Lecture ~~8~~ 6 | With some Thermodynamics warm-up — 2021 nov 07 / 10- (2023 dec 21)

Cosmic Microwave Background ~~& Cosmic Temperature~~

2) Planck Law and Generalized Wien's Law

3) Integrated Planck Law

1) λ, ν, E representation of specific intensity ~~of Planck's Law~~

3) DEBRA & CMB
= Diffuse Extragalactic Background & the Cosmic Microwave Background

4) Proof ~~that~~ Planckian. ~~Radiation~~ Stays Planckian with cosmological Expansion (Really any Expansion ~~that has~~ $\lambda \propto a$)

5) Photon to Baryon Ratio

6) Simplified origin of CBR (Cosmic BackGround Radiation)

7) Sunt ~~? ~~ ... maybe preview them

Note $\rho_M \propto \frac{1}{a^3}$, $\rho_r \propto \frac{1}{a^4}$

rest mass

$\rho_{rem} \propto \frac{1}{a^0}$

So Radiation kept matter hot to decoupling

'0-002

1) Plank Law & Generalized Wien Law

$$B_\nu = \frac{2hc^2}{\lambda^5}\frac{1}{e^x-1} \quad \text{or} \quad B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^x-1}$$

$$x = \frac{hc}{kT\lambda} \qquad x = \frac{h\nu}{kT}$$

$$B_\nu \, d\nu = -B_\lambda \, d\lambda$$

-ve since $d\lambda < 0$ given $d\nu > 0$

$$B_\nu = B_\lambda \frac{c}{\nu^2} d\nu$$

$$= \frac{2hc^2\nu^5}{c^5}\frac{1}{e^x-1}\frac{c}{\nu^2}$$

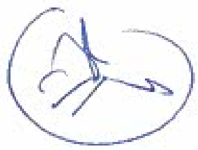
$$= \frac{2h\nu^3}{c^2}\frac{1}{e^x-1}$$

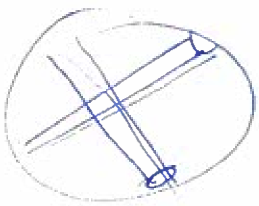
$\lambda = \frac{c}{\nu}$

$d\lambda = -\frac{c}{\nu^2} d\nu$

$$B_\lambda = \frac{2hc^3}{\left(\frac{hc}{kT}\right)^5}\frac{x^5}{e^x-1}$$

B_ν



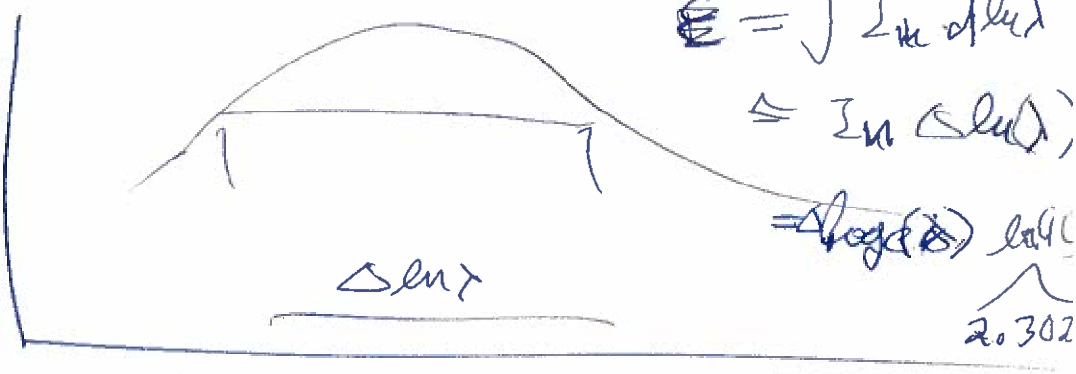


③ Also if you plot

Intergrated energy not photons

10 − 005

I_H



$E = \int I_\nu \, d\ln\nu$

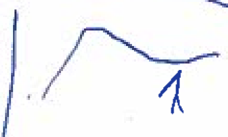
$\simeq I_\nu \, \Delta\ln\nu$

$= \Delta \log(\nu) \, \ln 10$

2.302

$\Delta \ln\nu$

②



Tends to flatten $d\ln\nu$ and so make νI_ν smaller plot vertical range (good for visualization)

$\ln\nu$ or $\ln\nu$ or $\ln E$

all the same

c) So DEBRA: CMB $\sim 100 \cdot 2 = 200$ ~~10.7~~

Rest of DEBRA $30.3 + .03 \cdot 5 + .003$ ... $\simeq 30$

Diffuse ~~extragalactic~~ extragalactic Background radiation

So CMB is main EMR component of DEBRA

Make expand on optical depth due to reionization

Show image sometime from the Lec 10 surf

d) Optical Depth from decoupling since we are discussing specific intensity

D) CMB → from CBR at decoupling

↳ patchy over large patches

$\tau \simeq .05 - .1$ (NASA page 2021)

in free electron Thomson scatter

Optical depth from decoupling to now

In Int gal Med IGM ioniz where

10-006

$\tau << .1$ is good otherwise initial patchiness of CBR would be washed away and all that information about initial density fluctuations would be erased which is so vital for our initial conditions of large scale structure formation

$$I_{Decou} e^{-\tau} \simeq I_{decoupl}(1-\tau) = I_{decay}(.95 \leftrightarrow .9) = I_{now}$$

ΔT fluctuation

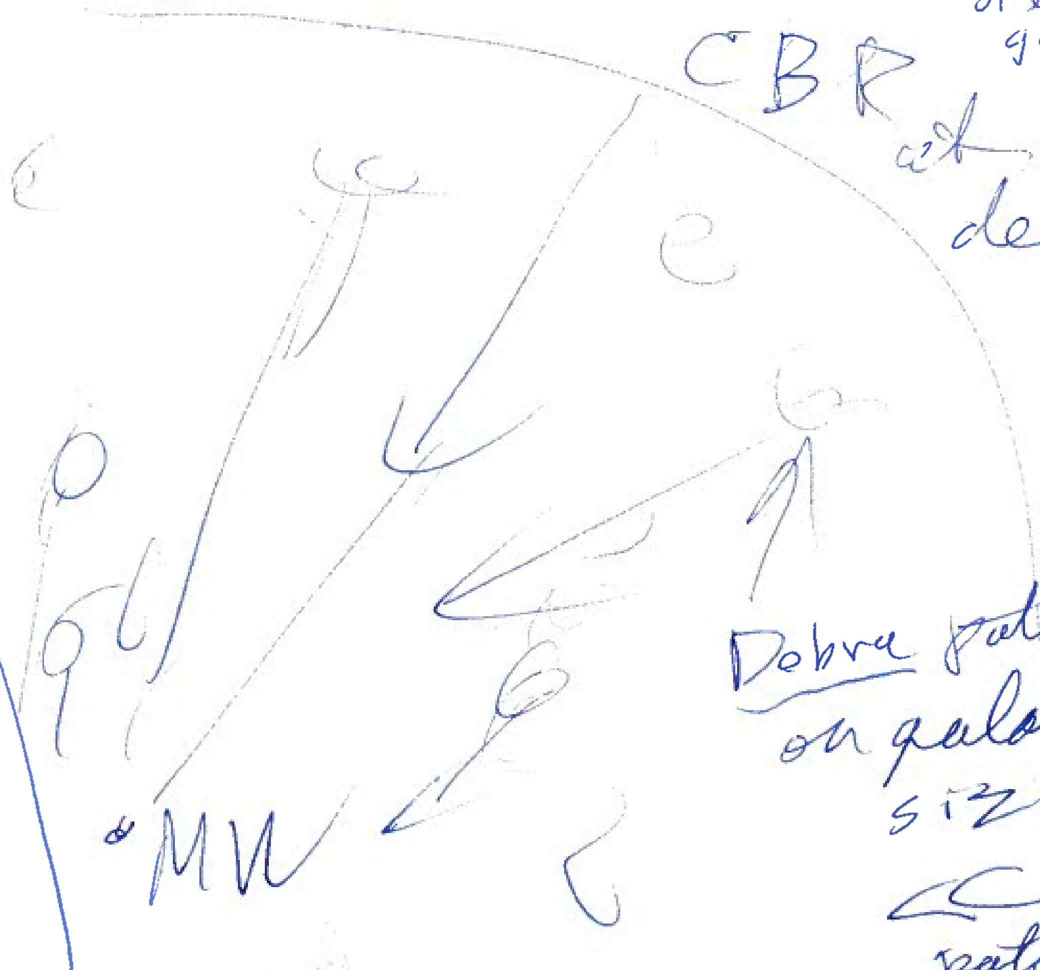
↓

$\delta \rho$ fluct

↓

initial conditions for LSS

Would be $\tau \le \tau_{decoupling}$ for rest of DEBRA (at least once it escapes galaxies where it might scatter somewhat)

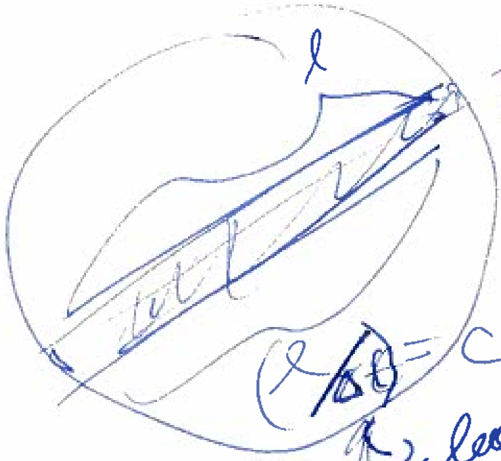


Debra patches on galaxy size $<<$ CBR patchiness

But can't find how much now.

(10-007

e) Energy density & Specific Intensity



$$I_\lambda dA = \mathcal{E}_\lambda dA \left(\frac{\ell}{\Delta t}\right) \quad dV$$

$$\frac{E}{\Delta\lambda \, A \cdot T \cdot \text{solid angle}} \qquad E$$

$(\ell / \Delta t = c)$ crossing time

$$I_\lambda = c\,\mathcal{E}$$

$I_\nu = c\mathcal{E}_\nu$

energy density per solid angle

$$\mathcal{E}_\lambda = \oint \mathcal{E}_\lambda \, d\Omega$$

Integrated over solid angle

$$= \frac{4\pi}{c} \oint \frac{I_\lambda}{4\pi} d\Omega$$

$$= \frac{4\pi}{c} J_\lambda$$

mean intensity

Special Case of interest

$$\mathcal{E}_{\text{planckian}} = \frac{4\pi}{c} B_T$$

energy density

Planck Law

specific intensity

10-008

2) Planck Law

& Generalized Wiens Law

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^x-1}, \quad B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^x-1}$$

$$= \frac{2hc^2}{\left(\frac{hc}{kT}\right)^5}\frac{x^5}{e^x-1} \qquad = \frac{2h}{c^2\left(\frac{h}{kT}\right)^3}\frac{x^3}{e^x-1}$$

$$x = \frac{hc}{kT\lambda} \qquad \Rightarrow \quad x = \frac{h\nu}{kT}$$

$B_\nu \to \frac{x^3}{e^x-1}$

$B_\lambda \to \frac{x^5}{e^x-1}$

$B_H \Rightarrow \frac{x^4}{e^x-1}$

$B_{\nu^2} \Rightarrow \frac{x^2}{e^x-1}$

$$B_\nu = B_\lambda\left(\frac{d\lambda}{d\nu}\right)$$

$$= B_\lambda \frac{c}{\nu^2}$$

$$= 2hc^3\frac{\nu^5}{c^5}\frac{1}{\nu^2}\frac{1}{e^x-1}$$

$$= \frac{2h\nu^3}{c^2}\frac{1}{e^x-1} \quad \text{confirmed}$$

$$B_H = \lambda B_\lambda = \nu B_\nu$$

$$= \underbrace{\frac{2hc^2}{\lambda^4}}\frac{1}{e^x-1} = \frac{2h\nu^4}{c^2}\frac{1}{e^x-1} = C\frac{x^4}{e^x-1}$$

$$\frac{c^2}{\lambda^4} = \frac{\nu^4}{c^2}$$

explicitly the same.

(10-003

1) λ, ν, E (photon energy), and Hybrid Representations of Specific intensity

& Miscellaneous Topic

a) $I_i = \dfrac{E \text{ flow}}{(\text{per Area})(\text{per } i)(\text{per time})(\text{per solid})}$

($i = \lambda, \nu, E$)

perpendicular to direction of flow

There must be the same energy per corresponding $d\lambda$, $d\nu$, dE interval

$-d\lambda$ is positive if $d\lambda < 0$

$-$ve sign be decrease in λ corresponds to increase in ν and

$$I_E\, dE = I_\nu\, d\nu = I_\lambda (-d\lambda)$$

(so $-d\lambda > 0$)

$E = h\nu = \dfrac{hc}{\lambda}$, $\nu = \dfrac{c}{\lambda}$, $\lambda = \dfrac{c}{\nu} = \dfrac{hc}{E}$

$$\therefore I_\nu = I_\lambda \left(-\frac{d\lambda}{d\nu}\right) = I_\lambda \frac{c}{\lambda^2} = I_\lambda \frac{\nu^2}{c}$$

$$\therefore \underbrace{I_E = I_\nu \frac{d\nu}{dE} = \underbrace{I_\nu}_{\text{per } \nu} \frac{1}{h}}_{\text{per energy}}$$

10-004

Now

$$E I_E \frac{dE}{E} = \nu I_\nu \frac{d\nu}{\nu} = \lambda I_\lambda \left(-\frac{d\lambda}{\lambda}\right)$$

$$E I_E \, d\ln E = \nu I_\nu \, d\ln\nu = \lambda I_\lambda (-d\ln\lambda)$$

but $\ln\nu = \ln c - \ln\lambda$

$$d\ln\nu = -d\ln\lambda$$

and $\ln E = \ln h + \ln\nu$

$$d\ln E = d\ln\nu = -d\ln\lambda$$

$$\therefore \; E I_E = \nu I_\nu = \lambda I_\lambda = I_H$$

The hybrid representation.

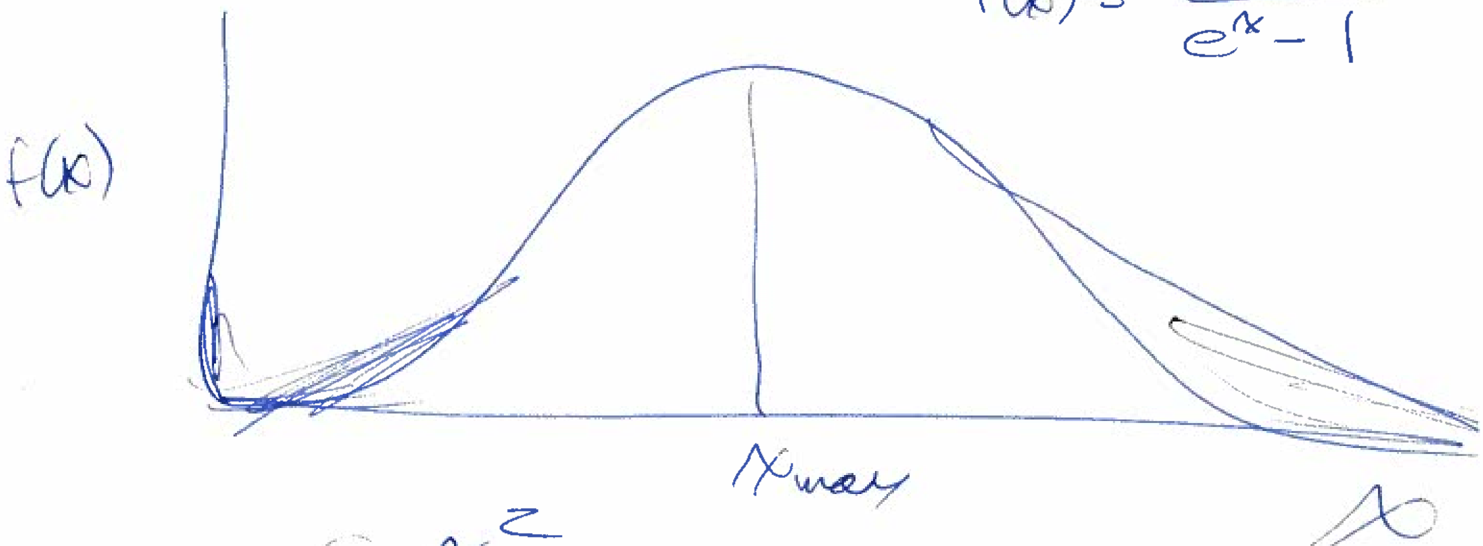
My own name but you see this form a lot these days.

b) Why? ① It's the same value written as a function of E, ν, λ.

Probably main reason

~~Both~~ All Planck law representations 10 – ~~0H~~ 00:
have the form

$$f(x) = \frac{x^z}{e^x - 1}$$



$$f(x) = \frac{x^z}{e^x - 1} \simeq \frac{x^z}{x} \text{ for small } x$$

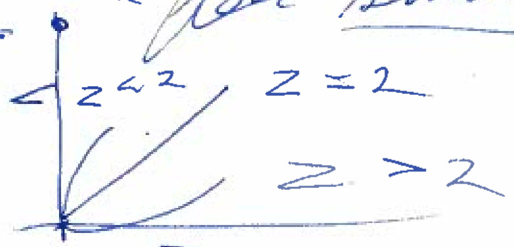
general

$z < 1$ (we won't bother with this case)

for small x

x^{z-1}

$z = 1$ case



stationary point in this case

$\underline{z > 2}$.

$x^z e^{-x}$ for $x >> 1$

$$\lim_{x \to \infty} \frac{x^z}{e^x} = 0$$ by L'Hôpital's rule for any z

So in fact ~~x~~ $x = \infty$ is a stationary minimum.

10-01(2)

For maximum

$$\frac{df}{dx} = \frac{z x^{z-1}}{e^x-1} - \frac{x^z e^x}{(e^x-1)^2} = 0$$

already know $x=0$ gives a zero
(but only stationary for $z>2$)

and $x=\infty$ is a stationary minimum

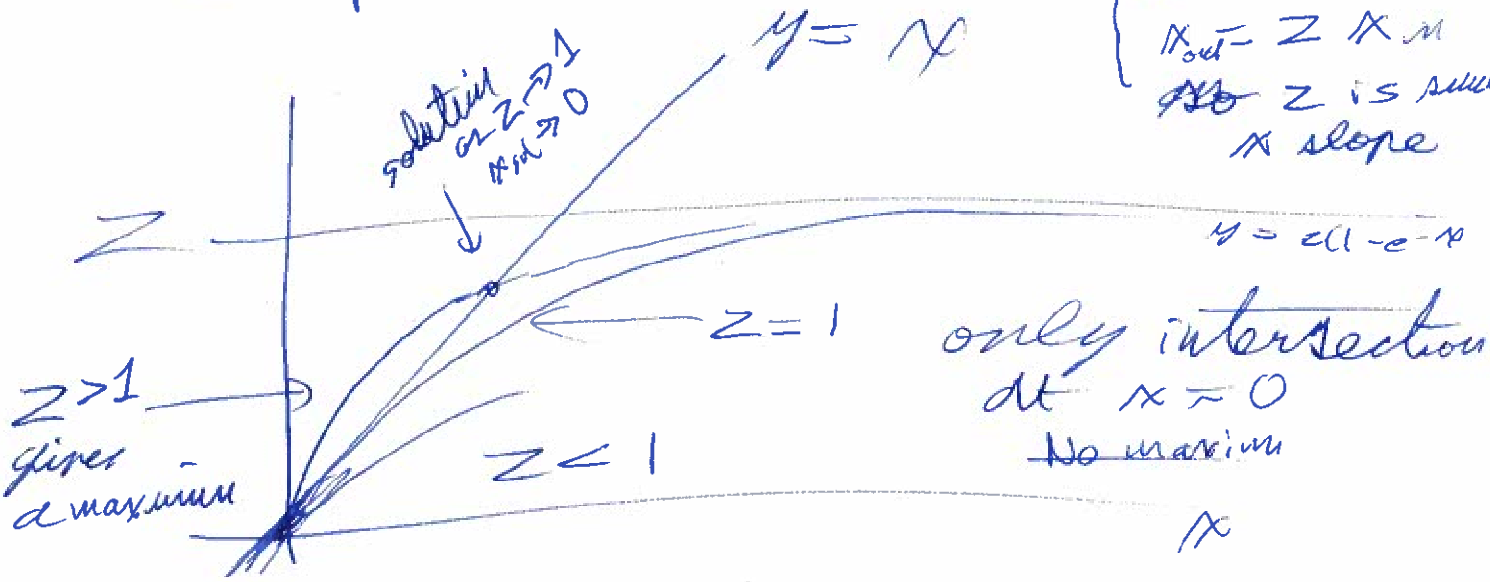
$$z(e^x-1) - xe^x = 0$$

Iteration formula for solution

$$x_{max\,out} = z(1-e^{-x_{in}})$$

Graphical solution

from $x \ll 1$: $x_{out} = z\, x_{in}$, z is small x slope



10-01

In fact for $z > 0$

$x = z(1-e^{-x})$ is an iteration formula that always converges which I won't prove

$x_2 = z(1-e^{-x_1})$

- fast for $z >> 1$
- slower as $z \to 1$ $\Rightarrow$ very slow

In fact for $z >> 1$, $x = z$ ($x_{max} \cong 3$ for $z = 3$...) , $x = z(1-e^{-z})$ are asymptotically good.

Can one find a good solution for $z = 1 + \Delta z$ with $\Delta z << 1$ and can one find an excellent interpolation formula for x_{max}.

Yes & yes.

Substitute with $1 + \Delta z$ and expand in small x

$$x = (1+\Delta z)\left[1 - \left(1 - x + \tfrac{1}{2}x^2 - ..\right)\right]$$

$$x = (1+\Delta z)\left(x - \tfrac{1}{2}x^2 + \ldots\right)$$

$$= x + \Delta z\, x - \tfrac{1}{2}x^2 - \Delta z \tfrac{1}{2}x^2 + \ldots$$

$$1 = 1 + \Delta z - \tfrac{1}{2}x - \Delta z\left(\tfrac{1}{2}x\right) + \ldots$$

$$0 = \Delta z - \tfrac{1}{2}x - \Delta z\left(\tfrac{1}{2}x\right) + \ldots$$

10-U1

Keeping only 1st order terms in small x & Δz

$\Rightarrow \frac{x^z}{e-1}$ maximizing x recall

$$x = 2\Delta z$$

is the 1st order in small Δz solution.

Then by some uncanny insight (and some fooling around) I deduced

$$x_{int} = z\left(1 - e^{-(z - \frac{1}{z})}\right)$$

is a good interpolation solution

$$x_{int}(z \gg 1) = z(1 - e^{-z})$$

the asymptotic large solution

$$x_{int}(\Delta z \ll 1) = (1+\Delta z)\left[1 - e^{-\left[(1+\Delta z - \frac{1}{1+\Delta z})\right]}\right]$$

geometric serie

$1 - \Delta z + \dots$

$$(1+\Delta z) - (1-\Delta z) = 2\Delta z$$

and $1 - e^{-2\Delta z} = 2\Delta z$ to 1st order

10-D13

$x_{int}(\Delta z \ll 1) = (1+\Delta z)(2\Delta z)$

$= 2\Delta z$ to 1st order

$x_{int} = z\left(1 - e^{-(z-\frac{1}{z})}\right)$

asymptotically exact

as $z \to 1$

or $z \to \infty$

Always a slight underestimate elsewhere

with maximum relative error ~3% at $z \doteq 1.5$

Improved version relevancy = −.132 001e at z = 2.1

0.132%

~~But the~~ Bdl polynomial one does orders of mag better or $z \to \infty$

~~[illegible]~~ /adlib/math3.f region !

Greatest error for intermediate region ~~$\Delta z \ll 1$~~ ~~But~~ where Not $\Delta z \ll 1$ and Not $z \gg 1$

So $\Delta z \doteq \frac{1}{2}$ and $z \doteq 2$ between region

Excellent for a simple formula

One can even do better and fit ~~high~~ more low order and high order expansion series terms exactly

and get an interpolation formula asymptotically correct for $z \to 1$ and $z \to \infty$

10-01~~4~~

and maximum error $\sim \frac{3}{10^4}$ at $z \stackrel{?}{=} 1.8$

(/aalib/math3.A)

And you learn about Bell's polynomials too.

(This is real thrilling part of this exercise in ~~over~~ numerical overkill

The educational point was to find a high accuracy way of interpolating between ~~a low value~~ ~~expansion~~ low and high parameter expansions.

Results

z	x_{max} (double precision 1e-18 relative error)	Comment
2	1.593624....	photon density
3	2.8~~21~~439....	τ representation
4	3.920690...	hybrid representation
5	4.965114...	λ representation

Wanting to know this value really ~~was~~ the origin of this long tedious project

10-01

$$x_{max} = \frac{h\nu}{kT} = \frac{hc}{kT\lambda}$$

and so find $\nu_{max} = \frac{kT}{h} x_{max}(z=3)$

Famous Wien law (or Wein Displacement law)

$$\lambda_{max} = \frac{hc}{kT} \frac{1}{x_{max}(z=5)}$$

$$= \frac{2.897771955...}{T_K} \times 10^{-3}\ m$$

$$\lambda_{max} = \frac{2.897.771...\ \mu m\, K}{T}$$

micron

exact since h, c, k are now defined as exact

and $x_{max}(z=5)$ is exact But irrational

$k = 1.380\,649 e^{-23}\ J\,K^{-1}$

$h = 6.626\,070\,1 ...\ J \cdot s \times 10^{-34}$

$c = 2.99$

(NIST) all constants

0-018

3) Integrated Planck Law (10-019)

↳ i.e. integrated over all frequency

a) **Recall** $B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^x - 1}$ (w.r.t.)

Recall the connection between specific intensity (or like quantities and density is $\frac{4\pi}{c}$ factor (see p. 10-017)

B_ν, B_λ and B_{mixed} (with differential $d\ln\nu$)

all yield the same ~~energy integrated~~ ~~in~~integrated energy, of course.

Photon ~~density~~ specific intensity $\cancel{B}\ \cancel{B_\gamma} = \frac{1}{h\nu} B_\nu$ and yields a different value

10-0

b) So in general we need to integrate

$$F = \int_0^\infty \frac{x^z}{e^x - 1}\,dx$$

$$= \int_0^\infty x^z e^{-x} \sum_{l=0}^\infty (e^{-x})^l \,dx$$

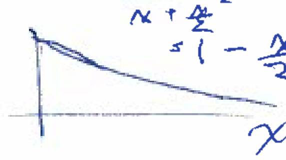
using the geometric series (which I find infinitely useful)

Debye function integrated aside from constants (Wiki)

$z > 1$

x

$z = 1$ $\frac{x}{x + \frac{x^2}{2}} \approx 1 - \frac{x}{2}$



x

$z < 1$

x

Does this converge? Uninteresting question even to me

$$= \sum_{l=0}^\infty \int_0^\infty x^z e^{-(l+1)x}\,dx$$

$$= \sum_{l=0}^\infty \frac{1}{(l+1)^{z+1}} \int_0^\infty x^z e^{-x}\,dx$$

$x_{new} = (l+1)x_{old}$

Arf-542 factorial function

$$\cancel{= \sum_{l=0}^\infty \frac{1}{l+1}}$$

$$= z! \cancel{\sum_{l=1}^\infty \frac{1}{l}}$$

$$= z! \sum_{l=1}^\infty \frac{1}{l^{z+1}} = z!\,\zeta(z+1)$$

Riemann Zeta function Arf-332

c) Cases of interest

~~Energy density~~

$$B = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x-1}\,dx$$

$$\eta = \frac{2}{c^2}\left(\frac{kT}{h}\right)^3 \int_0^\infty \frac{x^2}{e^x-1}\,dx$$

$$N = \frac{4\pi}{c}\eta = \frac{4\pi}{c}\frac{2}{c^2}\left(\frac{kT}{h}\right)^3 2!\,\zeta(3)$$

1.2020569032

...

Apé-332

No π factors from the odd Riemann-Zeta functions

$$B = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 3!\,\underbrace{\zeta(4)}_{\frac{\pi^4}{90}}$$

$$= \frac{\pi^4}{15}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4$$

$$\mathcal{E} = \frac{4\pi}{c}B$$

$$= \frac{4\sigma}{c}\frac{\pi^4}{15}\frac{2h}{c^2}\left(\frac{k}{h}\right)^4 T^4$$

$$\sigma = \frac{\pi^5}{15}\frac{2h}{c^2}\left(\frac{k}{h}\right)^4$$

With Stefan-Boltzmann Law confirms

$$= a_r T^4$$

Radiation constant (with Stefan-Boltzmann Law).

$a_r = 7.5657 \times 10^{-16}$ $\frac{J}{m^3 K^4}$

$$a = \frac{4\sigma}{c}$$

U-02

d) Entropy & Pressure of Planck Photon gas

Energy not energy per unit volume

$\mathcal{E}$ energy itself

$$d\mathcal{E} = T\,dS - P\,dV + \underbrace{\mu\, dN}_{\mu = 0}$$

$\mu = 0$ chemical potential zero for photon gas since number conservation not imposed in establishing thermodynamic equilibrium

$$\left(\frac{\partial \mathcal{E}}{\partial V}\right)_S = -P, \quad \left(\frac{\partial \mathcal{E}}{\partial S}\right)_V = T$$

Natural variables $\mathcal{E}(T,V)$

But we have $\mathcal{E} = aT^4 V = EV$

$$= \mathcal{E}(T, V)$$

$$dS = \left(\frac{d\mathcal{E}}{T}\right)_V$$ constant volume

$$= \frac{d\mathcal{E}}{\left(\frac{\mathcal{E}}{aV}\right)^{1/4}}$$

$$S = (aV)^{1/4}\, \mathcal{E}^{3/4}\Big|_0^{\mathcal{E}} + \text{Constant}$$

Constant = set to zero. No reason not to.

$$\mathcal{E} = \left(\frac{S}{(aV)^{1/4}}\right)^{4/3} = \frac{S^{4/3}}{(aV)^{1/3}}$$

$$\left(\frac{\partial \mathcal{E}}{\partial V}\right)_S = -\frac{1}{3}\frac{\mathcal{E}}{V} \quad \text{and so } P = \frac{1}{3}aT^4 = \frac{1}{3}E$$

a famous formula

$$= -\frac{1}{3}E$$

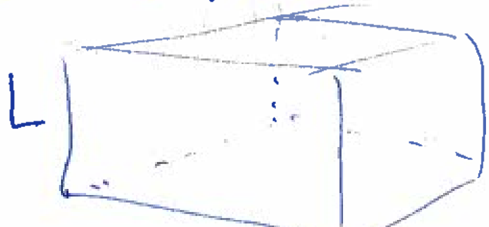
$$E = aT^4$$

This result can also be derived from classical particles and quantum mechanical particles — a nice consistency.

6024) f) Quantum Derivation

Imagine an infinite square well ~~that~~ a cube of side length L

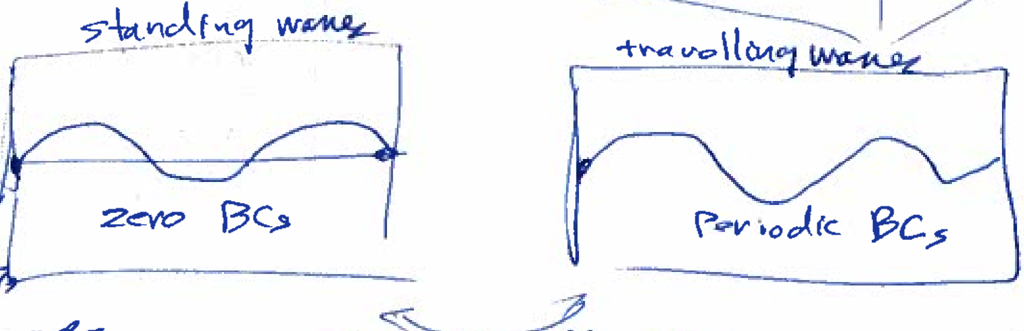
Then $V = L^3$ or $L = V^{1/3}$



$L_x = n\frac{\lambda_x}{2}$

$k_x = \frac{2\pi}{\lambda_x} \propto \frac{1}{L_x} \propto \frac{1}{V^{1/3}}$

same for y and z



$L_x = n\lambda_x$

$k_x = 2\pi/\lambda_x$

$k_x \propto \frac{1}{L_x} \propto \frac{1}{V^{1/3}}$

Same for y, z.

These give the same answer for density of states.

Textbooks ~~argue~~ say the BCs shouldn't matter deep in interior, but since BCs matter in all other physics problems, textbooks seem too glib. I can understand why the shape of volume shouldn't matter for ~~some~~ high k (small λ) modes, maybe that is all textbooks mean. ~~Besides BCs are~~ What if you don't really have BCs like deep in a star or the universe as a whole. Textbooks pass over this crucial point in silence. Perhaps just a limiting argument. You get same answer asymptotically if you divide a volume up into cubes of any size, and so the answer must be the same even with no cubes — ~~or anything~~ or boundaries. Can't complete discussion here.

In any case

$$E \propto \sum_i k_i^q$$ where sum is over all particles in the completely delocalized states.

$q = 1$ for ER, $E_{\text{kin}} = p c \propto \frac{1}{V^{1/3}}$

$q = 2$ for NR, $E_{\text{kin}} = \frac{p^2}{2m} \propto \frac{1}{V^{2/3}}$

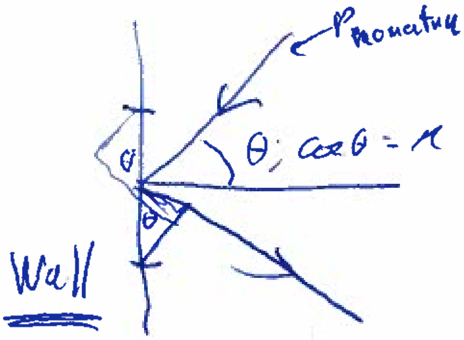
Expanding adiabatically $dE = -P\,dV$

$$\therefore \quad P = -\left(\frac{\partial E}{\partial V}\right)_S = \begin{cases} \frac{2}{3}\frac{E}{V} = \frac{2}{3}\varepsilon & \text{NR} \\ \frac{1}{3}\frac{E}{V} = \frac{1}{3}\varepsilon & \text{ER.} \end{cases}$$

Now we think of the particles so far in completely delocalized states. But they can be in compact wave packets ~~and~~ just ~~like~~ as discussed in section (e) on classical particles and results don't change.
P just depends on ε and not on the distribution, could be thermodynamic or anything.

e) Classical Particle Derivation (gases) $\left\{ \frac{p}{m} \right.$ ~~classic~~ NR limit

$\boxed{6023}$



$$\underset{\text{Pressure}}{d\mathcal{P}}\, dA = 2\, \underbrace{\mu p}_{\text{component of momentum toward surface}}\ \frac{v}{4\pi}\, n(p)\, dp\, d\Omega\, \mu\, dA$$

- 2: accounts for change in momentum of particle on bounces of rigid surface
- $\frac{v}{4\pi}$: Flow in one direction
- $n(p)$: number density of Particles ~~[illegible]~~ p ~~[illegible]~~ per ~~momentum~~ magnitude $|p|$
- dp: momentum
- $d\Omega$: solid angle
- $\mu\, dA$: perpendicular surface area

$$\mathcal{P} = \frac{2}{4\pi} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{1} \mu^2 d\mu}_{\frac{1}{3}} \int_0^{\infty} \frac{p^2}{m}\, n(p)\, dp$$

$$\mathcal{P} = \frac{1}{3} \int_0^{\infty} v\, p\, n(p)\, dp$$

Now NR limit (non-relativistic limit) $E_{kin} = \frac{p^2}{2m} = \frac{1}{2} v p$

ER limit (extreme relativistic limit) $E_{kin} = pc = vp$

$$\mathcal{P} = \begin{cases} \frac{2}{3}\mathcal{E} & \text{NR} \\ \frac{1}{3}\mathcal{E} & \text{ER} \end{cases}$$

where $\mathcal{E}$ is energy per unit volume: i.e., energy density.

Note these results do not depend on $n(p)$: i.e., on how the particles are distributed in momentum magnitude, just on our ~~the~~ instantaneous energy density. So the gas doesn't have to be in thermodynamic equilibrium.

If the gas volume is changed adiabatically:

No heat energy change ~~added~~ and no particle number change (or zero chemical potential);

$$dE = -\mathcal{P}\, dV$$

$E = E(V,S)$ and ~~[illegible]~~ $\Rightarrow \mathcal{P} = \mathcal{P}(E)$

Note we have called the particles but they could also be compact wave packets in QM.

$\Big\{$ $\mathcal{P}$ just depends on instantaneous energy not on ~~[illegible]~~ its energy derivative (even though it is one $\mathcal{P} = -\left(\frac{\partial E}{\partial V}\right)_S$)

(2021 nov 07)

10-023

4) Proof That Planchion Radiation Field stays Planckian Under Universal Expansion (or actually Adiabatic expansion in general)

No significant heat sources or sinks and so adiabatic expansion ~~Itisnot necessarily~~ required individual photon conservation though that has been nearly true since decoupling → see 10-005, 10-006

Recall cosmological redshift quick derivation

$$\frac{d\lambda}{\lambda} = \frac{+dv}{c} = \frac{+H\,dr}{c} = \frac{\frac{\dot a}{a}\,c\,dt}{c} = \frac{da}{a}$$

Somewhat mysteriously turning an approximation into a differential equation

Well dv isn't just the difference in infinitesimal velocity — It corresponds to an infinitesimaly change in velocity and space between photon locations,

observer ← source

$$\therefore \ln\lambda \Big| = \ln a \Big|$$

$$\lambda = \lambda_1\left(\frac{a}{a_1}\right) \quad \text{or} \quad \lambda_1 = \lambda\left(\frac{a_1}{a}\right)$$

where 1 is some fiducial epoch in the past say

Still pondering the correct interpretation (see lecture ~~4~~ notes p.4036ff ↳ for my best argument)

or

$$\nu = \nu_1\left(\frac{a_1}{a}\right)$$

$$\nu_1 = \nu\left(\frac{a}{a_1}\right)$$

0-024

conversion from specific I to density

In parallel
Photon number Density & Specific Intensity

Aug. 10-2008

$$n_\nu d\nu = \frac{4\pi}{c}\frac{1}{h\nu}B_\nu d\nu$$

$$n_\lambda d\lambda = \frac{4\pi}{c}\frac{1}{\frac{hc}{\lambda}}B_\lambda d\lambda$$

$$\cancel{n_{\lambda_1}d\lambda_1 = \frac{4\pi}{c}\frac{2c}{\lambda^4}\frac{1}{e^{x_1}-1}d\lambda_1}$$

$$\cancel{= \frac{4\pi}{c}\frac{2c}{\lambda^4\left(\frac{a_1}{a}\right)^4}\frac{1}{e^{x_1}-1}d\lambda\left(\frac{a_1}{a}\right)}$$

$$n_{\lambda_1}d\lambda_1 = \frac{4\pi}{c}\frac{2c}{\lambda_1^4}\frac{1}{e^{x_1}-1}d\lambda_1$$

$$= \frac{4\pi}{c}\frac{2c}{\lambda^4\left(\frac{a_1}{a}\right)^4}\frac{1}{e^{x_1}-1}d\lambda\left(\frac{a_1}{a}\right)$$

$$= \left(\frac{a}{a_1}\right)^3\frac{4\pi}{c}\frac{2c}{\lambda^4}\frac{d\lambda}{e^{x_1}-1}$$

$$B_\nu d\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^x-1}d\nu$$

$$B_\lambda d\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^x-1}d\lambda$$

$$\cancel{B_{\lambda_1}d\lambda_1 = \frac{2hc^2}{\lambda^5}\frac{1}{e^{x_1}-1}d\lambda_1}$$

$$\cancel{= \frac{2hc^2}{\lambda\left(\frac{a_1}{a}\right)^5}\frac{1}{e^{x_1}-1}d\lambda\left(\frac{a_1}{a}\right)}$$

Go from past to present or reverse: Same

$$B_{\lambda_1}d\lambda_1 = \frac{2hc^2}{\lambda_1^5}\frac{1}{e^{x_1}-1}d\lambda_1$$

$$= \frac{2hc^2}{\lambda^5\left(\frac{a_1}{a}\right)^5}\frac{1}{e^{x_1}-1}d\lambda\left(\frac{a_1}{a}\right)$$

$$= \left(\frac{a}{a_1}\right)^4\frac{2hc^2}{\lambda^5}\frac{d\lambda}{e^{x_1}-1}$$

As mathematicians would say nothing forbid us from setting

$$x_1 = x$$

$$\frac{hc}{kT_1\lambda_1} = \frac{hc}{kT\lambda}$$

$$T_1\lambda_1 = T\lambda$$

$$T = T_1\frac{\lambda_1}{\lambda}$$

$$T = T_1\left(\frac{a_1}{a}\right)$$

So given $T = T_1\left(\frac{a_1}{a}\right)$

for later or earlier we have

$$\mathcal{N}_\lambda d\lambda = \left(\frac{a_1}{a}\right)^3 \mathcal{N}_{\lambda_1} d\lambda_1 \qquad B_\lambda d\lambda = \left(\frac{a_1}{a}\right)^4 B_{\lambda_1} d\lambda_1$$

The distributions still have Planckian shape but do they have the overall right scale? (Maybe obviously in some sense)

Integrate to check

$$\int \mathcal{N}_{\lambda} d\lambda = \left(\frac{a_1}{a}\right)^3 \int \mathcal{N}_{\lambda_1} d\lambda_1 \qquad \int B_\lambda d\lambda = \left(\frac{a_1}{a}\right)^4 \int B_{\lambda_1} d\lambda_1$$

$$\tilde{\mathcal{N}} = \left(\frac{a_1}{a}\right)^3 \mathcal{N}_1 \qquad \tilde{B} = \left(\frac{a_1}{a}\right)^4 B_1$$

Not assuming the correct scale.

Not assuming the correct scale

~~Now~~ since ~~volume~~ ~~density~~ scales $\left(\frac{a_1}{a}\right)^3$ and energy density scales $\left(\frac{a_1}{a}\right)^3\left(\frac{a_1}{a}\right)$

for volume scaling

for cosmological redshift

For a photon gas of our distribution

$$\mathcal{N} = \left(\frac{a_1}{a}\right)^3 \mathcal{N}_1 \qquad B = \left(\frac{a_1}{a}\right)^4 B_1$$

$$\therefore \tilde{\mathcal{N}} = \mathcal{N} \qquad \tilde{B} = B$$

(10-2028) and so the scaled Planckian gas is Planckian ~~itt~~ shape and scale and defining

$z = \frac{a_0}{a} - 1$

$T = T_0 \left(\frac{a_0}{a}\right)$

$$T = T_1 \left(\frac{a_1}{a}\right)$$

must be

$T_0 = 2.7260$ keep. 2030

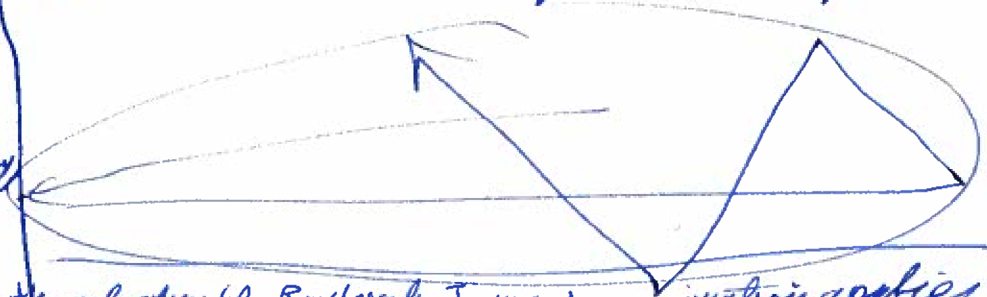
exactly right $\longrightarrow$ since an independently ~~thermodyn~~ prepared Planckian gas at the other time would necessarily have $T_{\text{independent}} = T = T_1 \left(\frac{a_1}{a}\right)$

So the other time gas is just Planckian and we've found the adiabatic uniform expansion law for a Planckian gas where ~~photons~~ only change energy by redshifting

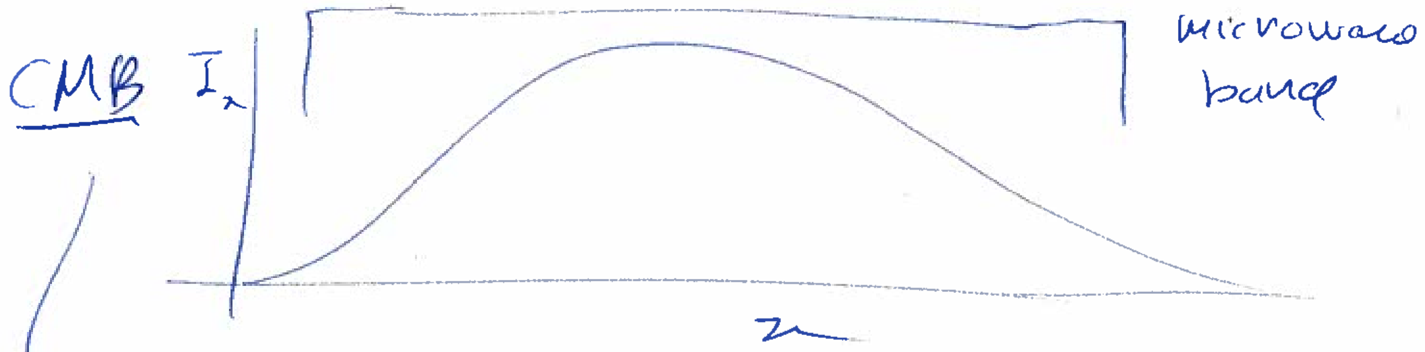
omit

Actually how Wein derived his approximation to the Planck law valid for high frequency but fails at low ones i.e. Long wavelength and where the Rayleigh-Jeans Approximation applies.

— Could be universe or an ideal expanding reflecting container where photons progressively redshift



5) There is a Cosmic Temperature (602.



is one of the most perfect blackbody spectra in nature — maybe most perfect in the microwave band

— other bands have D.E.B.R.A — diffuse extrapolated background radiation

From point source — but scattered by electrons often emerging from galaxies and ~~other~~ point sources unresolved

CMB = cosmic microwave background at cosmic present,

CBR = cosmic background radiation for general time

Since recombination era ($t \cong 377$ kyr) — more exactly it has streamed freely cooling adiabatically

030

and retaining its Planckian shape and scale

with $T = T_0 \left(\frac{a_0}{a}\right)$ $a_0 = 1$

cosmic present

$T_0 = 2.7260(13)\,K$

(Fixsen 2013)

Some matter interaction

$I = I_{rec} e^{-\tau}$

$\tau = .05 - .1$

~~and~~ but small ~~though~~ not negligible

- but mostly Thomson scattering off free electrons in IGM = Intergalactic medium

that doesn't change frequency and so doesn't perturb blackbody shape → in fact is adiabatic scattering or elastic scattering

→ no coupling to matter thermal state

Of course, some CBR photon do ~~get~~ absorbed in matter → ~~they~~ hit a planet or a detector.

→ But this is a small perturbation.

So back to recombination era
(~~6~~377 kyr)
the CBR temperature is fairly
called the cosmic temperature
⟶ what you deduce
just looking at space.

What of before?

In fact it is the temperature
of everything as far back
in time as we can certainly
go.
In inflation era there may
($t \lesssim 10^{-32}$ s) have been
"Reheating"
(but I know nix
about that)

10-029

~~Quit Doorman 10-023~~

~~It's still hazy~~ to me

by $\lambda = \lambda_1 \sqrt{\frac{1+\beta}{1-\beta}}$

for the Doppler shift

between two

initial frames

gives $\lambda = \lambda_1 (1 + \frac{1}{2}\beta)(1 + \frac{1}{2}\beta)$

$= \lambda_1 (1 + \beta)$

$\frac{\lambda - \lambda_1}{\lambda_1} = \beta = \frac{v}{c}$

as a first order approximation

which is NOT a differential

can be turned into

$\frac{d\lambda}{\lambda} = \frac{dv}{c} = \frac{H dr}{c} = \frac{(\frac{\dot{a}}{a}) dc}{c}$

$= \frac{da}{a}$

which is a

differential

equation for

cosmological redshift or

any general expansion but

$\frac{d\lambda}{\lambda} = \frac{dv}{c}$ is ~~but~~ NOT for the original exact

formula for finite velocity ~~shift~~.

What is the argument? The ~~cogent~~ argument

10-030

But what of the ~~only~~ energy change ~~is~~ ~~NOT~~ by more ~~than~~ redshifting?

Well as long as energy changing random processes exist (and they do NOT have to be further specified) and sufficient time is allowed (reactions fast enough), then TE is established for an photon gas with B_λ don B_ν dn

Or any gas of non-conserved extreme relativistic particles (even if they have rest mass)
→ Because negligible and the Gas is always Planckian and Not ideal gas or degenerate gas.

So very early Universe could have all kinds of energy changing processes — pair creation + annihilation and radioactive decays of unstable particles.

Since energy was conserved (aside from red shift loss) the gas ineluctably stayed at T E (or kept returning to it from any deviation)

with $T = T_1 \left(\frac{a_1}{a}\right)$

$T \propto \frac{1}{a}$

$T \propto \frac{1}{a} \propto T^{-\frac{1}{2}}$ Rad
$\propto T^{-2/3}$ Mat
$\propto e^{-H_\Lambda t}$ Λ

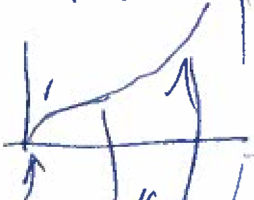
This is true it is thought going back to Planck time ~~or reheating or inflation~~ or inflation time or reheating

To theory does Not matter what exotic particles + quarks hadrons mesons neutrinos

dark matter or violation of 2nd law of Thermo over photon gas

peculiar vacuum state perhaps chosen at random in multiverse

We do know a(t) pretty well from Λ-CDM which must be approximately true



Rad Matter

Λ era

Recall $\frac{a_0}{a} = z+1$

So $\frac{a_0}{a} = \frac{T}{T_0}$

and $z_{dc} \simeq \frac{a_0}{a} = \frac{T}{T_0} \simeq \frac{3000K}{2.7K} \simeq 1100$

(Wiki)

[10-B2] – The plot shows reheating as big cooling ~~drop~~

Reheating an era in some models of inflation (Wikipedia)

But if energy conserved (aside from redshifting) TE must be restored when processes ~~...~~ randomize the energy again and $T = T_1 \left(\frac{a_1}{a}\right)$ ~~...~~ evolution re-established

And the universe stays homogeneous and isotropic enough as plot shows

Which people have wondered about

but the cosmological principle still seems adequate — at least for the photon gas — the cosmic background radiation

But as the universe cooled $T \propto \frac{1}{a}$, eventually pair creation turned off as higher rest mass particles could not be created without energetic enough photons.

— unstable particles decayed

— matter-antimatter mostly mutually annihilated

→ matter won maybe barely (so we don't meet antimatter people

[if matter-antimatter perfectly symmetric — we wouldn't be here. → so Anthropic principle "explains" why there must be an asymmetry but not what is ~~also~~ why it is a certain level.

antipeople
it wouldn't end well if we did meet them

10:03)

Less asymmetry and the universe would more empty and maybe life would even more rare

(on the far shore of Dennis Sciama's island of biophilia)

more asymmetry and maybe a universe full of stars even between (odd-looking galaxies)

Eventually density so low and stable neutrinos decoupled and went free streaming (t = 1 s from point origin of Λ-CDM model)

and ~~then~~ just photons, electrons, protons, neutrons, and

during frozen out

dark matter
(particles or primordial Black holes)

10 - 03

Now free neutrons are unstable ← 2021 measurement

t_e is e-folding time or mean lifetime

$t_{\frac{1}{2}} = t_e \ln 2$

$= 610\,s$

$t_e = 877.75\,s$ (± .1 s?)

$= 14\,m\ 37.75\,s$

alas a different kind of gives ~ 888 s (± 2 s)

So which experiment is right if either → or the discrepancy one of ~~those~~ "new physics" ~~things~~ people go on and on about

So the neutrons were close to all decaying away too lent in the nick of time

Big Bang nucleosynthesis occurred $t = 10 - 1200\,s$

the photon gas cool enough not to photodisintegrate nuclei.

(036)

BN
ay different
in reaction
on stops
r several
easons
But
keyone
is there
re free
eutrons
hich there
re Not in
tar
clear
urning

If there had been fewer neutrons, there would've been no BBN since you need neutrons for stable particles

and I believe ~~th~~

P → P PP chain reaction would ~~the~~ would've been ~~too~~ slow

before densities ~~fell~~ and photon fell to create much deuterium

PP → P, N

e^+ ν_e

PP chain

a weak reaction is needed and that is very slow in some sense and

is the rate-determining step | 10-37
in Main-sequence
hydrogen burning.

If diprotons were stable, maybe
stars would all be short lived
and a sterile universe.

But what if the neutron had
much lower abundance
BBN would've not
have happened and the Big Bang
would've left a universe starting
from pure hydrogen → This is
OK → stars can make
all the elements from
hydrogen.

But what if the neutrons had
been much more abundant
or the nuclear binding force
a touch stronger?
BBN might have gone to all
He-4

0038

Disastrous

→ no H_2O → no liquid water → life as we know it needs liquid water.

We evolved to live out of the ocean — but only by having an ocean inside

An old saying, you can take the buoy out of the ocean, but not the ocean out of the boy.

Brit. Boy
Am. Buoy
pronounciation

Now for a key point

→ baryons (protons & bound neutrons are stable at least before proton decay

($t_{\frac{1}{2}} \geq 1.67 \times 10^{34}$ years (Wiki) if at all.

So all baryons since before BBN are still around

(10-03.

Well nearly.

— some fall into Black holes where they may or may not be conserved.

What of photons → ever since the freeze out of mass particle they are mostly conserved too since cosmic temperature continued $T \propto \frac{1}{a}$

The baryons eventually starting being nonrelativistic (NR) and acting like an ideal gas

and $p_{baryon} \propto \frac{1}{\lambda^2}$

meaning $KE_{baryon} \propto p^2 \propto \frac{1}{\lambda^2}$ when NR

and so they lost KE faster with expansion than photons

But before decoupling (~380 kyr cosmic time)

the thermal energy of photons dominated totally and

Matter (i.e., baryon dominated energy after $t \hat{=} 50$ kyr but almost all that energy is rest mass energy. a frozen out reserve that doesn't effect thermal state. No process left for rest mass energy to change and exchange with thermal energy ↳ photons, particle KE and atomic level populations.

(B-040)

baryon interaction and cooling was a small perturbation (though maybe really detailed calculations account for it) on the thermal state of the photons.

So $T_{photon} \propto \frac{1}{a}$

and $T_{baryon} \propto T_{photon} \propto \frac{1}{a}$

until decoupling

So CBR photons are mostly conserved ~~too~~ since baryon freeze out.

→ Occasionally ~~they~~ run ~~into~~ stars, planets, detectors, etc but ~~those~~ are perturbations (which some may account for in detailed work).

The key upshot is baryon-to-photon ratio

$\eta =$ see p. 10-042

has been conserved ~~through~~ from well before BBN nucleosynthesis to now and cosmic temperature has scaled as

$$T = T_0 \left(\frac{a_0}{a(t)}\right)$$

from that time too.

$T_0 \doteq 2.725$ K the CMB temperature in cosmic present

Reheating in inflation era is different, but is really $t < 10^{-32}$ s so at least back to then there has been a cosmic temperature $T = T_0\left(\frac{a_0}{a}\right)$ ~~with~~ be present

So we can fairly easily just run the clock back and track BBN through $t_{cosmic} = 10 - 1200$ s or maybe a bit longer ~20

(Steven Weinberg in the 1970s wrote a famous book The First Three Minutes but it is more like the first 20 minutes

The photon to baryon ratio constant since baryons froze out — proton no longer could be created and destroyed via antimatter-matter annihilation — pair creation and destruction

10 - 042) In any case
at BBN era η is
a key parameter -
(we know T(t) is fixed)

$$\eta = \frac{n_b}{n_\gamma} = 6.16 \times 10^{-10}$$

Planck 2018

$$\eta^{-1} = \frac{10^{10}}{6} \simeq 1.67 \times 10^{9}$$

directly known from CMB Temperature

$$\Omega_b h^2 = 0.02237(15) \quad \text{Planck}$$

$$\Omega_b = \frac{\rho}{\rho_{crit}} = \frac{\rho}{\frac{3(100)^2}{8\pi G}} \frac{1}{h^2}$$

$\Omega_b h^2 =$ is what we know

but $h^2 \simeq (.7)^2 \simeq \underline{0.49} \simeq .5$

$$\therefore \Omega_b = 0.04474 \simeq 5\%$$

of critical density

$$\eta = \frac{n_b}{n_\gamma} = 2.75 \times 10^{-8}\, \Omega_b h^2$$

and one can use either η or $\Omega_b h^2$
as a BBN synthesis parameter.

We'll look at plots of BBN later but for now [10-04]

we can say for $\eta \triangleq 6 \times 10^{-10}$

the right amounts of H, D, He-3, He-4 and Li-7

good agreement with observation

not so good

~3 time to high Liddle 97

Cosmological lithium problem.

But lithium can be created ~~or destroyed~~ or moved around even in very old stars and so most believe the cosmological lithium problem is due star physics and Not due wrong BBN.

But a longstanding annoying problem.

However 6η doesn't give the right abundances and so the dark matter can't be baryonic

)-044

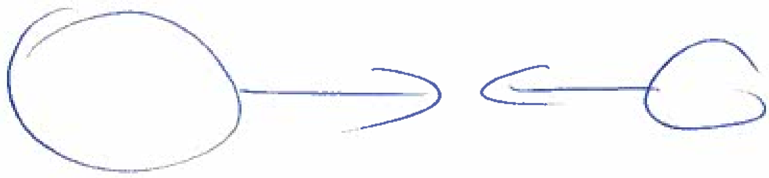
So some stable ~~dark matter~~ particle from freeze out → or primordial black hole from pre-BNN

Maybe
will
be
the
right
or
none

— or both
— or neither → MOND
↳ modified Newtonian dynamics
— MOND has been a good counter theory to dark matter → accounts well for galaxy rotation
— less well for galaxy cluster
— not ~~at all~~ ~~at all fo~~ well for Bullet cluster and similar cases

Also dark matter ~~rep~~ reproduces the large scale structure in simulations.

to clusters went through each other



gas and dark matter left clumped it seems which MOND predicts NOT.

(2023 nov 28) 6045

5) The Saha Equation, the Ionization History of the universe and Recombination

Falling ~~Cosmic~~ temperature increase recombination — fewer ionizing photons

But decreasing density decreases recombination since electrons and ions interact less.

Decreasing temperature wins at $T_{cosmic} \triangleq 3000\,K$ and the recombination era at $t = 377$ kyr

Reionization due to UV from stars happens $t \sim 1$ Gyr and after that decreasing density seems to have won leaving the intergalactic medium ionized.

Recombination — only Hydrogen & Helium significant

H

$E_{ion} = 13.5984\ldots\,eV$

$\neq E_{Rydberg}$ slightly smaller due to reduced mass effect

$E_{Ryd} = 13.6056\ldots\,eV$ is for an infinite mass nucleus

He

$E_{ion1} = 24.587\ldots\,eV$

$E_{ion2} = 54.4717\ldots\,eV$

So Helium recombines first and to crude approximation can be neglected

$E_{ion\,H^-} = 0.754\,eV$ and has significant bf and ff opacity HK-206-208. Important in solar and cooler stars.

H^- ion was probably super negligible — always a minor species, but very high opacity ff & bf makes it important

046

Recombination happens nearly in Thermodynamic equilibrium (TE) and so the Saha equation (TE result) applies. Deviations from TE are probably important in advanced analysis — and including He too of course.

What are the abundances of H & He

Mass fractions $\left.\begin{matrix} \\ \end{matrix}\right\}$ $X_H = 0.75$, $X_{He} = 0.25$

fiducial numbers not counting Li.

But for the Saha analysis we need number fractions and number abundances.

In general for species i

~~$n_{ell} =$~~

$$\cancel{n_i = \frac{X_i A_b a_{amu} n_b}{A_i a_{amu}}}$$

$$n_i = \frac{X_i \rho}{A_i a_{amu}}$$ ← mass ~~in species~~ density in species i

← mass of species i particle

$$= \frac{X_i (A_b a_{amu} n_b)}{A_i a_{amu}}$$

$$= X_i \frac{A_b}{A_i} n_b$$

$$\frac{n_H}{n_{He}} \doteq \frac{3/4}{\frac{1}{4} \cdot \frac{1}{4}} \doteq 12$$

$n_H \doteq 0.75 \frac{1}{1} n_b$ $\qquad$ $n_{He} \doteq 0.25 \frac{1}{4} n_b$

$= 0.75 n_b$ $\qquad$ $= \frac{1}{16} n_b = 0.0625 n_b$

6047

Apparently the ~~stan~~ conventional definition of recombination — a fiducial value is

$$x = \frac{n_e}{n_b} = 0.1$$

$$\approx \frac{n_e}{n_H} \approx 0.1$$

Ratio of free electrons to baryons or hydrogen

— some one must have seen ~~that this is the~~ point where ~~optical depth~~ the mean free path of photons is long enough for T_{photon} and T_{matter} to diverge significantly

— Decoupling is when is essentially complete.

Actually H has 3 ionization stages H, H^- H^- is ~~[illegible]~~ probably always tiny but it is important in stars such like ours because of high opacity CHM-206-[illegible] There is an exact cubic solution but when is it given

meon free path to free electron scattering which traps photons enough for other reactions; ionizations and excitations

For H and H^+ alone the Saha ~~funct~~ equation is simple and analytically solvable

Derived from statistical mechanics and n_e is a free parameter

Saha Equation itself

$$\frac{N_{H0}}{N_{H+}} = n_e \Phi$$

$$= n_e/\Psi$$

$$\frac{N_{H+}}{N_{H0}} = \frac{\Psi}{n_e}$$

$n_e = N_{H^+}$ by charge conservation

Impose charge conservation (universe neutral)

$$= N_H \frac{N_{H^+}}{N_H} = N_H \left(\frac{N_{H^+}}{N_{H^0} + N_{H^+}} \right)$$

$$= N_H \left(\frac{n_e}{N_{H^0} + n_e} \right) = N_H \left(\frac{1}{N_{H0}/N_{H^+} + 1} \right)$$

$$= N_H \left(\frac{1}{n_e \Phi + 1} \right) = N_H \left(\frac{1}{n_e/\Psi + 1} \right)$$

6048

$\frac{N_{H0}}{N_{H+}} = n_e \bar{\Phi}$ but $\frac{N_{H+}}{N_{H0}} = \frac{\Psi}{n_e}$

since $\Psi = \frac{1}{\Phi}$ has the intelligible behavior; as $\Psi \uparrow$, ionization $\uparrow$

Ψ and Φ have exponential dependence on inverse temperature which makes ionization very temperature sensitive in LTE. Usually one stage completely dominates in number, but often 2 stages are important for opacity.

$$x \equiv \frac{n_e}{N_H} = \frac{1}{\underbrace{\frac{n_e}{N_H}}_{=x}\underbrace{\frac{N_H}{\Psi}}_{\equiv 1/X} + 1}$$

$$= \frac{1}{x/X + 1}$$

$$\frac{x^2}{X} + x = 1$$

$$\frac{x^2}{X} + x - 1 = 0$$

$$x = \frac{2}{1 + \sqrt{1 + 4/X}}$$

using the alternative quadratic equation solution that has no cancellation between nearly equal values

(see Press 1992 p. 178)

$X \ll 1$, low ionization

$$\cancel{\frac{2}{1+\ldots}}$$

$$\sqrt{X}$$

$$\frac{2(-1+\sqrt{1+4/X})}{-1+1+4/X} = \frac{-1+\sqrt{1+4/X}}{2/X}$$

$X \gg 1$

$$\frac{2}{2 + 2/X} = \frac{1}{1 + 1/X} \cong 1 - \frac{1}{X}$$

$$\therefore \quad \alpha = \frac{n_e}{N_H} = \sqrt{X}$$ if assume X is small → already low ionization

$$= \sqrt{\frac{\Psi}{N_H}}$$

$$\propto e^{-\frac{E_{ion}}{2kT}}$$ 2 from taking square root.

$$= e^{-\frac{13.6\,eV}{2 * 10^{-4}\,eV\,\cancel{kT}}}$$

$$= e^{-\frac{7.0 \times 10^{4}\,K}{T}}$$

so $T < 7 \times 10^{5}$, $T = T_0 \left(\frac{a_0}{a}\right)$

there is an ~~exponential~~ rapid exponential decrease in $\alpha = \frac{n_e}{N_H}$

Now $\frac{1}{\sqrt{N_H}}$ gets bigger as $\sqrt{N_H} = \sqrt{N_H\left(\frac{a_0}{a}\right)^3}$

$$= \left(\frac{a}{a_0}\right)^{3/2}$$

but this weak compared to the decrease due ~~with~~ to the exponential.

040 | 2023dec21

The hydrogenic solution is more general ~~than~~ it looks, since all LTE atomic gases can be approximated as hydrogenic.

a) One limit all baryons treated as H. This would tend to overestimate ionization since most electrons are much more tightly bound to their nuclei ~~than~~ hydrogen

b) Another limit all ~~nuclei~~ atoms treated as H, This would tend to underestimate ionization since only one free electron per nuclei.

However the hydrogen ionization energy is probably too high ~~for most atoms~~ 1st ionization for most atoms so case a) is not guaranteed to be an upper limit, nor case (b) a lower limit.

Still n_e is bounded between $n_e = N_b$ and $n_e = 0$

Note even when there are negative ions the free electrons can't be less than zero and conserve charge

There might be some interpolation N_{atom} & $E_{ionization}$ values that would give a good n_e estimate in most cases or at least a good initial ~~estimate~~ for a numerical solution

To actually get $x(t)$,
we need to go back from now.

[10-049] (should renumber to 6051)

$$n_B = \left(\frac{a_0}{a}\right)^3 n_{B_0} \simeq \left(\frac{a}{a_0}\right)^3 (.26\times 10^{-6}\ \text{cm}^{-3})$$

Li-80

$$T = T_0 \left(\frac{a_0}{a}\right) = 3000\text{K}$$

2.725 (Li 79

$.26 \times 10^{-6}$
$* 1,3 \times 10^{9}$
$= .4 \times 10^{3}\ \text{cm}^{-3}$

$$\frac{a_0}{a} = z+1 \simeq 1100 \quad \text{(wik)}$$

from $\frac{3000\text{K}}{2.7\text{K}}$ {assuming the answer

$$a = \left(\frac{\Omega_{m0}}{\Omega_{\Lambda}}\right)^{1/3} \sinh^{2/3}\left[\frac{3}{2}\sqrt{\Omega_\Lambda} H_0 t\right]$$ (Les 5 p.5018)

the Λ-CDM result
valid back to
Radiation era ~50 kyr ; ~150 kyr

Rad-mass equality

change in function solution natural time

But
$t_{recombination}$
$= 370\,000$ yr
$= 370.0$ kyr

Recoupling 377.7 kyr in Λ-CDM Model
~~Planck 2018~~
(wik, 2011)

So well into matter dominated ~~stage~~ era

0-050

but ~~that~~

is total mass-energy
for dominance.

In thermal energy
the photons dominates
until recombination
~~the close to decoupling~~

The exponential dependence on inverse temperature see p.60A8

So

$$x = \sqrt{\frac{\frac{(1/2)}{2\times 10^{-16}}(3000)^{3/2}\, e^{-\frac{13.6}{10^{-4}\cdot 3000}}}{.4\times 10^{+3}}}$$

$$= \sqrt{\frac{\frac{1}{4}\times 10^{16}(125\times 10^{3})}{.4\times 10^{+3}}}\; e^{-\frac{13.6}{.3}}$$

$$= \sqrt{\frac{30}{4}\times 10^{19}\cdot 10^{-18}\cdot 2.5\times 10^{3}}$$

$$e^{-4.5\times 10^{1}}$$

$$= e^{-45}$$

$$= 10^{-(.4\cdot 45)}$$

$$= 10^{-18}$$

$= \sqrt{3004}$

$= 55$

bad calculation

$$= \sqrt{20\cdot 10^{-2}}$$

$$= \sqrt{.2}$$

$\sqrt{.4} = 4\times 10^{-1} = .4$

Li ~ 83 ret value

$> .1$

$= 0.4$

$.1$

10-051

For a rough calculation, not so bad

~~If~~

But if one wants the ~~TE~~ LTE

prediction of ionization

one just runs back the clock $a(t) = \ldots$

(see p. 10-093)

and calculates for H 75%

He 25%

Li-7 — small, negligible

solve the 2 element Saha charge conservation equation.

However after recombination to decoupling, one needs at same point

NLTE ⇒ non local thermodynamic equilibrium.

10-052 which has doubtless been done including all kinds of fine details

Some comments on LTE
ionization solved for from the charge conservation Saha eq. + NLTE

1) ~~Often~~ Usually one stage of atom is ~~dominant~~ overwhelmingly dominant in abundance and there may be a second adjacent stage that is important in radiative transfer (RT) via opacities of not abundance

Of course a very small temperature range of near equality

But have 3 stages & significant in RT is ~~pretty~~ rare given the sensitivity to temperature via the exponential and the fact that ionization energies increase rapidly with ionization stage.

Two nonadjacent stages important in

10-053

almost impossible

but H^{1-}, H^{0+}, H^{1+}

~~may be an exception~~

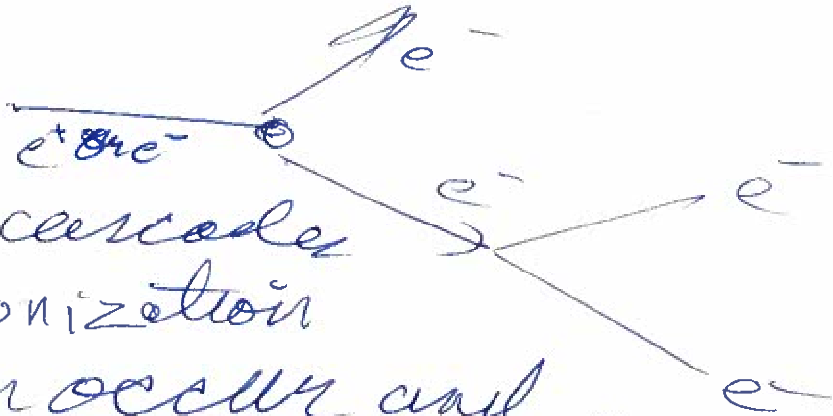
But probably not.

but only because H^- has ~~large~~ opacity even of a trace ~~?~~ amount in solar type & cooler stars (Mihalas p. ?)

Low Thomson scatter since $\sigma_T \propto$ [illegible] and $m_H \simeq 20$[illegible]

2) ~~?~~ nonLTE ionization

— say fast electron/positrons from a radioisotope



then cascades of ionization can occur and you can have multiple ~~important~~ ionization stages e.g. Fe I, Fe II, III, IV in nebular SNe Ia spectra?

10-056

I made a plot (with some effort) to illustrate cosmic evolution of a, T, HII H ionization using fiducial values

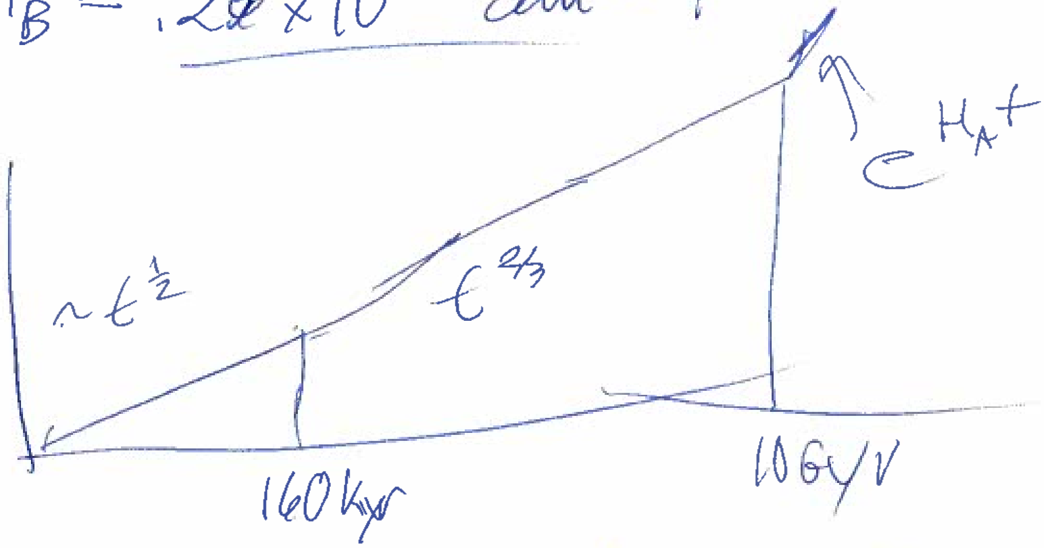
$\Omega_m = .3$

$\Omega_\Lambda = .7$

$\Omega_R = 8.47 \times 10^{-5}$

} Fiducial values Galanti & Roncadelli

~~a~~ $n_B = .25 \times 10^{-6}\ \mathrm{cm}^{-3}$ ✓ ← cosmic_a_gen.f



$a(t) \Rightarrow$ Galanti and Roncadelli 2021

↳ quasi exact.

joined RM exact ML exact

in middle of matter era when

Radiation negligible 10-057

= the GR formula is beautiful
↪ interesting symmetries
and ~~sinh cosh~~, sinh, cosh
behavior ~~that we somewhat~~
elucidated in lect. 5.

But by eye, you can't
easily see how it behaves
So I came up with a
reasonable and simple
approximate formula
↪ an interpolation formula

$$a = a_0 \left\{ \left[2\sqrt{\Omega_R} H_0 t_f \right]^{1/2} + \left[\frac{3}{2}\sqrt{\Omega_M} \left(\frac{\left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/2} \sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda} H_0 (t-t_f)\right)}{\left(\Omega_M \frac{9}{4}\right)^{1/2}} \right) \right]^{2/3} \right\}$$

1

Radiation part

matter Lambda part

and $t_f = t_{RME} \tanh(t/t_{RME}) = \begin{cases} t_{RME}, & t \to \infty \\ t & \text{as } t \to 0 \end{cases}$

Not where $\rho_{rad} = \rho_{matter}$
where $\rho_{matter}/\rho_{rad} = 2$

Functional behavior suggests this is the good characteristic transition time

~~Q~~ 8) I ~~anyalyzed~~
~~this~~ would work
reasonably well.
and it does

$t \to 0$, (asymptotically
$t \to \infty$) exactly correct

and reasonable ~~in~~ between.

the t_c causes ~~the~~
radiation term to
~~saturate at about~~
~~the right~~ starting level
for ~~the~~ Matter - Λ growth
and ~~delays the start~~
~~of~~ Matter - Λ growth ~~to~~
~~about then~~.
It's not bad, at worst
about 10% in error.
No need to improve since ~~the~~
exact solution exists
Unless some other form
gives more insight

It ~~the~~ seems
the Radiation
does NOT
have a
long
perturbation
~~into the~~
Matter
& Lambda
era
It largely
Just sets
the initial
size
for the
matter-Lambda
~~exp~~
expansion
era

10-059

b) The $T = T_0 \left(\frac{a_0}{a}\right)$

→ so curvy since $t = 10$ Gyr when exponential ~~grow~~ expansion started.

→ But linear on a log-log plot before. Radn $T \propto \frac{1}{t^{1/2}}$, Matter era $T \propto \frac{1}{t^{2/3}}$

→ ~~in fact it should have slope -2 in both radiation and matter era~~

~~$a \propto t^{1/2}$ radiation, $\rho \propto \rho$~~

$a \propto t^{2/3}$ matter

→ There is a change in slope at the radiation-matter transition if you look really closely

c) TFE H drop really fast at $t \cong 370$ kyr

→ very sensitive temperature dependence.

— $T\uparrow$ $I\uparrow$, $T\downarrow$ $I\downarrow$

— actually an density $N_H\downarrow$, $I\uparrow$

(0-050)

But in the observable universe the decoupling temperature beats the decoupling density

Of course, after about recombination LTE fails

~~fails~~

The ~~universe~~ did get reionized at $t = 150 - 1000$ Myr

H all reionized

He?

$z = 6 - 20$

(cosmic dawn 150 Myr) by hard photons from ~~stars~~ stars and/or AGN

in $\frac{1}{10}\frac{1000}{10}$

$a_{reion} \simeq \frac{1}{10}$ a_{nce}

But by then density was so low ~~that~~ even with free electrons the CBR and DEBRA still don't couple much to matter

(So reionization did not recouple the CBR to matter)

$\tau_{since\ decoupling} \simeq .05$

$\therefore I = I_{last\ sca} e^{-\tau_s} = I(.95)$

$\Rightarrow$ significant but small

10-061

<u>So</u> even if Recombination never happened and ~~the~~ universe stay ionized eventually ~~the~~ CBR would have <u>decoupled</u> eventually

The universe steep reionized
- density so low recombination is slow
- continued UV & X-ray photons from stars/AGN

<u>Will</u> ~~the~~ CBR photons <u>ever</u> all electron scatter erasing ~~the~~ angular information about density fluctuation of universe at decoupling:

<u>Depends</u>: ~~decouple travelling~~ ~~further will~~ mean free path grow faster than distance travele

10-062

by CBR photons?
Someone must have done
a calculation for the
Λ-CDM model, but
I've not found the result
yet — one could probably
do it myself (if I had a
moment)