Cosmology & Galaxies Name:

Homework 2: Miscellaneous Math Problems: All Questions

1. The ancient Egyptian mathematicians thought there was something unfundamental about non-unit fractions (those not of the form $1/n$) though they made a bit of an exception for $2/3$ (Boyer-13–14). So they thought it a good idea to expand non-unit fractions as sums of unit fractions (those of the form $1/n$).

   There are parts a,b,c.

   a) Show the general rational number m/n can be expanded into infinitely many possible unit fraction expansions. **Hint:** This is trivial.

   b) The ancient Egyptians apparently thought some kinds of unit-fraction expansions were good, but have not left us any definite rules (Boyer-14). Probably they never formulated any. However, we can formulate a rule/algorithm. Specify an rule/algorithm for expanding general m/n in unit fractions

   $$m/n = \sum_{i=1}^{I} \frac{k_i}{n_i}$$

   where the denominators n_i are all divisors of n in increasing order of size, there are I divisors in total, and k_i are all zero or 1, except that k_I can be greater than 1. **Hint:** The proof just requires some subtraction using a recurrence relation.

   c) Use your rule/algorithm from the part (b) answer to expand 601/360 in unit fractions. You could do this by hand or write a small computer progam do to it. Note that 360 has 24 divisors which is probably one of the main reasons why the ancient Babylonian astronomers chose it for the division of the circle—they wanted easy division. The other main reason was probably to get angle unit nearly equal to the distance the Sun moved every day on the celestial sphere. **Hint:** If you write a computer code, make it find the divisors with the mod function for you then it will be general for any denominator n. Try your code out on 1170/360.

   d) Consider m/n and an expansion in the harmonic series with omissions:

   $$\frac{m}{n} = \sum_{\ell=2}^{K} \frac{k_\ell}{\ell} ,$$

   where $k_i = 1$ or zero and K is in general ∞. Why is it always possible to make this expansion? Can the series truncate with K finite? I will give one buck to the first person who finds out by themselves or from some source whether or not the expansion truncates to finite K always.

2. A general cone is 3-dimensional shape formed from a planar base and continuum of line segments from the base's perimeter to a vertex (or apex) not in the plane of the base. The height of the cone is the length of the perpendicular from the base plane to the vertex. A general frustum—or, tripping off the tongue erroneously, frustrum—is a general cone with the top sliced off parallel to the base.

   The ancient Egyptian mathematicians were very interested in frustums because of the topless pyramid kind—they are were always designing and building things like that. They even knew the rule for the volume of square pyramidal frustum which in modern formula form is

   $$V = \frac{\Delta h}{3}(a^2 + ab + b^2) ,$$

   where Δh is the height of the frustum (not the height of the pyramid it's cut from), a is the base square side length, and b is the top square side length. The ancient Egyptians probably deduced this rule by constructing a square pyramidal frustum from simpler parts (Boyer-21).

   There are parts a,b.

   a) By the power of pure guess, generalize the volume formula to that of a general frustum with base area A and top area B.

   b) Prove your generalization from the part (a) answer. **Hint:** Note the following factoids. Factoid 1: You can approximately replace any cone/frustum by a **SET** of equal-base-area square cones/frustums with their base-parallel slices slid appropriately: just picture it. Factoid 2: If you slide parallel slices of 3-dimensional shape, you don't change the volume of the shape (e.g., for parallelopiped obviously).

c) Now derive the volume of a general cone with base area A and height h without using the equation in the preamble or the formula found in the parts (a) and (b) answers. **Hint:** The area of any base-parallel slice A_z is proportional to the square of the distance from the vertex to the slice z along the perpendicular from the base plane to the vertex. This is obvious if you envisage the slice as covered by a grid: each grid line obviously scales as z.

d) Now what is the volume of a frustum with base of area A and height to the invisible vertex h, and top with area B and height to the invisible vertex h_B?

e) Given $\Delta h = h - h_B$, derive the formula found in the part (a) answer from the formula found in the part (d) answer. **Hint:** You will have to express h and h_B in terms of Δh, A, and B making use of the integrand used in the part (d) answer, and do some mildly tricky algebra which is accelerated by using the sum/difference of cubes formula:

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2) \ .$$

f) Who is responsible for

...
Come, every frustum longs to be a cone,
And every vector dreams of matrices.
Hark to the gentle gradient of the breeze:
It whispers of a more ergodic zone.
...
I see the eigenvalue in thine eye,
I hear the tender tensor in thy sigh.
Bernoulli would have been content to die,
Had he but known such $a^2 \cos(2\phi)$.

3. The Pythagorean theorem was known to the ancient Babylonians, but not as far as known the ancient Egyptians, long before Pythagoras (c. 570–c. 495 BCE) (Boyer-42). But it is likely the ancient Babylonians never gave a general proof: they just did not think in terms of general proofs. The ancient Greek mathematicians may or may not have learnt of the Pythagorean theorem from the ancient Babylonians. However, they probably gave the first general proof. Late reports say Pythagoras himself proved it and hence its name. This may just be legend (Boyer-54; Wikipedia: Pythagorean theorem: History). Euclid (fl. 300 BCE) gives the first proof on the historical record. We will not attempt his proof, but something simpler. By the way, no one wrote equations like we do before circa 1600—they used other klutzy ways of expressing formulae (see Wikipedia: History of mathematical notation: Symbolic stage).

   Assume a Euclidean 2-dimensional space. Since the space is Euclidean or flat, a square (a 4-sided polygon with sides of equal length and right-angle vertices) has area $A = d^2$ where d is the length of a side. Prove the Pythagorean theorem for this Euclidean space. **Hint:** Draw a square with side length $a + b$ and an inscribed square of side length c where the vertices of the inscribed square touch the first square sides at the points that divide the sides into parts of length a an b.

4. In 2-dimensional Euclidean space (i.e., 2-dimensional flat space), we have a simple area principle. If you draw a general closed contour, you can tile it without overlap with squares of equal size with side length a. We define a^2 as the area of the squares. The sum of areas a^2 for closed contour in the limit that $a \to 0$ and number of squares goes to infinity is the area A of the closed contour. An identical closed contour anywhere in the space has the same area A and if you scale any the linear dimension of the contour by f, the area scales by f^2. Somewhat obviously, the area of two general closed contours (joined or separated) must equal the sum of areas of the two general closed contours since the tiled areas just equal the count of squares of area a^2 times aread a^2 before you take the limit.

   The area principle implies the Pythagorean theorem and consequently the metric of 2-dimensional flat space: $ds^2 = dx^2 + dy^2$, where x and y are general perpendicular coordinates and ds is the distance or interval between two points with coordinates that differ by dx and dy.

There are parts a,b,c,d. The parts can be done independently, and so do not stop if you cannot do a part.

a) Use the area principle to prove the area of a right triangle with sides of length a and b forming the right angle is $ab/2$. **Hint:** Imagine little squares of side length e and tile a rectangle with them, count the squares, find the area of the rectangle as $e \to 0$ and the number of squares goes to infinity, and then use symmetry.

b) Draw a diagram of a square with sides of length $a+b$ and an inscribed square with side of length c with corners touching the sides of the first square (which is the circumscribed square) at points a from each corner of the first square.

c) Use answers from the parts (a) and (b) to prove the Pythagorean theorem: i.e., $c^2 = a^2 + b^2$.

d) Prove the metric $ds^2 = dx^2 + dy^2$ holds for a 2-dimensional flat space. **Hint:** This is easy.

5. Can we prove the Pythagorean theorem semi-rigorously? Yes.

There are parts a,b,c,d,e,f,g,h,i. The parts can be done independently, and so do not stop if you cannot do a part.

a) Assume an homogeneous, isotropic (homist) 2-dimensional space. Assume there is a geodesic rule: i.e., there is a rule for measuring distance and for measuring the stationary distance between two points. Draw intersecting equal length geodesics that intersect at their midpoints and that have 4-fold rotational symmetry about their intersection point. A full rotation about the intersection point is measured as 360°. How would you describe size of the angles subtended at the intersection point separating the crossed geodesic arms and why would you say this? Note draw the geodesics vertical and horizontal, so that the descriptions in the following parts are consistent with the diagram.

b) Now draw geodesics between the endpoints of your crossed geodesics, but note we are not assuming Euclidean (i.e., flat space) so that these geodesics could bend outward/inward from intersection point in some projection or another. You now have a square (but not necessarily a Euclidean square). Call it square 1. Now copy square 1 to square 2 and translate square 2 to the upper right so that the lower left corner endpoints of square 2 lie on the upper right corner endpoints of square 1. Is there a space between geodesics of the two squares joining common endpoints? Why or why not?

c) Now copy square 2 to square 3 and translate square 3 to the lower right, but otherwise with the same intructions as in part (b). Now copy square 3 to square 4 and translate square 4 to the lower left, but otherwise with the same intructions as in part (b). Does square 4 necessarily share a common geodesic with the original square 1? Why or why not?

d) The answer to part (c) was no. However, if there is a common geodesic then the space is a Euclidean plane and, at the common corner of the 4 squares, the angles between the geodesics that meet there are all 90°. Postulating that they are 90° is equivalent to Euclid's 5th postulate. For long ages mathematicians wondered if 5th postulate was derivable from Euclid's first 4 postulates. The answer is no. Even somewhat obviously no since, among other things, geodesics that are parallel on a sphere at the equator (i.e., separated by a mutually perpendicular geodesic there) meet at the poles.

Assuming a Euclidean plane, prove that ines (as we now call geodesics) parallel at one location (i.e., separated by a mutually perpendicular line) stay the same perpendicular distance apart no matter how extended. There are probably many ways of proving this, but one path is to start by noting that equal squares of any size can tile the whole Euclidean plane without overlap which actually is an immediate consequence of our considerations above.

e) The fact that one can tile the Euclidean plane completely with squares without overlap suggests an area concept. Consider differential rectangles of side lengths dx and dy. Define their area to be $dx\,dy$. We define area to be countable in the sense that the area of N rectangles is $N\,dx\,dy$. You can tile completely any region surrounded by a closed curve with equal differential rectangles with no rectangles wholly out of the region. We define the area of the region by

$$A = \lim_{N\to\infty,\, dx\,dy\to 0} N\,dx\,dy \ .$$

That such limit exists in general requires a rigorous proof that we will not do here. However, one can prove the limit exists in special cases easily and those special cases they also show why defining the area of the differential rectangles in terms of the lengths of their sides is reasonable since finite regions of sufficient symmetry also have areas specified by their defining lengths. An important point is that area is independent of the ordering of the adding up the differentials areas. As a nonce expression, we call this independence the area principle.

Determine the area of a large rectangle of sides a and b in terms of differential rectangles and take the limit so that the properties of the differential rectangles vanish.

f) Prove that the area of a right triangle with sides forming the right angle being of length a and b is $ab/2$. **Hint:** You do need to use the area principle.

g) Draw a diagram of a square with sides of length $a+b$ and an inscribed square with side of length c with corners touching the sides of the first square (which is the circumscribed square) at points a from each corner of the first square.

h) Use the area principle to prove the Pythagorean theorem: i.e., $c^2 = a^2 + b^2$.

i) Prove the metric $ds^2 = dx^2 + dy^2$ holds for a Euclidean plane. **Hint:** This is easy.

6. The golden ratio ϕ is a special number known since Greco-Roman antiquity. But there's nothing especially special about it. There are many special numbers: all small natural numbers $(0, 1, 2, \ldots)$, all small prime numbers $(2, 3, 5, 7, 11, 13, \ldots)$, $e = 2.71828\ldots$, pi $= \pi = 3.14159\ldots$, the Euler-Mascheroni constant $\gamma = 0.57721566\ldots$, etc. Here we will investigate the golden ratio just a bit.

   There are parts a,b.

a) Draw a line segment of length a and divide into two parts of of lengths b and c: thus $a = b + c$. The golden ratio is just the ratio when

$$\frac{a}{b} = \frac{b}{c} \ .$$

b) Let's do a general investigation of ratios of the form

$$\frac{a}{b} = g\frac{b}{c} \ ,$$

where $a = b + c$. Solve for the positive case of the ratio a/b as a function of g only. Find the cases for $g = \infty, 1, 0$. The case $g = 1$ gives the golden ratio itself.

c) Prove that

$$\frac{1}{\phi} = \phi - 1 \ .$$

d) In 1202, Fibonacci, perhaps independantly of Indian mathematics, discovered the Fibonnaci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ which has an interesting connection to the golden ratio.

The discovery was from the problem of reproducing pairs of rabbits. A pair takes 1 month to mature from birth and reproduces a new pair after maturity every one month: so the first reproduction happens 2 months after birth. Consider times t_i separated by 1 month periods. Say you at time t_{i-1} you had n_{i-1} adult pairs. However, only the adult pairs n_{i-2} existing at time t_{i-2} can reproduce at t_{i-1} since the new baby pairs at time t_{i-2} have only just matured at t_{i-1}. So at t_{i-1}, the old adult pairs n_{i-2} produce n_{i-2} babies who mature to be adult pairs at time t_i. So the number of adult pairs at time t_i is

$$n_i = n_{i-1} + n_{i-2}$$

which is, of course, a recurrence relation valid for $i \geq 2$.

Starting with 1 baby pair and no adult pairs at time zero, compute by inspection the Fibonnaci sequence until you get bored.

e) In the limit $i \to \infty$, the ratio of adjacent numbers following from Fibonacci recurrence relation

$$n_i = n_{i-1} + n_{i-2}$$

for $i \geq 2$, $n_0 \geq 0$, and $n_1 > 0$ obeys

$$R_i = \frac{n_i}{n_{i-1}} \to \phi \ .$$

Note we are allowing more general initial n_i values than for Fibonacci sequence. In fact, the R_i's alternate with every increment of i between being too high and too low compared to ϕ as $i \to \infty$ and they go to ϕ exactly for finite i in only one special case. Prove the above statements. **Hint:** Start from the Fibonacci recurrence relation, use the definition $\epsilon_i = (R_i - \phi)/\phi$, and remember the part (c) result.

f) Find a reasonable approximate asymptotic formula for the n_i from part (e) as $i \to \infty$. It should be exactly correct in one special case.

g) The recurrence relation

$$n_i = n_{i-1} + n_{i-2}$$

can be turned into a differential equation by changing i into continuous variable t expanding n_t and $n(t-2)$ about $t-1$ to 1st order. Make the transformation and solve the differential equation. How does the solution compare to the approximate asymptotic formula of part (f)?

7. The quadratic formula (which is the solution of the quadratic equation) is an infamous example of case where the standard analytic form (which is what everyone remembers) is numerically rotten. The equation and formula in standard form are, respectively,

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

The numerical rottenness occurs if $|4ac| << b^2$: in this case, one of the roots can become affected by severe round-off error. We'll see how to fix the problem in this problem.

NOTE: There are parts a,b,c,d,e,f. The parts cannot be done independently, but parts (a) and (b) are not so hard and the later parts are just intricate.

a) Solve the quadratic equation for the standard quadratic formula using completing the square. Note we assume that a, b, and c are pure real numbers.

b) The root of the numerical problem is the sign of b. Note if $b = 0$, there is no problem at all:

$$x = \pm\sqrt{\frac{-c}{a}} .$$

So the crucial insight to avoiding a derivation by clairvoyance for the numerically robust quadratic formula is to isolate sign of b: i.e., to factorize b into its sign and absolute value. Rewrite the standard solution in the form

$$x_\pm = \frac{-\mathrm{sgn}(b)|b| \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{1}{2}\mathrm{sgn}(b)\left(\frac{|b| \pm \sqrt{b^2 - 4ac}}{a}\right) ,$$

where we note that the $(-1)(\pm 1) = (\pm 1)$ if the (± 1) is uncorrelated with the (-1), using a bit of clairvoyance for a nice formula we put the factor of $1/2$ where it's been put, and the sign function is given by

$$\mathrm{sgn}(b) = \begin{cases} 1 & \text{for } b > 0. \\ 1 & \text{for } b = 0 \text{ which is unlike the usual definition of 0.} \\ -1 & \text{for } b < 0. \end{cases}$$

As now written, we can see that solution x_+ is numerically robust, but solution x_- is not. But you can make solution x_- robust by using the a difference of squares factor. Write the numerically robust quadratic formula for solution x_- in terms

$$q = -\frac{1}{2}\mathrm{sgn}(b)\left(|b| + \sqrt{b^2 - 4ac}\right)$$

when the moment is right. **Hint:** Recall the difference of squares formula:

$$(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2 .$$

c) What can you say about the robust solutions when the discriminant $(b^2 - 4ac) < 0$ and what can you say about q, a, b, and c in this case?

d) What can you say about the robust solutions when $a = 0$ and $q \neq 0$, and what can you say about q, b, and c in this case?

e) What can you say about the robust solutions when $a \neq 0$ and $q = 0$, and what can you say about a, b, and c in this case?

f) What can you say about the robust solutions when $a = 0$ and $q = 0$, and what can you say about b and c in this case?

8. Consider the following linear 1st order differential equation (DE):

$$x' = A - kx \ ,$$

where t is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant.
There are parts a,b,c,d. Parts (a) and (b) can be done independently at least.

a) Solve for the constant solution x_A. **Hint:** This is easy.

b) We can now write the DE as

$$x' = k\,(x_A - x) \ .$$

Without solving for non-constant solution describe what it must look like as a function of t for arbitrary initial value $x_0 = x(t = 0)$. In particular, where are its stationary points if any? **Hint:** Consider the continuity of all orders of derivative of x.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$, $x'(t)$, and the 1st order in small t solution $x_{\rm 1st}(t)$. **Hint:** You can use an integrating factor, but there is a more straightforward way.

d) What is the e-folding time t_e of your solution and what does it signify? What is the $x(t_e)$? What is the $x_{\rm 1st}(t_e)$? What is remarkable about $x_{\rm 1st}(t_e)$?

9. Consider the following linear 1st order differential equation:

$$x' = A - kx \ ,$$

where t is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant.
There are parts a,b,c,d. Parts (a) and (b) can be done independently at least.

a) Solve for the constant solution x_A. **Hint:** This is easy.

b) Where is it possible for a non-constant solution of a 1st order differential equation to have a stationary point? Will there be stationary points at those t locations for the particular differenital equation of the preamble? **Hint:** Consider the differential equation written in the form

$$x' = k\left(\frac{A}{k} - x\right)$$

and consider what happens to the solution as $t \to \infty$ and remember that if the solution becomes constant, it stays constant. It helps to think graphically.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$ and the 1st order in small t solution $x_{\rm 1st}(t)$. **Hint:** You can use an integrating factor, but there is a more straightforward way.

d) What is the e-folding constant t_e and what does it signify? What is the $x(t_e)$? What is the $x_{\rm 1st} t_e)$? What is remarkable about $x_{\rm 1st}(t_e)$?

10. Consider the 1st order nonlinear differential equation

$$x' = a \prod_{i=1}^{n} (x - x_i) \ ,$$

where t (which may or may not be time) is the independent variable, a is constant, and the x_i are the roots of the polynomial on the right-hand side: the roots are increase monotonically with index i: i.e., they obey $x_1 \leq x_2 \leq ... \leq x_n$.

a) Solve the equation for the general solution for $n = 0$: i.e., when $x' = a$.

b) Solve the equation for the general solution for $n = 1$: i.e., when $x' = a(x - x_1)$. Since this is a warm-up question, a solution by inspection is not adequate.

c) Qualitatively and compactly describe the solutions of the differential equation in all regions for $n \geq 2$. **Hint:** The equation is a 1st order differential equation and the right-hand side is infinitely differentiable everywhere. There are 4 cases to consider. Don't forget to describe the stability of the constant solutions: i.e., does a sufficiently small perturbation lead to a restoration to the constant solution or a permanent departure from it.

d) Consider distinct roots x_{j-1} and x_j for the case with $n \geq 2$. Find an approximate interpolation solution which has the correct values at $t = \pm\infty$. The approximate solution should contain the function element ge^{-ht} where h can be positive or negative, but not zero and $g > 0$ always. The values of h and g are determined in part (e) just below. **Hint:** This is pretty easy.

e) Continuing with the problem from part (d), determine h by requiring that the approximate solution satisfy the differential equation at the midpoint $x = (x_j + x_{j-1})/2$ and g by requiring that it pass through the point (t_0, x_0), where $x_0 \in (x_{j-1}, x_j)$. **Hint:** This is a lot easier than it seems at first.

f) Continuing with the problem from part (d), show that the approximate formula is, in fact, the exact solution for the case of $n = 2$. This solution is called the logistic function. **Hint:** Simplify the formula for h and then differentiate the solution for $n = 2$ and keep substituting the solution for $n = 2$ to eliminate the h and ge^{-ht} function elements.

g) Now solve the equation for the general solution for general $n \geq 2$ and all roots the same x_r: i.e., for $x_i = x_r$ for all i. **Hint:**

11. Consider the 1st order (ordinary, autonomous) differential equation

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. The 1st order DE rule (as yours truly calls it) applies to this DE. We have $f(x_i) = 0$ and therefore x_i yields a constant solution and a stationary point at either of $\pm\infty$.

NOTE: There are parts a,b.

a) Assuming $(df/dx)(x_i) \neq 0$, solve without words for the 1st order perturbation solution in small $\Delta x = x - x_i$. Let Δx_0 be the initial perturbation, time zero is 0, and $R_1 = (df/dx)(x_i)$ for compactness. What is the condition for convergence/divergence in the future to the constant solution? What is the condition for convergence/divergence in the past to the constant solution? **Hint:** Recall the antiderivative of $1/y$ is always $\ln(|y|)$.

b) Now assume the lowest order nonzero coefficient in the expansion of $f(x)$ in small δx is $(d^k f/dx^k)(x_i)$ where $k \geq 2$. The write the solution only in terms of $|\Delta x|$ and $|\Delta x_0|$ since that seems most clear and start from the differential form

$$\frac{d|\Delta x|}{|\Delta x|^k} = hR_k \, dt \ ,$$

where for k even $h = \pm 1$ with upper case for $\Delta x > 0$ and lower case for $\Delta x < 0$ and for k odd $h = 1$, and $R_k = (d^k f/dx^k)(x_i)$ for compactness. Show why this differential form is correct before you use it.

c) What happens as $hR_k t$ **INCREASES/DECREASES** from 0? At what time t is there an infinity?

12. The logistic function (called that for a darn good reason) turns up in many contexts looking like:

$$f(x) = \begin{cases} \dfrac{f_{\mathrm{M}}}{1 + e^{-r(x-x_0)}} = \dfrac{f_{\mathrm{M}}}{1 + (f_{\mathrm{M}}/f_0 - 1)e^{-rx}} & \text{in general form;} \\[2ex] \dfrac{1}{1 + e^{-x}} = \dfrac{e^x}{e^x + 1} = \dfrac{1}{2}\left[\tanh(x/2) + 1\right] & \text{in natural or reduced form.} \end{cases}$$

In this question, we only use the natural form for simplicity and elegance.

There are parts a,b,c,d.

a) Determine f' (which is, in fact, called the logistic distribution), f'' (also write it as an explicitly even function which it is), the antiderivative of f (easy if you write f in terms of e^x), and the integral of f' from $-x$ to x. Use the natural form of the function.

b) Determine stationary points of f and f' and the values of f and f' at those points. Use the natural form of the function.

c) The logistic function can be used as a smooth replacement for the Heaviside step function:

$$H(x) = \begin{cases} 0 & x < 0; \\ 1/2 & x = 0; \\ 1 & x > 0. \end{cases}$$

Show that logistic function becomes the that Heaviside step function with the appropriate limiting procedure. **Hint:** This is really easy.

d) The logistic function is actually the solution of a 1st order nonlinear differential equation. This equation shows up, for example, in population dynamics. Say you have population N that grows at rate (per population) r with unlimited resources. However, the rate with resources limited by carry capacity (or maximum population) K is modeled as $r(1 - N/K)$ which is zero when $N \to K$. The growth differential equation for N, sometimes called the Verhulst-Pearl equation, is

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N \ ,$$

Reduce this equation to natural form and find the solution. Then write the solution out in population-dynamics form for general initial population N_0 at $t = 0$ and show the small N/K and $t \to \infty$ asymptotic limiting cases explicitly. **Hint:** You'll need a table integral.

13. Consider the 1st order (ordinary, autonomous) differential equation

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. The 1st order DE rule (as yours truly calls it) applies to this DE. We have $f(x_i) = 0$ and therefore x_i yields a constant solution and a stationary point at either of $\pm\infty$.

NOTE: There are parts a,b.

a) Assuming $(df/dx)(x_i) \neq 0$, solve without words for the 1st order perturbation solution in small $\Delta x = x - x_i$. Let Δx_0 be the initial perturbation, time zero is 0, and $R_1 = (df/dx)(x_i)$ for compactness. What is the condition for convergence/divergence in the future to the constant solution? What is the condition for convergence/divergence in the past to the constant solution? **Hint:** Recall the antiderivative of $1/y$ is always $\ln(|y|)$.

b) Now assume the lowest order nonzero coefficient in the expansion of $f(x)$ in small δx is $(d^k f/dx^k)(x_i)$ where $k \geq 2$. The write the solution only in terms of $|\Delta x|$ and $|\Delta x_0|$ since that seems most clear and start from the differential form

$$\frac{d|\Delta x|}{|\Delta x|^k} = hR_k \, dt \ ,$$

where for k even $h = \pm 1$ with upper case for $\Delta x > 0$ and lower case for $\Delta x < 0$ and for k odd $h = 1$, and $R_k = (d^k f/dx^k)(x_i)$ for compactness. Show why this differential form is correct before you use it.

c) What happens as $hR_k t$ **INCREASES/DECREASES** from 0? At what time t is there an infinity?

14. A 1st order homogeneous differential equation, linear or nonlinear, of the form

$$f' = g(f) \ ,$$

(with independent variable t which g has **NO** explicit dependence on) at points where it is infinitely differentiable only has solutions that are strictly in/decreasing or that are constant. Note that differentiable at a point means there is a finite derivative of the same value taken from above or below the point and there is no singularity at the point (which is usually implied by the first condition). Also note that strictly in/decreasing means there are no have stationary points and constant means constant for a finite region. The constant solutions are often stable/unstable in the sense that small perturbations from them lead to convergent/divergent behavior with increasing independent variable.

The rule actually requires the extra condition that higher derivatives of the differential equation $f^{\langle n\rangle}$ (where we use angle brackets do indicate differentiation order when primes will not do) do **NOT** generate zero-over-zero cases: i.e., cases where a f' on the right-hand side of the equation is multiplied by a factor that cancels the zero at stationary point making the higher order derivative on the left-hand side of the equation non-zero. Such a non-zero $f^{\langle n\rangle}$ means that a Taylor expansion around the stationary point will show curvature. That zero-over-zero cases occur will be proven showing important examples.

There are parts a,b,c,d,e,f,g. The parts can all be done independently, and so do not stop if you cannot do a part.

a) Prove the rule given in the preamble for a $g(f)$ that does **NOT** generate zero-over-zero cases. **Hint:** Use proof by induction using the general Leibniz rule (which is the generalization of the product rule):

$$(rs)^{\langle n\rangle} = \sum_{k=0}^{n} \binom{n}{k} r^{\langle n-k\rangle} s^{\langle k\rangle} \ ,$$

where r and s are general functions (Ar-667; Wikipedia: General Leibniz rule). Note $s^{\langle 0\rangle} = s$ not 1.

b) For this part, the preamble is long, the answer is short—have patience.

The zero-over-zero case can (but not necessarily will) occur when we have

$$(f')^p = g(f) \qquad \text{or, equivalently,} \qquad f' = e^{i\phi} g(f)^{1/p}$$

where $e^{i\phi}$ is a phase factor (and we only consider its pure real values) and where $g(f)$ does not itself lead to the zero-over-zero case. The zero-over-zero case will when

$$g^{1/p-(n-1)}(f')^(n-1) = Q \neq 0 \ ,$$

where Q is a constant and $n > 2$ and $[1/p-(n-1)]$ and $(n-1)$ are powers, **NOT** derivative orders. Note that when $n = 1$, we have

$$f' = e^{i\phi} g^{1/p} = e^{i\phi} Q$$

which means $f = at + b$ which has no stationary points and is not zero-over-zero case.

To prove the exception, we differentiate the differential equation $In - 1$ times to get

$$f^{\langle In\rangle} = Ag^{1/p-(n-1)}(f')^(n-1)f^{\langle I-1\rangle} + Bg^{1/p-(n-2)}(f')^(n-2)f^{\langle (I-1)n+1\rangle} + \ldots \ ,$$

where A and B are constants whose values are of no interest and $\{(I-1)n+1\}$ is a derivative order. Note that every term must have the sum of derivative orders equal to $In-1$: e.g., $(n-1)+(I-1)n = In-1$ and $(n-2)+(I-1)n+1 = In-1$. an **INHOMOGENEOUS** 1st order differential equation does not have to obey the rule stated in the preamble. **Hint:** Find a trivial counterexample. Think trigonometry.

b) Prove that a homogenous 1st order differential equation can have a stationary point at $\pm\infty$. **Hint:** Find a trivial example.

c) Prove the rule given in the preamble and discuss why exceptions can occur. **Hint:** Use proof by induction to show that if $x(t)$ has a stationary point where $x' = f(x)$ are infinitely differentiable that the function is constant at that point: i.e., all orders of derivatives of x are zero a that point.

d) Prove that a solution can be nonmonotonic if there is point t where $x' = f(x))$ is not infinitely differentiable. **Hint:** Find a simple example of a 1st order differential equation such a solution. Yours truly suggests differential equation with solution $x = 1/t$.

e) Prove that a solution can have a stationary point at a point t where $x' = f(x))$ is not infinitely differentiable. **Hint:** Find a simple example of a 1st order differential equation such a solution. Yours truly suggests differential equation with solution $x = |t|^3$.

15. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P \ ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to x. Note $g(x)$ may go negative as a function of x, but we assume it does not negative as function of t at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm[g(x)]^{1/2} \ ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

NOTE: There are parts a,b,c,d.

a) Given x_i (or in the time variable t_i) is a stationary point of $x' = f(x)$ (i.e., $x'(x_i) = f(x_i) = f[x(t_i)] = 0$), prove without words that $x''(x_i) = 0$.

b) The part (a) answer gives the base case (or 1st step) for a proof by induction that all orders of derivative of x with respect to t at x_i (or in the time variable t_i) are zero. The proof follows by inspection if your math intuition is good enough. However, do a formal proof by induction. **Hint:** For the proof, you do **NOT**, in fact, need the full general Leibniz rule for the derivative of a product (Ar-558)

$$\frac{d^m(fg)}{dx^m} = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dx^k} \frac{d^{m-k} g}{dx^{m-k}} \ .$$

Using it actually makes the proof a bit more tricky to follow. But you do need to know that the nth order derivative of x (i.e., $x^{(n)}$) is obtained by applying the general Leibniz rule for $m = n - 2$ to the result of the part (a) answer and that highest derivative of x on the right-hand side of that application is $x^{(n-1)}$. Note that $f(x)$ is general to the degree specified in the preamble, and so the proof is unchanged if any order of derivative $f(x)$ with respect to x is zero at x_i.

c) Given the part (b) result, give an argument for why the stationary point t_i must be all points (i.e., is actually a static solution) or at time equals infinity.

d) A 1st order DE system given a small perturbation from a static solution either asymptotically goes back to it (i.e., is asymptotic to it at positive infinity, and so is called stable) or grows away from it (i.e., is asymptotic to it at negative infinity, and so is called unstable). Assuming the df/dx is nonzero at x_i, prove without words that a 1st order DE system given a small perturbation (i.e., a perturbation Δx_0 which requires only 1st order expansion of $f(x)$ in small $\Delta x = x - x_i$) varies exponentially and determine the condition for stability.

16. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P \ ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to x. Note $g(x)$ may go negative as a function of x, but we assume it does not negative as function of t at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm\sqrt{g(x)} \ ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

NOTE: There are parts a,b,c,d,e.

a) Given x_i (or in the time variable t_i) is a stationary point of $x' = \pm\sqrt{g(x)}$ (i.e., $x'(x_i) = \pm\sqrt{g(x_i)} = \pm\sqrt{g[x(t_i)]} = 0$), prove without words that $x''(x_i) \neq 0$ for $g(x_i) \neq 0$.

b) What does the part (a) answer imply about x_i? What does the part (a) answer imply about x_i given the sign of $dg/dx(x_i)$?

c) Given $(dg/dx)(x_i) = 0$, prove by induction that for general $n \in [1\infty]$ that $x^{(n)}(x_i) = 0$. **Hint:** Consider $x^{(4)}(x_i) = 0$ as step 1 (i.e., the base case) of the proof. Note that the right-hand side of the expressions in the proof will always have a derivative of x two orders lower than the left-hand side.

d) Given $(dg/dx)(x_i) = 0$, what does the part (c) answer imply about x_i?

e) Given $(dg/dx)(x_i) = 0$, and therefore there is a static solution $x = x_i$ for all time t, we can consider what the lowest order solution is for a small perturbation from the static solution. The expansion of the differential equation in small $\Delta x = x - x_i$ is

$$\frac{d\Delta x}{dt} = \pm\sqrt{\sum_{k=\ell}^{\infty} \Delta x^k \left[\frac{d^k g}{dx^k}(x_i)\right]} \ ,$$

where ℓ is the lowest power for which there is a nonzero coefficient $(d^\ell g/dx^\ell)(x_i)$. What possible signs can Δx when ℓ is even and $(d^\ell g/dx^\ell)(x_i) > 0$? What possible signs can Δx when ℓ is even and $(d^\ell g/dx^\ell)(x_i) < 0$? What possible signs can Δx when ℓ is odd?

17. In a Monte Carlo simulation, you want want to sample a random variable x drawn from a probability density function (pdf) $\rho(x)$. The trick is to set another random variable

$$y = P(x) = \int_0^x \rho(x')\,dx'$$

where P(x) is the cumulative probability distribution function (cdf). You then generate y values from a computer random number generator that gives them with uniform probability over the range $(0, 1)$. You then obtain the sample random variables x from

$$x = P^{-1}(y)$$

where P^{-1} is the inverse function of P. The probability of y values in general range Δy is exactly the probability of x values in the corresponding range Δx since

$$\Delta y = \Delta P = \int_{\Delta x} \rho(x')\,dx' \ .$$

An odd point is that random number generators generate y values completely deterministically. So the y values are deterministic relative to source, but, for a good random number generators such as those discussed by Pr-191ff, the y values are random to all useful statistical tests relative to receiver. This fact invites the philosophical question: Is there any fundamental difference between a deterministic universe that mimics some amount of intrinsic randomness to all detection and one that has some intrinsic randomness as quantum mechanics as ordinarily discussed posits?

In any case, let's investigate how to do Monte Carlo sampling for photons for a couple of interesting cases.

There are parts a,b.

a) A stream of photons in a certain direction is scattered out that direction obeying

$$dN = -N\,d\tau$$

where N is the number of photons traveling in the direction and τ is the optical depth. What is the cdf for photon being scattered by general τ if it started at $\tau = 0$? What is the pdf?

b)

18. To determine geodesics (stationary paths through spaces) one needs to apply variational calculus in general which in the end amounts to solving a differential equation. The most famous variational calculus differential equation is Euler's equation (or Euler's equations if the plural is needed). Euler's equation can be used to find geodesics and it can be specialized to the Euler-Lagrange equations of classical mechanics whose use is justified by Hamilton's principle. We will derive Euler's equation now.

You have integral

$$I = \int_a^b f(x_i, \dot{x}_i, t)\,dt$$

where the set of coordinate functions $x_i = x_i(t)$ constitute a path through space with path parameter t and f is general function for its arguments. We want to determine the path $x_i(t)$ that makes the integral stationary for fixed endpoints $x(a)$ and $x(b)$. Note that following a general relativity convention, the subscript i means that x_i is one of set of coordinates and that it stands for all of them if that is what the context means.

We define

$$x_i(t, \alpha) = x_i(t) + \alpha\eta_i(t) \ ,$$

where $x_i(t)$ is the stationary path, $x_i(t, \alpha)$ is the varied path, α is a variational parameter, and η_i is a general function of t except that it vanishes at the endpoints of the integral. It is helpful to think of η_i as any little blip deviation from the stationary path you care to think of. Since η_i is general it and is derivative $\dot{\eta}_i$ can be varied independently, and thus x_i and $\dot{x}_i$ can be treated as independent in the variation. We now determine the condition on the stationary path as follows:

$$\begin{aligned} 0 = \frac{dI}{d\alpha} &= \int_a^b \left(\frac{\partial f}{\partial x_i}\eta_i + \frac{\partial f}{\partial \dot{x}_i}\dot{\eta}_i \right) dt \\ &= \int_a^b \left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i} \right) \right] \eta_i\,dt + \left. \frac{\partial f}{\partial \dot{x}_i}\eta_i \right|_a^b \\ &= \int_a^b \left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i} \right) \right] \eta_i\,dt \\ 0 &= \frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i} \right) \end{aligned}$$

where repeated indices in a product means summed over all index values (which is Einstein's summation rule), where we have used integration by parts, and the last line follows since the only way the integral (including all the Einstein summed terms) can be zero in general for general η_i is if the bracketed expression in the second to last line vanishes everywhere. Euler's equations (regarding subscript i as indicating a set of equations) are, in fact,

$$\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i} \right) = 0 \ .$$

There are certain special cases. First is the case when f has no dependence on a particular x_k (which does not stand for the set of coordinate functions x_i). In this case, Euler's equation for x_k reduce to

$$\frac{\partial f}{\partial \dot{x}_k} = C_k \ ,$$

where C_k is a constant of integration. Second is the case when f has no dependence on a particular $\dot{x}_k$. In this case, Euler's equations reduce to

$$\frac{\partial f}{\partial x_k} = 0$$

which implies that f is independent of the particular x_k. This result may have a profound significance that altogether escapes yours truly.

Third is the case when f has no intrinsic dependence on t: i.e., f is just $f(x_i, \dot{x}_i)$, and so $\partial f/\partial t = 0$. To progress, we invoke the Einstein-when-off-track-contract rule and contract Euler's equation with the clairvoyantly chosen $\dot{x}_i$ (i.e., multiply by $\dot{x}_i$ and Einstein sum on i):

$$\begin{aligned} 0 &= \dot{x}_i \frac{\partial f}{\partial x_i} - \dot{x}_i \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right) \\ 0 &= \frac{df}{dt} - \ddot{x}_i \frac{\partial f}{\partial \dot{x}_i} - \frac{\partial f}{\partial t} - \left[\frac{d}{dt}\left(\dot{x}_i \frac{\partial f}{\partial \dot{x}_i}\right) - \ddot{x}_i \frac{\partial f}{\partial \dot{x}_i}\right] \\ 0 &= -\frac{\partial f}{\partial t} + \frac{d}{dt}\left(f - \dot{x}_i \frac{\partial f}{\partial \dot{x}_i}\right) . \end{aligned}$$

The last equation is the single-alternative Euler's equation. Because of the sum on i it can only replace one of the set of Euler's equations for x_i. But if there is only one coordinate function x_i, then the single-alternative Euler's equation can be useful. The single-alternative Euler's equation is mostly likely to be useful (no matter how many function coordinates x_i there are) when f has no intrinsic dependence on t (i.e., when $\partial f/\partial t = 0$) which is the case we have been working toward in this paragraph. So when $\partial f/\partial t = 0$, we obtain

$$f - \dot{x}_i \frac{\partial f}{\partial \dot{x}_i} = C ,$$

where C is a constant of integration. Now if, in fact, there is only one coordinate function x_i, the last equation is likely to be very useful.

There are parts a,b.

a) The metric for a Euclidean space is

$$ds^2 = \sum_j dx_j^2 ,$$

where we have not used Einstein summation—we turn it on and off as convenient. Using Euler's equation, prove that the stationary path between any two points is a straight line. **Hint:** First, find what the function f is in this case.

b) What kind of a stationary path is the answer from part (a): global minimum, local minimum, global maximum, local maximum, inflection? Explain your answer.

c) The metric for the surface of sphere of radius R is

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) .$$

Using Euler's equations, prove that the stationary path between any two points is a great circle (i.e., a circle that cuts the sphere in half). **Hint:** First, find what the function f is in this case. Second, without loss of generality you can choose one endpoint to be the pole (i.e., the place where $\theta = 0$). Third, find the Euler equation result for ϕ first and check its behavior at pole.

d) What kind of a stationary path is the answer from part (c)? Note there are two cases. Explain your answer.

19. The laws of reflection and refraction can be proven from the modern version Fermat's principle (HZ-69; Wikipedia: Fermat's principle)—which yours truly for some reason keeps thinking of as Fermat's last principle. Fermat's principle states that a light ray traveling between two points follows a path that is stationary in optical path length which is defined by the differential ds/λ, where ds is differential physical length and λ is local wavelength. In the wave theory of light, Fermat's principle follows from the idea that along stationary paths multiple coherent wave fronts are in phase to 1st order, and so an add constructively: along other paths the multiple coherent wave fronts cancel out by destructive interference virtually totally.

There are parts a,b.

a) Write down the laws of reflection and refraction.

b) Give an argument why the stationary optical path must be in a perpendicular plane to the interface of reflection/transmission for the two laws. This plane is called the plane of incidence (AKA incidence plane) in optics jargon.

c) Draw a diagram in of incidence plane with a reflection/transmission interface. Mark point 1, a source, at (x_1, y_1), and point 2, a receiver, at (x_2, y_2). For niceness, x_1 is measured to the left from the origin at the point of reflection/transmission, x_2 is measured to the right from the origin at the point of reflection/transmission, and y_2 can be on either side of the interface and is positive either way.

d) Continuing with the part (c) setup, consider the source and receiver points as fixed, but the origin as free to vary along the interface in the incidence plane. Now write down the formula for h which is the optical path length between source and receiver for reflection and transmission plus a Lagrange multiplier term.

e) Solve for the stationary point of the formula of part (d) and show that it is a minimum.

f) Now complete the proof of the laws of reflection and refraction.

20. The first variational principle in physics was discovered by Hero of Alexandria (10?–70? CE) (Wikipedia: Hero of Alexandria: Inventions). He noted that the law of reflection followed from the idea that a light ray traveled the shortest path of light from source to receiver during a reflection of a planar surface. In equation form the law of reflection is

$$\theta_1 = \theta_2 \ ,$$

where θ_1 is the angle of incidence and θ_2 is the angle of reflection both measured in the plane of incidence (i.e., the plane defined by the source and the normal to the surface). Pierre de Fermat (1607–1665) generalized the Hero's idea by saying a light ray traveled the shortest time between source to receiver and from this idea was able to prove the law of refraction as well as the law of reflection. In modern form, the law of refraction is

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \ ,$$

where θ_1 is the angle of incidence and θ_2 is the angle of transmission both measured from the normal to the surface between the media, the $n_i = c/v_i$ are the refractive indices of the media v_i is the light speed in the media, and angles are both in the plane of incidence.

Fermat's idea in modern form is called Fermat's principle and states that a light ray moves along a stationary path in optical path length (i.e., length divided by wavelength). Fermat's principle and the earlier notions of Hero and Fermat himself are variational principles in that variations from the stationary path are used to find it. In fact, a key law of classical mechanics is a variational principle: the principle of least action—more accurately, the principle of stationary action. The classical principel of least action is actually derivable from quantum mechanics. Particles propagate as waves and phase variation tend to cause destructive interference, except for the stationary path for action which is the wave phase itself (Ba-69ff). In the macroscopic limit, the destructive interference causes virtually complete cancellation of propagation, except along the stationary path. Actually, Fermat's principle is, we can now see, the special case for light of the principle of least action.

There are parts a,b.

a) Draw a diagram with a source P_1 a distance y away from a planar surface and a general receiver P_2 that is y above the surface for reflection and y below for refraction. The separation along the direction parallel the planar surface is ℓ. A light ray from the source hits the surface at the origin 0. Draw a normal to the surface at origin 0. The incident angle is θ_1 and the reflection/refraction angle is θ_2. The incident wavelength is λ_1 and the reflection/refraction is λ_2.

b) What is the ray optical path length s from P_1 to P_2 expressed in terms of y, θ_1, θ_2 λ_1, and λ_2?

c) The elegant way to prove the laws of reflection and refraction is to use Lagrange multipliers. The general form is

$$L = f + \alpha g \ ,$$

where L is called the Lagrangian function, f is the function whose contrained stationary point you want find, α is the Lagrange multiplier, and g is the contraint function: i.e., $g =$ constant when the constraint is imposed. Write down the Lagrangian function for the optical path length case. Find the formula for θ_i that makes s stationary consistent with the constraint.

d) From the results of part (c), prove the laws of reflection and refraction.

e) Why can't the stationary path be outside of the planet of incidence?

21. The Euler equations (Ar-928) (AKA the Euler-Lagrange equations: Go45) are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where t is a general independent variable (but we are already thinking of specializing it to time), i the representative index for a set of indices j, q_j is set of unknown functions that one solves Euler equations for (but we are already thinking of them as being generalized coordinates in classical mechanics), $\dot{q}_j$ are the t partial derivatives of the q_j, and $L = L(q_j, \dot{q}_j, t)$ is a known function.

Now whence the Euler equations and what for their solutions. The solutions of Euler equations, are the functions that make the functional (i.e., function of functions)

$$S(q_j) = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)$$

stationary with respect to general small varitions in q_i: i.e., unchanging to 1st order in a variational parameter that actually never needs to be specified. The Euler equations themselves are obtained by variational calculus on S. Note we are already thinking of specializing S to the action in physics jargon in which case L is the Lagrangian for a system (which is a function of the generalized coordinates q_j) and the Euler-Lagrange equations become the Lagrange equations of motion for the system (Go-45). That stationarized S yields the equations of motions is a variational principle called the principle of least action (though more precisely of stationary action). The specific version of the principle of least action that yields the Lagrange equations is formally called Hamilton's principle (Go-34), but I think most people refer to it just by generic name principle of least action.

There are parts a,b.

a)

b)