt{AB} + B) = \frac{2}{3}\left(\frac{A+B}{2}\right) + \frac{1}{3}\sqrt{AB} \ ,$$

where the last expression shows that the Heronian mean is a weighted mean of the arithmetic mean and the geometric mean.

b) We approximately divide the general frustum (base area A and top area B) into as set of I small slid square frustums. The volume of the ith small frustum is

$$V_i = \frac{\Delta h}{3}(A_i + \sqrt{A_i B_i} + B_i) \ ,$$

where A_i and B_i are, respectively areas of base and top squares of small frustums: all A_i are equal, all B_i are equal. Note A_i and B_i go to zero as $I \to \infty$. Now the volume of the general frustum is

$$V = \lim_{I\to\infty} \sum_i \frac{\Delta h}{3}(A_i + \sqrt{A_i B_i} + B_i) = \frac{\Delta h}{3}\left(A + \lim_{I\to\infty}\sum_i \sqrt{A_i B_i} + B\right)$$
$$= \frac{\Delta h}{3}\left(A + \lim_{I\to\infty} I\sqrt{A_i B_i} + B\right) = \frac{\Delta h}{3}(A + \sqrt{AB} + B) : \text{QED.}$$

c) Behold:

$$V = \int_0^h A_z \, dz = A\int_0^h \left(\frac{z}{h}\right)^2 dz = Ah\int_0^1 x^2 \, dx = \frac{Ah}{3} \ .$$

d) By inspection:

$$V = \frac{1}{3}(Ah - Bh_B) \ .$$

e) Behold:

$$B = A\left(\frac{h_B}{h}\right)^2 = A\left(\frac{h - \Delta h}{h}\right)^2 \qquad \sqrt{\frac{B}{A}} = 1 - \frac{\Delta h}{h}$$
$$h = \frac{\Delta h}{1 - \sqrt{B/A}} \ .$$

and

$$B = A\left(\frac{h_B}{h}\right)^2 = A\left(\frac{h_B}{h_B+\Delta h}\right)^2 \qquad \sqrt{\frac{B}{A}} = \frac{1}{1+\Delta h/h_B} \qquad \sqrt{\frac{A}{B}} = 1+\Delta h/h_B$$

$$h_B = \frac{\Delta h}{\sqrt{A/B}-1} = \frac{\Delta h\sqrt{B/A}}{1-\sqrt{B/A}} \ .$$

Now we substitute for h and h_B into the formula found in the part (d) answer to get

$$V = \frac{\Delta h}{3}\left(\frac{A}{1-\sqrt{B/A}} - \frac{B}{\sqrt{A/B}-1}\right) = \frac{\Delta h}{3}\left(\frac{A\sqrt{A}}{\sqrt{A}-\sqrt{B}} - \frac{B\sqrt{B}}{\sqrt{A}-\sqrt{B}}\right)$$
$$= \frac{\Delta h}{3}\left(\frac{A\sqrt{A}-B\sqrt{B}}{\sqrt{A}-\sqrt{B}}\right) = \frac{\Delta h}{3}(A+\sqrt{AB}+B) \ ,$$

where we have used the difference of cubes to get the last formula: QED.

f) Stanislaw Lem (1921–2006) and translator Michael Kandel (1941–) in *The Cyberiad* (1965, translation 1974).

Redaction: Jeffery, 2018jan01

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002 qfull 00210 1 3 0 easy math: Pythagorean theorem I

3. The Pythagorean theorem was known to the ancient Babylonians, but not as far as known the ancient Egyptians, long before Pythagoras (c. 570–c. 495 BCE) (Boyer-42). But it is likely the ancient Babylonians never gave a general proof: they just did not think in terms of general proofs. The ancient Greek mathematicians may or may not have learnt of the Pythagorean theorem from the ancient Babylonians. However, they probably gave the first general proof. Late reports say Pythagoras himself proved it and hence its name. This may just be legend (Boyer-54; Wikipedia: Pythagorean theorem: History). Euclid (fl. 300 BCE) gives the first proof on the historical record. We will not attempt his proof, but something simpler. By the way, no one wrote equations like we do before circa 1600—they used other klutzy ways of expressing formulae (see Wikipedia: History of mathematical notation: Symbolic stage).

   Assume a Euclidean 2-dimensional space. Since the space is Euclidean or flat, a square (a 4-sided polygon with sides of equal length and right-angle vertices) has area $A = d^2$ where d is the length of a side. Prove the Pythagorean theorem for this Euclidean space. **HINT:** Draw a square with side length $a+b$ and an inscribed square of side length c where the vertices of the inscribed square touch the first square sides at the points that divide the sides into parts of length a an b.

SUGGESTED ANSWER:

You will have to imagine the diagram. Clearly, we have the area equality and following equations:

$$c^2 + 4(ab/2) = (a+b)^2$$
$$c^2 + 2ab = a^2 + b^2 + 2ab$$
$$c^2 = a^2 + b^2 \ ,$$

where the last result is the Pythagorean theorem: QED.

Yours truly learnt the above proof from Bertrand Russell's book *Wisdom of the West* (1959).

Redaction: Jeffery, 2018jan01

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002 qfull 00220 1 3 0 easy math: Pythagorean theorem II with area rule

4. In 2-dimensional Euclidean space (i.e., 2-dimensional flat space), we have a simple area principle. If you draw a general closed contour, you can tile it without overlap with squares of equal size with side length a. We define a^2 as the area of the squares. The sum of areas a^2 for closed contour in the limit that $a \to 0$ and number of squares goes to infinity is the area A of the closed contour. An identical closed contour anywhere in the space has the same area A and if you scale any the linear dimension of the contour by f, the area scales by f^2. Somewhat obviously, the area of two general closed contours

(joined or separated) must equal the sum of areas of the two general closed contours since the tiled areas just equal the count of squares of area a^2 times aread a^2 before you take the limit.

The area principle implies the Pythagorean theorem and consequently the metric of 2-dimensional flat space: $ds^2 = dx^2 + dy^2$, where x and y are general perpendicular coordinates and ds is the distance or interval between two points with coordinates that differ by dx and dy.

There are parts a,b,c,d. The parts can be done independently, and so do not stop if you cannot do a part.

a) Use the area principle to prove the area of a right triangle with sides of length a and b forming the right angle is $ab/2$. **HINT:** Imagine little squares of side length e and tile a rectangle with them, count the squares, find the area of the rectangle as $e \to 0$ and the number of squares goes to infinity, and then use symmetry.

b) Draw a diagram of a square with sides of length $a + b$ and an inscribed square with side of length c with corners touching the sides of the first square (which is the circumscribed square) at points a from each corner of the first square.

c) Use answers from the parts (a) and (b) to prove the Pythagorean theorem: i.e., $c^2 = a^2 + b^2$.

d) Prove the metric $ds^2 = dx^2 + dy^2$ holds for a 2-dimensional flat space. **HINT:** This is easy.

SUGGESTED ANSWER:

a) Say you had a rectangle with side lengths a and b. You tiled it without overlap with little squares of side length e. For finite e, the area covered would be approximately $\text{Int}(a/e + 1/2) \times \text{Int}(b/e + 1/2) \times e^2 \approx ab$, where "Int" is the round-off-to-integer function. In the limit $e \to 0$ and the number of squares goes to infinity, the rectangle area is clearly ab. Form two identical right triangles by a diagonal bisecting the rectangle. By symmetry, the area of each right triangle is $(ab)/2$

b) You will have to imagine the diagram or view it at Wikipedia: Pythagorean theorem: Pythagorean proof.

c) Behold:

$$\begin{aligned} 4(ab/2) + c^2 &= (a + b)^2 \\ 2ab + c^2 &= a^2 + b^2 + 2ab \\ c^2 &= a^2 + b^2 \ , \end{aligned}$$

with the last line being the Pythagorean theorem itself: QED.

d) If Pythagorean theorem $c^2 = a^2 + b^2$ holds for finite perpendicular distances a and b, then it must hold for differential coordinate differences dx and dy. Thus, $ds^2 = dx^2 + dy^2$ is the metric for 2-dimensional flat space, QED.

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002 qfull 00230 1 3 0 easy math: Pythagorean theorem III with area rule with Euclid's 5th postulate

5. Can we prove the Pythagorean theorem semi-rigorously? Yes.

There are parts a,b,c,d,e,f,g,h,i. The parts can be done independently, and so do not stop if you cannot do a part.

a) Assume an homogeneous, isotropic (homist) 2-dimensional space. Assume there is a geodesic rule: i.e., there is a rule for measuring distance and for measuring the stationary distance between two points. Draw intersecting equal length geodesics that intersect at their midpoints and that have 4-fold rotational symmetry about their intersection point. A full rotation about the intersection point is measured as 360°. How would you describe size of the angles subtended at the intersection point separating the crossed geodesic arms and why would you say this? Note draw the geodesics vertical and horizontal, so that the descriptions in the following parts are consistent with the diagram.

b) Now draw geodesics between the endpoints of your crossed geodesics, but note we are not assuming Euclidean (i.e., flat space) so that these geodesics could bend outward/inward from intersection point in some projection or another. You now have a square (but not necessarily a Euclidean square). Call it square 1. Now copy square 1 to square 2 and translate square 2 to the upper right so that the lower left corner endpoints of square 2 lie on the upper right corner endpoints of square 1. Is there a space between geodesics of the two squares joining common endpoints? Why or why not?

c) Now copy square 2 to square 3 and translate square 3 to the lower right, but otherwise with the same intructions as in part (b). Now copy square 3 to square 4 and translate square 4 to the lower left, but otherwise with the same intructions as in part (b). Does square 4 necessarily share a common geodesic with the original square 1? Why or why not?

d) The answer to part (c) was no. However, if there is a common geodesic then the space is a Euclidean plane and, at the common corner of the 4 squares, the angles between the geodesics that meet there are all 90°. Postulating that they are 90° is equivalent to Euclid's 5th postulate. For long ages mathematicians wondered if 5th postulate was derivable from Euclid's first 4 postulates. The answer is no. Even somewhat obviously no since, among other things, geodesics that are parallel on a sphere at the equator (i.e., separated by a mutually perpendicular geodesic there) meet at the poles.

Assuming a Euclidean plane, prove that ines (as we now call geodesics) parallel at one location (i.e., separated by a mutually perpendicular line) stay the same perpendicular distance apart no matter how extended. There are probably many ways of proving this, but one path is to start by noting that equal squares of any size can tile the whole Euclidean plane without overlap which actually is an immediate consequence of our considerations above.

e) The fact that one can tile the Euclidean plane completely with squares without overlap suggests an area concept. Consider differential rectangles of side lengths dx and dy. Define their area to be $dx\,dy$. We define area to be countable in the sense that the area of N rectangles is $N\,dx\,dy$. You can tile completely any region surrounded by a closed curve with equal differential rectangles with no rectangles wholly out of the region. We define the area of the region by

$$A = \lim_{N\to\infty,\, dx\,dy\to 0} N\,dx\,dy \ .$$

That such limit exists in general requires a rigorous proof that we will not do here. However, one can prove the limit exists in special cases easily and those special cases they also show why defining the area of the differential rectangles in terms of the lengths of their sides is reasonable since finite regions of sufficient symmetry also have areas specified by their defining lengths. An important point is that area is independent of the ordering of the adding up the differentials areas. As a nonce expression, we call this independence the area principle.

Determine the area of a large rectangle of sides a and b in terms of differential rectangles and take the limit so that the properties of the differential rectangles vanish.

f) Prove that the area of a right triangle with sides forming the right angle being of length a and b is $ab/2$. **HINT:** You do need to use the area principle.

g) Draw a diagram of a square with sides of length $a+b$ and an inscribed square with side of length c with corners touching the sides of the first square (which is the circumscribed square) at points a from each corner of the first square.

h) Use the area principle to prove the Pythagorean theorem: i.e., $c^2 = a^2 + b^2$.

i) Prove the metric $ds^2 = dx^2 + dy^2$ holds for a Euclidean plane. **HINT:** This is easy.

SUGGESTED ANSWER:

a) Imagine the diagram and the angles would be described as 90° since the 4-fold symmetry implies they are equal and each a quarter of 360°.

b) There is no gap, because the corners of the original and copy are joined by geodesics in our homist 2-dimensional space and those two geodesics must the same geodesic by the homist properties of the space.

c) Square 4 and square 1 do not necessarily share a common geodesic. They share a common corner point with each other and the other squares, but angles between the geodesics meeting at this corner do not have to be 90° in general.

d) If you can tile the whole plane without overlap by squares of any size, then you can tile one square by four smaller squares. Consider one side of the big square. It is a line between the corners of the big square. The two small squares that fill between those both have sides that are lines coincident with the line of the big square. Thus, in general squares arranged in a row have sides that form two lines because any two points on one of those lines have their shortest distance apart along those lines and the squares can have any size we like and this must still be true. Those lines are parallel at every point no matter how extended the row and they are always the same perpendicular distance apart.

What about the converse? If two lines are not perpendicular at some location (i.e., there is a location where they are not separated by a mutually perpendicular line), must they intersect? The answer is yes, but I cannot think of a concise proof at the moment.

e) We make the differential rectangles similar to the large rectangle such that N differential rectangles span the x direction and N span the y direction. In fact, N is just an integer scaling factor. The area of the large rectangle is

$$A = N^2\,dx\,dy = N^2\left(\frac{a}{N}\right)\left(\frac{b}{N}\right) = ab\ .$$

Thus, the area of the large rectangle does not, in fact, depend on the size or number of the little rectangles, but just on its own lengths and, in fact, the product of those lengths. This suggests that defining differential area by a product of lengths is rational for the reason given in the question.

f) Just bisect a rectangle of sides a and b by a diagonal to get two triangles fitting the specifications. The two parts must have equal area by symmetry. The area principle then implies that those areas must be the rectangle area divided by 2 since that area is independent of how the differential areas are ordered in adding process to get area. Thus, the area of each triangle is $ab/2$.

g) You will have to imagine the diagram or view it at Wikipedia: Pythagorean theorem: Pythagorean proof.

h) Behold:

$$\begin{aligned} 4(ab/2) + c^2 &= (a+b)^2 \\ 2ab + c^2 &= a^2 + b^2 + 2ab \\ c^2 &= a^2 + b^2\ , \end{aligned}$$

with the last line being the Pythagorean theorem itself: QED.

i) If Pythagorean theorem $c^2 = a^2 + b^2$ holds for finite perpendicular distances a and b, then it must hold for differential coordinate differences dx and dy. Thus, $ds^2 = dx^2 + dy^2$ is the metric for the Euclidean plane, QED.

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002 qfull 00310 1 3 0 easy math: Golden Ratio and Fibonacci sequence, golden ratio

6. The golden ratio ϕ is a special number known since Greco-Roman antiquity. But there's nothing especially special about it. There are many special numbers: all small natural numbers $(0, 1, 2, \ldots)$, all small prime numbers $(2, 3, 5, 7, 11, 13, \ldots)$, $e = 2.71828\ldots$, pi $= \pi = 3.14159\ldots$, the Euler-Mascheroni constant $\gamma = 0.57721566\ldots$, etc. Here we will investigate the golden ratio just a bit.

   There are parts a,b.

a) Draw a line segment of length a and divide into two parts of of lengths b and c: thus $a = b + c$. The golden ratio is just the ratio when

$$\frac{a}{b} = \frac{b}{c}\ .$$

b) Let's do a general investigation of ratios of the form

$$\frac{a}{b} = g\frac{b}{c}\ ,$$

where $a = b + c$. Solve for the positive case of the ratio a/b as a function of g only. Find the cases for $g = \infty, 1, 0$. The case $g = 1$ gives the golden ratio itself.

c) Prove that

$$\frac{1}{\phi} = \phi - 1 \ .$$

d) In 1202, Fibonacci, perhaps independantly of Indian mathematics, discovered the Fibonnaci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ which has an interesting connection to the golden ratio.

The discovery was from the problem of reproducing pairs of rabbits. A pair takes 1 month to mature from birth and reproduces a new pair after maturity every one month: so the first reproduction happens 2 months after birth. Consider times t_i separated by 1 month periods. Say you at time t_{i-1} you had n_{i-1} adult pairs. However, only the adult pairs n_{i-2} existing at time t_{i-2} can reproduce at t_{i-1} since the new baby pairs at time t_{i-2} have only just matured at t_{i-1}. So at t_{i-1}, the old adult pairs n_{i-2} produce n_{i-2} babies who mature to be adult pairs at time t_i. So the number of adult pairs at time t_i is

$$n_i = n_{i-1} + n_{i-2}$$

which is, of course, a recurrence relation valid for $i \geq 2$.

Starting with 1 baby pair and no adult pairs at time zero, compute by inspection the Fibonnaci sequence until you get bored.

e) In the limit $i \to \infty$, the ratio of adjacent numbers following from Fibonacci recurrence relation

$$n_i = n_{i-1} + n_{i-2}$$

for $i \geq 2$, $n_0 \geq 0$, and $n_1 > 0$ obeys

$$R_i = \frac{n_i}{n_{i-1}} \to \phi \ .$$

Note we are allowing more general initial n_i values than for Fibonacci sequence. In fact, the R_i's alternate with every increment of i between being too high and too low compared to ϕ as $i \to \infty$ and they go to ϕ exactly for finite i in only one special case. Prove the above statements. **HINT:** Start from the Fibonacci recurrence relation, use the definition $\epsilon_i = (R_i - \phi)/\phi$, and remember the part (c) result.

f) Find a reasonable approximate asymptotic formula for the n_i from part (e) as $i \to \infty$. It should be exactly correct in one special case.

g) The recurrence relation

$$n_i = n_{i-1} + n_{i-2}$$

can be turned into a differential equation by changing i into continuous variable t expanding n_t and $n(t-2)$ about $t-1$ to 1st order. Make the transformation and solve the differential equation. How does the solution compare to the approximate asymptotic formula of part (f)?

SUGGESTED ANSWER:

a) You will have to imagine the diagram.

b) Behold:

$$\frac{a}{b} = g\frac{b}{c} \qquad a^2 - ab = gb^2 \qquad 0 = a^2 - ab - gb^2 \qquad a = \frac{b \pm \sqrt{b^2 + 4gb^2}}{2}$$

$$\frac{a}{b} = \begin{cases} \dfrac{1+\sqrt{1+4g}}{2} & \text{in general;} \\ g \to \infty & \text{for } g \to \infty; \\ \dfrac{1+\sqrt{5}}{2} = 1.6180\ldots = \phi & \text{for } g = 1. \text{ This is the golden ratio;} \\ 1 & \text{for } g = 0. \end{cases}$$

c) Behold:

$$\frac{1}{\phi} = \left(\frac{2}{1+\sqrt{5}}\right)\left(\frac{\sqrt{5}-1}{\sqrt{5}-1}\right) = \frac{2(\sqrt{5}-1)}{4} = \frac{\sqrt{5}-1}{2} = \frac{2\phi - 1 - 1}{2} = \phi - 1 \ ,$$

and so

$$\frac{1}{\phi} = \phi - 1 \qquad \text{and} \qquad \phi = 1 + \frac{1}{\phi} \ .$$

d) Behold:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$$

and so boredom.

e) Behold:

$$n_i = n_{i-1} + n_{i-2} \qquad R_i = 1 + \frac{1}{n_{i-1}/n_{i-2}} \qquad (\epsilon_i + 1)\phi = 1 + \frac{1}{R_{i-1}}$$

$$\epsilon_i + 1 = \frac{R_{i-1} + 1}{R_{i-1}\phi} \qquad \epsilon_i = \frac{R_{i-1} + 1 - R_{i-1}\phi}{R_{i-1}\phi} \qquad \epsilon_i = \frac{R_{i-1} + 1 - R_{i-1}(1 + 1/\phi)}{R_{i-1}\phi}$$

$$\epsilon_i = -\frac{\epsilon_{i-1}}{R_{i-1}\phi}$$

which, of course, is only valid for $i \geq 2$. Note $R_i > 1$ for $i \geq 2$, but

$$R_1 = \frac{n_1}{n_0} \in (0, \infty] \qquad \text{implying} \qquad \epsilon_1 \in (-1, \infty] \ ,$$

and so the ϵ_i formula can be indeterminate for $i = 2$ if one has $n_0 = 0$ implying $R_1 = \infty$ and $\epsilon_1 = \infty$. However, ϵ_i for $i \geq 2$ are all finite, and so one just considers the ϵ_i formula for $i \geq 3$ if $n_0 = 0$. Note also if $n_0 = 0$, then $R_2 = 1$ and $\epsilon_2 = (1 - \phi)/\phi = -1/\phi^2 \neq \phi$.

From the ϵ_i formula, the ϵ_i's change sign with every increment of i, and thus R_i must alternate about ϕ as i increases. Since, $R_{i-1} > 1$ for $i > 3$ for sure and the ϵ_i formula is always determinate for $i > 3$ for sure, clearly the absolute value of ϵ_i must decreases to zero as $i \to \infty$ impying $R_i \to \phi$ as $i \to \infty$. Since R_i never goes to infinity for finite $i > 1$, the only way the sequence of ϵ_i's can go to zero for finite i is if $\epsilon_1 = 0$ implying $R_i = \phi$ for all $i \geq 1$. Note $\epsilon_1 = 0$ means $R_1 = n_1/n_0$.

Note if $n_0 = 0$, then $\epsilon_2 \neq \phi$ as shown above and ϵ_i again cannot go to zero for finite i.

f) Behold the approximate asymptotic formula

$$n_i^* = n_2\phi^{i-2} = (n_1 + n_0)\phi^{i-2} \ ,$$

where the asterisk $*$ means asymptotic. The formula gives the correct asymptotic $R_i^* = \phi$, is exactly correct for $i = 2$, and is exactly correct in the special case that $R_1 = \phi$.

To go beyond the required answer, does the asymptotic formula have the correct asymptotic behavior for the n_i as $i \to \infty$ in general? No as we show in the special case of the Fibonacci sequence below.

First, is there an exact asymptotic formula

$$n_i^{**} = C\phi^i$$

(where C is a constant) such that asymptotic n_i^{**} is exact as $i \to \infty$? Probably since *Wikipedia: Fibonacci number: Closed-form expression* says that all linear reccurence relations with constant coefficients have a closed-form solutions. For the Fibononacci sequence, this closed-form solution is

$$n_i = \frac{\phi^i - \phi^{-i}}{\sqrt{5}}$$

which leads to the exact asymptotic formula for $i \to \infty$

$$n_i^{**} = \frac{\phi^i}{\sqrt{5}} \ .$$

So here the coefficient $C = 1/\sqrt{5}$. Corresponding coefficient for our approximate asymptotic formula above in the case of the Fibonancci sequence is

$$1 \times \phi^{-2} = \frac{1}{\phi^2} = \frac{1}{\phi(1 + 1/\phi)} = \frac{1}{\phi + 1} = \frac{1}{2.618\ldots} < \frac{1}{\sqrt{5}} = \frac{1}{2.236\ldots}$$

g) Behold:

$$n_i = n_{i-1} + n_{i-2} \qquad n(t) = n(t-1) + n(t-2)$$

$$n(t-1) + 1 \times \left.\frac{dn}{dt}\right|_{(t-1)} = n(t-1) + n(t-1) - 1 \times \left.\frac{dn}{dt}\right|_{(t-1)} \qquad \text{to 1st order.}$$

$$2\left.\frac{dn}{dt}\right|_{(t-1)} = n(t-1)) \qquad \text{to 1st order.}$$

$$\frac{dn}{dt} = \frac{1}{2}n \qquad \text{is the approximate differential equation.}$$

$$n(t) = n_0 e^{t/2} \qquad \text{is its solution.}$$

Note that

$$\frac{n(t)}{n(t-1)} = e^{1/2} = 1.64872\ldots \gtrsim \phi = 1.61803\ldots \ ,$$

and so the differential equation approximation to the recurrence relation is only so-so good. Clearly, truncating to 1st order is not adequate for the differential equation to yield the exact asymptotic ratio.

Fortran-95 Code

```
      print*
      coef=((sqrt(5._np)+1._np)/2._np)**2
      print*,'sqrt(5._np),coef'
      print*,sqrt(5._np),coef
! 2.2360679774997896964          2.6180339887498948483
      phi=(sqrt(5._np)+1._np)*.5_np
      con=exp(0.5_np)
      print*,'phi,con'
      print*,phi,con
! 1.6180339887498948482          1.6487212707001281469
```

Redaction: Jeffery, 2018jan01

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002 qfull 00410 1 3 0 easy math: quadratic formula made numerically robust

7. The quadratic formula (which is the solution of the quadratic equation) is an infamous example of case where the standard analytic form (which is what everyone remembers) is numerically rotten. The equation and formula in standard form are, respectively,

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$

The numerical rottenness occurs if $|4ac| << b^2$: in this case, one of the roots can become affected by severe round-off error. We'll see how to fix the problem in this problem.

NOTE: There are parts a,b,c,d,e,f. The parts cannot be done independently, but parts (a) and (b) are not so hard and the later parts are just intricate.

a) Solve the quadratic equation for the standard quadratic formula using completing the square. Note we assume that a, b, and c are pure real numbers.

b) The crucial insight is that root cause of the numerical problem is the sign of b. If $|4ac| << b^2$, then the standard formula gives numerically good solution for one sign of b and numerically bad one for the other. Note if $b = 0$, there is no problem at all:

$$x = \pm\sqrt{\frac{-c}{a}} \ .$$

So the trick to getting a numerically robust quadratic formula is to isolate sign of b: i.e., to factorize b into its sign and absolute value. Rewrite the standard formula in the form

$$x_\pm = \frac{-\text{sgn}(b)|b| \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{1}{2}\text{sgn}(b)\left(\frac{|b| \pm \sqrt{b^2 - 4ac}}{a}\right) \ ,$$

where we note that the $\text{sgn}(b)(\pm 1) = (\pm 1)$ if the (± 1) is uncorrelated with the $\text{sgn}(b)$, using a bit of clairvoyance for a nice formula we put the factor of $1/2$ where it's been put, and the sign function is given by

$$\text{sgn}(b) = \begin{cases} 1 & \text{for } b > 0. \\ 1 & \text{for } b = 0 \text{ which is unlike the usual definition of 0.} \\ -1 & \text{for } b < 0. \end{cases}$$

As now written, we can see that solution x_+ is numerically robust, but solution x_- is not. But you can make solution x_- robust by using the a difference of squares factor. Write the numerically robust quadratic formula for solution x_- in terms

$$q = -\frac{1}{2}\text{sgn}(b)\left(|b| + \sqrt{b^2 - 4ac}\right)$$

when the moment is right. **HINT:** Recall the difference of squares formula:

$$(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2 \ .$$

c) What can you say about the robust solutions when the discriminant $(b^2 - 4ac) < 0$ and what can you say about q, a, b, and c in this case?

d) What can you say about the robust solutions when $a = 0$ and $q \neq 0$, and what can you say about q, b, and c in this case?

e) What can you say about the robust solutions when $a \neq 0$ and $q = 0$, and what can you say about a, b, and c in this case?

f) What can you say about the robust solutions when $a = 0$ and $q = 0$, and what can you say about b and c in this case?

SUGGESTED ANSWER:

a) Assuming a is nonzero, we proceed as follows:

$$1)\quad 0 = ax^2 + bx + c \qquad 2)\quad 0 = x^2 + \frac{b}{a}x + \frac{c}{a} \qquad 3)\quad 0 = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$$

$$4)\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \qquad 5)\quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \qquad 6)\quad x + \frac{b}{2a} = \pm\frac{1}{|2a|}\sqrt{b^2 - 4ac}$$

$$7)\quad x + \frac{b}{2a} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac} \qquad 8)\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$

Note that we have made use of the fact that $(\pm)(\pm) = (\pm)$ and $(\pm)(\mp) = (\pm)$ if the two cases of upper and lower cases are not correlated. Thus $\pm|2a| = (\pm)(\pm 2a) = \pm 2a$. Of course, if two cases of upper and lower cases are correlated, then $(\pm)(\pm) = 1$ and $(\pm)(\mp) = -1$.

b) Behold:

$$\begin{aligned} x_\pm &= -\frac{1}{2}\text{sgn}(b)\left(\frac{|b| \pm \sqrt{b^2 - 4ac}}{a}\right) \\ &= \begin{cases} \dfrac{q}{a} & \text{for the upper case.} \\ -\dfrac{1}{2}\text{sgn}(b)\left[\dfrac{-|b|^2 + b^2 - 4ac}{a\left(-|b| - \sqrt{b^2 - 4ac}\right)}\right] & \text{for the lower case.} \end{cases} \\ &= \begin{cases} \dfrac{q}{a} & \text{for the upper case.} \\ -\dfrac{1}{2}\text{sgn}(b)\left[\dfrac{-4ac}{a\left(-|b| - \sqrt{b^2 - 4ac}\right)}\right] & \text{for the lower case.} \end{cases} \\ &= \begin{cases} \dfrac{q}{a} & \text{for the upper case.} \\ \dfrac{1}{[-(1/2)\text{sgn}(b)]}\left(\dfrac{c}{|b| + \sqrt{b^2 - 4ac}}\right) & \text{for the lower case.} \end{cases} \end{aligned}$$

$$= \begin{cases} \dfrac{q}{a} & \text{for the upper case.} \\ \dfrac{c}{q} & \text{for the lower case.} \end{cases}$$

We now see that this version quadratic formula is numerically robust because q is not subject to round-off error when $|4ac| << b^2$ since it involves only an addition of $|b|$ and $\sqrt{b^2 - 4ac}$. Recall

$$q = -\frac{1}{2}\text{sgn}(b)\left(|b| + \sqrt{b^2 - 4ac}\right) \ .$$

c) If discriminant $(b^2 - 4ac) < 0$, there are two complex solutions. All you can say about about q, a, b, and c in this case is that q is complex, $|b| < 2\sqrt{ac}$, and neither a and c can be zero and both must be positive or both must be negative.

d) In this case, x_+ is indeterminate, $q = -b$, and $x_- = -c/b$ is the only solution. Note $b \neq 0$ since $q \neq 0$ and c is unconstrained. Note also that the $x_- = -c/b$ solution is what you get directly from the quadratic equation with $a = 0$, and so is exactly correct.

e) In this case, x_- is indeterminate and $x_+ = 0$ is the only solution. Note since $q = 0$, we must have $b = 0$ (since $\sqrt{b^2 - 4ac}$ can only contribute a positive value or an imaginary value to q and neither of them can cancel $|b| \neq 0$) and then $c = 0$ for $q = (1/2)\text{sgn}(b)\sqrt{-4ac} = 0$ with $a \neq 0$.

f) In this case, the both x_+ and x_- are indeterminate and there are no solutions. Since $q = 0$, we have $b = 0$ (since $\sqrt{b^2 - 4ac}$ can only contribute a positive value or an imaginary value to q and neither of them can cancel $|b| \neq 0$). Also, $c = 0$ for a consistent quadratic equation.

Redaction: Jeffery, 2018jan01

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002 qfull 00510 1 3 0 easy math: simple 1st order DE solution

8. Consider the following linear 1st order differential equation (DE):

$$x' = A - kx \ ,$$

where t is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant.

There are parts a,b,c,d. Parts (a) and (b) can be done independently at least.

a) Solve for the constant solution x_A. **HINT:** This is easy.

b) We can now write the DE as

$$x' = k\left(x_A - x\right) \ .$$

Without solving for non-constant solution describe what it must look like as a function of t for arbitrary initial value $x_0 = x(t = 0)$. In particular, where are its stationary points if any? **HINT:** Consider the continuity of all orders of derivative of x.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$, $x'(t)$, and the 1st order in small t solution $x_{1\text{st}}(t)$. **HINT:** You can use an integrating factor, but there is a more straightforward way.

d) What is the e-folding time t_e of your solution and what does it signify? What is the $x(t_e)$? What is the $x_{1\text{st}}(t_e)$? What is remarkable about $x_{1\text{st}}(t_e)$?

SUGGESTED ANSWER:

a) The constant solution has $x' = 0$ everywhere. Therefore

$$x_A = \frac{A}{k} \ .$$

temb) Consider intelligently

$$x' = k\left(x_A - x\right) \ .$$

If x_0 is less/greater than x_A, then x' is greater/less than 0, and then x must increase/decrease until $x = x_A$, where $x' = 0$. Now since $x^{(n)} = -kx^{(n-1)}$ for all $n \geq 2$, all orders of derivative must go to zero at the same time t without discontinuities. But for any finite time, there must be a discontinuity in some derivative for them all to go to zero at the same time since the function goes perfectly flat at that time. Therefore, x' can only go to zero at infinity: i.e., asymptotically as $t \to \infty$. It follows at once that the only stationary point is at infinity: it's a maximum/minium for x_0 is less/greater than x_A.

c) Behold:

$$x' = k(x_A - x) \qquad \frac{dx}{x_A - x} = k \qquad -\ln(x_A - x) = kt + C \qquad x_A - x = (x_A - x_0)e^{-kt}$$
$$x = x_0 e^{-kt} + x_A(1 - e^{-kt}) \ ,$$

where the first term is the transient solution (i.e., small t solution) and the second, the asymptotic solution (i.e., large t solution). The solution $x(t)$ matches the description of part (b).

The derivative is

$$x' = k(x_A - x_0)e^{-kt}$$

and to 1st order in small t, we have

$$x_{\rm 1st} = x_0(1 - kt) + x_A kt \ .$$

Just for completeness, using an integrating factor, one obtains the solution thusly:

$$x' = A - kx \qquad x' + kx = A \qquad gx' + gkx = gA$$
$$(gx)' = gx' + g'x \qquad g' = gk \qquad g = e^{kt} \qquad (gx)' = gA$$
$$e^{kt}x|_{t=0}^{t} = (A/k)e^{kt}|_{t=0}^{t} \qquad e^{kt}x - x_0 = x_A(e^{kt} - 1) \qquad x = x_0 e^{-kt} + x_A(1 - e^{-kt}) \ .$$

d) Behold: $t_e = 1/k$. Well $t = t_e$ is the fiducial time for transient solution to start vanishing exponentially and the asymptotic solution to start approaching the asymptotic value x_A. At $t = t_e$ for the solution and 1st order solution, we have, respectively,

$$x(t = t_e) = x_0 e^{-1} + x_A(1 - e^{-1}) \qquad \text{and} \qquad x_{\rm 1st}(t = t_e) = x_A \ .$$

Remarkably, $x_{\rm 1st}(t = t_e)$ is independent of x_0 and equals the asymptotic value x_A.

Redaction: Jeffery, 2018jan01

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002 qfull 00520 1 3 0 easy math: simple 1st order DE solution variant: conflate with 00510?

Extra keywords: Has part (b) of the original 01010. Is it worth anything?

9. Consider the following linear 1st order differential equation:

$$x' = A - kx \ ,$$

where t is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant.

There are parts a,b,c,d. Parts (a) and (b) can be done independently at least.

a) Solve for the constant solution x_A. **HINT:** This is easy.

b) Where is it possible for a non-constant solution of a 1st order differential equation to have a stationary point? Will there be stationary points at those t locations for the particular differenital equation of the preamble? **HINT:** Consider the differential equation written in the form

$$x' = k\left(\frac{A}{k} - x\right)$$

and consider what happens to the solution as $t \to \infty$ and remember that if the solution becomes constant, it stays constant. It helps to think graphically.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$ and the 1st order in small t solution $x_{\rm 1st}(t)$. **HINT:** You can use an integrating factor, but there is a more straightforward way.

d) What is the e-folding constant t_e and what does it signify? What is the $x(t_e)$? What is the $x_{\rm 1st}t_e)$? What is remarkable about $x_{\rm 1st}(t_e)$?

SUGGESTED ANSWER:

a) The constant solution has $x' = 0$ everywhere. Therefore

$$x_A = \frac{A}{k} \ .$$

b) A non-constant solution of a 1st order differential equation can (but not necessarily will) have stationary points where it is not infinintely differentiable and will certainly have stationary points where its 1st derivative is zero, but some higher order derivatives are not zero because of a zero-over-zero cancellation when the differential equation is differentiated A common example of the first kind of stationary point is at $\pm\infty$ which are points where the solution is not formally differentiable in the strict sense. For example, the solution of logisitic differential equation (i.e., the logistic function) clearly has stationary points at $\pm\infty$:

$$x' = x(1-x) \qquad x = \frac{1}{1+e^{-t}} = \begin{cases} 1 - e^{-t} & \text{for } t \to \infty; \\ e^t & \text{for } t \to -\infty. \end{cases}$$

A common example of the second kind of stationary point is as follows:

$$x' = \pm\sqrt{1-x^2} \qquad x'' = \mp\frac{xx'}{\sqrt{1-x^2}} = -x \qquad x = \sin(t) \ .$$

Since $x' = A - kx$ is infinitely differentiable everywhere (except at $\pm\infty$, of course) and has no points where a zero 1st derivative is canceled out to give non-zero higher order derivatives, there can only potentially be stationary points at $t = \pm\infty$. From the hint, we see that as $t \to \infty$, the solution converges to $x = A/k$ and after which it stays constant with $x' = 0$. But if the solution is constant at a finite time, it is always constant. Therefore, it can only reach $x = A/k$ at $t = \infty$. So there is a stationary point at $t = \infty$. On the other hand, as $t \to -\infty$, the solution diverges from $x = A/k$ and the derivative x' never goes to zero, and so there is no stationary point at $t = -\infty$.

c) Behold:

$$x' = A - kx \qquad \frac{dx}{A-kx} = 1 \qquad \frac{1}{(-k)}\ln(A-kx) = t + C \qquad A - kx = Ce^{-kt}$$

$$x = Ce^{-kt} + \frac{A}{k} = Ce^{-kt} + x_A \qquad x_0 = C - x_A \qquad x = x_0 e^{-kt} + x_A(1 - e^{-kt}) \ ,$$

where the first term is the transient solution (i.e., small t solution) and the second, the asymptotic solution (i.e., large t solution). To 1st order in small t, we have

$$x_{\rm 1st} = x_0(1 - kt) + x_A kt \ .$$

Using an integrating factor, one obtains the solution thusly:

$$x' = A - kx \qquad x' + kx = A$$

$$gx' + gkx = gA \qquad (gx)' = gx' + g'x \qquad g' = gk \qquad g = e^{kt} \qquad (gx)' = gA$$

$$e^{kt}x|_{t=0}^{t} = (A/k)e^{kt}|_{t=0}^{t} \qquad e^{kt}x - x_0 = x_A(e^{kt} - 1) \qquad x = x_0 e^{-kt} + x_A(1 - e^{-kt}) \ .$$

d) Behold: $t_e = 1/k$. Well $t = t_e$ is the fiducial dividing point between the transient and asymptotic solutions. At $t = t_e$ for the solution and 1st order solution, we have, respectively,

$$x(t = t_e) = x_0 e + x_A(1 - e) \qquad \text{and} \qquad x_{\rm 1st}(t = t_e) = x_A \ .$$

Remarkably, $x_{\rm 1st}(t = t_e)$ is independent of x_0 and equals the asymptotic value x_A.

Redaction: Jeffery, 2018jan01

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002 qfull 00550 1 5 0 tough math: Solving a 1st-order polynomial DE

Extra keywords: (Tu2003-12) Not very relevant to cosmology and needs reworking

10. Consider the 1st order nonlinear differential equation

$$x' = a\prod_{i=1}^{n}(x - x_i) \ ,$$

where t (which may or may not be time) is the independent variable, a is constant, and the x_i are the roots of the polynomial on the right-hand side: the roots increase monotonically with index i: i.e., they obey $x_1 \le x_2 \le ... \le x_n$.

a) Solve the equation for the general solution for $n = 0$: i.e., when $x' = a$.

b) Solve the equation for the general solution for $n = 1$: i.e., when $x' = a(x - x_1)$. Since this is a warm-up question, a solution by inspection is not adequate.

c) Qualitatively and compactly describe the solutions of the differential equation in all regions for $n \ge 2$. **HINT:** The equation is a 1st order differential equation and the right-hand side is infinitely differentiable everywhere. There are 4 cases to consider. Don't forget to describe the stability of the constant solutions: i.e., does a sufficiently small perturbation lead to a restoration to the constant solution or a permanent departure from it.

d) Consider distinct roots x_{j-1} and x_j for the case with $n \ge 2$. Find an approximate interpolation solution which has the correct values at $t = \pm\infty$. The approximate solution should contain the function element ge^{-ht} where h can be positive or negative, but not zero and $g > 0$ always. The values of h and g are determined in part (e) just below. **HINT:** This is pretty easy.

e) Continuing with the problem from part (d), determine h by requiring that the approximate solution satisfy the differential equation at the midpoint $x = (x_j + x_{j-1})/2$ and g by requiring that it pass through the point (t_0, x_0), where $x_0 \in (x_{j-1}, x_j)$. **HINT:** This is a lot easier than it seems at first.

f) Continuing with the problem from part (d), show that the approximate formula is, in fact, the exact solution for the case of $n = 2$. This solution is called the logistic function. **HINT:** Simplify the formula for h and then differentiate the solution for $n = 2$ and keep substituting the solution for $n = 2$ to eliminate the h and ge^{-ht} function elements.

g) Now solve the equation for the general solution for general $n \ge 2$ and all roots the same x_r: i.e., for $x_i = x_r$ for all i. **HINT:**

SUGGESTED ANSWER:

a) Behold:

$$x = at + x_0 \ ,$$

where x_0 is a constant of integration. So we have a linear solution: i.e., a straight line.

b) Behold:

$$\begin{aligned} x' &= a(x - x_1) \\ \frac{dx}{(x - x_1)} &= a \\ \ln(x - x_1) &= at + c \\ x &= x_0 e^{at} + x_1 \ , \end{aligned}$$

where x_0 is a constant of integration. So we have an exponential function solution.

c) There are 4 cases:

1 There are n constant solutions: $x = x_i$. Nondistinct roots have the same constant solutions. We will describe the stability of these solutions after considering the 2nd case.

2 Between the general distinct roots x_{i-1} and x_i, there is a monotonic solution that has its only stationary points at infinity (SPIs) $t = \pm\infty$. These stationary points

are the maximum and minimum of the solution. If $a > 0$ and $i_{\text{count down}} = (n-i)+1)$ is even/odd, then the solution is monotonically increasing/decreasing and the minimum/maximum is at x_{i-1} and the maximum/minimum is at x_i. If $a < 0$ and $i_{\text{count down}} = (n-i)+1)$ is odd/even, then the solution is monotonically increasing/decreasing and the minimum/maximum is at x_{i-1} and the maximum/minimum is at x_i.

We can see that the constant solution $x = x_i$ is stable/unstable if $a > 0$ and $i_{\text{count down}} = (n-i)+1)$ is even/odd and stable/unstable if $a < 0$ and $i_{\text{count down}} = (n-i)+1)$ is odd/even.

3 For $x > x_n$ and $a > 0$ ($a < 0$), there is a monotonically increasing (decreasing) solution with minimum x_n at $-\infty$ (∞) and which goes to ∞ at finite t which we know from part (f) or clairvoyance.

3 For $x < x_1$ and $a > 0$ ($a < 0$), there is a monotonically increasing (decreasing) solution with maximum x_1 at ∞ (-∞) and which goes to $-\infty$ at finite t which we know from part (f) or clairvoyance.

d) By pure thought the approximate solution is

$$x_{\text{approx}} = \begin{cases} \dfrac{x_j - x_{j-1}}{1 + ge^{-ht}} + x_{j-1} & \text{in general;} \\ \dfrac{x_j + x_{j-1}}{2} & \text{for } ge^{-ht} = 1; \\ x_j & \text{for } t \to \infty \text{ and } h > 0; \\ x_{j-1} & \text{for } t \to -\infty \text{ and } h > 0; \\ x_{j-1} & \text{for } t \to \infty \text{ and } h < 0; \\ x_j & \text{for } t \to -\infty \text{ and } h < 0. \end{cases}$$

e) Behold:

$$\begin{aligned} x'_{\text{approx}} &= \left[\frac{x_j - x_{j-1}}{(1+ge^{-ht})^2}\right] hge^{-ht}\Bigg|_{\text{midpoint}} \\ &= \left[\frac{x_j - x_{j-1}}{1+ge^{-ht}}\right]\left[\frac{hge^{-ht}}{1+ge^{-ht}}\right]\Bigg|_{\text{midpoint}} \\ &= \left[\frac{x_j - x_{j-1}}{2}\right]\frac{h}{2} . \end{aligned}$$

Evaluating the differential equation at the midpoint gives

$$x' = a\prod_{i=1}^{n}\left[\frac{x_j + x_{j-1}}{2} - x_i\right] .$$

Equating the two expressions for x' and solving for h gives

$$h = 2a\prod_{i=1, i\neq j}^{n}\left[\frac{x_j + x_{j-1}}{2} - x_i\right] .$$

Note that $h > 0$ for a monotonically increasing solution and $h < 0$ for a monotonically decreasing one.

Yours truly will spare you the algebra and just give the formula for g:

$$g = e^{ht_0}\left(\frac{x_j - x_0}{x_0 - x_{j-1}}\right) ,$$

where we note that $g > 0$ always since $x_0 \in (x_{j-1}, x_j)$.

f) For $n = 2$, we have

$$h = -a(x_2 - x_1)$$

and

$$x = \frac{x_2 - x_1}{1 + ge^{-ht}} + x_1$$

(dropping the "approx" which no longer applies) which differentiates to

$$\begin{aligned} x' &= \left[\frac{x_2 - x_1}{(1 + ge^{-ht})^2} \right] hge^{-ht} \\ &= (x - x_1) \left(\frac{hge^{-ht}}{(1 + ge^{-ht})} \right) \\ &= h \left[\frac{(x - x_1)^2}{x_2 - x_1} \right] ge^{-ht} \\ &= h \left[\frac{(x - x_1)(x_2 - x)}{x_2 - x_1} \right] \\ &= a(x - x_1)(x - x_2) \end{aligned}$$

which is the original differential equation. Thus, the $n = 2$ solution is exact as advertised.

Redaction: Jeffery, 2016jan01

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002 qfull 00560 1 3 0 easy math: perturbation solutions for 1st order DEs

11. Consider the 1st order (ordinary, autonomous) differential equation

$$x' = f(x) ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. The 1st order DE rule (as yours truly calls it) applies to this DE. We have $f(x_i) = 0$ and therefore x_i yields a constant solution and a stationary point at either of $\pm\infty$.

NOTE: There are parts a,b.

a) Assuming $(df/dx)(x_i) \neq 0$, solve without words for the 1st order perturbation solution in small $\Delta x = x - x_i$. Let Δx_0 be the initial perturbation, time zero is 0, and $R_1 = (df/dx)(x_i)$ for compactness. What is the condition for convergence/divergence in the future to the constant solution? What is the condition for convergence/divergence in the past to the constant solution? **HINT:** Recall the antiderivative of $1/y$ is always $\ln(|y|)$.

b) Now assume the lowest order nonzero coefficient in the expansion of $f(x)$ in small δx is $(d^k f/dx^k)(x_i)$ where $k \geq 2$. The write the solution only in terms of $|\Delta x|$ and $|\Delta x_0|$ since that seems most clear and start from the differential form

$$\frac{d|\Delta x|}{|\Delta x|^k} = hR_k \, dt ,$$

where for k even $h = \pm 1$ with upper case for $\Delta x > 0$ and lower case for $\Delta x < 0$ and for k odd $h = 1$, and $R_k = (d^k f/dx^k)(x_i)$ for compactness. Show why this differential form is correct before you use it.

c) What happens as $hR_k t$ **INCREASES/DECREASES** from 0? At what time t is there an infinity?

SUGGESTED ANSWER:

a) Behold:

$$1) \quad \frac{d\Delta x}{dt} = \Delta x R_1 \qquad 2) \quad \frac{d\Delta x}{\Delta x} = R_1 \, dt \qquad 3) \quad \ln\left(\left| \frac{\Delta x}{\Delta x_0} \right| \right) = R_1 t$$

$$4) \quad |\Delta x| = |\Delta x_0| \exp(R_1 t) \qquad 5) \quad \Delta x = \Delta x_0 \exp(R_1 t) .$$

As expression (5) shows convergence (divergence) in the future is given for $R_1 < 0$ ($R > 0$). As expression (5) shows convergence (divergence) in the past is given for $R_1 > 0$ ($R < 0$).

b) Behold:

$$1)\quad \frac{d\Delta x}{dt} = \Delta x^k R_k \qquad 2)\quad \frac{d\Delta x}{\Delta x^k} = R_k\,dt \qquad 3)\quad \frac{d(\pm\Delta x)}{(\pm\Delta x)^k} = hR_k\,dt$$

$$4)\quad \frac{d|\Delta x|}{|\Delta x|^k} = hR_k\,dt \qquad 5)\quad \left.\frac{|\Delta x|^{-k+1}}{-k+1}\right|_{\Delta x_0}^{\Delta x} = hR_k t$$

$$6)\quad |\Delta x|^{-k+1} = |\Delta x_0|^{-k+1} - (k-1)hR_k t$$

$$7)\quad |\Delta x| = \left[\frac{1}{1/|\Delta x_0|^{k-1} - (k-1)hR_k t}\right]^{1/(k-1)} .$$

Note that if k is even, then $(\pm\Delta x)^k = \Delta x^k$ and in order to turn the differential $d\Delta x$ into $(\pm\Delta x)$ we need to multiply the other side of the equation by $h = \pm 1$. If k is odd, then $(\pm\Delta x)^k = \pm\Delta x^k$ and in order to turn the differential $d\Delta x$ into $(\pm\Delta x)$ we just need to multiply top and bottom of $d\Delta x/\Delta x^k$ by ± 1 and in this case $h = 1$.

c) As $hR_k t$ increases/decreases from 0, Δx diverges/converges relative to the constant solution. In fact, the diverging solution goes to $+\infty$ at

$$t = \frac{1}{(k-1)hR_k|\Delta x_0|^{k-1}} .$$

Redaction: Jeffery, 2018jan01

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002 qfull 00590 1 3 0 easy math: logistic function

12. The logistic function (called that for a darn good reason) turns up in many contexts looking like:

$$f(x) = \begin{cases} \dfrac{f_{\rm M}}{1+e^{-r(x-x_0)}} = \dfrac{f_{\rm M}}{1+(f_{\rm M}/f_0 - 1)e^{-rx}} & \text{in general form;} \\[2ex] \dfrac{1}{1+e^{-x}} = \dfrac{e^x}{e^x+1} = \dfrac{1}{2}\left[\tanh(x/2)+1\right] & \text{in natural or reduced form.} \end{cases}$$

In this question, we only use the natural form for simplicity and elegance.

There are parts a,b,c,d.

a) Determine f' (which is, in fact, called the logistic distribution), f'' (also write it as an explicitly even function which it is), the antiderivative of f (easy if you write f in terms of e^x), and the integral of f' from $-x$ to x. Use the natural form of the function.

b) Determine stationary points of f and f' and the values of f and f' at those points. Use the natural form of the function.

c) The logistic function can be used as a smooth replacement for the Heaviside step function:

$$H(x) = \begin{cases} 0 & x < 0; \\ 1/2 & x = 0; \\ 1 & x > 0. \end{cases}$$

Show that logistic function becomes the that Heaviside step function with the appropriate limiting procedure. **HINT:** This is really easy.

d) The logistic function is actually the solution of a 1st order nonlinear differential equation. This equation shows up, for example, in population dynamics. Say you have population N that grows at rate (per population) r with unlimited resources. However, the rate with resources limited by carry capacity (or maximum population) K is modeled as $r(1-N/K)$ which is zero when $N \to K$. The growth differential equation for N, sometimes called the Verhulst-Pearl equation, is

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N ,$$

Reduce this equation to natural form and find the solution. Then write the solution out in population-dynamics form for general initial population N_0 at $t = 0$ and show the small N/K and $t \to \infty$ asymptotic limiting cases explicitly. **HINT:** You'll need a table integral.

SUGGESTED ANSWER:

a) Behold:

$$f(x) = \frac{1}{1+e^{-x}}$$

$$f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} = \frac{1}{(e^{x/2}+e^{-x/2})^2} = \left(\frac{1}{4}\right)\left[\frac{1}{\cosh^2(x/2)}\right] \geq 0 \qquad \text{which is the logistic distribution;}$$

$$f''(x) = \frac{2e^{-2x}}{(1+e^{-x})^3} - \frac{e^{-x}}{(1+e^{-x})^2} = \frac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3} \leq 0$$

$$\int f(x)\,dx = \int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)$$

$$\int_{-x}^{x} f'(x)\,dx = \frac{1}{1+e^{-x}} - \frac{1}{1+e^{x}} = \frac{e^{x/2}}{e^{x/2}+e^{-x/2}} - \frac{e^{-x/2}}{e^{x/2}+e^{-x/2}} = \tanh(x/2) = \begin{cases} 1 & \text{for } x=\infty; \\ 0 & \text{for } x=0. \end{cases}$$

b) Behold:

$$f(x) = \begin{cases} \dfrac{1}{1+e^{-x}} & \text{in general;} \\ 0 & \text{for } f \text{ minimum at } x=-\infty; \\ 1 & \text{for } f \text{ maximum at } x=\infty; \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{e^{-x}}{(1+e^{-x})^2} = \dfrac{1}{(e^{x/2}+e^{-x/2})^2} \geq 0 & \text{in general;} \\ 0 & \text{for } f \text{ stationary points at } x=\pm\infty; \\ 0 & \text{for } f' \text{ minima at } x=\pm\infty; \\ \dfrac{1}{4} & \text{for } f' \text{ maxima at } x=0; \end{cases}$$

$$f''(x) = \begin{cases} \dfrac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3} \leq 0 & \text{in general;} \\ 0 & \text{for stationary points at } x=0 \text{ and } x=\pm\infty; \end{cases}$$

c) Behold:

$$\lim_{r\to\infty} f(x) = \lim_{r\to\infty} \frac{1}{1+e^{-rx}} = \left\{ \begin{array}{ll} 0 & x<0; \\ 1/2 & x=0; \\ 1 & x>0 \end{array} \right\} = H(x) \ .$$

d) Let $x = N/K$ and $\tau = rt$. The Verhulst-Pearl equation now reduced form and solution follow:

$$\frac{dx}{d\tau} = x(1-x) \qquad \frac{dx}{x(1-x)} = d\tau$$

$$\ln\left(\frac{x}{1-x}\right) = \tau - C \qquad \frac{x}{1-x} = Ce^{\tau} \qquad x(1+Ce^{\tau}) = Ce^{\tau} \qquad x = \frac{1}{1+Ce^{-\tau}}$$

$$N = \begin{cases} \dfrac{K}{1+(K/N_0-1)e^{-rt}} & \text{in general;} \\ N_0e^{rt} & \text{for } N_0/K < N/K << 1 \text{ which is exponential growth;} \\ K[1-(K/N_0-1)e^{-rt}] & \text{asymptotically as } t\to\infty. \end{cases}$$

Redaction: Jeffery, 2018jan01

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002 qfull 00560 1 3 0 easy math: perturbation solutions for 1st order DEs

13. Consider the 1st order (ordinary, autonomous) differential equation

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. The 1st order DE rule (as yours truly calls it) applies to this DE. We have $f(x_i) = 0$ and therefore x_i yields a constant solution and a stationary point at either of $\pm\infty$.

NOTE: There are parts a,b.

a) Assuming $(df/dx)(x_i) \neq 0$, solve without words for the 1st order perturbation solution in small $\Delta x = x - x_i$. Let Δx_0 be the initial perturbation, time zero is 0, and $R_1 = (df/dx)(x_i)$ for compactness. What is the condition for convergence/divergence in the future to the constant solution? What is the condition for convergence/divergence in the past to the constant solution? **HINT:** Recall the antiderivative of $1/y$ is always $\ln(|y|)$.

b) Now assume the lowest order nonzero coefficient in the expansion of $f(x)$ in small δx is $(d^k f/dx^k)(x_i)$ where $k \geq 2$. The write the solution only in terms of $|\Delta x|$ and $|\Delta x_0|$ since that seems most clear and start from the differential form

$$\frac{d|\Delta x|}{|\Delta x|^k} = hR_k\,dt \ ,$$

where for k even $h = \pm 1$ with upper case for $\Delta x > 0$ and lower case for $\Delta x < 0$ and for k odd $h = 1$, and $R_k = (d^k f/dx^k)(x_i)$ for compactness. Show why this differential form is correct before you use it.

c) What happens as $hR_k t$ **INCREASES/DECREASES** from 0? At what time t is there an infinity?

SUGGESTED ANSWER:

a) Behold:

$$1)\quad \frac{d\Delta x}{dt} = \Delta x R_1 \qquad 2)\quad \frac{d\Delta x}{\Delta x} = R_1\,dt \qquad 3)\quad \ln\left(\left|\frac{\Delta x}{\Delta x_0}\right|\right) = R_1 t$$
$$4)\quad |\Delta x| = |\Delta x_0|\exp(R_1 t) \qquad 5)\quad \Delta x = \Delta x_0 \exp(R_1 t) \ .$$

As expression (5) shows convergence (divergence) in the future is given for $R_1 < 0$ ($R > 0$). As expression (5) shows convergence (divergence) in the past is given for $R_1 > 0$ ($R < 0$).

b) Behold:

$$1)\quad \frac{d\Delta x}{dt} = \Delta x^k R_k \qquad 2)\quad \frac{d\Delta x}{\Delta x^k} = R_k\,dt \qquad 3)\quad \frac{d(\pm\Delta x)}{(\pm\Delta x)^k} = hR_k\,dt$$
$$4)\quad \frac{d|\Delta x|}{|\Delta x|^k} = hR_k\,dt \qquad 5)\quad \left.\frac{|\Delta x|^{-k+1}}{-k+1}\right|_{\Delta x_0}^{\Delta x} = hR_k t$$
$$6)\quad |\Delta x|^{-k+1} = |\Delta x_0|^{-k+1} - (k-1)hR_k t$$
$$7)\quad |\Delta x| = \left[\frac{1}{1/|\Delta x_0|^{k-1} - (k-1)hR_k t}\right]^{1/(k-1)} \ .$$

Note that if k is even, then $(\pm\Delta x)^k = \Delta x^k$ and in order to turn the differential $d\Delta x$ into $(\pm\Delta x)$ we need to multiply the other side of the equation by $h = \pm 1$. If k is odd, then $(\pm\Delta x)^k = \pm\Delta x^k$ and in order to turn the differential $d\Delta x$ into $(\pm\Delta x)$ we just need to multiply top and bottom of $d\Delta x/\Delta x^k$ by ± 1 and in this case $h = 1$.

c) As $hR_k t$ increases/decreases from 0, Δx diverges/converges relative to the constant solution. In fact, the diverging solution goes to $+\infty$ at

$$t = \frac{1}{(k-1)hR_k|\Delta x_0|^{k-1}} \ .$$

Redaction: Jeffery, 2018jan01

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002 qfull 00610 1 3 0 hard math: 1st order DE rule I (version II better, conflate?)

Extra keywords: This version may be completely obsolete due to the 640 version

14. A 1st order homogeneous differential equation, linear or nonlinear, of the form

$$f' = g(f) \ ,$$

(with independent variable t which g has **NO** explicit dependence on) at points where it is infinitely differentiable only has solutions that are strictly in/decreasing or that are constant. Note that differentiable at a point means there is a finite derivative of the same value taken from above or below the point and there is no singularity at the point (which is usually implied by the first condition). Also note that strictly in/decreasing means there are no have stationary points and constant means constant for a finite region. The constant solutions are often stable/unstable in the sense that small perturbations from them lead to convergent/divergent behavior with increasing independent variable.

The rule actually requires the extra condition that higher derivatives of the differential equation $f^{\langle n\rangle}$ (where we use angle brackets to indicate differentiation order when primes will not do) do **NOT** generate zero-over-zero cases: i.e., cases where a f' on the right-hand side of the equation is multiplied by a factor that cancels the zero at stationary point making the higher order derivative on the left-hand side of the equation non-zero. Such a non-zero $f^{\langle n\rangle}$ means that a Taylor expansion around the stationary point will show curvature. That zero-over-zero cases occur will be proven showing important examples.

There are parts a,b,c,d,e,f,g. The parts can all be done independently, and so do not stop if you cannot do a part.

a) Prove the rule given in the preamble for a $g(f)$ that does **NOT** generate zero-over-zero cases. **HINT:** Use proof by induction using the general Leibniz rule (which is the generalization of the product rule):

$$(rs)^{\langle n\rangle} = \sum_{k=0}^{n} \binom{n}{k} r^{\langle n-k\rangle} s^{\langle k\rangle} \ ,$$

where r and s are general functions (Ar-667; Wikipedia: General Leibniz rule). Note $s^{\langle 0\rangle} = s$ not 1.

b) For this part, the preamble is long, the answer is short—have patience.

The zero-over-zero case can (but not necessarily will) occur when we have

$$(f')^p = g(f) \qquad \text{or, equivalently,} \qquad f' = e^{i\phi} g(f)^{1/p}$$

where $e^{i\phi}$ is a phase factor (and we only consider its pure real values) and where $g(f)$ does not itself lead to the zero-over-zero case. The zero-over-zero case will when

$$g^{1/p-(n-1)}(f')^{(}n-1) = Q \neq 0 \ ,$$

where Q is a constant and $n > 2$ and $[1/p-(n-1)]$ and $(n-1)$ are powers, **NOT** derivative orders. Note that when $n = 1$, we have

$$f' = e^{i\phi} g^{1/p} = e^{i\phi} Q$$

which means $f = at + b$ which has no stationary points and is not zero-over-zero case.

To prove the exception, we differentiate the differential equation $In - 1$ times to get

$$f^{\langle In\rangle} = Ag^{1/p-(n-1)}(f')^{(}n-1)f^{\langle I-1\rangle} + Bg^{1/p-(n-2)}(f')^{(}n-2)f^{\langle (I-1)n+1\rangle} + \ldots \ ,$$

where A and B are constants whose values are of no interest and $\{(I-1)n+1\}$ is a derivative order. Note that every term must have the sum of derivative orders equal to $In - 1$: e.g., $(n-1) + (I-1)n = In - 1$ and $(n-2) + (I-1)n + 1 = In - 1$. an **INHOMOGENEOUS** 1st order differential equation does not have to obey the rule stated in the preamble. **HINT:** Find a trivial counterexample. Think trigonometry.

b) Prove that a homogenous 1st order differential equation can have a stationary point at $\pm\infty$. **HINT:** Find a trivial example.

c) Prove the rule given in the preamble and discuss why exceptions can occur. **HINT:** Use proof by induction to show that if $x(t)$ has a stationary point where $x' = f(x)$ are infinitely differentiable that the function is constant at that point: i.e., all orders of derivatives of x are zero a that point.

d) Prove that a solution can be nonmonotonic if there is point t where $x' = f(x))$ is not infinitely differentiable. **HINT:** Find a simple example of a 1st order differential equation such a solution. Yours truly suggests differential equation with solution $x = 1/t$.

e) Prove that a solution can have a stationary point at a point t where $x' = f(x))$ is not infinitely differentiable. **HINT:** Find a simple example of a 1st order differential equation such a solution. Yours truly suggests differential equation with solution $x = |t|^3$.

SUGGESTED ANSWER:

a) Proof by induction:

i) Differentiating $f' = g(f)$ gives

$$f'' = g'f' \; ,$$

where g' is a derivative with respect to f and the chain rule has been used. Given f is stationary at a point (i.e., $f' = 0$), we have $f'' = 0$ at that stationary point.

ii) We differentiate the $f'' = g'f'$ equation $(n-2)$ times to get

$$f^{\langle n-1\rangle} = (f'g')^{\langle n-2\rangle} = \sum_{k=0}^{n-2} \binom{n-2}{k} (g')^{\langle (n-2)-k\rangle}(f')^{\langle k\rangle} \; .$$

We assume all $(f')^{\langle i\rangle} = 0$ at the statioanry point for $i \leq n-2$ which implies $f^{\langle n-1\rangle} = 0$ at the stationary point.

iii) We differentiate the $f'' = g'f'$ equation $n-1$ times to get

$$f^{\langle n\rangle} = (f'g')^{\langle n-1\rangle} = \sum_{k=0}^{n-1} \binom{n-1}{k} (g')^{\langle (n-1)-k\rangle}(f')^{\langle k\rangle} \; .$$

By step 2, we have $f^{\langle i\rangle} = 0$ at the stationary point for all $i \leq n-1$, and so we have $f^{\langle n\rangle} = 0$ at the stationary point.

Since n is general we have proven that $f^{\langle n\rangle} = 0$ for all $n > 0$ at the stationary point. This implies f is a constant (which means all points are stationary points). If there are no stationary points (i.e., no points where $f' = 0$), f is strictly increasing/decreasing. This completes the proof: QED.

a) Behold the counterexample:

$$x' = \omega A \cos(\omega t)$$

which has solution

$$x = A \sin(\omega t)$$

which certainly has stationary points where it is infinitely differentiable.

b) Behold the example:

$$x' = kx$$

which has solution

$$x = x_0 e^{kt} \; .$$

If k is positive/negative, there is stationary point at negative/positive infinity. So one can have a varying solution with stationary points if they are at $\pm\infty$.

c) Proof by induction:

i) Differentiating $x' = f(x)$ gives

$$x'' = f'(x)x' \; ,$$

where f' is a derivative with respect to x and the chain rule has been used. If x is stationary at point t, $x' = 0$ and $x'' = 0$ too.

ii) We differentiate the $x' = f(x)$ equation $(n-1)$ times to get

$$x^{((n-1))} = g(\{x^{(i)}\}) \; ,$$

where $\{x^{(i)}\}$ stands for the set of all $x^{(i)}$ with $i \in [1, n-2]$. Clearly, every term in g has at least one power of $x^{(i)}$. We assume $x^{((n-1))} = 0$ at stationary points of x.

iii) We differentiate the $x' = f(x)$ equation n times to get

$$x^{(n)} = g(\{x^{(i)}\}) ,$$

where $\{x^{(i)}\}$ stands for the set of all $x^{(i)}$ with $i \in [1, n-1]$. Clearly, every term in g has at least one power of $x^{(i)}$. Since $x^{((n-1))} = 0$ by (ii), we have $x^{(n)} = 0$.

Since $x'(t) = 0$ implies $x^{(n)} = 0$, we have proven the rule. If $x'(t) = 0$, we have a constant solution. If there is no t such that $x'(t) = 0$, then the solution is strictly in/decreasing. This completes the proof: QED.

Note we can have all orders of derivatives zero at $\pm\infty$ and still have a non-constant solution as proven in the part (b) answer. The paradox seems resolvable this way. All orders of derivatives zero implies a region of constant solution. This would mean that there has to be a singularity between the constant solution at $\pm\infty$ and the varying solution not at $\pm\infty$. However, $\pm\infty$ is $\pm\infty$ and the mythical singularity is pushed off to $\pm\infty$ and never turns up. There must a rigorous way to describe this, but yours truly kens it not.

Note also the proof fails if $x'' = f'(x)x'$ is not zero when x' is zero because of a zero over zero cancellation with $f'(x)$. An actual example in later problem shows one get a stationary point for a differential equation of the form $x' = f(x)$ if this cancellation happens.

d) Consider $x = 1/t$. The function behavior: 1) at $t = -\infty$, $x = 0$, 2) as t increases, x strictly decreases, 3) as $t \to 0$, x goes to $-\infty$, 4) at $t = 0$ the function is undefined and is undifferentiable, 5) as t increases above 0, x strictly decreases, 6) as $t \to \infty$, x goes to zero. The function is not strictly decreasing everywhere, and so is not monotonic everywhere because of the infinite discontinuity at $t = 0$.

Differentiating $x = 1/t$, we get $x' = -1/t^2 = -x^2$. Thus, $x = 1/t$ satisfies a 1st order linear differential equation $x' = -x^2$ and we have have proven what was asked: QED.

e) Consider $x = |t|^3$. The function is zero at $t = 0$ and rises strictly going in the negative/positive t direction. The derivative of function and its differential equation (1st order, homogeneous, and nonlinear) are given by

$$x' = \begin{cases} 3t^2 = 3x^{2/3} & \text{for } t \geq 0; \\ -3t^2 = -3x^{2/3} & \text{for } t \leq 0. \end{cases}$$

We see that x has a stationary point at $t = 0$ which is, in fact, a minimum. We differentiate x' twice to get

$$x''' = \begin{cases} 6 & \text{for } t > 0; \\ \text{undefined} & \text{for } t = 0; \\ -6 & \text{for } t < 0. \end{cases}$$

So we see that x' is not infinitely differentiable at $t = 0$ which is nevertheless a stationary point, but this is allowed by the rule stated in the preamble.

Redaction: Jeffery, 2018jan01

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002 qfull 00620 1 3 0 easy math: 1st order DE rule II (better version?)

Extra keywords: This version may be completely obsolete due to the 640 version

15. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to x. Note $g(x)$ may go negative as a function of x, but we assume it does not negative as function of t at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm[g(x)]^{1/2} ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

NOTE: There are parts a,b,c,d.

a) Given x_i (or in the time variable t_i) is a stationary point of $x' = f(x)$ (i.e., $x'(x_i) = f(x_i) = f[x(t_i)] = 0$), prove without words that $x''(x_i) = 0$.

b) The part (a) answer gives the base case (or 1st step) for a proof by induction that all orders of derivative of x with respect to t at x_i (or in the time variable t_i) are zero. The proof follows by inspection if your math intuition is good enough. However, do a formal proof by induction. **HINT:** For the proof, you do **NOT**, in fact, need the full general Leibniz rule for the derivative of a product (Ar-558)

$$\frac{d^m(fg)}{dx^m} = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dx^k} \frac{d^{m-k} g}{dx^{m-k}} .$$

Using it actually makes the proof a bit more tricky to follow. But you do need to know that the nth order derivative of x (i.e., $x^{(n)}$) is obtained by applying the general Leibniz rule for $m = n - 2$ to the result of the part (a) answer and that highest derivative of x on the right-hand side of that application is $x^{(n-1)}$. Note that $f(x)$ is general to the degree specified in the preamble, and so the proof is unchanged if any order of derivative $f(x)$ with respect to x is zero at x_i.

c) Given the part (b) result, give an argument for why the stationary point t_i must be all points (i.e., is actually a static solution) or at time equals infinity.

d) A 1st order DE system given a small perturbation from a static solution either asymptotically goes back to it (i.e., is asymptotic to it at positive infinity, and so is called stable) or grows away from it (i.e., is asymptotic to it at negative infinity, and so is called unstable). Assuming the df/dx is nonzero at x_i, prove without words that a 1st order DE system given a small perturbation (i.e., a perturbation Δx_0 which requires only 1st order expansion of $f(x)$ in small $\Delta x = x - x_i$) varies exponentially and determine the condition for stability.

SUGGESTED ANSWER:

a) Behold:

$$1) \quad x' = f(x) \qquad 2) \quad x'' = \frac{df}{dx} x' \qquad 3) \quad x''(x_i) = \frac{df}{dx} x'(x_i) = 0 .$$

b) Part (a) gave the first step of the proof by induction: i.e., that $x''(x_i) = 0$. The second step is assuming $x^{(j)}(x_i) = 0$ for all $j \in [1, n-1]$ and then for the third step expanding

$$\begin{aligned}
x^{(n)} &= \frac{d^{n-2}\left[(df/dx)x'\right]}{dt^{n-2}} = \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{d^k(df/dx)}{dt^k} \frac{d^{n-2-k}x'}{dt^{n-2-k}} \\
&= \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{d^{n-2-k}(df/dx)}{dt^{n-2-k}} \frac{d^k x'}{dt^k} \\
&= \frac{d^{n-2}(df/dx)}{dt^{n-2}} x' + \ldots + \frac{df}{dx}(x')^{(n-2)} = \frac{d^{n-1}f)}{dx^{n-1}}(x')^{n-1} + \ldots + \frac{df}{dx}(x')^{(n-2)} \\
&= \text{terms all with factors of } (x')^{(j)} \text{ with } j \in [1, n-2] \\
&= \text{terms all with factors } x^{(j)} \text{ with } j \in [1, n-1] \\
&\qquad \text{which are all zero for } x = x_i \text{ by assumption} \\
x^{(n)}(x_i) &= 0 \qquad \text{QED.}
\end{aligned}$$

Since the result is for general n, we have $x^{(n)}(x_i) = 0$ for all $n \geq 1$.

c) If all orders of derivative are zero at t_i, the solution of x must be constant to $\pm\infty$ with value x_i (i.e., must be a static solution x_i) or it is asymptotically constant at one of $\pm\infty$ where it is asymptotic to asymptote $x = x_i$.

d) Behold:

$$x' = f(x) = f(x_i) + \Delta x \frac{df}{dx}\bigg|_{x_i} + \ldots = 0 + \Delta x \frac{df}{dx}\bigg|_{x_i} + \ldots = \Delta x \frac{df}{dx}\bigg|_{x_i} + \ldots ,$$

where $\Delta x = x - x_i$ and hereafter we set $R = df/dx|_{x_i}$ for niceness. For perturbation Δx_0 sufficiently small, we have the approximate 1st order DE and solution

$$1)\quad \frac{d\Delta x}{dt} = \Delta x R \qquad 2)\quad \frac{d\Delta x}{\Delta x} = R\,dt \qquad 3)\quad \frac{d(\pm\Delta x)}{(\pm\Delta x)} = \frac{d(|\Delta x|)}{|\Delta x|} = R\,dt$$

$$4)\quad \ln\left(\left|\frac{\Delta x}{\Delta x_0}\right|\right) = Rt \qquad 5)\quad |\Delta x| = |\Delta x_0| e^{Rt} \qquad 6)\quad \Delta x = \Delta x_0 e^{Rt}$$

where the upper case is for $\Delta x_0 > 0$ and the lower case is for $\Delta x_0 < 0$. Note we did not need the upper/lower case stuff if we just knew that the antiderivative of $1/y$ is always $\ln(|y|)$. From expression (5), we see that the exponential variation is away from the static solution for $R > 0$ and toward the static solution if $R < 0$. Thus, the condition for stability is $R < 0$ and the condition for instability is $R > 0$. If $R = 0$, then one must check what happens for the first higher order expansion term n of $f(x)$ where the nth order derivative coefficient $(d^n f/dx^n)|_{x_i} \neq 0$.

Redaction: Jeffery, 2018jan01

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002 qfull 00630 1 3 0 easy math: main exception to the 1st order DE rule

Extra keywords: This version may be completely obsolete due o 640 version

16. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to x. Note $g(x)$ may go negative as a function of x, but we assume it does not negative as function of t at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm\sqrt{g(x)} ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

NOTE: There are parts a,b,c,d,e.

a) Given x_i (or in the time variable t_i) is a stationary point of $x' = \pm\sqrt{g(x)}$ (i.e., $x'(x_i) = \pm\sqrt{g(x_i)} = \pm\sqrt{g[x(t_i)]} = 0$), prove without words that $x''(x_i) \neq 0$ for $g(x_i) \neq 0$.

b) What does the part (a) answer imply about x_i? What does the part (a) answer imply about x_i given the sign of $dg/dx(x_i)$?

c) Given $(dg/dx)(x_i) = 0$, prove by induction that for general $n \in [1\infty]$ that $x^{(n)}(x_i) = 0$. **HINT:** Consider $x^{(4)}(x_i) = 0$ as step 1 (i.e., the base case) of the proof. Note that the right-hand side of the expressions in the proof will always have a derivative of x two orders lower than the left-hand side.

d) Given $(dg/dx)(x_i) = 0$, what does the part (c) answer imply about x_i?

e) Given $(dg/dx)(x_i) = 0$, and therefore there is a static solution $x = x_i$ for all time t, we can consider what the lowest order solution is for a small perturbation from the static solution. The expansion of the differential equation in small $\Delta x = x - x_i$ is

$$\frac{d\Delta x}{dt} = \pm\sqrt{\sum_{k=\ell}^{\infty} \Delta x^k \left[\frac{d^k g}{dx^k}(x_i)\right]} \ ,$$

where ℓ is the lowest power for which there is a nonzero coefficient $(d^\ell g/dx^\ell)(x_i)$. What possible signs can Δx when ℓ is even and $(d^\ell g/dx^\ell)(x_i) > 0$? What possible signs can Δx when ℓ is even and $(d^\ell g/dx^\ell)(x_i) < 0$? What possible signs can Δx when ℓ is odd?

SUGGESTED ANSWER:

a) Behold:

$$1)\quad x' = \pm\sqrt{g} \qquad 2)\quad x'' = \frac{1}{2}\frac{1}{(\pm\sqrt{g})}\frac{dg}{dx}x' \qquad 3)\quad x'' = \frac{1}{2}\frac{1}{(\pm\sqrt{g})}\frac{dg}{dx}(\pm\sqrt{g})$$

$$4)\quad x'' = \frac{1}{2}\frac{dg}{dx} \qquad 5)\quad x''(x_i) = \frac{1}{2}\frac{dg}{dx}(x_i) \neq 0 \ ,$$

given that $g(x_i) \neq 0$.

b) The point x_i (or t_i in the time variable) is a stationary point of $x(t)$. If $dg/dx(x_i)$ is positive/negative, the stationary point is a minimum/maximum.

c) From part (a), we obtain

$$1)\quad x^{(3)} = \frac{1}{2}\frac{d^2 g}{dx^2}x' \qquad 2)\quad x^{(4)} = \frac{1}{2}\left[\frac{d^3 g}{dx^3}(x')^2 + \frac{d^2 g}{dx^2}x''\right] \ ,$$

where expressions (1) and (2) are zero for $x = x_i$ since $x'(x_i) = 0$ by hypothesis and $x''(x_i) = 0$ by part (a) plus the hypothesis that $(dg/dx)(x_i) = 0$. Expression (1) is actually the first step of the proof since it implies every higher derivative $x^{(n)}$ can be obtained if you know all the derivatives between $x^{(1)}$ and $x^{(n-2)}$. In any case, we explicitly differentiate expression (1) $(n-3)$ times to obtain

$$x^{(n)} = \frac{1}{2}\left[\frac{d^{n-1} g}{dx^{n-1}}(x')^{n-2} + \ldots + \frac{d^2 g}{dx^2}(x')^{(n-2)}\right] \ .$$

All the terms on the right-hand side have factors of $(x')^j$ with $j \in [1, n-2]$. As the second step for the proof, we assume all $(x')^j(x_i) = 0$ for $j \in [1, n-2]$. The third step for the proof is by noting that given the first two steps the last expression gives $x^{(n)}(x_i) = 0$ for $n \in [1, \infty]$.

d) Since $x^{(n)}(x_i) = 0$ for $n \in [1, \infty]$, $x(t)$ must be constant to $\pm\infty$ with value x_i (i.e., must be a static solution x_i) or it is asymptotically constant at one of $\pm\infty$ where it is asymptotic to asymptote $x = x_i$.

e) When ℓ is even and $(d^\ell g/dx^\ell)(x_i) > 0$, Δx can be either positive or negative. This is actually the case for small perturbations from the Einstein universe and the radiation-positive curvature-Λ universe (which is the radiation analogue to the Einstein universe which is the matterpositive curvature-Λ universe).

When ℓ is even and $(d^\ell g/dx^\ell)(x_i) < 0$, there are no possible perturbation solutions for real numbers. There is just the static solution itself isolated in solution land. An example of this case is when $g(x) = -\Delta x^2$ which implies $\ell = 2$

When ℓ is odd and $(d^\ell g/dx^\ell)(x_i) > 0$, we can only have $\Delta x > 0$. An example of this case is when $g(x) = \Delta x^3$ which implies $\ell = 3$.

When ℓ is odd and $(d^\ell g/dx^\ell)(x_i) < 0$, we can only have $\Delta x < 0$. An example of this case is when $g(x) = -\Delta x^3$ which implies $\ell = 3$.

Redaction: Jeffery, 2018jan01

002 qfull 00640 1 3 0 easy math: 1st order autonomous DE and stationary points

Extra keywords: This is the definitive version as of 2025mar09

17. First order autonomous ordinary differential equations (FAODEs), linear or nonlinear, only have solutions with stationary points at infinity (SPIs), (except for special cases which are not all that rare) and constant solutions. Actually, each SPI corresponds to a constant solution which could also be viewed as a continuum of stationary points. Note an autononous differential equation depends only on functions of the dependent variable, and so has no explicit dependence on the independent variable.

To investigate the SPI behavior of FAODEs consider the (somewhat general) FAODE

$$x^{(1)} = [f(x)]^{1/k} \ ,$$

where t (not necessarily time) is the independent variable, the superscript (1) means 1st derivative with respect to t, $f(x)$ is an infinitely differentiable function with zeros at set of values $\{x_i\}$, and $k > 0$. We limit k to being greater than zero to avoid uninteresting generality. Since $f(x)$ is infinitely differentiable at (general) x_i, we can expand $f(x)$ about x_i with some radius of convergence: i.e.,

$$f(\Delta x) = \sum_{j=\ell}^{\infty} \Delta x^j f_j = \Delta x^\ell f_\ell + \dots \ ,$$

where $\Delta x = x - x_i$, the f_j are expansion constants, and $\ell > 0$ is the lowest (nonzero) order in the expansion. Note $\ell \neq 0$ since we have assumed x_i is a zero of $f(x)$: i.e., $f(x_i) = 0$.

We will primarily be examining the lowest order solutions in Δx, and so we will be dealing with $\Delta x^{\ell/k} f_\ell^{1/k}$ and related expressions. Mathematically, if ℓ/k is not an integer, complex numbers can arise in these expressions. However, we are only interested FAODEs and their solutions corresponding to physical systems involving real numbers. In these systems, the solutions just never evolve into the complex number realm. So we are not going to concern ourselves with question what happens mathematically if some our expressions can give rise to complex numbers. They never give rise to complex numbers physically.

NOTE: There are parts a,b,c,d,e,f,g,h,i,j,k. On exams, do **ONLY** parts i,j.

a) What is the behavior of x as a function of t between the points in the set $\{x_i\}$.

b) In this question we are only interested in the SPI behavior and constant solution behavoir, and so we are only interested in the behavior of $x(t)$ when it is arbitrarily close to x_i where SPI and constant solutions occur. Therefore expand the FAODE about x_i with dependent variable Δx to lowest order in the exponent.

c) Determine the formula $p(n)$ for the exponent of Δx in the n derivative of Δx (for the lowest order of the FAODE) with respect to t. **HINT:** Drop all constants that turn up in the differentiations.

d) What is behavior of the t derivatives of Δx when $x = x_i$ for $\ell/k \geq 1$? What solutions $x(t)$ are implied by $\ell/k \geq 1$?

e) What is behavior of the t derivatives of Δx for $f(x_i)$ for $\ell/k < 1$ assuming the formula $p(n)$ never equals zero? What solution $x(t)$ behavior is implied by $\ell/k < 1$ in this case? Only a short answer is expected to the last question.

f) If $\ell/k < 1$ and the formula $p(n)$ goes to zero for a stopping $n_{\rm st}$, what is the formula for ℓ/k as a function of $n_{\rm st}$ and what are the values of ℓ/k for the set $n_{\rm st} = 1, 2, 3, \dots, \infty$ and what do the $n_{\rm st} = 1$ and $n_{\rm st} = \infty$ cases mean? What is the formula $n_{\rm st}$ as a function of ℓ/k? What is this formula good for?

g) What is implied by a stopping $n_{\rm st} \in [2, \infty)$ (i.e., an actual integer $n_{\rm st}$ in this range)? Give the solution for small $\Delta x(t)$ with with initial condition $\Delta x(t = 0) = 0$. Describe the function behavior at $\Delta x(t = 0) = 0$: i.e., maximum or minimum stationary point or rising or falling inflection point.

h) What would you expect the two likeliest values for ℓ to be for physically relevant FAODEs? What would you expect the two likeliest value for $k \neq 1$ to be for physically relevant FAODEs?

i) Now we intuited for the case of $\ell/k \geq 1$ that the stationary point would be a stationary point at infinity (i.e., an SPI), but we did not prove this directly. To prove directly, we need to show that the small Δx (meaning small in absolute value) solutions of

$$\Delta x^{(1)} = \Delta x^{\ell/k} f_\ell^{1/k}$$

that go to zero only do so as $t \to \infty$. Solutions that go to zero are convergent solutions. This means that the constant solutions they correspond to are stable solutions: small perturbations from the constant solutions damp out. Those that do not go to zero are divergent solutions. This means that the constant solutions they correspond to are unstable solutions: small perturbations from the constant solutions cause non-stopping divergence from the constant solutions.

Here consider the $\ell/k = 1$ case and the solutions for $\Delta x(t)$ starting from $t = t_0$ and $\Delta x = \Delta x_0$ as initial conditions. Determine the solutions and under what conditions they are convergent/divergent. Does the convergent solution, in fact, have a SPI? **HINT:** Let $y = \pm\Delta x$ where the upper/lower case is for positive/negative Δx_0.

j) Repeat part (i) for the case of $\ell/k > 1$.

k) An optional continuation of the discussion of the part (h) answer.

SUGGESTED ANSWER:

a) Since $f(x)$ has no zeros between points in the set $\{x_i\}$, $x(t)$ has no stationary points there and must either increase or decrease always. Since $f(x)$ is infinitely differentiable, it seems intuitively clear that $x(t)$ cannot reach any x_i, except as $t \to \infty$ since that would require some kind of singularity in some order of t derivative of $x(t)$. However, a proof is needed verify that the intuition is true. Of course, the exponent $1/k$ could cause a singularity in some order of t derivative of $x(t)$, but we are not going concern ourselves with what happens in those cases.

b) Behold:

$$\Delta x^{(1)} = \Delta x^{\ell/k} f_\ell^{1/k} \ .$$

c) Behold:

$$\begin{aligned}
\Delta x^{(1)} &\propto \Delta x^{\ell/k} \\
\Delta x^{(2)} &\propto \Delta x^{\ell/k-1}\Delta x^{(1)} = \Delta x^{\ell/k-1}\Delta x^{\ell/k} = \Delta x^{2\ell/k-1} \\
\Delta x^{(3)} &\propto \Delta x^{3\ell/k-2} \\
&\vdots \\
\Delta x^{(n)} &\propto \Delta x^{n\ell/k-(n-1)} = \Delta x^{(\ell/k-1)n+1} \ ,
\end{aligned}$$

where the generalization to the n derivative with respect to t is by inspection. The exponent formula is

$$p(n) = n\ell/k - (n-1) = (\ell/k - 1)n + 1 \ .$$

d) If $\ell/k \geq 1$, then exponent of the t derivatives of Δx strictly increases with n (if $\ell/k > 1$) or is constant $p = 1$, and so in either case the t derivatives of Δx and $x(t)$ are zero for $x = x_i$. This implies that for $\ell/k \geq 1$, there is a constant solution $x(t) = x_i$. We intuit that any other solution of x_i will converge to x_i as $t \to \infty$ or diverge from x_i. However, a proof is needed verify that the intuition is true.

e) If $\ell/k < 1$ and $p(n)$ never equals zero, then exponent of the t derivatives of Δx strictly decreases with n, and so there must be derivative $\Delta x^{(n)}(x = x_i) = \infty$: i.e., a derivative with a singularity. A singularity may cause a solution $x(t)$ to reach x_i in a finite time, but what happens probably depends on the detailed behavior of $f(x)$ and goes beyond the scope of this question. Yours truly guesses that most real physical solutions are unlikely to have such a singularity, except in idealized cases. Note that the derivatives of Δx for n lower than the one that gives the singularity will all be zero for $x = x_i$.

f) The formula for ℓ/k and the values for set $n_{\rm st} = 1, 2, 3, \ldots, \infty$ are given by

$$\frac{\ell}{k} = 1 - \frac{1}{n_{\rm st}} = \frac{n_{\rm st} - 1}{n_{\rm st}} = 0, \frac{1}{2}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots, 1 \ .$$

The $n_{\rm st} = 1$ case is ruled out since that gives $\ell = 0$ which implies $f(x_i) \neq 0$ (i.e., $f_0 \neq 0$) which we assumed is not the case. The $n_{\rm st} = \infty$ case just gives $\ell/k = 1$ which is the case already described in the part (d) answer. The formula $n_{\rm st}$ as a function of ℓ/k is

$$n_{\rm st} = \frac{1}{1 - \ell/k}$$

which is useful to see if a particular $\ell/k < 1$ gives an actual stopping $n_{\rm st}$ which must be an integer greater than 1. If there is no actual stopping $n_{\rm st}$, then we have the case discussed in the part (e) answer.

g) An actual stopping $n_{\rm st}$ implies the function has a stationary point not at infinity for $x = x_i$. The small Δx solution with initial condition $\Delta x(t = 0) = 0$ is

$$\Delta x = Ct^{n_{\rm st}} \ ,$$

where C is a constant of integration and there can be no lower powers of t since they would give lower order derivatives than $\Delta x^{(n_{\rm st})}$ that are nonzero at $t = 0$ where $x = x_i$. If $n_{\rm st}$ is even $\Delta x(t = 0)$ is a minimum/maximum for C positive/negative. If $n_{\rm st}$ is odd $\Delta x(t = 0)$ is a rising/falling inflection point (which is also a stationary point) for C positive/negative.

h) There is no absolutely right answer to this part. For the two likeliest ℓ values given that $f(x_i) = 0$, yours truly thinks they will be in order of decreasing likelihood 1 and 2 since that is about the order they seem to turn up for in expansions of physics formulae about a point. For the two likeliest k values, yours truly thinks they will be in order of decreasing likelihood 1 and 2. Most of physics formulae have no overall root function (i.e., $k = 1$) and of those that do it seems the square root (i.e., $k = 2$) is probably the most common.

i) Behold:

$$y^{(1)} = yf_1^{1/k} \qquad \frac{dy}{y} = f_1^{1/k}\,dt \qquad \ln\left(\frac{y}{y_0}\right) = f_1^{1/k}(t - t_0) \qquad y = y_0 e^{f_1^{1/k}(t-t_0)}$$

$$\Delta x = \Delta x_0 e^{f_1^{1/k}(t-t_0)} \ .$$

If $f_1^{1/k} < 0$, we have a convergent solution. If $f_1^{1/k} > 0$, we have a divergent solution. By the nature of the exponential function, it is clear that the convergent solution has a SPI.

j) Behold:

$$y^{(1)} = y^{\ell/k}(\pm 1)^{\ell/k-1} f_\ell^{1/k} \qquad y^{(1)} = y^\gamma C \qquad \text{where we have simplified the notation,}$$

$$\frac{dy}{y^\gamma} = C\,dt \qquad \frac{y^{-\gamma+1} - y_0^{-\gamma+1}}{-\gamma+1} = C(t - t_0)$$

$$y^{-\gamma+1} = (-\gamma+1)C(t - t_0) + y_0^{-\gamma+1}$$

$$y = \frac{1}{\left[(-\gamma+1)C(t-t_0) + y_0^{-\gamma+1}\right]^{1/(\gamma-1)}}$$

$$\Delta x = \pm\frac{1}{\left[(-\ell/k+1)(\pm 1)^{\ell/k-1} f_\ell^{1/k}(t-t_0) + |\Delta x_0|^{-\ell/k+1}\right]^{1/(\ell/k-1)}} \ .$$

If $(-\ell/k+1)(\pm 1)^{\ell/k-1} f_1^{1/k} > 0$, we have a convergent solution. If $(-\ell/k+1)(\pm 1)^{\ell/k-1} f_1^{1/k} < 0$, we have a divergent solution. In fact, the $(-\ell/k+1)(\pm 1)^{\ell/k-1} f_1^{1/k} < 0$ case leads to divergence to an infinity in finite time. By the nature of the function, it is clear that the convergent solution has a SPI.

Note that convergence/divergence depends on the sign of Δx_0 if ℓ/k is even. So if ℓ/k is even and perturbations are not somehow restricted in sign, the constant solution will be unstable in general since some perturbations will always put the system into a divergent solution.

k) Continuation of the part (h) answer: Going beyond, the required answer, the Friedmann equation is a FAODE with $k = 2$. Many Friedmann equation cases have no x_i points, and so have no stationary points, including no SPIs. There are, however, Friedmann equation cases with one x_i point and $\ell = 1$. These have $n_{\rm st} = 2$ and stationary points at finite time t. The ones your truly knows of have cosh-like solutions with minima (bounce universes with two inverse-power density components, the lowest inverse-power one being positive and the highest being negative) and sine-like solutions with maxima (those that expand and then contract with two inverse power density components, the lowest inverse-power one being negative and the highest being positive). The latter case includes the matter-positive-curvature universe. There are Friedmann equation cases with one x_i point and $\ell = 2$. These include matter-positive-curvature-Lambda universes with density component constants adjusted to set the $\ell = 1$ expansion constant set to zero. The constant solution for this kind of matter-positive-curvature-Λ universe is the Einstein universe (i.e., static universe). If the positive curvature density component constant is made larger in absolute value (recall it is a negative quantity) than the Einstein universe case, then you have a forbidden zone in x (i.e., cosmic scale factor). Above the forbidden zone is a cosh-like solution and below is a sine-like solution. If the positive curvature density component constant is made smaller in absolute value (recall it is a negative quantity) than the Einstein universe case, then you have a Lemaître universe which has an inflection point centering a low slope region that when adjusted to have a very low slope gives the Lemaître universe an Einstein universe phase. For these solutions or related solutions, see Jeffery (2026, Exact Two-Density Component Solutions for the Cosmic Scale Factor From a General Approach Including a Simplified Exact Solution Formula for the Radiation-Matter Universe).

Redaction: Jeffery, 2018jan01

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002 qfull 00650 1 3 0 easy math: a FAODE with a stationary point that is not a SPI

18. In this problem, we will get some more insight into first order autonomous ordinary differential equations (FAODEs) with stationary points that are not stationary points at infinity (SPIs) by examining a solution beyond solution to lowest (nonzero) order around the stationary points. Consider the FAODE

$$x^{(1)} = f(x) ,$$

where $f(x_i) = 0$ (i.e., $x = x_i$ gives a stationary point of some kind) and the independent variable is t (not necessarily time). However,

$$x^{(2)} = \frac{df}{dx}x^{(1)} = \frac{df}{dx}f(x) \neq 0$$

for $x = x_i$. This means the stationary point is not a SPI.

NOTE: There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c.

a) Let

$$g(x) = \frac{df}{dx}f(x)$$

and determine a formal solution for $f(x)$.

b) Assume $x(t)$ has maximum and minimum at, respectively, x_i and $-x_i$. Now invent the simplest $f(x)$ you can starting from the part (a) answer, except it has a general constant coefficient so as to give a general scale to the derivative $x^{(1)}$.

c) Now solve for $x(t)$ given the part (b) answer. **HINT:** You could do this by integrating $x(t)$, but differentiating $x(t)$ lead to solution by inspection.

d) Say a FAODE is given by

$$x^{(1)} = [f(x)]^{1/k} ,$$

where t is the independent variable (not necessarily time), $k > 0$, $f(x)$ is infinitely differentiable, and $f(x) = \Delta x^{\ell} f_{\ell} + \ldots$ is the expansion of $f(x)$ around the stationary point x_i with $\Delta x = x - x_i$ starting with the lowest nonzero order. Then the lowest order FAODE is

$$\Delta x^{(1)} = x^{\ell/k} f_{\ell}^{1/k} ,$$

In order for a solution of the FAODE to have stationary point that is not a SPI, there must be a stopping (derivative order) $n_{\rm st}$ given the formula

$$n_{\rm st} = \frac{1}{1 - \ell/k}$$

where an actual stopping $n_{\rm st}$ must be an integer. If the formula gives a non-integer value, then there is a singularity in the behavior of some order of derivative of $x(t)$ at $x = x_i$ and that behavior takes some analysis to determine. An actual stopping $n_{\rm st}$ gives the only nonzero derivative order of $x(t)$ at $x = x_i$. What are the ℓ and k values for the FAODE used in the part (c) and are they consistent with a nonzero derivative order $n = 2$ which is what we imposed in the preamble?

SUGGESTED ANSWER:

a) Behold:

$$g(x) = \frac{df}{dx} f(x) \qquad \frac{1}{2}\frac{d(f^2)}{dx} = g(x) \qquad f(x) = \pm\sqrt{2\int g(x) + C} \ ,$$

where the integral is an indefinite integral (i.e., an antiderivative) and C is a constant of integration.

b) There is no absolutely right answer, but yours truly thinks the simplest $f(x)$ that can be invented is

$$f(x) = \pm A\sqrt{x_i^2 - x^2} \ .$$

Note that the maximum and minimum of $x(t)$ must be at, respectively, x_i and $-x_i$.

c) Behold:

$$x^{(1)} = \pm A\sqrt{x_i^2 - x^2} \qquad x^{(2)} = (\pm A)\frac{(-x)}{\sqrt{x_i^2 - x^2}} x^{(1)} = -A^2 x = -\omega^2 x$$

$$x = x_i \cos\left[\omega(t - t_0)\right] \ ,$$

where we recognized the simple harmonic oscillator differential equation and rewrote A as ω for consistency with usual symbol usage. So the nonlinear FAODE is actually equivalent to a 2nd order linear differential equation.

d) Behold:

$$x^{(1)} = \pm A\sqrt{x_i^2 - x^2} = \pm A\sqrt{x_i^2 - (x_i + \Delta x)^2} = \pm A\sqrt{-2\Delta x + \ldots} \ ,$$

where we require $\Delta x \leq 0$. Clearly, $\ell = 1$ and $k = 2$, and so the stopping $n_{\rm st} = 2$ which is consistent with the imposed nonzero derivative order $n = 2$.

Redaction: Jeffery, 2018jan01

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002 qfull 00900 1 3 0 easy math: Monte Carlo sampling

19. In a Monte Carlo simulation, you want want to sample a random variable x drawn from a probability density function (pdf) $\rho(x)$. The trick is to set another random variable

$$y = P(x) = \int_0^x \rho(x')\,dx'$$

where P(x) is the cumulative probability distribution function (cdf). You then generate y values from a computer random number generator that gives them with uniform probability over the range $(0, 1)$. You then obtain the sample random variables x from

$$x = P^{-1}(y)$$

where P^{-1} is the inverse function of P. The probability of y values in general range Δy is exactly the probability of x values in the corresponding range Δx since

$$\Delta y = \Delta P = \int_{\Delta x} \rho(x')\,dx' \ .$$

An odd point is that random number generators generate y values completely deterministically. So the y values are deterministic relative to source, but, for a good random number generators such as those discussed by Pr-191ff, the y values are random to all useful statistical tests relative to receiver. This fact

invites the philosophical question: Is there any fundamental difference between a deterministic universe that mimics some amount of intrinsic randomness to all detection and one that has some intrinsic randomness as quantum mechanics as ordinarily discussed posits?
In any case, let's investigate how to do Monte Carlo sampling for photons for a couple of interesting cases.

There are parts a,b.

a) A stream of photons in a certain direction is scattered out that direction obeying

$$dN = -N\,d\tau$$

where N is the number of photons traveling in the direction and τ is the optical depth. What is the cdf for photon being scattered by general τ if it started at $\tau = 0$? What is the pdf?

b)

SUGGESTED ANSWER:

a)

b)

Fortran-95 Code

Redaction: Jeffery, 2018jan01

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002 qfull 01010 1 3 0 easy math: variational calculus and Euler's equation

20. To determine geodesics (stationary paths through spaces) one needs to apply variational calculus in general which in the end amounts to solving a differential equation. The most famous variational calculus differential equation is Euler's equation (or Euler's equations if the plural is needed). Euler's equation can be used to find geodesics and it can be specialized to the Euler-Lagrange equations of classical mechanics whose use is justified by Hamilton's principle. We will derive Euler's equation now.

You have integral

$$I = \int_a^b f(x_i, \dot{x}_i, t)\,dt$$

where the set of coordinate functions $x_i = x_i(t)$ constitute a path through space with path parameter t and f is general function for its arguments. We want to determine the path $x_i(t)$ that makes the integral stationary for fixed endpoints $x(a)$ and $x(b)$. Note that following a general relativity convention, the subscript i means that x_i is one of set of coordinates and that it stands for all of them if that is what the context means.

We define

$$x_i(t, \alpha) = x_i(t) + \alpha\eta_i(t) \ ,$$

where $x_i(t)$ is the stationary path, $x_i(t, \alpha)$ is the varied path, α is a variational parameter, and η_i is a general function of t except that it vanishes at the endpoints of the integral. It is helpful to think of η_i as any little blip deviation from the stationary path you care to think of. Since η_i is general it and is derivative $\dot{\eta}_i$ can be varied independently, and thus x_i and $\dot{x}_i$ can be treated as independent in the variation. We now determine the condition on the stationary path as follows:

$$\begin{aligned}
0 = \frac{dI}{d\alpha} &= \int_a^b \left(\frac{\partial f}{\partial x_i}\eta_i + \frac{\partial f}{\partial \dot{x}_i}\dot{\eta}_i \right) dt \\
&= \int_a^b \left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i} \right) \right] \eta_i\,dt + \left. \frac{\partial f}{\partial \dot{x}_i}\eta_i \right|_a^b \\
&= \int_a^b \left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i} \right) \right] \eta_i\,dt \\
0 &= \frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i} \right)
\end{aligned}$$

where repeated indices in a product means summed over all index values (which is Einstein's summation rule), where we have used integration by parts, and the last line follows since the only way the integral (including all the Einstein summed terms) can be zero in general for general η_i is if the bracketed expression in the second to last line vanishes everywhere. Euler's equations (regarding subscript i as indicating a set of equations) are, in fact,

$$\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = 0 \ .$$

There are certain special cases. First is the case when f has no dependence on a particular x_k (which does not stand for the set of coordinate functions x_i). In this case, Euler's equation for x_k reduce to

$$\frac{\partial f}{\partial \dot{x}_k} = C_k \ ,$$

where C_k is a constant of integration. Second is the case when f has no dependence on a particular $\dot{x}_k$. In this case, Euler's equations reduce to

$$\frac{\partial f}{\partial x_k} = 0$$

which implies that f is independent of the particular x_k. This result may have a profound significance that altogether escapes yours truly.

Third is the case when f has no intrinsic dependence on t: i.e., f is just $f(x_i, \dot{x}_i)$, and so $\partial f/\partial t = 0$. To progress, we invoke the Einstein-when-off-track-contract rule and contract Euler's equation with the clairvoyantly chosen $\dot{x}_i$ (i.e., multiply by $\dot{x}_i$ and Einstein sum on i):

$$\begin{aligned}
0 &= \dot{x}_i \frac{\partial f}{\partial x_i} - \dot{x}_i \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right) \\
0 &= \frac{df}{dt} - \ddot{x}_i \frac{\partial f}{\partial \dot{x}_i} - \frac{\partial f}{\partial t} - \left[\frac{d}{dt}\left(\dot{x}_i \frac{\partial f}{\partial \dot{x}_i}\right) - \ddot{x}_i \frac{\partial f}{\partial \dot{x}_i}\right] \\
0 &= -\frac{\partial f}{\partial t} + \frac{d}{dt}\left(f - \dot{x}_i \frac{\partial f}{\partial \dot{x}_i}\right) \ .
\end{aligned}$$

The last equation is the single-alternative Euler's equation. Because of the sum on i it can only replace one of the set of Euler's equations for x_i. But if there is only one coordinate function x_i, then the single-alternative Euler's equation can be useful. The single-alternative Euler's equation is mostly likely to be useful (no matter how many function coordinates x_i there are) when f has no intrinsic dependence on t (i.e., when $\partial f/\partial t = 0$) which is the case we have been working toward in this paragraph. So when $\partial f/\partial t = 0$, we obtain

$$f - \dot{x}_i \frac{\partial f}{\partial \dot{x}_i} = C \ ,$$

where C is a constant of integration. Now if, in fact, there is only one coordinate function x_i, the last equation is likely to be very useful.

There are parts a,b.

a) The metric for a Euclidean space is

$$ds^2 = \sum_j dx_j^2 \ ,$$

where we have not used Einstein summation—we turn it on and off as convenient. Using Euler's equation, prove that the stationary path between any two points is a straight line. **HINT:** First, find what the function f is in this case.

b) What kind of a stationary path is the answer from part (a): global minimum, local minimum, global maximum, local maximum, inflection? Explain your answer.

c) The metric for the surface of sphere of radius R is

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \ .$$

Using Euler's equations, prove that the stationary path between any two points is a great circle (i.e., a circle that cuts the sphere in half). **HINT:** First, find what the function f is in this case. Second, without loss of generality you can choose one endpoint to be the pole (i.e., the place where $\theta = 0$). Third, find the Euler equation result for ϕ first and check its behavior at pole.

d) What kind of a stationary path is the answer from part (c)? Note there are two cases. Explain your answer.

SUGGESTED ANSWER:

a) In this case, the function

$$f(x_i,\dot{x}_i,t)=f(\dot{x}_i)=\sqrt{\sum_j \dot{x}_j^2}\ .$$

Applying Euler's equation gives

$$\frac{\dot{x}_i}{\sqrt{\sum_j \dot{x}_j^2}}=C_i\ .$$

After thinking about it for 20 minutes or so, we must have for all i

$$x_i=x_i(a)+\left[\frac{x_i(b)-x_i(a)}{b-a}\right]g(t)\ ,$$

where $g(a)=0$ and $g(b)=b-a$. The set of x_i constitute a straight line that passes throught the endpoints. This is clear in vector form

$$\vec{x}=\vec{x}(a)+\vec{s}g(t)\ ,$$

where $\vec{s}$ is a vector formed from the formulae for the x_i's. The simplest $g(t)$ is $g(t)=t-a$. But any function g with the right endpoints will do. It doesn't have to be monotonic or anything. Any back and forth motion along the path just cancels out in the integral.

b) Well any blips added to straight-line path make the path longer, and so the straight line is a minimum. Since there is no other minimum, the straight line is either a global minimum or a local minimum if the path length can go to $-\infty$. Since the path length can only go to zero and not $-\infty$, the straight line is a global minimum. There are no other stationary solutions and clearly there is no maximum path length other than $+\infty$.

c) In this case, the function

$$f(x_i,\dot{x}_i,t)=f(\theta,\dot{\theta},\dot{\phi})=\sqrt{\dot{\theta}^2+\sin^2\theta\dot{\phi}^2}\ .$$

Applying Euler's equation for the ϕ coordinate with one endpoint at the pole and the other general gives

$$\frac{\sin^2\theta\dot{\phi}}{\sqrt{\dot{\theta}^2+\sin^2\theta\dot{\phi}^2}}=C_\phi\ .$$

At the pole (i.e., $\theta=0$), we find that $0=C_\phi$ whatever the value of $\dot{\theta}$, nonzero or zero. Thus, $\dot{\phi}=0$ and ϕ coordinate function must be a constant. So the geodesic must be meridian of the sphere and meridians are great circles. So we've proven the stationary path must be a great circle.

What about the θ coordinate function? Well $\theta(t)$ can be anything as long as $\theta(a)$ and $\theta(b)$ are the endpoints. What would Euler's equation give for θ? It would have to be indeterminate since $\theta(t)$ can be anything except for the endpoints. We can prove this explicitly from single-alternative Euler's equation (which has $\partial f/\partial t=0$ in this case):

$$\begin{aligned}\sqrt{\dot{\theta}^2}-\frac{\dot{\theta}^2}{\sqrt{\dot{\theta}^2}}-0&=C_\theta\\ \pm\dot{\theta}-(\pm\dot{\theta})&=C_\theta\\ 0&=C_\theta\ ,\end{aligned}$$

where the upper/lower case is for $\dot{\theta}$ positive/negative. So the constant of integration is 0 and $\theta(t)$ is indeterminate. In fact, we know it can be anything, except for the endpoints.

d) Well there are two great circle paths between the endpoints: shorter and longer. The shorter one is clearly a global mininum since any blips added to it make it longer and the path length cannot go to $-\infty$ since can be no shorter than zero. The longer great circle path is a maximum path when one is confined to the great circle. However, any blip added to it makes it longer, and so it is a local minimum. One can imagine distorting the longer great circle path into the shorter one. There are is no maximum path since you can imagine a spiral path between the endpoints of any length.

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002 qfull 01110 1 3 0 easy math: law of reflection/refraction from Fermat's principle I

21. The laws of reflection and refraction can be proven from the modern version Fermat's principle (HZ-69; Wikipedia: Fermat's principle)—which yours truly for some reason keeps thinking of as Fermat's last principle. Fermat's principle states that a light ray traveling between two points follows a path that is stationary in optical path length which is defined by the differential ds/λ, where ds is differential physical length and λ is local wavelength. In the wave theory of light, Fermat's principle follows from the idea that along stationary paths multiple coherent wave fronts are in phase to 1st order, and so an add constructively: along other paths the multiple coherent wave fronts cancel out by destructive interference virtually totally.

There are parts a,b.

a) Write down the laws of reflection and refraction.

b) Give an argument why the stationary optical path must be in a perpendicular plane to the interface of reflection/transmission for the two laws. This plane is called the plane of incidence (AKA incidence plane) in optics jargon.

c) Draw a diagram in of incidence plane with a reflection/transmission interface. Mark point 1, a source, at (x_1, y_1), and point 2, a receiver, at (x_2, y_2). For niceness, x_1 is measured to the left from the origin at the point of reflection/transmission, x_2 is measured to the right from the origin at the point of reflection/transmission, and y_2 can be on either side of the interface and is positive either way.

d) Continuing with the part (c) setup, consider the source and receiver points as fixed, but the origin as free to vary along the interface in the incidence plane. Now write down the formula for h which is the optical path length between source and receiver for reflection and transmission plus a Lagrange multiplier term.

e) Solve for the stationary point of the formula of part (d) and show that it is a minimum.

f) Now complete the proof of the laws of reflection and refraction.

SUGGESTED ANSWER:

a) The laws of reflection and refraction are

$$\theta_1 = \theta_2 \qquad \text{and} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2 \ ,$$

where 1 is incident, 2 is reflected/transmitted, θ_i is an angle, and n_i is an index of refraction:

b) Consider an optical path between two points not in the incidence plane. The optical path can always be made shorter by moving the contact point on the interface closer to the incidence plane along a line perpendicular to the incidence plane. There can be no minimum along this perpendicular line until you reach the incidence plane which is obviously a minimum along the line—but not in general a minimum for optical in incidence plane itself, of course. There can be no other stationary point along the perpendicular line since there is no special feature to cause one. Thus, the only place where there can be stationary points is in the incidence plane.

c) You will have to imagine the diagram.

d) Behold:

$$h = \frac{\sqrt{x_1^2 + y_1^2}}{\lambda_1} + \frac{\sqrt{x_2^2 + y_2^2}}{\lambda_2} + \alpha(x_1 + x_2) \ ,$$

where α is the Lagrange multiplier and $(x_1 + x_2) = C$ (where C is a constant) is the constraint on the x_i variables.

e) The 1st and 2nd derivatives of h are

$$\frac{\partial h}{\partial x_i} = \frac{1}{\lambda_i}\frac{x_i}{\sqrt{x_i^2+y_i^2}} + \alpha \qquad \text{and} \qquad \frac{\partial^2 h}{\partial x_i^2} = \frac{1}{\lambda_i}\frac{y_i^2}{\sqrt{x_i^2+y_i^2}} \ ,$$

where i is either of 1 and 2. For the stationary point, we find

$$\frac{\sin\theta_i}{\lambda_i} = -\alpha \qquad \text{and} \qquad \left.\frac{\partial^2 h}{\partial x_i^2}\right|_{\text{stationary point}} \geq 0 \ ,$$

where the equality in the last expression only holds for $y_i = 0$ which an irrelevant case since we implicitly assumed $y_i > 0$. Since the 2nd derivative is greater than zero, the stationary point is a minimum.

f) For the case of reflection where $\lambda_1 = \lambda_2$, we find from the part (e) answer the law of reflection

$$\theta_1 = \theta_2 \ .$$

Now the index of refraction obeys

$$n_i = \frac{c}{v_i} = \frac{\lambda}{\lambda_i} \ ,$$

where c is the vacuum light speed, v_i is the medium light speed, and λ is the vacuum wavelength of the light. Thus, from the part (e) answer, we find the law of refraction

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \ .$$

So QED.

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002 qfull 01120 1 3 0 easy math: law of reflection/refraction from Fermat's principle II

22. The first variational principle in physics was discovered by Hero of Alexandria (10?–70? CE) (Wikipedia: Hero of Alexandria: Inventions). He noted that the law of reflection followed from the idea that a light ray traveled the shortest path of light from source to receiver during a reflection of a planar surface. In equation form the law of reflection is

$$\theta_1 = \theta_2 \ ,$$

where θ_1 is the angle of incidence and θ_2 is the angle of reflection both measured in the plane of incidence (i.e., the plane defined by the source and the normal to the surface). Pierre de Fermat (1607–1665) generalized the Hero's idea by saying a light ray traveled the shortest time between source to receiver and from this idea was able to prove the law of refraction as well as the law of reflection. In modern form, the law of refraction is

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \ ,$$

where θ_1 is the angle of incidence and θ_2 is the angle of transmission both measured from the normal to the surface between the media, the $n_i = c/v_i$ are the refractive indices of the media v_i is the light speed in the media, and angles are both in the plane of incidence.

Fermat's idea in modern form is called Fermat's principle and states that a light ray moves along a stationary path in optical path length (i.e., length divided by wavelength). Fermat's principle and the earlier notions of Hero and Fermat himself are variational principles in that variations from the stationary path are used to find it. In fact, a key law of classical mechanics is a variational principle: the principle of least action—more accurately, the principle of stationary action. The classical principal of least action is actually derivable from quantum mechanics. Particles propagate as waves and phase variation tend to cause destructive interference, except for the stationary path for action which is the wave phase itself (Ba-69ff). In the macroscopic limit, the destructive interference causes virtually complete cancellation of propagation, except along the stationary path. Actually, Fermat's principle is, we can now see, the special case for light of the principle of least action.

There are parts a,b.

a) Draw a diagram with a source P_1 a distance y away from a planar surface and a general receiver P_2 that is y above the surface for reflection and y below for refraction. The separation along the direction parallel the planar surface is ℓ. A light ray from the source hits the surface at the origin 0. Draw a normal to the surface at origin 0. The incident angle is θ_1 and the reflection/refraction angle is θ_2. The incident wavelength is λ_1 and the reflection/refraction is λ_2.

b) What is the ray optical path length s from P_1 to P_2 expressed in terms of y, θ_1, θ_2 λ_1, and λ_2?

c) The elegant way to prove the laws of reflection and refraction is to use Lagrange multipliers. The general form is

$$L = f + \alpha g \ ,$$

where L is called the Lagrangian function, f is the function whose contrained stationary point you want find, α is the Lagrange multiplier, and g is the contraint function: i.e., $g =$ constant when the constraint is imposed. Write down the Lagrangian function for the optical path length case. Find the formula for θ_i that makes s stationary consistent with the constraint.

d) From the results of part (c), prove the laws of reflection and refraction.

e) Why can't the stationary path be outside of the planet of incidence?

SUGGESTED ANSWER:

a) You will have to imagine the diagram.

b) Behold:

$$s = \sum_j \frac{y}{\lambda_j \cos\theta_j}$$

c) The constraint equation is

$$g = \ell = \sum_j y \tan\theta_j \ ,$$

and so the Lagrange function is

$$L = \sum_j \left(\frac{y}{\lambda_j \cos\theta_j} + \alpha y \tan\theta_j \right) \ .$$

Differentiating with respect to θ_i and equating to zero gives

$$\begin{aligned} 0 &= \frac{y \sin\theta_i}{\lambda_i \cos^2\theta_i} + \frac{\alpha y}{\cos^2\theta_i} \\ 0 &= \frac{\sin\theta_i}{\lambda_i} + \alpha \\ -\alpha &= \frac{\sin\theta_i}{\lambda_i} \ , \end{aligned}$$

d) In the case of reflection, $\lambda_1 = \lambda_2$, and so the law of reflection from the part (c) answer is

$$\theta_1 = \theta_2 \ ,$$

QED. In the case of refraction, From the part (c) answer, we now find

$$\begin{aligned} \frac{\sin\theta_1}{\lambda_1} &= \frac{\sin\theta_2}{\lambda_2} \\ \frac{\sin\theta_1}{v_1} &= \frac{\sin\theta_2}{v_2} \\ n_1 \sin\theta_1 &= n_2 \sin\theta_2 \ , \end{aligned}$$

the law of refraction. Note we used the facts that $f\lambda = v$ and frequency f is constant across the interface or waves would pile up there which they don't.

e) Between P_1 and the origin 0 and the origin 0 and P_2, straight lines are the shortest optical paths. If you move the origin out of the plane of incidence, clearly those optical paths get longer.

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002 qfull 01130 1 3 0 easy math: Euler-Lagrange equations

23. The Euler equations (Ar-928) (AKA the Euler-Lagrange equations: Go45) are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where t is a general independent variable (but we are already thinking of specializing it to time), i the representative index for a set of indices j, q_j is set of unknown functions that one solves Euler equations for (but we are already thinking of them as being generalized coordinates in classical mechanics), $\dot{q}_j$ are the t partial derivatives of the q_j, and $L = L(q_j, \dot{q}_j, t)$ is a known function.

Now whence the Euler equations and what for their solutions. The solutions of Euler equations, are the functions that make the functional (i.e., function of functions)

$$S(q_j) = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)$$

stationary with respect to general small varitions in q_i: i.e., unchanging to 1st order in a variational parameter that actually never needs to be specified. The Euler equations themselves are obtained by variational calculus on S. Note we are already thinking of specializing S to the action in physics jargon in which case L is the Lagrangian for a system (which is a function of the generalized coordinates q_j) and the Euler-Lagrange equations become the Lagrange equations of motion for the system (Go-45). That stationarized S yields the equations of motions is a variational principle called the principle of least action (though more precisely of stationary action). The specific version of the principle of least action that yields the Lagrange equations is formally called Hamilton's principle (Go-34), but I think most people refer to it just by generic name principle of least action.

There are parts a,b.

a)

b)

SUGGESTED ANSWER:

a)

b)

Fortran-95 Code

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