2023 Oct 31

5001

Lecture 5

— on more advanced realistic Friedmann Equation Solutions

— including matter-Λ universe, & radiation-matter universe

— Λ-CDM solution a(t)'

— Lemaitre universe analog

— Λ-CDM universe analog

3003

1) Solution for Two density components

~~Today~~ we have FE form $\left\{ \Omega_{p0} + \Omega_q = 1 \right.$

$$\left(\frac{\dot{x}}{x}\right)^2 = \Omega_{p,0} x^{-p} + \Omega_{q,0} x^{-q}$$

$x = \frac{a}{a_0}$

$d\tau = H_0 dt$

where $p > q \geq 0$ without loss of generality

$(J_e - A2)$

~~But we will restrict $\Omega_{p,0}$~~

$$g = \frac{\Omega_{p0}}{|\Omega_{p0}|}, \quad h = \frac{\Omega_{q,0}}{|\Omega_{q,0}|}$$

either can be ~~negative~~ ± 1

since a +ve curvature universe has $\Omega_{k,0} < 0$

and $\Omega_\Lambda < 0$ in principle

~~though~~ we only consider $\Omega_\Lambda \geq 0$ usually.

5004

Not due the derivation which is tedious with many scalings but only 3 general solutions exist

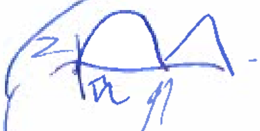
Not general linear combination

Recall FE is nonlinear, So you cannot add solutions in general. You can expand of the argument is η + Constant which sometimes useful.

$$Z = \begin{cases} \sinh(\hat{\eta}) & g = h = 1,\ \hat{\eta} \in [0, \infty] \\ \cosh(\hat{\eta}) & g = -1,\ h = 1,\ \hat{\eta} \in [-\infty, \infty] \\ \sin(\hat{\eta}) & g = 1,\ h = -1,\ \hat{\eta} \in [0, \pi] \end{cases}$$

($\cos(\hat{\eta})$ same)

where Z is scaled cosmic scale factor – a scaled $a(t)$



cyclic universe needs extra hypothesis

and $\hat{\eta}$ is a generalized conformal time

Inverses also exist

$$\hat{\eta} = \begin{cases} \operatorname{arcsinh}(z) = \ln(z + \sqrt{z^2+1}) & g = h = 1,\ z \in [0, \infty] \\ \operatorname{arccosh}(z) = \pm\ln(z + \sqrt{z^2-1}) & g = -1,\ h = 1,\ z \in [1, \infty] \\ \arcsin(z) & g = 1,\ h = -1,\ z \in [0, 1] \end{cases}$$

But many special cases follow, but only in a limited ~~number~~ set of cases is there exact solution

[5005]

in terms of variables
~~that~~ are just x and τ.

~~onto~~
The key parameter is $Q = \frac{q}{p-q} \geq 0$
and recall $p > q \geq 0$

If Q <u>not</u> an integer no such
solution exist
If Q = integer
then $\tau(x)$ solution always,

but $x = x(\tilde{\eta})$,

and so you still need
the generalized conformal
time as an auxiliary variable
<u>But I consider this an exact solut</u>

If $Q = 0, 1, 3,$ then
exact $\tau(x)$ and $x(\tau)$ exist
and the auxiliary $\hat{\eta}$ is
no longer needed.

5006)

We will just quote this result

$Q = 0$ is when $q = 0$ and so all cases where you have a cosmological constant have both $x(\tau)$ & $\tau(x)$ exact, including the matter-Lambda universe of Λ-CDM model after radiation is unimportant

We will derive as useful exercise

$Q = 3$ is interesting for $p = 4$, $q = 3$ whence $\frac{3}{4-3} = 3 = Q$

this is the radiation-matter universe which is the ~~and~~ Λ-CDM universe at early times ≤ 50 kyr when Λ was negligible but radiation dominated the mass-energy of the universe.

2) $Q = 0$ cases

so $p > 0$ and $q = 0$

$\sinh x = \frac{e^{x} - e^{-x}}{2}$, $\sinh x \Rightarrow x$ for x small, $\sinh x = e^{x}/2$ for $x \gg 1$

Note that nice formula conflates power-law and exponential beautifully, elegantly

$$x = \begin{cases} \left(\frac{\Omega_{p0}}{\Omega_{\Lambda 0}}\right)^{1/p} \sinh^{2/p}\left[\frac{p}{2}\sqrt{\Omega_{\Lambda 0}}\,\tau\right] & \text{general} \\ \text{small } \tau \quad \left(\frac{\Omega_{p0}}{\Omega_{\Lambda 0}}\right)^{1/p}\left[\frac{p}{2}\sqrt{\Omega_{\Lambda 0}}\,\tau\right]^{2/p} = \left[\frac{p}{2}\sqrt{\Omega_{p0}}\,\tau\right]^{2/p} & \\ \left(\frac{\Omega_{p0}}{\Omega_{\Lambda 0}}\right)^{1/p} \frac{e^{(\sqrt{\Omega_{\Lambda 0}})\tau}}{2} & \text{for large } \tau, \text{ asymptotic de Sitter universe} \end{cases}$$

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Note $\Omega_{P,0}$ has not cancelled out for $\tau >> 1$.

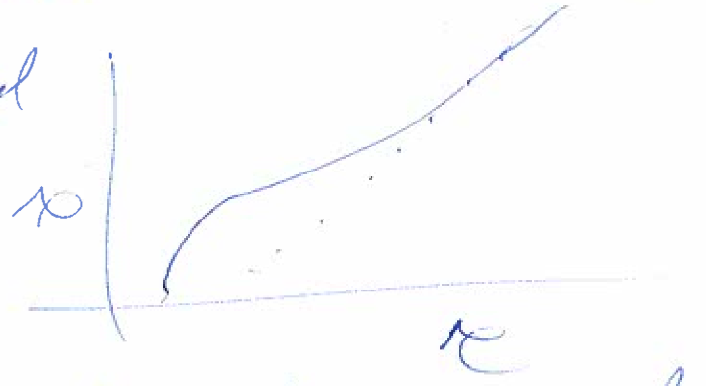
If we'd started just with the FE with just one component $\Omega_{\Lambda,0}$,

then $x = x_0 e^{(\sqrt{\Omega_{\Lambda 0}})\tau}$

where x_0 is a constant to be fitted at $\tau = 0$

— a real de Sitter universe —

Not cancelled since it sets the scale



The inverse can also be found

$$\tau = \frac{1}{\frac{P}{2}\sqrt{\Omega_{\Lambda 0}}} \text{arcsinh}[y]$$

$$= \frac{1}{\frac{P}{2}\sqrt{\Omega_{\Lambda,0}}} \ln\left[y + \sqrt{y^2+1}\right]$$

where $y = \left[\frac{x}{\left(\frac{\Omega_{P0}}{\Omega_{\Lambda 0}}\right)^{1/P}}\right]^{P/2} = \frac{x^{P/2}}{\sqrt{\frac{\Omega_{P0}}{\Omega_{\Lambda 0}}}}$

5008

So the age ~~the~~ of the universe for the Λ-CDM model ~~not~~ leaving the radiation era as negligible period follows from setting

$p = 3$ for matter

$x = x_0 = 1$

$\Omega_{p,0} = \Omega_{M,0} = 0.3$

$\Omega_{\Lambda,0} = 0.7$

$H_0 = 70 \frac{km/s}{Mpc} = \frac{1}{13.968 \text{ Gyr}}$

Fiducial values.

cosp/cosmic_4_gen.x

$t_0 = 13.467$ Gyr

but my fiducial values aren't the absolute best and Planck-2018 gives

$t_0 = 13.797(23)$ Gyr

Using ~~their~~ Planck-2018 $h = 67.36$, ~~and~~ $\Omega_{M,0} = 0.3153$, $\Omega_\Lambda = 0.6847$

gives $t_0 = 13.800$ Gyr. I can't quite account for the remaining discrepancy but we know the observational people have their little tricky corrections that are hidden deep.

3) Q = 3 case

with p = 4 , q = 3
(p = 4: radiation; q = 3: matter)

So this is for the early universe.

Here we will not start from my general approach, but from the FE itself — it's more educational and a sanity check too. (2023 jan 01 notes p.1)

a) Scaled Friedmann equation

General Scaling

$$H^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_{p=0}^{4} \Omega_{p,0} x^{-p}$$

where we've constraint

$$\sum_{p=0}^{4} \Omega_{p,0} = 1$$

by our scaling

$$d\tau = \frac{dx}{x\sqrt{\sum_{p=0}^{4} \Omega_{p,0} x^{-p}}}$$

Often one solves for $\tau(x)$ most easily analytically and numerically always clearly.

5010

b) Specializing to radiation-matter universe

$$d\tau = \frac{dx}{x\sqrt{\Omega_{M0}x^{-3} + \Omega_{R0}x^{-4}}}$$

$Q = \frac{p}{p-q} = \frac{3}{4-3} = 3$ ... are and no exact solution $\tau(x)$ exists and $x(\tau)$ too.

$$= \frac{x\,dx}{x^2\sqrt{\dots}} = \frac{x\,dx}{\sqrt{\Omega_{M0}x + \Omega_R}}$$

$$\sqrt{\Omega_{R0}}\,d\tau = \frac{x\,dx}{\sqrt{\frac{x}{(\Omega_{R0}/\Omega_{M0})} + 1}}$$

Now we need to do special case rescaling

rescaled scale factor

$$y \equiv \begin{cases} \frac{x}{x_e} & \text{where } \Omega_{M0}x_e^{-3} = \Omega_{R0}x_e^{-4} \quad \text{(equality } x\text{)} \\ 1 & \text{where radiation and matter are equal.} \end{cases}$$

$\Omega_{M0}x = \Omega_{R0}$

$x_e = \frac{\Omega_{R0}}{\Omega_{M0}}$

$x_e << 1$ since $\Omega_{R,0} << \Omega_{M,0}$

$$\frac{\sqrt{\Omega_{R0}}}{x_e^2}\,d\tau = \frac{y\,dy}{\sqrt{y+1}}$$

$= dW$ rescaled cosmic time

$$dW = \frac{\sqrt{\Omega_{R0}}\,d\tau}{x_0^2} = \frac{\Omega_{M0}^2\,d\tau}{\Omega_{R0}^{3/2}} = \frac{d\tau}{\tau_{scale}}$$

where $\tau_{scale} = \frac{\Omega_{R0}^{3/2}}{\Omega_M^2} = x_0^2/\sqrt{\Omega_{R0}}$

Solving rescaled FE exactly.

c) $dw = \frac{y\,dy}{\sqrt{y+1}}$

Table integral Hudson - 5 eq (78)

$$W = \int_0^y \frac{y\,dy}{\sqrt{y+1}} = \frac{2}{3}(y-2)\sqrt{y+1}\Big|_0^y$$

$$W = \frac{2}{3}(y-2)\sqrt{y+1} + \frac{4}{3}$$

in general $W(y)$ grows monotonically with W: $\frac{dw}{dy} = \frac{y}{\sqrt{y+1}}$

0 for $y = 0$

$$W_e = -\frac{2}{3}(-1)\sqrt{2} + \frac{4}{3} \quad \text{for } y = 1 \text{ the equality time}$$

$$= \frac{4}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= 0.39052\ldots$$

$$= \frac{4}{3} \quad y = 2$$

$\approx (3.414\ldots) W_e$ twice the equality time.

$\frac{2}{3}(1)\sqrt{3+1} + \frac{4}{3} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3} = 2.67$ for $y = 3$ (a key time see p. 5…)

$\frac{2}{3} y^{3/2}$ $y \gg 1$

rescale of scale factor

asymptotic large y

$\therefore y = \left[\frac{3}{2} W\right]^{2/3}$

$x = \left[\frac{3}{2} \frac{\tau}{x_e^2 \sqrt{\Omega_{R_0}}}\right]$

$x = \left[\frac{3}{2} \frac{\sqrt{\Omega_{R_0}}}{x_e^{1/2}} \tau\right]^{2/3}$

$= \left[\frac{3}{2} \sqrt{\Omega_{M_0}}\, \tau\right]^{2/3}$

which is exactly the Lecture 3 p. 308 limit result for small τ limit of the matter universe. (Becomes a pure matter universe as $\Omega_{M0} = 1$)

Actually the exact solution has a numerical problem as $y \to 0$ since one is subtracting nearly equal values and so round off error

$t_0 \sim 6.5$ times equality time

5012

1) Series expansion of W(y)

— to avoid round-off error one expand W(y) in a power series in small y and ~~When~~ keep enough terms so ~~that~~ your W(y) values has all the accuracy you want:

~~$W_{series}(y)$ = $W_{simple}(y)$ = 10^{-m}~~

~~$W_{general}(y)$~~

~~Being finicky m is machine precision~~

say m = 18 for 10^{-18} = 10^{-18} FIT double precision on my computer

Calculate a table

y	W_{series}	W_{exact}	diff = $\left(\frac{W_s - W_e}{W_e}\right)$
0	.	.	.
:	:	:	:
y_{diff}	—	—	— smallest $diff_m$

and for $y \le y_{diff_m}$ use series
$y > y_{diff_m}$ use exact

but what if $diff_m \ge 10^{-m}$ where m is machine precision

5013

Well if you can live with it, it's OK.

If not keep ~~computing terms~~ ~~until~~ adding terms to the series until

$diff_m \lesssim 10^{-m}$

and use ~~that~~ corresponding y_{difm} as the transition value.

How do we get the series in our case

$$w = \frac{2}{3}(y-2)\sqrt{1+y} + \frac{4}{3}$$

$$= \frac{2}{3}(y-2)\left(\sum_{\ell=0}^{\infty} a_\ell y^\ell\right) + \frac{4}{3}$$

$$\sqrt{1+y} = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{1}{16}y^3 - \frac{5}{128}y^4 + \frac{7}{256}y^5 - \frac{21}{1024}y^6 + \ldots$$

Wolfram Taylor series to x^5

$$= 1 + \frac{y}{2} + \sum_{\ell=2}^{\infty} (-1)^{\ell+1} \frac{(2\ell-3)!!}{2^\ell \ell!} y^\ell$$

powers in den $\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\ldots(-\frac{7}{2})\ldots$

alternating series

these $\frac{1}{2}$ in the denom 32, 64, 128, 256, 512,

Confirmed by me numerically

So $\ell = 6$

$$(-1)^7 \frac{(12-3)!!}{2^6 6!} = (-1)\frac{9\cdot7\cdot5\cdot3\cdot1}{2^6\cdot6\cdot5\cdot4\cdot3\cdot2} = -\frac{\ldots}{2\cdot6\cdot5}$$

$$= -\frac{9\cdot7}{2^6\cdot6\cdot4\cdot2} = -\frac{21}{2^{10}} = -\frac{21}{1024}$$

5014

$$\frac{3}{2}W = (y-2)\left(\sum_{\ell=0}^{\infty} a_\ell y^\ell\right) + 2$$

$$= \sum_{\ell=0}^{\infty} a_\ell y^{\ell+1} - 2\sum_{\ell=0}^{\infty} a_\ell y^\ell + 2$$

$$= \sum_{\ell=1}^{\infty} a_{\ell-1} y^\ell - 2\sum_{\ell=0}^{\infty} a_\ell y^\ell + 2$$

$$= -2a_0 + 2 + \sum_{\ell=1}^{\infty} (a_{\ell-1} - 2a_\ell) y^\ell$$

($a_0 = 1$)

$$= 0 + \sum_{\ell=3}^{\infty}\left[\frac{(-1)^\ell(2\ell-5)!!}{2^{\ell-1}(\ell-1)!} - 2\frac{(-1)^{\ell+1}(2\ell-3)!!}{2^\ell \ell!}\right]y^\ell$$

$$= 0 + (1-2(\tfrac{1}{2}))y + (\tfrac{1}{2} - 2(\tfrac{-1}{8}))y^2$$

$$= \frac{3}{4}y^2 + \sum_{\ell=3}^{\infty} 2(-1)^\ell\left[\frac{\ell(2\ell-5)!! + (2\ell-3)!!}{2^\ell \ell!}\right]y^\ell$$

$$W = \frac{y^2}{2} + \frac{4}{3}\sum_{\ell=3}^{\infty}(-1)^\ell\left[\frac{\ell(2\ell-5)!!+(2\ell-3)!!}{2^\ell \ell!}\right]y^\ell$$

$$= \frac{y^2}{2} + \frac{4}{3}\sum_{\ell=3}^{\infty}(-1)^\ell\left[\frac{\ell+(2\ell-3)}{2^\ell \ell!}\right](2\ell-5)!!\, y^\ell$$

$$= \frac{y^2}{2} + 4\sum_{\ell=3}^{\infty}(-1)^\ell\left[\frac{(\ell-1)(2\ell-5)!!}{2^\ell \ell!}\right]y^\ell$$

$\ell = 3$, $4\cdot\frac{(-1)\,2\cdot 1}{2^3\cdot 6} = -\frac{1}{6}$ ✓

$\ell = 4$ $4\cdot(-1)^4\frac{3\cdot 3}{2^4\cdot 4\cdot 3\cdot 2} = \frac{3}{32}$ ✓

Correct by [2023jun01 p.5]

Convergence
I'll spare you
Raabe test (Art - 287)
for convergence
but (Te - 3S) gives

5015
The Ratio test fails for $y = 1$

$y \leq 1$ has absolute convergence

$y > 1$ divergence in all senses even though an alternating series.

e) We have $w(y)$ What of $y(w)$

$$w = \frac{2}{3}(y-2)\sqrt{y+1} + \frac{4}{3}$$

Let's rearrange:

$$\left(w - \frac{4}{3}\right) = \frac{2}{3}(y-2)\sqrt{y+1}$$

$$w^2 - \frac{8}{3}w + \frac{16}{9} = \frac{4}{9}(y^2 - 4y + 4)(y+1)$$

$$9w^2 - 24w + 16 = 4(y^3 - 3y^2 + 0 + 4)$$

$$9w^2 - 24w = 4y^3 - 12y^2 \quad \{\text{Galcalc} - 5\}$$

$$\frac{9}{4}w^2 - 6w = y^3 - 3y^2 \quad [\text{2023 jun 01 notes p.3}$$

This is a cubic equation for y.

5016

It can be solved and Galanti & Roncadelli (2021) = GR did solve with they said a lot of labor and got a rather complex 3 branch solution.

They said really and so maybe the mathematicians would

Actually in my general approach solution for 2-density components I got a different cubic in terms of u rather than y

I would never have guessed this substitution

Let $y = u^2 - 1$ (2021dec15 notes p. 129)

$$\frac{9}{4}w^2 - 6w = (u^2-1)^3 - 3(u^2-1)^2$$
$$= u^6 - 3u^4 + 3u^2 - 1 - 3(u^4 - 2u^2 + 1)$$
$$= u^6 - 6u^4 + 9u^2 - 2$$
$$= (u^3 - 3u)^2 - 2$$

$$\left(\frac{9}{4}w^2 - 6w + 2\right)^{1/2} = u^3 - 3u$$

$$\pm\left(\frac{3}{2}w - 2\right) = u^3 - 3u$$

Recall $w(y)$: w must grow monotonically with u and so the +ve root applies

$$0 = u^3 - 3u + \left(2 - \frac{3}{2}w\right)$$

(Je-33,973)

still a cubic, but it's a depressed cubic (a very sad cubic)

This is what my general approach gave naturally.

and then I reversed it to recoverd the Galbuti & Roncadelli cubic. (GR)

But the great thing is depressed cubics are much easier to solve

← Just look up cubic in Numerical Recipes Press et al 1992 p. 179

I won't repeat the steps

$$y(w) = \begin{cases} \left\{ 2\cos\left[\frac{\arccos(\mathcal{W})}{3}\right]\right\}^2 - 1 & \text{for } w \in [0, 8/3] \\ \left(A + \frac{1}{A}\right)^2 - 1 & \text{for } w \geq 8/3 \end{cases}$$

this -1 means round off er at W → ~w =0, ... be ... p...

Recall from p. 5011 $w(y=3) = 8/3$ Triple equali... Ye scale factor rescal scali facto...

~~[illegible]~~ 2 phases

Where $\mathcal{W} = \frac{3}{4}w - 1$

and $A = \left(\mathcal{W} + \sqrt{\mathcal{W}^2 - 1}\right)^{1/3}$ (Je-40)

— amazing this solution doesn't look obviously like GR solution which has arcsin + sin and cosh and arccosh and 3 ~~[illegible]~~ phases

5 D18)

I can't even imagine the direct steps to go from one to the other → only by ~~starting~~ starting from the same cubic is it shown — and GR said they took a lot of insight to extract their formula

But they are mathematically equivalent.

I've verified numerically to agree within machine precision 18 (10^{-18})

until very small y

$w \lesssim 0.002$ (Je -42) where round-off errors became a problem and you needed $w = \frac{1}{2}y^2$ and y soln the series solution

$y = \sqrt{2w}$

$= \sqrt{4 \times 10^{-3}}$

$\approx 6 \times 10^{-2}$

$y \lesssim 0.06$

My formula is simpler

1-term estimate suffices

Also the GR formula ~~stays~~ machine precision to a bit smaller w than mine.

So mine is a simpler easier derived formula but GR is a tiny bit less subject to round-off error

5019

To deal with the round-off error again we need a series solution. I thought directly would be very hard. But maybe it isn't now that I've learn about power of power series

So I tried inverting

$$w = \sum_{n=2}^{\infty} \tilde{a}_n \left(\sum_{k=1}^{\infty} b_k \sqrt{w}^{\,k} \right)^n$$

For $\tilde{a}_n = a_{n-2}$ Je-42

where $y = \sum_{k=1}^{\infty} b_k \sqrt{w}^{\,k}$

and there is a algorithm for Power series raised a power in Wikipedia (Formal power series) but it just gives numerical values not exact

But it was hard but it worked

and I did a few by hand (I know Mathematica not) → It was tricky above k=6

$b_1 = \sqrt{2}$ easy

$b_2 = \frac{1}{3}$

$b_3 = -\frac{7}{144}\sqrt{2}$

$b_4 = \frac{5}{216}$ ...

$b_5 = -\frac{2279\sqrt{2}}{345600}$

$b_6 = \frac{1}{243}$

I used the algorithm for numerical evaluation to confirm these and go much higher.

You may wonder if getting both w(y) and y(w) and the series solutions for both is worthwhile. Given the importance of the radiation-matter universe in cosmology, I argue that we should know it in detail just because we should. Someone should do it once for all. That seems to be me

5020

7) Λ-CDM scale factor a(t)

The scaled FE is

$$\left(\frac{\dot{x}}{x}\right)^2 = \Omega_{r0} x^{-4} + \Omega_{m0} x^{-3} + \Omega_\Lambda)$$

$d\tau = H_0 dt$

$x = \frac{a}{a_0}$

General scaling not w or that zero for special case resulting

There is no exact solution it seems

not $x(\tau)$ nor $\tau(x)$ nor $x(\hat{\eta})$ $\tau(a)$

Steiner (2008) in a unarXived conference proceeding only available on his own site gives an exact solution $a(\eta)$ for

$$\sum_{p=0}^{4} \Omega_{p,0} x^{-p}$$

but t has to be evaluated numerically

η is conformal time

$\to \frac{c\,dt}{a(t)}$

$a(\eta)d\eta = dt$

conformal time

$$t(\eta) = \int_0^\eta \frac{a(\eta)\, d\eta}{c}$$

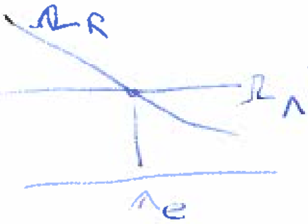
Steiner p.3

and the exact solution is in terms

of ~~the~~ Weierstraß elliptic ℘-function which is a transcendental special function and has be evaluated numerically itself (though maybe once for all).

However we can divide the ΛCDM $a(t)$ into two phases — a radiation-matter phase ~~piecewise scale factor~~ [a piecewise scale factor solutions] — a matter-Λ phase and ~~match the two~~ equate ~~join~~ the two solutions where matter completely dominates, and radiation and Λ are almost negligible and equal seems the best matching point.

~~Best join~~ Equality point is when



$$\Omega_{R0}\, a^{-4} = \Omega_{\Lambda 0} \quad \checkmark$$

Cahill p. 5

fiducial values →

$$a_e = \left(\frac{\Omega_{R0}}{\Omega_\Lambda}\right)^{1/4} = \left(\frac{9.0606\times10^{-5}}{.7}\right)^{1/4} \simeq 10^{-1} = 0.10666\ldots$$

Note Ω_{R0}, Ω_{M0} (.3), Ω_Λ (.7) $= 1$ but Ω_{R0} much less than the ~~error~~ error in Ω_{M0}, Ω_Λ

5022

So we don't need to correct Ω_{DM} $\Omega_{\Lambda 0}$ nor $\Omega_{R,0}$ unless we want to be finicky.

Equality Values:

$\Omega_{m\mathrm{eq}} = .3/(.1)^3$ $\Omega_{R\mathrm{eq}} = 10^{-4}/(.1)^4$

$= 300$ $= 1$ ✓

$= 297.22$ exact $= 0.6999...$ ✓

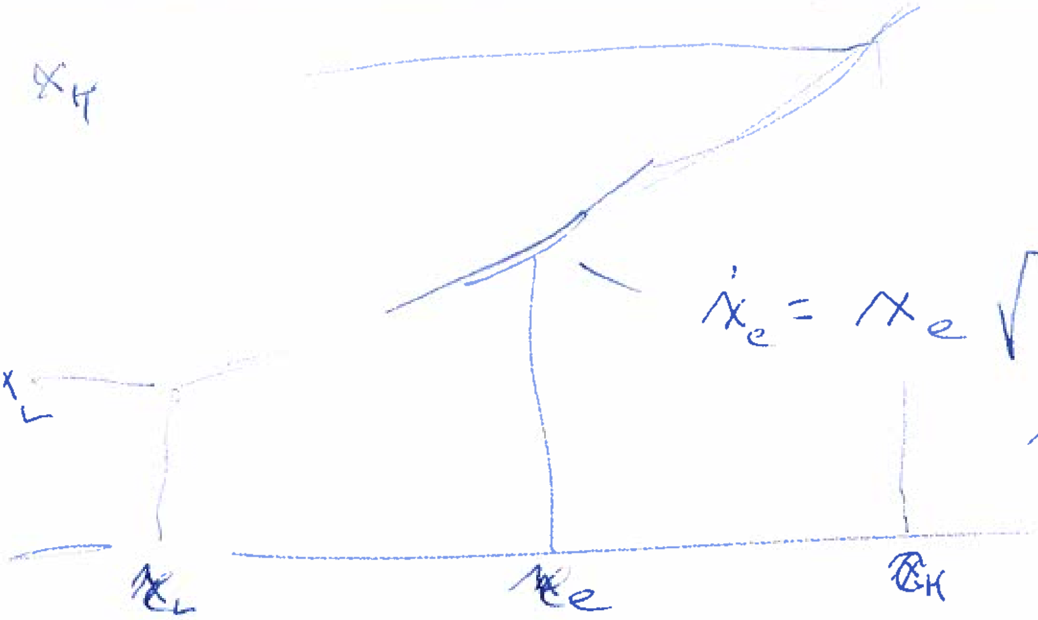
$$\Omega_{R\mathrm{eq}} = \frac{\Omega_{r0}}{\left(\frac{\Omega_{r0}}{\Omega_{\Lambda}}\right)^4}$$

So at ~~the~~ equality of Ω_R and Ω_Λ point $= \Omega_\Lambda = .7$ ✓

$\Omega_{m\mathrm{eq}} = 297.22 \gg \begin{cases} \Omega_R = .7 \\ \Omega_\Lambda = .7 \end{cases}$ of course since they are equal there and Ω_Λ is constant

a priori one would say at $x \cong .1$ ~~that the~~ radiation and Λ contributions to ~~the~~ the slope $\dot{x}$ are small about 0.3% close to negligible and also equal and ~~to change~~ to discontinuity in slope ~~are negligible~~ (of order $\frac{1}{2}$%) probably negligible but ideally this should be checked

5023



$$\dot{x}_e = x_e\sqrt{\Omega_{R0}x_e^{-4} + \Omega_{M0}x_e^{-3} + \Omega}$$

so here ~~and~~ ~~slop~~ nearby the slope would be a bit low

Would such a slope error cause ~~an~~ significant error?

One could do an super accurate numerical midpoint (Newton-Raphson) integration from

$$x_e^{RW} \ll x_L \ll x_e \ll x_H \ll x_{ML}$$

$$\dot{x} = x_e\sqrt{\Omega_{M0}x_e^{-3} + \Delta\Omega + \Delta}$$

~~$= x_e\sqrt{\Omega_{M0}x_e^{-3}}\left(1\right.$~~ $+\Delta\Omega$

$$\dot{x} = x_e\sqrt{\Omega_{M0}x_e^{-3} + \Delta\Omega}\left(1 + \frac{1}{2}\frac{\Delta\Omega}{\Omega_{M0}x_e^{-3} + \Delta\Omega}\right.$$

$$\frac{1}{2}\cdot\frac{.7}{248}$$

$$\approx \frac{1.4}{10^3} = 0.14\%$$

x_L to x_H to see if x_H

and see $x_{H\,\text{numerical}} = x_H$ match well enough.

But a priori I think the matching solution should be pretty good.

F024

I tried to think of approximate analytic improvements, but they all seemed to have uncontrolled error.

How does one do the matching at the equality x_e?

$\tau_{RM}(x_e) = \tau_e$ the equality time

then $\tau_{M\Omega}(x_e) = \tau_{asymptotic}$ where the M-Ω solution tracks into the RM solution (which is approximate solution at x_e)

$x_{R-M}(\tau) \in [0, \tau_e]$

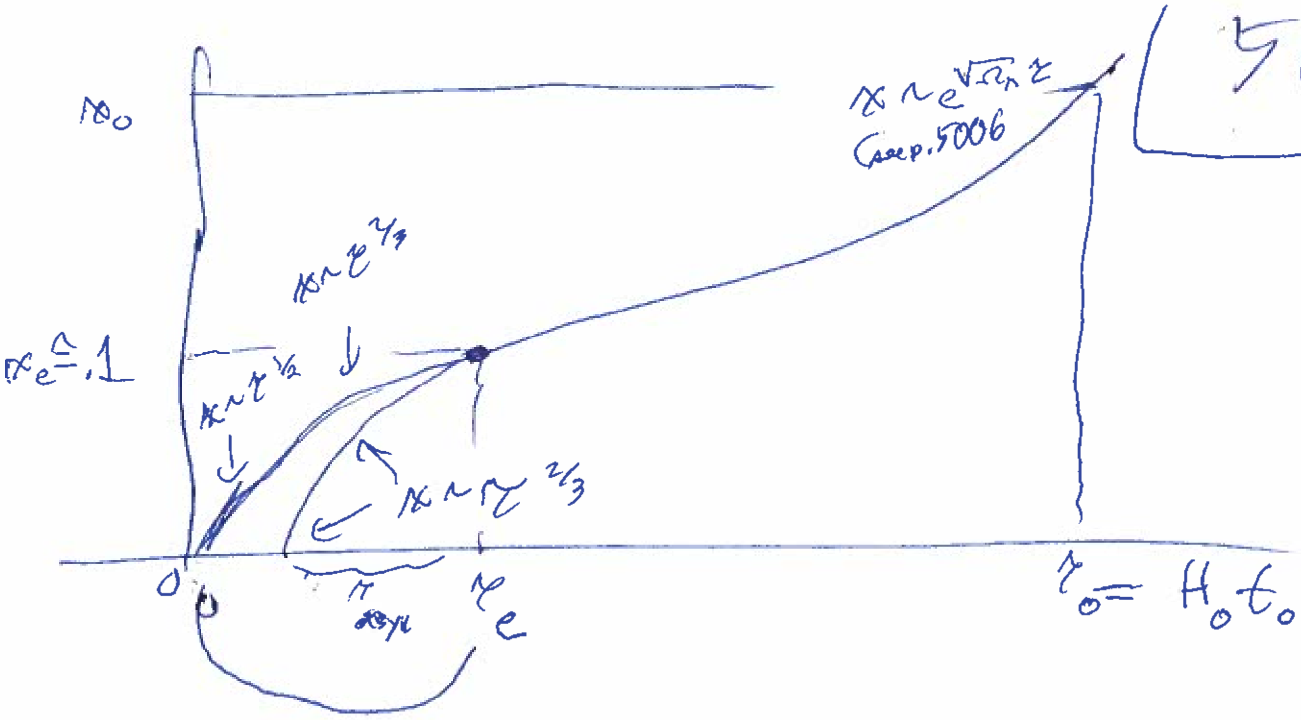
then for $[\tau_e, \infty]$,

$$x_{M-\Lambda} = x_{M-\Lambda}[\tau - \tau_e + \tau_{asym}]$$

- actual time
- subtract off equality time
- add τ_{asym}

$$\tau_{M\Lambda} = \tau - \tau_e + \tau_{asym}$$
$$= \tau - (\tau_e - \tau_{asym})$$

$\therefore \tau = \tau_{M\Lambda} + (\tau_e - \tau_{asym})$ > 0



$\tau_e > \tau_{asym}$ clearly

Age of universe $\tau_0 = \tau_{M\Lambda}(x_0 = 1) + (\tau_e - \tau_{asym})$

$\tau_{M\Lambda} = 13.467$ Gyr from fiducial values p. 5008

→ ~~13467 k~~
1.3467×10^7 kyr

I've not done the calculation but it would be of order the $\tau_{R\text{-}M\,equal} = 50$ kyr (Cahill p. 5.

I do not know if Planck-2018 or others even bother to add on $\tau_e - \tau_{asym}$ since it is probably smaller than error.

5026

502

Can you do the same thing for a general $\left(\frac{\dot{a}}{a}\right)^2 = \sum_{p=0}^{4} \Omega_{p,0}\, a^{-p}$ universe

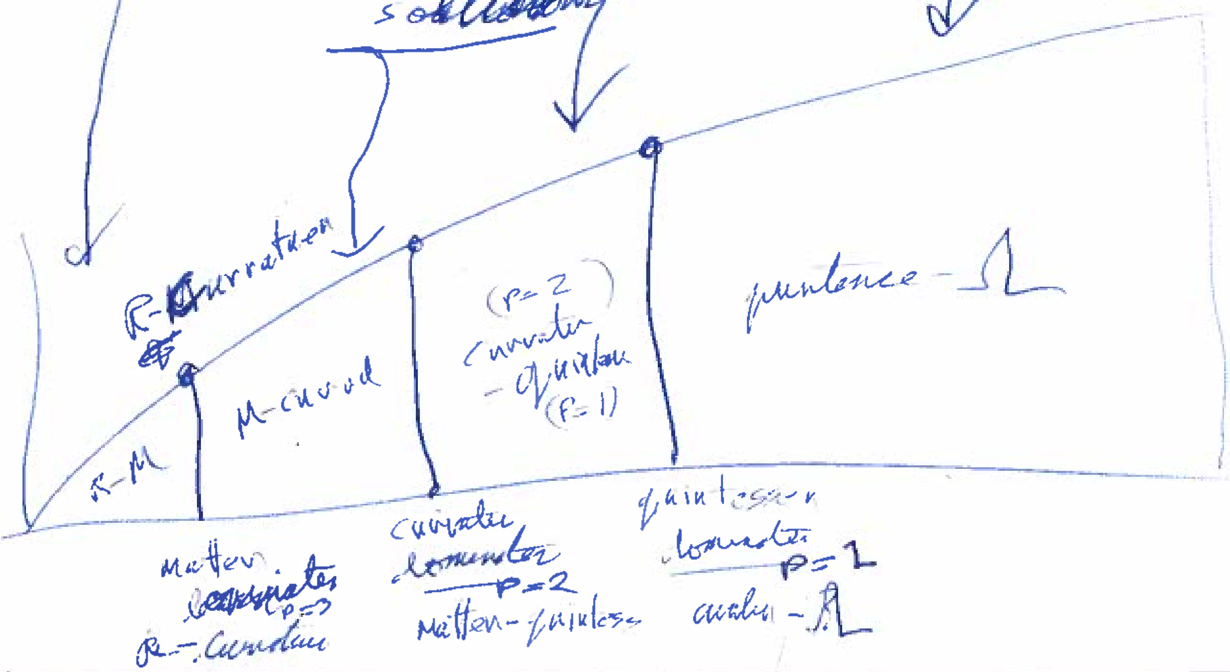
Yes but it may be useless

$Q = \frac{q}{p-q} = \frac{3}{4-3} = 3$

$Q = \frac{2}{3-2} = 2$ only using auxiliary conformal time u for exact solution

$Q = \frac{1}{2-1} = 1$

$Q = \frac{0}{1-0} = 0$



i 028

Yes, ~~should~~ this would be quite good, but you assume ~~that there are~~ points where matter, curvature, and quintessence overwhelmingly ~~clearly~~ dominates

- the piecewise approach fails if those points don't exist
- In fact, if in general one or more may never dominate.

Actually people ~~frequently~~ can use a 1 component piecewise solution (or so I'm told somewhere) → but may also be unclear

Recall $d\tau = \frac{dx}{x\sqrt{\Omega_{r,0}x^{-p}}}$

for a one component FE

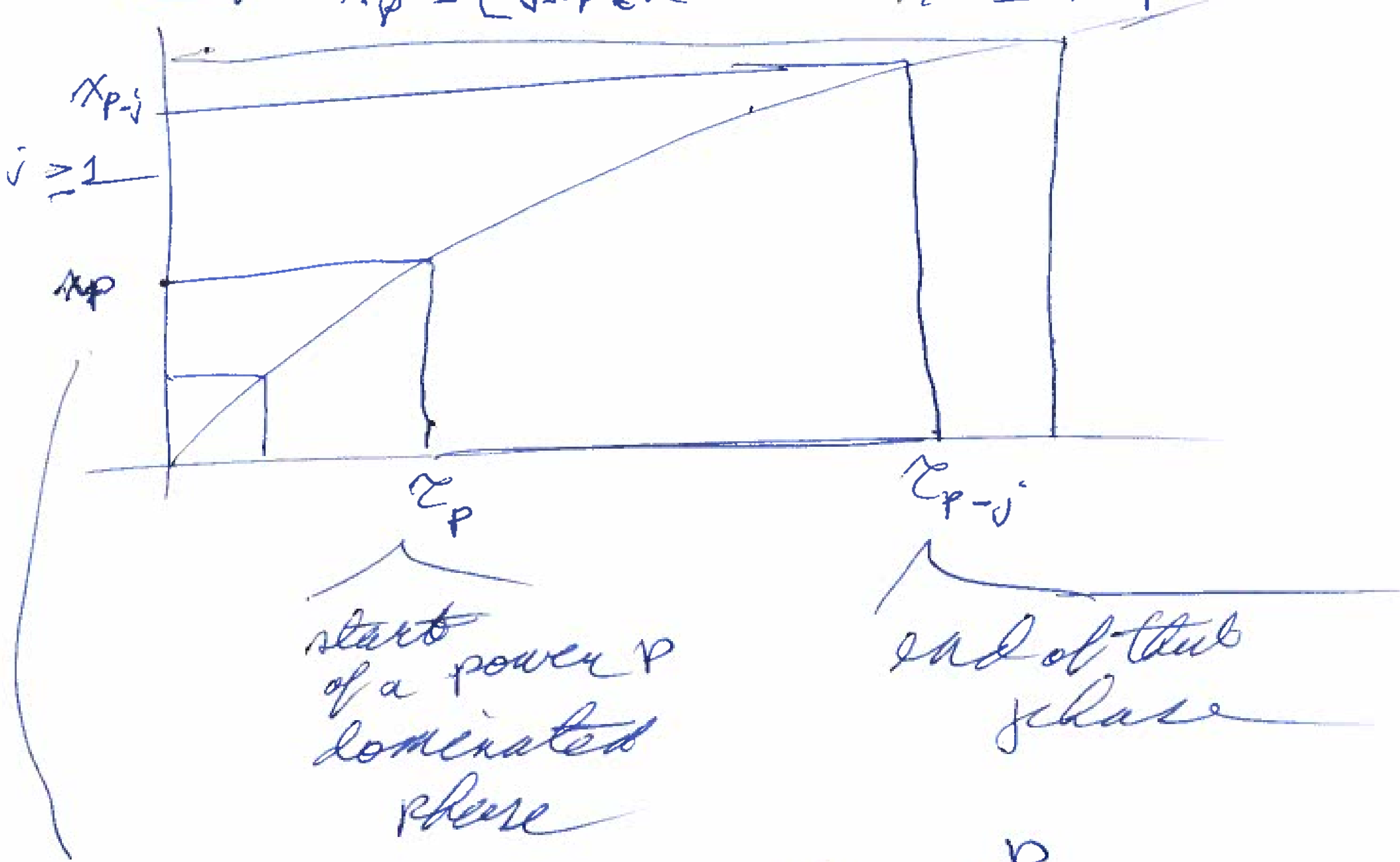
One ~~assume there is always a dominant~~ component.

5029

$$d\tau = \frac{1}{\sqrt{\Omega_p}} x^{\frac{p}{2}-1} dx$$

$$\tau \Big|_{\tau_p}^{\tau} = \frac{1}{\sqrt{\Omega_p}} \frac{x^{p/2}}{p/2} \Big|_{x_p}^{x} \quad \text{or} \quad \frac{1}{\sqrt{\Omega_{p=0}}} \ln(x/x_p)$$

$$x_p = \left[\sqrt{\Omega_p}\left(\frac{p}{2}\right)(\tau - \tau_p) + x_p^{p/2}\right]^{2/p} \qquad x = x_p e^{\sqrt{\Omega_{p0}}(\tau - \ldots}$$



$$\Omega_{p+i}\, x_p^{p+i} = \Omega_p x^p$$

and $i \geq 1$ $\qquad x_p = \left(\frac{\Omega_p}{\Omega_{p+i}}\right)^{\frac{1}{i}}$

usually $p = [0, 4]$ integer in ~~physically~~ relevant simple models

$$\tau_{p} = \frac{1}{\sqrt{\Omega_{p+i}}} \frac{x_p^{p+i}}{(p+i)/2}$$

< ~~where~~ if $p+i$ is the highest power

usually { $p+i = 4$ radiation, $p = 3$ matter

and then on contract up to the lowest power - usually $p=0$

4030

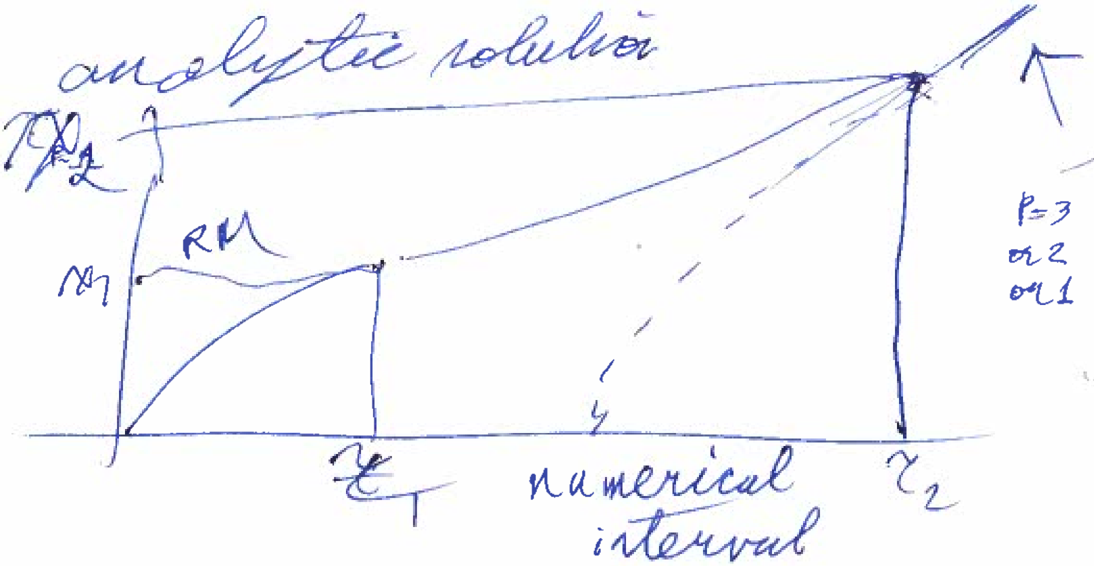
This could be OK sometimes

There may be some components ~~that~~ are never dominant.

If an analytic solution is not available and two-component or one component piecewise solution are inadequate or too finicky ~~then~~ a good numerically solution might be best.

However, you could start with the radiation-matter analytic solution

$x_2 = x(\tau_{eq}\Omega)$
$\tau_{asym}(x_2)$
$x_{P-\Omega} = x_{P-\Omega}(\tau - \tau_2 + \tau_{asym})$
similar to p. 4024



Track into P–Ω

P=3 or 2 or 1

two-component solution with Ω one component. may be ~~good~~ for really high accuracy.

For numerical solution

$$d\tau = \frac{dx}{x\sqrt{\sum_{p=0} \Omega_{p0}\, x^{-p}}}$$

but recall $\tau_{n+1} = \tau_n + \Delta x\, f(x_{n+\frac{1}{2}})$

midpoint method (Newton-Raphson method) or something more accurate like Runge-Kutta

Remember discretization error will increase the farther in Δx you go, and so you'll need numerical checks for accuracy
e.g., - decrease step size
- test by comparison to analytic solution.

5) Exact 3-Component Solutions

a) No derivations. They're not hard, just tedious and we've already done one derivation in detail (the radiation-matter universe; see p. 500)

50 32

One can get 3-component solutions

for $d\tau = \frac{dx}{x\sqrt{\Omega_{p,0}x^{p} + \Omega_{q,0}x^{-q} + \Omega_{r,0}x^{-r}}}$ if $r=0$ and $r=2q$

Special case of interest $p=4$, $q=2$

ie radiation curvature Lambda universe

$a = \Omega_{\Lambda,0}$

$r=0$

a is not the cosmic scale factor

$b = \Omega_{2,0}$ so curvature universe (or cosmic string or $R_c = kt$ universe)

$b<0$ for +ve curvature

$b>0$ for -ve curvature

$c = \Omega_{4,0}$ for radiation

I used a, b, c because the Ω's are klutzy symbols for somewhat intricate formulae and also the a, b, c are the usual symbol for a quadratic formula

$$\frac{d\tau}{V} = \frac{dz}{\sqrt{az^2+bz+c}}$$

{ which has a table integral exact solution }

where $V=2$, $z = x^{1/V} = x^{\frac{1}{2}}$

b) After a few tedious steps

$$X = \left(\frac{\mp b}{2a}\right)^{\frac{1}{2}}\left[\pm 1 + \sqrt{\frac{4ac}{b^2}-1}\;\sinh\left[2\sqrt{a}\,(\tau-\tau_{off})\right]\right]$$

where upper case is for

$b < 0$, +ve curvature

lower case

$b > 0$ -ve curvature

Offset τ

$$\tau_{off} = \pm\frac{1}{2\sqrt{a}}\,\operatorname{arcsinh}\left[\frac{1}{\sqrt{\frac{4ac}{b^2}-1}}\right]$$

where $\tau = 0$ gives $x = 0$

$\operatorname{arcsinh} x = \ln\left[x+\sqrt{x^2+1}\right]$

If $b = 0$, and so just a radiation-Λ universe

With

$$X = \left(\frac{\sqrt{4ac}}{2a}\right)^{\frac{1}{2}}\sinh^{\frac{1}{2}}\left[2\sqrt{a}\;\tau\right]$$

with $\tau_{off} = 0$

$$X = \left(\frac{\Omega_{R0}}{\Omega_{\Lambda 0}}\right)^{\frac{1}{2}}\sinh^{\frac{1}{2}}\left[2\sqrt{\Omega_\Lambda}\;\tau\right]$$

which is the same as the 2-component formula on p. 5006 with $p=4$

034

If $\frac{4ac}{b^2} - 1 = 0$

and $b < 0$

($b > 0$ then the formula on p. 5031 is $\left(\frac{b}{2a}\right)^{\frac{1}{2}}[-1]$ ~~gives~~
Ω_{02} and no solution)

$$X = \left(\frac{-b}{2a}\right)^{\frac{1}{2}}$$ a constant solution.

This is the ~~radiation curvature~~
radiation – +ve curvature – Λ ~~Einstein~~ static universe
which is analogue to
Einstein' Universe
is matter – +ve curvature – Λ universe

I think there is an exact formula
~~but I forgotten it at the moment~~ see p. 5045

c) But the static universe are unstable
– Any global perturbation
static sets them expanding
~~growing~~
or contracting

This is easy to prove

$$\left(\frac{\dot{X}}{X}\right)^2 = \sum_p \Omega_{p,0} X^{-p}$$

$$\frac{dX}{d\tau} = \pm X\sqrt{\sum_{p_0} \Omega_{p_0} X^{-p}}$$

Note there are two solutions — ~~ang~~ expanding and contracting branches. Since the observable universe is expanding we haven't bothered with the contracting case but it's there.

Say X_s gives $\sum_p \Omega_{p,0} X_s = 0$

which can only happen ~~for~~ if one $\Omega_{p,0}$ is negative.

i.e., a positive curvature $\Omega_{k,0} = \Omega_{2,0} < 0$

or $\Omega_{0,0} = \Omega_\Lambda < 0$ which is not an interesting case for us.

s for static or stationary solution.

503B

Say we let $x = x_s + \Delta x$

$$\frac{d\Delta x}{d\tau} = \pm\sqrt{\emptyset + C\Delta x^2} = \pm C(\pm \Delta x)$$

slope $\pm$

Δx can be $\pm$ve and -ve signifies when $\Delta x < 0$

to 1^{st} order in small Δx where C is some constant which could be +ve or -ve but we can assume +ve without loss of generality

Note we could have perturb Δx from $\Delta x = 0$ for a root of $\{\Omega_{p,0}\}$ or perturbed the set $\{\Omega_p\}$ creating a new $x_{s\,new}$ making $\Delta x = x_{s\,old} - x_{s\,new}$

If Δx, then there is a stationary point at some τ not static solution. See p. 5062

Must be Δx^2 or perturbation would become imaginary

$$\frac{d\Delta x}{d\tau} = \pm C[\pm\Delta x]$$

uncorrelated $\pm$

~~so there~~

$$\pm\frac{d(\pm\Delta x)}{\pm\Delta x} = \frac{d\Delta x}{\pm\Delta x} = \pm C\,d\tau$$

still ~~uncorrelated~~ correlated so $(\pm)(\mp 1) = \pm 1$

$$\pm \ln[\pm\Delta x]\Big|_{\Delta x_0}^{\Delta x} = \pm C\tau$$

where set $\tau_0 = 0$ without loss of generality

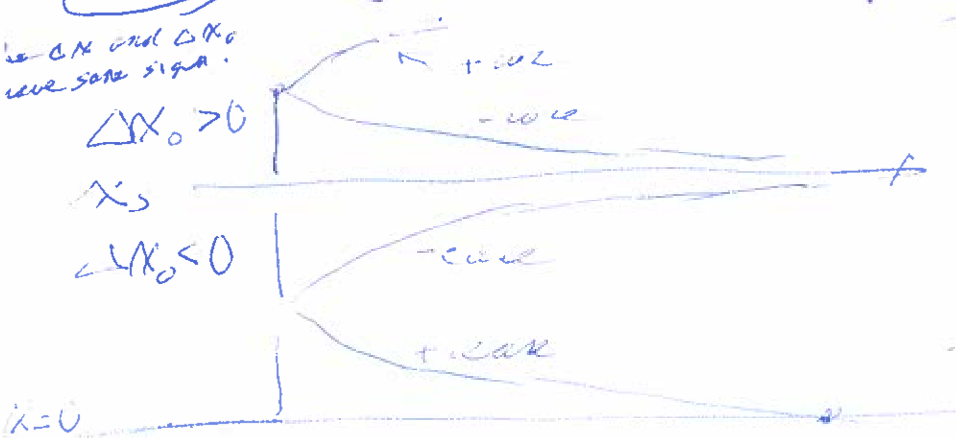
a small perturbation ~~where~~ the 1^{st} ~~perturb~~ expression is valid

$$\left|\frac{\pm\Delta x}{\pm\Delta x_0}\right| = e^{\pm C\tau}$$

$$\Delta x = \Delta x_0\, e^{\pm C\tau}$$

i.e. Δx and Δx_0 have same sign.

only the 1^{st} order solution.



and so the +ve case will become ~~[struck]~~ invalid for $\tau \to \infty$ but -ve case ~~[struck]~~

4 03

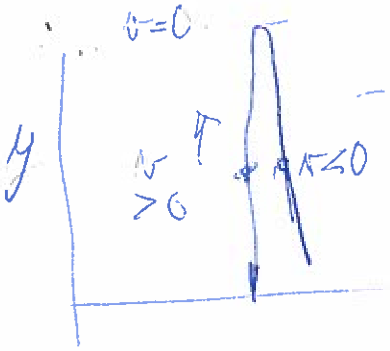
is valid as the solution asymptotically approaches x_s which it only reaches as $\tau \to \infty$,

The ~~For~~ +ve case shows the instability of static solutions.

The -ve case is stable.

But what chooses the case? Past history to Δx_0 since all physics we allow is wrapped into the Friedmann Eq.

The situation is analogous to the classical ball tossed vertically



$F = ma$

$-mg = ma$

$a = -g$

is acceleration but that doesn't tell direction

But conservation of mechanical energy gives

$$E = \frac{1}{2} m v^2 + mgy$$

$$v = \pm\sqrt{\frac{2E}{m} - 2gy}$$

For y you know the speed $|v|$, but not v.

It's the history that gets you to y that determines the sign of v.

5038

So the history that gets you to Δx_0 tells whether the +ve case or -ve case applies.

[The FE is really an energy balance equation and in using it to get universe models, we incorporate no physics outside of it. We can start it from point origin t = 0 [illegible] from de Sitter but those are assumed point origins. We don't say how the universe got there]

If you imagine vaguely random perturbations, then some will put you onto the +ve case, and so overall the static solution is unstable.

Einstein didn't notice the instability of the Einstein universe since he derived it from a klutzy path and not the FE which didn't derive

But the FE equation applies to the global scale factor $x(t)$ and global perturbations to the whole universe seem very unlikely. More likely is that there will be some distribution of perturbations → Let's defer perturbations for a moment. discussing

Based on physics with absolute homogeneity and isotropy built in and little variation ↓ deflation is out of this trap.

after the instability was noted in the 1920s, it was a consideration in his abandoning the Einstein universe which he'd come to hope was true I think despite that not being his original goal which was just to show that GR allowed self-contained universe model

~~1034~~ (1503?)

c) The radiation +ve curvature − Λ univers is the analogue of matter +ve curvature − Λ Lemaitre universe

$$x = \left(-\frac{b}{2a}\right)^{\frac{1}{2}}\left[1 + \sqrt{\frac{4ac}{b^2} - 1}\ \sinh\left[\frac{\tau - \tau_{off}}{\frac{1}{2\sqrt{a}}}\right]\right]^{\frac{1}{2}}$$

$$\tau_{off} = \frac{1}{2\sqrt{a}}\ \text{arcsinh}\left(\frac{1}{\sqrt{\frac{4ac}{b^2} - 1}}\right)$$

You can make ~~if you make~~ $\frac{4ac}{b^2} - 1$ as close to zero as you like and create as long a Einstein static phase as you like and a long τ_{off} too making $c = \Omega_{R,0}$ small has no effect on the Einstein phase $x_s = \left(\frac{-b}{2a}\right)^{1/2} = \sqrt{\frac{-\Omega_{k,0}}{2\Omega_{\Lambda 0}}}$

near $\tau \cong \tau_{off}$ expanding to 1st order

$$x - \left(-\frac{b}{2a}\right)^{\frac{1}{2}}\left[1 + \frac{1}{2}\sqrt{\frac{4ac}{b^2} - 1}\left(\frac{\tau - \tau_{off}}{1/(2\sqrt{a})}\right)\right]$$

and so linear growth for $\Delta\tau = \tau - \tau_{off}$

F040

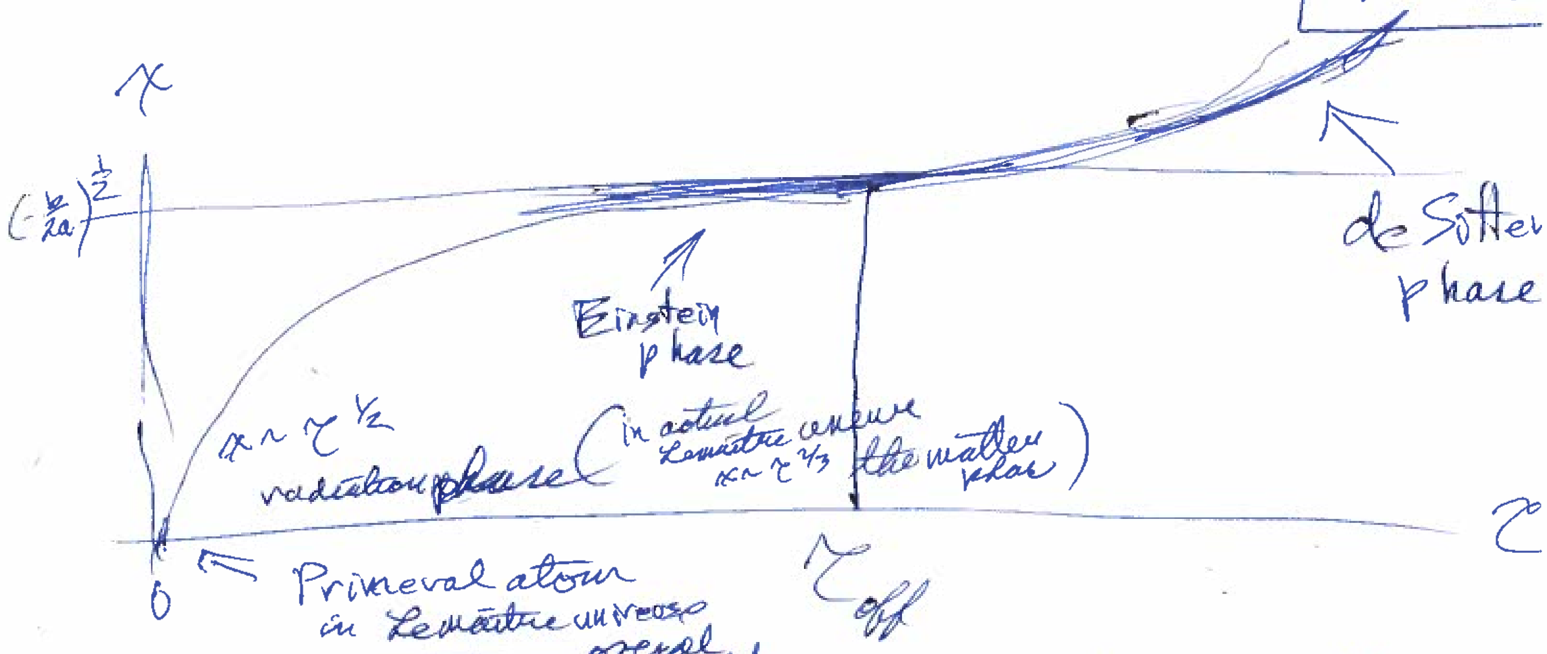
~~E033~~

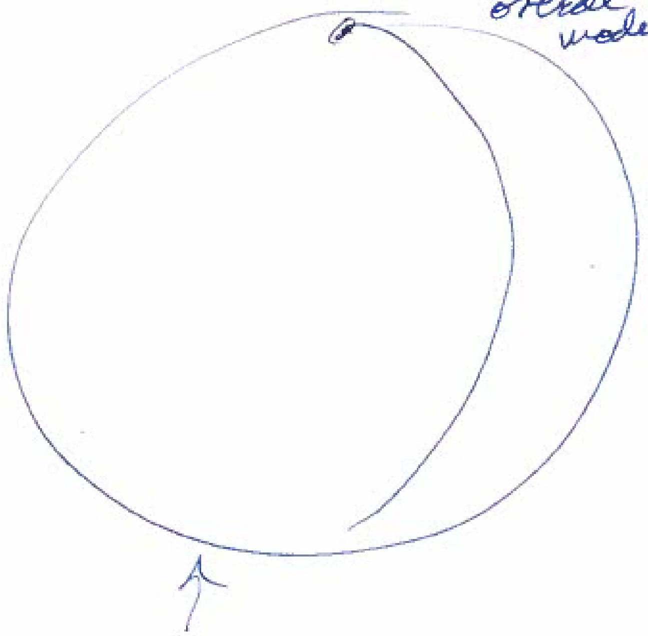
There is is easy to prove from ~~the~~ ~~F~~ Friedmamm equation but we'll not do so here. Einstein didn't see this since he didn't have ~~the~~ Friedmann equation.

Now global perturbations are critical.

What Lemaitre (1933) in the Lemaitre universe assumed was ~~that~~ the Einstein phase of his Lemaitre universe would have ~~the~~ overall expansion and local contractions which would collapse to galaxies — he did discover ~~the~~ Lemaitre-Tolmann metric in 1933 to account for local expansions and contractions in GR.

5041



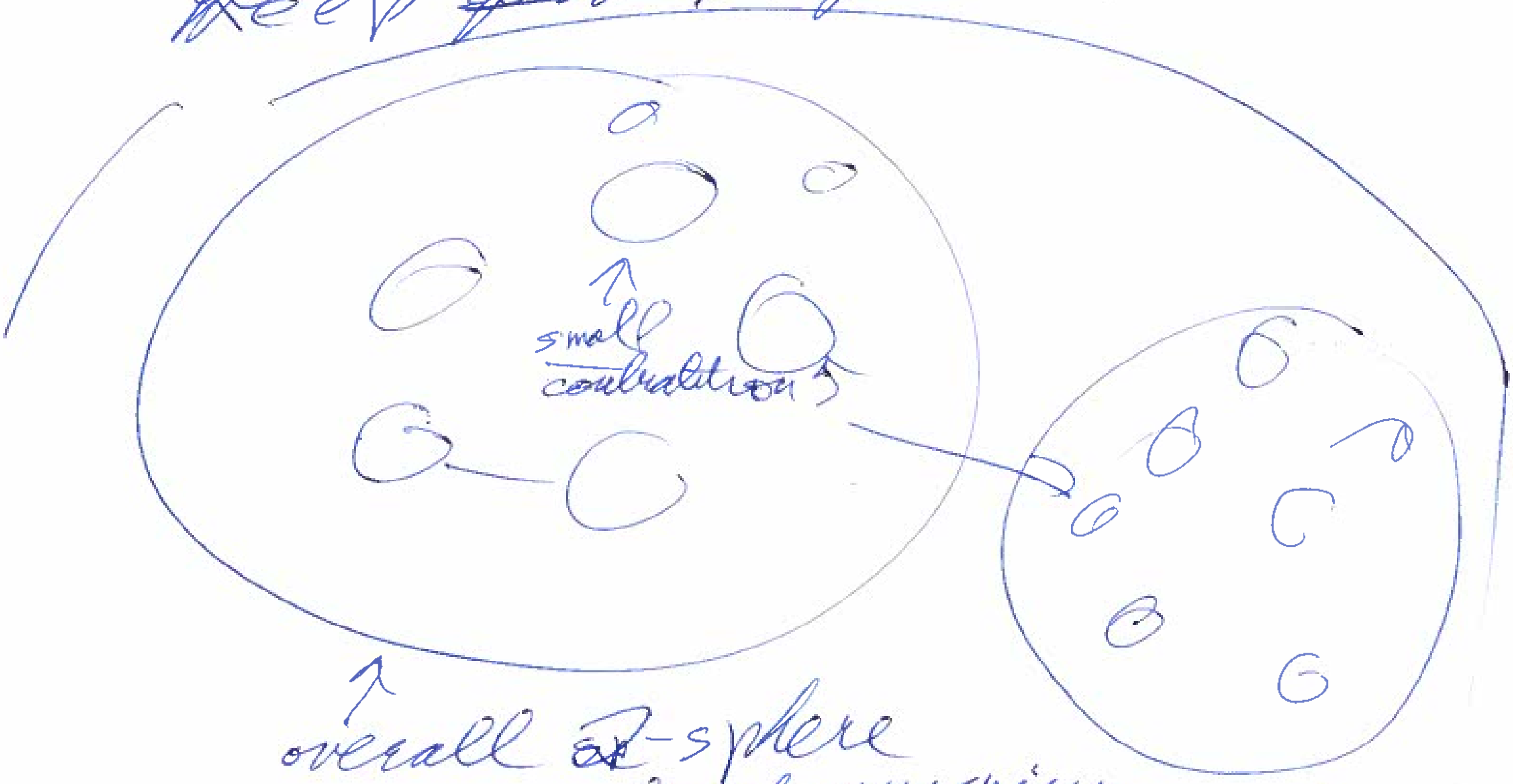


+ve curvature ~~is the~~ universe is a 3-sphere the curve 3-d space analogue of a 2-sphere (an ordinary sphere)

If you were just on the knife-edge with $x = (\frac{-b}{2a})^{\frac{1}{2}}$ in a true Einstein Universe what would happen? Presumably local density perturbation would cause local expansion and contraction and which would win overall

5042 would depend delicately on initial conditions..

Actually, I think Lemaitre just imagined a longish Einstein phase ~~that had was sufficient to hav~~ sufficiently growth to keep ~~quo~~ expanding overall



overall ~~a~~-sphere regions kept growing with the local inhomogeneities cancelling out.

Lemaitre thought you needed the Einstein phase for 2-reasons

5043

i) To avoid ~~the~~ age problem

$$\tau = \frac{1}{\sqrt{\Omega_{p,0}}} \frac{2}{p} x^{p/2}$$

one component model Lect. 3, p. 308?

Einstein-de Sitter universe (EdS universe)

$$t_0 = H_0^{-1} \frac{1}{\sqrt{1}} \frac{2}{p} x_0^{p/2} \text{ with } p=3$$

$$= \frac{2}{3}\left(\frac{13.968 \text{ Gyr}}{h_{70}}\right)$$

the 1990s age problem (But globular cluster $t \sim 13$ Gyr)

~ 9 Gyr for $H_0 \doteq 70$ as in early 1990s

~ 1.3 Gyr for $H_0 = 500 = 7 h_{70}$

as in 1930 age problem when radioactive dating already showed $t_{earth} \gtrsim 2$ Gyr

So Lemaitre posited

So matter (since there is matter)

the curvature which allows a static phase

and Λ which restores expansion which ~~verified by~~ proven by Hubble 1929.

Actually, Lemaitre in published work only consider $k=+1$ universe — he seemed to philosophically like them — finite, boundless hyperspherical universes.

5044 ii) The ~~long~~ Einstein phase, Lemaitre thought might be needed to get contractions to galaxies.

of course, modern computer simulations show contractions to galaxies in expansion phases.

— but neither computer simulations nor analytic work then were available to show collapse in the expansion phase.

— although you could have guessed it would happen with large enough density fluctuations

But Lemaitre did work out the GR Lemaitre–Tolman metric in 1933 to show regions of inhomogeneous collapse which he may have thought necessary to form galaxies.

5046

We define

$$\Omega_{k,s} \equiv -\frac{kc^2/a_s^2}{H_s^2}$$

where $x_s = 1$

To be consistent with the Robertson-Walker metric standard form we set $k = +1$ for +ve curvature

and so $a_s = \frac{c/H_0}{\sqrt{|\Omega_{k,0}|}}$

Recall H_0 is NOT the Hubble constant, is just a parameter set by our chosen ρ_0

so $\left(\frac{\dot{x}}{x}\right)^2 = x^{-3} + \Omega_{k,s} x^{-2} + \Omega_{\Lambda,s}$

which equals 0 for $x = 0$.

$\dot{x} = \frac{dx}{d\tau}$

But we have 2 unknowns still $\Omega_{k,0}$ and Ω_Λ

But we have another equation.

The acceleration equation.

6) Einstein Universe (1917) — 5045 Original idea

— a bit of analysis

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$ (Li-55)

ρ is only matter $P = w\rho c^2 = 0$ since $w = 0$.

Say LHS = RHS = 0

and ρ_0 is the actual density at that occasion where 0 means Einstein state not cosmic present.

We can set ρ_0 and then other physical values follow as we will see.

We define $H_0 = \sqrt{\frac{8\pi G\rho_0}{3}}$ (Li-51)

which is NOT the Hubble constant, but an inverse time parameter defined by ρ_0 (which is real Hubble constant when the sum of omegas = 1 but that's not the static pl (Li-56)

Hubble parameter for static phase $H = 0$ in fact.

or if only matter univer

Divide thru by H_0^2 and define $d\tau = H_0 dt$ / define $\equiv \Omega_\Lambda$

$$\left(\frac{\frac{dx}{d\tau}}{x}\right)^2 = \Omega_{0M} x^{-3} - \frac{kc^2}{H_0^2 x^2 a_0^2} + \frac{\Lambda}{3H_0^2}$$

$\Omega_M \equiv \frac{\rho}{\rho_0} = \Omega_{M0} x^{-3} = x^{-3}$ Define $x \equiv a/a_0$

L5047

The acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}$$

$\hookrightarrow 0$ for matter

Li-55

Scaling

$$\frac{\ddot{x}}{x} = -\frac{1}{2}x^{-3} + \Omega_\Lambda$$

The curvature term differentiated to zero.

$$\left(\frac{\dot{a}}{a}\right)^2 = \ldots \frac{k}{a^2} \ldots$$

$$\dot{a}^2 = \ldots k \ldots$$

$$2\dot{a}\ddot{a} = \ldots 0 \ldots$$

This cancels out nicely using the Fluid equation and k never comes back (see Li-26, 27)

For a static universe

It [illegible] is necessary for $\ddot{x}=0$ for static universe and we show this [illegible] for static solution of 1st order DEs on p. 5063

$$\ddot{x} = 0 \text{ at } x = x_0 = 1$$

$$\therefore \Omega_\Lambda = \frac{1}{2}$$

$\therefore$ From the Friedmann eq. at $x_0 = x_S = 1$

$$1 + \Omega_{k,S} + \frac{1}{2} = 0$$

$$\therefore \Omega_{k,S} = -\frac{3}{2} < 0$$

as it should be for +ve curvature.

$$\therefore \Omega_{m,S} = 1$$

$$\Omega_\Lambda = \frac{1}{2}$$

$$\Omega_{k,S} = -\frac{3}{2}$$

$\Omega_{mS} + \Omega_{kS} + \Omega_{\Lambda S} = 1 + \frac{1}{2} - \frac{3}{2} = 0 = H_S$

Not 1 and the usual Hubble constant for static [illegible]

which sets $a_S = \frac{c/H_S}{\sqrt{3/2}}$

$[a_0] = \left[\frac{L/T}{\sqrt{\frac{E}{M^2}L\frac{M}{L^3}}}\right] = \frac{L/T}{\sqrt{\left(\frac{L}{T}\right)^2 L^{-2}}} = L$

$\therefore$ dimensionally correct

$$a_S = \frac{c/\sqrt{\frac{8\pi G\rho_S}{3}}}{\sqrt{3/2}} = \frac{c}{\sqrt{4\pi G\rho_S}}$$

$\leftarrow$ a rather curious result

5048]

Since the Poisson equation for classical gravity is $\nabla \cdot \underset{\sim}{g} = -4\pi G \rho$

divergence of gravitational field.

$$\left(\frac{\dot{x}}{x}\right)^2 = x^{-3} - \frac{3}{2}x^{-2} + \frac{1}{2}$$

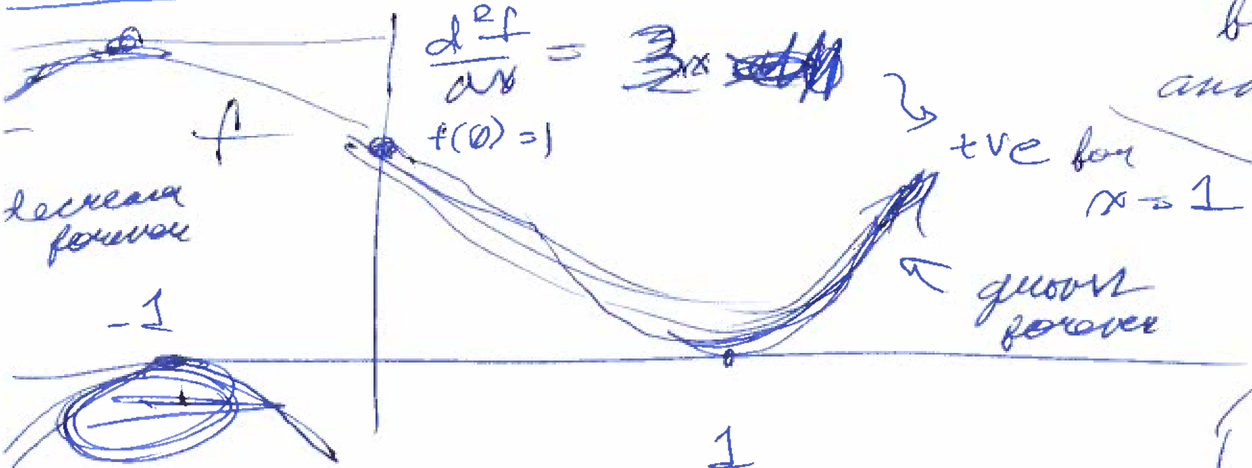
are there other zeros besides $x = 1$?

Let $f = 1 - \frac{3}{2}x + \frac{1}{2}x^3$

derivative with respect to x not τ

$$\frac{df}{dx} = -\frac{3}{2} + \frac{3}{2}x^2$$ which is 0 for $x = \pm 1$

$$\frac{d^2f}{dx^2} = 3x$$ → +ve for $x = 1$ and there are only two stationary points and no extremum



$x = -1$ is unphysical but is a maximum

$f(-1) = 1 + \frac{3}{2} - \frac{1}{2} = 2$

So only 2 stationary points and only one physically relevant zero of f and so of $\dot{x}$

Is only one Einstein universe per P_0 — **5049**

~~7) Iterative~~ Can one solve for $x(\tau)$ behavior when $x \neq x_{\text{static}} = x_c = 1$

Now
$$\left(\frac{\dot{x}}{x}\right)^2 = x^{-3} - \frac{3}{2}x^{-2} + \frac{1}{2}$$

$$d\tau = \pm \frac{x\,dx}{\sqrt{x - \frac{3}{2}x^2 + \frac{1}{2}x^4}}$$

This alas has no analytic solution.

But

$$\dot{x} = \pm\sqrt{x^{-1} - \frac{3}{2} + \frac{1}{2}x^2}$$

and let $x = x_0 + \Delta x = 1 + \Delta x$

and expend to 2nd order

~~$= \pm\sqrt{1 - \Delta x - \frac{3}{2} + \frac{1}{2}(1 + 2\Delta x)}$~~

$$\Delta\dot{x} = \pm\sqrt{\frac{1}{(1+\Delta x)} - \frac{3}{2} + \frac{1}{2}(1 + 2\Delta x + \Delta x^2)}$$

$1 - \Delta x + \Delta x^2 - \Delta x^3 + \Delta x^4 - \ldots$ (A&F-279 Geometric series)

Not no $\sqrt{\Delta x}$ term survives and this must be true for static solutions of 1st order DEs generally sup.

$$= \pm\sqrt{\frac{3}{2}\Delta x^2 - \Delta x^3 + \ldots}$$ But these are unrelated (uncorrelated)

$$\Delta\dot{x} = \pm\sqrt{\frac{3}{2}}\,|\Delta x| = \pm\sqrt{\frac{3}{2}}(\pm\Delta x)$$ ~~$= \pm\sqrt{\frac{3}{2}}\Delta x$~~

as anticipated on p. 5034

5040

So as anticipated on p. 5034

$\Delta x = \Delta x_1 e^{\pm\sqrt{\frac{3}{2}}\tau}$

where $\Delta x_1 \lessgtr 0$

Note

$\frac{d\ln|y|}{dy} = \frac{d\ln(\pm y)}{dy}$

$= \frac{d\ln|y|}{d|y|}\frac{d|y|}{dy}$

$= \frac{1}{|y|}(\pm 1)$

$= \frac{1}{\pm y}$

$\ln|y|\Big|_{y_1}^{y} = \int \frac{dy}{\pm y}$

$\frac{d\Delta x}{\pm\Delta x} = \pm C d\tau$

$\ln|\Delta x|\Big|_{\Delta x_1}^{\Delta x} = \pm C d\tau$

$\left|\frac{\Delta x}{\Delta x_1}\right| = e^{\pm C\tau}$

$\Delta x = \Delta x_1 e^{\pm C}$

$x_0 = 1$

what happens

at $x = 0$

$\dot{x} = -\sqrt{\frac{1}{0}}$

$= -\infty$

This ansatz is ~~not help~~

indeterminate

$\Delta x_a = 1$, $\Delta\dot{x} = \sqrt{\frac{3}{2}\Delta x^2 - \Delta x^3 + \ldots}$ $\dot{x} =$

$= \sqrt{\frac{1}{2}\Delta x^2} = \sqrt{\frac{1}{2}}$ strong change

$\ln\frac{1}{\Delta x_1} = \sqrt{\frac{3}{2}}\tau_a$

$\tau_a = \sqrt{\frac{2}{3}}\ln\left(\frac{1}{\Delta x_1}\right) > 0$

since $\Delta x_1 \ll 1$

But rather crudely

$x = 1 + \Delta x_1 e^{(\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}})\tau}$

$x = 1 + \Delta x_1 e^{\frac{1}{\sqrt{2}}\tau}\left[1 + g \tanh\left(\frac{\tau}{g}\right)\right]$

so $\tau \ll g$

$\sqrt{\frac{3}{2}}\tau$ exponential

and at $\tau \gg g$

$e^{\frac{1}{\sqrt{2}}}[1+g]$

asymptotically

$\Delta x = \sqrt{\frac{1}{2}\Delta x^2}$

$\Delta\dot{x} = \sqrt{\frac{1}{2}}\,x$

$x = x_{asym}\, e^{\sqrt{\frac{1}{2}}\tau}$

So an interpolation formula

hard to know what this will without a numerically calculation.

$x = 1 + \Delta x_1 e^{\frac{1}{\sqrt{2}}\tau}\left[1 + g\,\tau_a \tanh\left(\frac{\tau}{\tau_a}\right)\right]$

$x_{asym} = \Delta x_1 [1 + g\tau_a]$

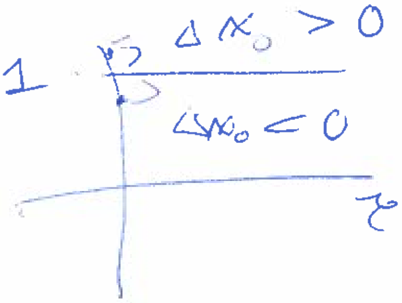
Solving $\Delta \dot{x} = \pm\sqrt{\frac{3}{2}}\,(\pm \Delta x)$

uncorrelated

± logarithmic slope or e-folding constant

± for sign of perturbation

$\sqrt{x^2} = \pm \Delta x$ must be +ve



$$\frac{d\Delta x}{\pm \Delta x} = \pm\sqrt{\frac{3}{2}}\,d\tau$$

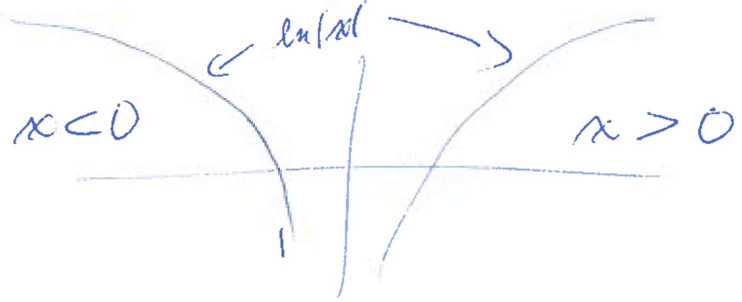
$$\pm\,\frac{d(\pm\Delta x)}{\pm \Delta x} = \pm\sqrt{\frac{3}{2}}\,d\tau \quad \text{still uncorrel.}$$

The d quantity and quantity must be the same for integration

[Note $\dfrac{d\ln|x|}{dx} = \dfrac{d\ln|x|}{d|x|}\dfrac{d|x|}{dt}$ using chain rule

$$= \frac{1}{|x|}(\pm 1)$$ upper case for $x>0$, lower case $x<0$

$$\frac{d\ln|x|}{dx} = \frac{1}{x}$$



]

$$\therefore \quad \ln\left|\frac{\Delta x}{\Delta x_0}\right| = (\pm)(\pm)\sqrt{\frac{3}{2}}\,\tau$$

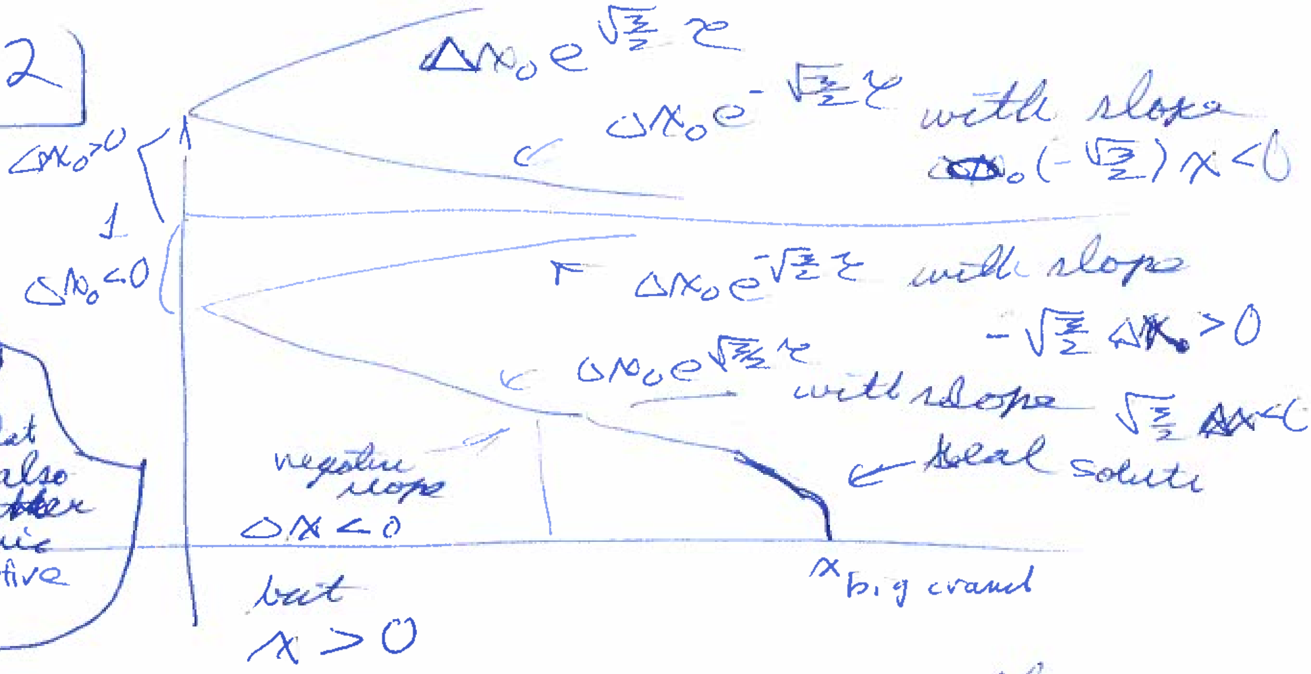
$(\pm 1)(\pm 1) = \pm$ if uncorrel.

$$\pm\Delta x = \pm\Delta x_0\, e^{\pm\sqrt{\frac{3}{2}}\tau}$$

$$\Delta x = \Delta x_0\, e^{\pm\sqrt{\frac{3}{2}}\tau}$$

5052

...s
...e
...gued
...p. 5036-5038
...e notion that
...t the Δx_0 also
...ts the whether
the logarithmic
slope is positive
or negative



As we proved on p. 5048, the Einstein parameters have only one static/stationary x.

∴ therefore the ~~solution~~ full solution for $\Delta x_0 < 0$ with $\Delta x_0 e^{\sqrt{\frac{\Xi}{2}}\tau}$ -ve slope (not just small $\Delta x_0 e$ solution) must keep getting smaller and then?

Not Δx

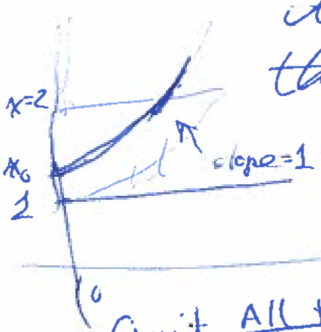
$$\dot{x} = -\sqrt{\frac{1}{x} - \frac{3}{3} + \frac{1}{2}\Lambda^2} \simeq -\frac{1}{\sqrt{x}}$$

at $x \to 0$.

∴ ~~also~~ as $x \to 0$, the slope goes to $-\infty$ and the solution crashes vertically into the $x = 0$ line

7) Lemaître-Eddington Universe (5053

a) Not the Lemaître universe but it starts from a perturbation from the Einstein universe, then expands exponentially. See p 5057 for history.



$$\dot{x} = 1 + \Delta\dot{x} = \sqrt{x^{-1} - \tfrac{3}{2} + \tfrac{1}{2}x^2} \quad \text{(see p 5049)}$$

$$\dot{x}(x=2) = \sqrt{\tfrac{1}{2} - \tfrac{3}{2} + 2} = 1$$

$\Delta x = 1$

Omit All to be revised to p.5061

small Δx

$$\dot{x} = 1 + \Delta x$$

$$\Delta\dot{x} = \sqrt{\tfrac{3}{2}\Delta x^2 - \Delta x^3 + \Delta x^4 \ldots}$$

See p.5049

$$\Delta\dot{x} = \sqrt{\tfrac{3}{2}\Delta x^2 - \Delta x^3\left(\tfrac{1}{1+\Delta x}\right)}$$

Lowest order

Ant 279

$$\Delta x = \Delta x_0 e^{\sqrt{\frac{3}{2}}\tau} \approx \Delta x_0(1+\sqrt{\tfrac{3}{2}}\tau)$$

feed in

$$\frac{\Delta\dot{x}}{\sqrt{\frac{3}{2}}\Delta x} = \sqrt{1 - \frac{2}{3}\frac{\Delta x}{1+\Delta x}}$$

$$= \left[1 - \frac{1}{3}\frac{\Delta x_0(1+\sqrt{\frac{3}{2}}\tau)}{(1+\Delta x_0 \ldots)}\right]$$

$$\Delta x_0(1+\sqrt{\tfrac{3}{2}}\tau)(1-\Delta x_0(\ \ )]$$

$$\Delta x_0[1-\Delta x_0 + (\sqrt{\tfrac{3}{2}})(1-\Delta x_0)\tau]$$

$$\ln\left(\frac{\Delta x}{\Delta x_0}\right) = \sqrt{\tfrac{3}{2}}\left[\ \tfrac{1}{3}\Delta x_0(1-\Delta x_0)\tau\right.$$

$$\left. + \frac{\Delta x_0}{3}\sqrt{\tfrac{3}{2}}(1-\Delta x_0)\frac{\tau^2}{2} + \ldots\right]$$

Too finicky

So an of order Δx_0 term with τ^2

$$\Delta x = \Delta x_0 e^{\tau A}$$

$$A = \sqrt{\tfrac{3}{2} - \tfrac{\Delta x_0}{1+\Delta x_0}}$$

exact at $\tau = 0$ for $\Delta x_0 < 1$ beyond which series diverges

large Δx

i) $\dot{x} = \sqrt{-\tfrac{3}{2} + \tfrac{1}{2}x^2}$

$$\frac{dx}{\sqrt{\frac{1}{2}x^2 - \frac{3}{2}}} = d\tau \quad \text{Hu-6, eq (45)}$$

$$\frac{1}{\sqrt{1/2}}\ln\left(\sqrt{\tfrac{1}{2}}x + \sqrt{\tfrac{1}{2}x^2 - \tfrac{3}{2}}\right)\Big|^{x}_{x_a} = \tau \quad \text{(Hu-6 eq (45)}$$

$\leftarrow x_a$ not x_0 asymptotic value

$$\sqrt{\tfrac{1}{2}}x + \sqrt{\tfrac{1}{2}x^2 - \tfrac{3}{2}} = Ce^{\tau}$$

– a quadratic and so can be solved for $x(\tau)$, but seems unpromising. You ~~will not get an~~ analytic solution ~~for x_a~~ or ~~any obvious~~ simple ~~approximate~~ ...

ii) $\dot{x} = \sqrt{\tfrac{1}{2}}x$

$$x = x_a e^{\sqrt{\frac{1}{2}}\tau}$$

$$(1+\Delta x) = (1+\Delta x_a)e^{\sqrt{\frac{1}{2}}\tau}$$

all true but wrong path?

feed in

iii) $\dot{x} = \sqrt{\tfrac{1}{2}x^2 - \tfrac{3}{2}}$

$$\dot{x} = \sqrt{\tfrac{1}{2}}x\sqrt{1 - \frac{3}{2}\frac{1}{\frac{1}{2}x_a^2}e^{-\tau/2}}$$

$$= \sqrt{\tfrac{1}{2}}x\left(1 - \frac{3}{2}\frac{1}{x_a^2}e^{-\tau/2}\right)$$

integrate

$$\ln\frac{x}{x_a} = \sqrt{\tfrac{1}{2}}\left(\tau + 3\frac{1}{x_a^2}e^{-\tau/2}\right)$$

$$x = x_a e^{\sqrt{\frac{1}{2}}\left[\tau + \frac{3}{x_a^2}e^{-\tau/2}\right]}$$

starts above x_a at $1+\Delta x_0$

x(τ) = x_0 ... But ... No ...

054

So $\Delta x(\tau=0) = \Delta x_0 A$ which is exactly right.

Ansatz $\Delta x = \Delta x_0 e^{\left[\sqrt{\frac{1}{2}}\tau + B\tau_a f\left(\frac{\tau}{\tau_a}\right)\right]}$

where $B = A - \frac{1}{\sqrt{2}} \hat{=} 1.2 - .7 = .5$

$B = \sqrt{\frac{3}{2} - \frac{\Delta x_0}{1+\Delta x_0}} - \frac{1}{\sqrt{2}}$

$\Delta x_a = \Delta x_0 e^{B\tau_a}$

and $f(z) = \begin{cases} z & \text{for } z \ll 1 \\ 1 & \text{for } z \gg 1 \end{cases}$

$\frac{z}{1+z} \approx \begin{cases} z - z^2, & z \ll 1 \\ 1 - \frac{1}{z}, & z \gg 1 \end{cases}$

lacks the exponential decline to asymptotic one

$1 - e^{-z} = \begin{cases} \sum_{l=1}^{\infty} (-1)^{l-1} \frac{z^l}{l!} \\ = z - \frac{z^2}{2} \text{ for } z \ll 1 \\ 1 - e^{-z} \text{ for } z \gg 1 \end{cases}$

Has a τ^2 term

it simpler than tanh

$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ so pure odd

$= \begin{cases} z - \frac{z^3}{3} + \frac{2}{15} z^5 & z \ll 1 \\ \frac{1 - e^{-2z}}{1 + e^{-2z}} \hat{=} 1 - 2e^{-2z} & \text{for } z \gg 1 \end{cases}$

But look all powers of τ occur in the small τ expansion and one can hope τ_a given them something like the right ripple in overall behavior.

$\therefore f(z) = 1 - e^{-z}$ is the choice.

then

$1 + \Delta x_0 = x(\tau = 0) = x_a e^{\sqrt{\frac{1}{2}}\left[0 + \frac{3}{x_a^2}\right]}$

$x_a = (1 + \Delta x_0) e^{-\sqrt{\frac{1}{2}}\left[3/x_a^2\right]}$

Maybe tiny Δx_0 but even $\Delta x_0 = 0$

$x_a = 1 - \sqrt{\frac{1}{2}} \frac{3}{\Delta x_a^2}$

Can't expand $x_a < 1 + \Delta x_0$ $e^{+\ldots}$ and so $\frac{1}{\Delta x_a}$ is not small

$x_a = e^{-\sqrt{\frac{1}{2}} \frac{3}{x_a^2}}$

say $x_a = \ldots$

$x_a = e^{-2/x_a^2} = e^{-2}$ no

$\sqrt{\frac{1}{2}} \frac{3}{x_a^2} = \ln\left[\frac{x_a}{1+x_0}\right]$

$x_a = \sqrt{\frac{\sqrt{\frac{1}{2}}\,3}{\ln\left(\frac{x_a}{1+x_0}\right)}}$

$= \frac{1+\Delta x_0}{x_a}$

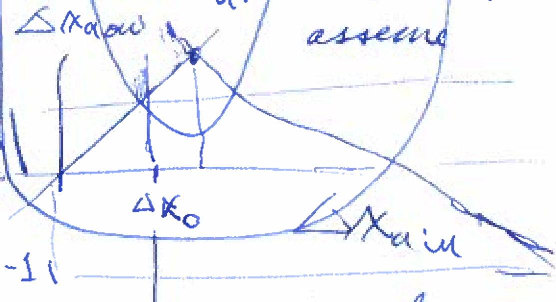
$\sqrt{\frac{1}{2}}\left[\frac{3}{x_a^2}\right] = \ln\left[\frac{x_a}{1+\Delta x_0}\right]$

$x_a = \sqrt{\frac{\sqrt{\frac{1}{2}}\,3}{\ln\left(\frac{x_a}{1+\Delta x_0}\right)}}$

$\Delta x_a = -1 + \sqrt{\frac{\sqrt{\frac{1}{2}}\,3}{\ln\left(\frac{1+\Delta x_a}{1+\Delta x_0}\right)}}$

Note there is no way to build slow initial phase in this solution (see p. 5060)

$\Delta x_a \geq \Delta x_0$ did assume



So one can get a solution for Δx_a and x_a numerically, but the fit would not have right slope at $\tau = 0$ and in klutz x

Can we find an interpolation formula that spans early to late time? — 5055

b) Interpolation formula

$$x = 1 + \Delta x_1 e^{\left[\sqrt{\frac{1}{2}}\tau + B\tau_a f\left(\frac{\tau}{\tau_a}\right)\right]} = \begin{cases} 1+\Delta x_1 e^{A\tau} & \tau \text{ small} \\ 1+\Delta x_a e^{\sqrt{\frac{3}{2}}\tau} & \tau \text{ large} \end{cases}$$

where $B = A - \sqrt{\frac{1}{2}} = \sqrt{\frac{3}{2}}\,(1 - \sqrt{\frac{1}{2}})$ $= 1.2(.3) = .5$

$A = \sqrt{\frac{3}{2}}\sqrt{1-\ldots}$

$B = A - \sqrt{\frac{1}{2}} \triangleq \sqrt{\frac{3}{2}} - \ldots = 1.2 - \ldots = .5$

and $f(z) = \begin{cases} z & \text{or } z \to 0 \\ 1 & \text{or } z \to \infty \end{cases}$

$\frac{z}{1+z}$ one idea $\cong z(1-z+z^2\ldots) = z - z^2$

$1-e^{-z}$ another $= z - \frac{z^2}{2}$

$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ ← pure odd / pure even

↑ pure odd

so no z^2 term but

$e^{\sqrt{3/2}\,\tau} \triangleq 1 + \sqrt{\frac{3}{2}}\tau + \frac{(\sqrt{3/2}\,\tau)^2}{2!}$

has a negative coefficient τ like first time

and so a priori $\tanh(z)$ has a defect and I exclude it

I can't even be perfectly right to order τ^2 in small $\tau \ll 1$

τ_a is the time scale of transition which must happen over $\Delta x \triangleq 1$ or $x = 2$ since the expansion in small Δx

$$\frac{1}{1+\Delta x} = \sum_{\ell=0}^{\infty}(-\Delta x)^{\ell}$$

fails for $\Delta x = 1$ (Arf–279 and see p. 5049)

And x_a is the asymptotic coefficient

We have

$$\Delta x_a = \Delta x_1 e^{B\tau_a}$$

comparing the large τ form and the interpolation formula

alas 1 equation and 2 unknowns to fit for.

(But turns out this is not a good idea (see p. 5056) specious)

(Not so good see p. 5054)

One could guess $\Delta x_a \triangleq 1$ the deduced transition point between small and large $\tau_a = \frac{1}{B}\ln\left(\frac{1}{\Delta x_1}\right) \triangleq 2\ln\left(\frac{1}{\Delta x_1}\right)$

505[illegible]

However the transition occurs near ~~for~~ $\Delta x = 1$, $x = 2$, $\Delta x = 2$ — exactly is defined as ... when

$\Delta x_a = \Delta x_0 e^{[B\tau_a]}$

since no τ dependence in this factor

$\Delta x_0 = \Delta x_0 e^{\ln(\frac{1}{\Delta x_0})}$ see p. D057

$= (\Delta x_0)^{-1} e$

$$\dot{x} = \sqrt{\frac{1}{2} - \frac{3}{2} + \frac{4}{2}} = 1$$

$$\dot{x} = \Delta\dot{x} = 1 = \Delta x_p e^{[\ldots]} \left[\frac{1}{\sqrt{2}} + B\tau_0 \frac{df}{dz}\frac{1}{\tau_a}\right]$$

Omit algebra

$$\left[\sqrt{\tfrac{1}{2}}\, z + B f(z)\right]\tau_a = \ln\left(\frac{1}{\Delta x_p}\right)$$

derivative of the exponent

this we fit to 1

(where $z = \frac{\tau}{\tau_a}$)

Then this must also be 1

$$1 = \left[\frac{1}{\sqrt{2}} + B\frac{df}{dz}\right]$$

$$f = \frac{z}{1+z}$$

$$\frac{df}{dz} = \frac{1}{1+z} - \frac{z}{(1+z)^2} = \frac{1}{(1+z)^2}$$

$$1 = \frac{1}{\sqrt{2}} + B\frac{1}{(1+z)^2}$$

$$\frac{B}{(1+z)^2} = 1 - \frac{1}{\sqrt{2}}$$

$$1+z = \sqrt{\frac{B}{1-\frac{1}{\sqrt{2}}}}$$

$$z = \sqrt{\frac{B}{1-\frac{1}{\sqrt{2}}}} - 1 \simeq \sqrt{\frac{.5}{.3}} - 1 \simeq 1.3 - 1 \simeq 0.3$$

$$f = 1 - e^{-z}$$

$$\frac{df}{dz} = e^{-z}$$

$$1 = \frac{1}{\sqrt{2}} + Be^{-z}$$

$$Be^{-z} = 1 - \frac{1}{\sqrt{2}}$$

$$e^{z} = \frac{B}{1-\frac{1}{\sqrt{2}}}$$

$$z = \ln\left(\frac{B}{1-\frac{1}{\sqrt{2}}}\right) \simeq \ln\left(\frac{B}{.3}\right) \simeq \ln(3B) \simeq \ln(1.5) \simeq .4$$

← very similar

$B = \sqrt{\frac{3}{2} - \frac{\Delta x}{1+z}} - \frac{1}{\sqrt{2}}$

$B_{max} = \sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}} = .5$

$B_{min} = \sqrt{1} - \sqrt{.5} = .3$

Clark - 30

[5059]

$$f(z) = \frac{z}{1+z} = \frac{0.3}{1.3} \simeq 0.25$$

$$f = 1 - e^{-z} = 1 - \frac{1}{e^z} \simeq 1 - \frac{1}{3B} = 1 - \frac{1}{1.5} \simeq \frac{1}{3}$$

$$f = 1 - e^{-z} = 1 - \frac{B}{1-\sqrt{2}}$$

rather similar
Not much distinction

$\tau_a = 0$ or 1 for $\Delta x_0 = 1$ and -ve for $\Delta x_0 > 1$ and so $\Delta x_0 < 1$ for ...

$$\ln\left(\frac{1}{\Delta x_i}\right) = \left[\sqrt{\tfrac{1}{2}}\, z + B f(z)\right] \tau_a$$

$$\tau_a = \frac{1}{\left[\sqrt{\tfrac{1}{2}}\, z + B f(z)\right]} \ln\left(\frac{1}{\Delta x_0}\right) \text{ for general } f.$$

$$= \frac{1}{\left[\sqrt{\tfrac{1}{2}}\, z + B - (1+\sqrt{2})\right]} \ln\left(\frac{1}{\Delta x_0}\right)$$

Neither ⇒ $\tau_a = \ln\left(\frac{1}{\Delta x_0}\right) / B$

$$\tau_a = \frac{\ln\left(\frac{1}{\Delta x_0}\right)}{\sqrt{\tfrac{1}{2}}\, z + B\left(\frac{z}{1+z}\right)} \simeq \frac{\ln\left(\frac{1}{\Delta x_0}\right)}{0.2 + .5\,\frac{1}{4}} \simeq \frac{\ln\left(\frac{1}{\Delta x_0}\right)}{0.3} \simeq 3 \ln\left(\frac{1}{\Delta x_0}\right)$$

$$\tau_a = \frac{\ln\left(\frac{1}{\Delta x_0}\right)}{\left[\sqrt{\tfrac{1}{2}}\, z + B(1 - e^{-z})\right]} = C \ln\left(\frac{1}{\Delta x_0}\right) = \frac{\ln\left(\frac{1}{\Delta x_1}\right)}{0.3 + .5(1/3)} = 2 \ln\left(\frac{1}{\Delta x_1}\right)$$

so again rather similar

$$\tau = z\tau_a = 0.7 \ln\left(\frac{1}{\Delta x_0}\right) \qquad \tau = z\tau_a = 0.8 \ln\left(\frac{1}{\Delta x_0}\right)$$

and so our guess on p. 5053 wasn't so good

$$\Delta x_a = \Delta x_0 e^{C \ln\left(\frac{1}{\Delta x_i}\right)} \quad \text{taken } \xi = \frac{B}{\ldots} = \frac{.5}{0.4} = 1.2$$

$= e^{\ldots} = 3.5$ which is rather bigger than guess of 2.

405 8

$\Delta x_a = \Delta x_0 e^{B \tau_a} = \Delta x_0 e^{BC \ln(\frac{1}{\Delta x_0})} = \frac{1}{\Delta x_0^{BC}} \doteq 1$

$= \Delta x_0 e^{~~2 \ln(\Delta x_0)~~}$

But

$\Delta x_a = \Delta x_0 \left(\frac{1}{\Delta x_0}\right)^{BC} \leq \Delta x_0^{1-C}$

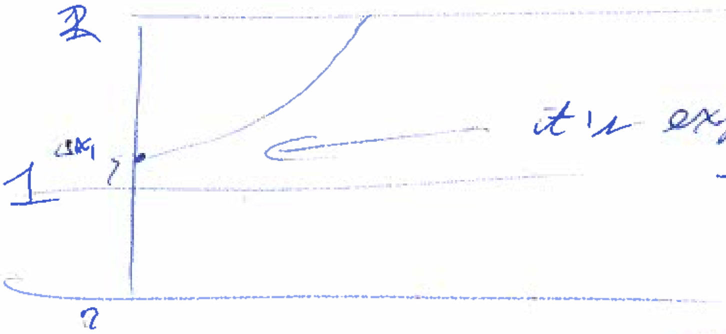
$= \frac{1}{\Delta x_0^{\ddagger}}$

which seems absurd.

~~At~~ least I ~~thought~~ it was ~~absurd~~ absurd.

But $\frac{1}{x_a} \to \infty$

$x_1 \to 0$, it ... but ... static solution maybe right, must be right



it's exponential growth – the smaller Δx_1 ~~the longer the~~ growth phase.

But $x_a = \Delta x_1 e^{B\tau_a}$

$x_a = \Delta x_1 e^{B\left(\frac{\ln(\frac{1}{\Delta x_1})}{B}\right)}$

$= 1$

This was my original idea

but I was supposing $x_a \doteq 1$ was a given

~~then $f(\ldots) = \frac{1}{B} \ln(\frac{1}{\Delta x_1})$~~

$f(z \to \infty) = \frac{1}{B} \ln(\frac{1}{\Delta}$

then $\tau_a = \frac{\ln(\frac{1}{\Delta x_1})}{B}$

In fact specious since $\Delta x_a \sim \frac{1}{\Delta x}$ ...

a useful ansatz but not a best bit.

If I've done all algebra correctly, then the interpolation formula

$x = 1 + \Delta x_1 e^{[\ldots \tau + B \tau_a f(\tau/\tau_a)]}$

has the exactly right solope at $\Delta x = \Delta x_1$

So it's probably not so bad.

$\tau_a \sim 2.4 \ln(\frac{1}{\Delta x})$

$\tau_2 = 0.7 \ln(\frac{1}{\Delta x})$

$\Delta x_a \leq \frac{1}{\Delta x^{1/4}}$

$\Delta x = 1$, $x = 2$

$x \to \infty$

$$\Delta X_a = \Delta X_0 \, e^{B\tau_a} = \Delta X_0 \, e^{BC \ln\left(\frac{1}{\Delta X_0}\right)}$$

505?

since $\tau \to \infty$
and $f(\tau/\tau_a) \to 1$

$$\Delta X_a = \Delta X_0^{1-BC}$$

$\tau_a \doteq 3 \ln\left(\frac{1}{\Delta X_0}\right)$ | $\tau_a = 2 \ln\left(\frac{1}{\Delta X_0}\right)$

and $B \doteq .5$

$$\Delta X_a = \Delta X_0^{1-1.5} = \Delta X_0^{-1/2}$$

$$\Delta X_a \doteq \Delta X_0^{1-1} \doteq 1$$

Is this result exact

$$B\tau_a = \frac{\sqrt{\frac{3}{2} - \frac{\Delta X_0}{1+\Delta X_0}} - \sqrt{\frac{1}{2}}}{\left[\sqrt{\frac{1}{2}}\, z + B(1-e^{-z})\right]} \ln\left(\frac{1}{\Delta X_0}\right)$$

Nah, but if $z \doteq 0$
$BC = 1$, but
$z = .4$

$$\frac{B\tau_a}{\ln\left(\frac{1}{\Delta X_0}\right)} = \frac{.5}{[.3 + .5/3]} \doteq$$

$$\frac{B\tau_a}{\ln\left(\frac{1}{\Delta X_0}\right)} = \frac{B}{\sqrt{\frac{1}{2}} \ln\left(\frac{B}{1-\sqrt{\frac{1}{2}}}\right) + B\left(1 - \frac{1-\sqrt{\frac{1}{2}}}{B}\right)}$$

$1 - \frac{.3}{.5}$

$$\frac{B\tau_a}{\ln\left(\frac{1}{\Delta X_0}\right)} = \frac{B}{\sqrt{\frac{1}{2}}\, z + Bz}$$

if $z \ll 1$

$$= \frac{B}{(\sqrt{.5} + B) z} = \frac{B}{(.7 + B)\, .4} = \frac{.5}{.5} = 1$$

Note the p. 505 ≥ asymptotic approach to the asymptote has $e^{-\tau/2}$ and the interpolation formula

$$e^{-\tau/\tau_a} = e^{-\frac{\tau}{c \ln\left(\frac{1}{\Delta X_0}\right)}}$$

$c = 2$ for the exponential $f = 1 - e^{-z}$

So they are different and no simple way to bring to agreement.

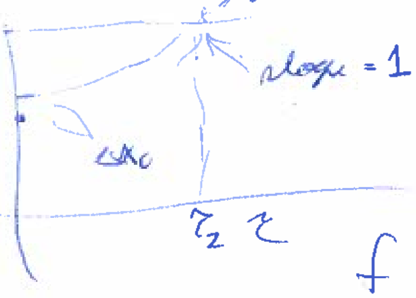
so just a lucky coincidence

5060

So to summarize we use interpolation formula

$$\Delta x = \Delta x_0 e^{\left[\frac{1}{\sqrt{2}}\tau + B\tau_a f(\tau/\tau_a)\right]}$$

where $B = \sqrt{\frac{3}{2} - \frac{\Delta x_0}{1+\Delta x_0}} - \frac{1}{\sqrt{2}} \doteq 1.2 - .7 \doteq .5$



$= \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}$ fiducial value $\Delta x_0 \to 0$

$= .5176\ldots$

$$f = 1 - e^{-z} = \begin{cases} \sum_{l=1}^{\infty} (-1)^{l-1} \frac{z^l}{l!} & \text{series} \\ z - \frac{z^2}{2} & \text{small} \\ 1 - e^{-z} & \text{large} \end{cases}$$

at $x = 2$, $\Delta x = 1$

$\Delta \dot{x} = 1$

at $z = z_2$, $\tau = \tau_2$ where $x = 2$

$$f = 1 - e^{-z_2} = 1 - \frac{1 - \sqrt{\frac{1}{2}}}{B} \simeq 1 - \frac{.3}{.5}$$

$= .43417\ldots$ fiducial $= 0.4$

$$z_2 = \ln\left(\frac{B}{1 - \sqrt{\frac{1}{2}}}\right) = \ln(3B) = \ln(1.5) \doteq 0.4$$

$= .5694\ldots$ fiducial

Note the initial growth will be small if you make by making Δx_0 smaller and smaller.

$$\tau_a = \frac{\ln\left(\frac{1}{\Delta x_0}\right)}{\left[\frac{1}{\sqrt{2}} z_2 + B(1 - e^{-z_2})\right]} = \begin{cases} 2 \ln\left(\frac{1}{\Delta x_0}\right) \\ 1.5938\ldots \text{ fiducial} \end{cases}$$

$= \ln(1/\Delta x_0) / \left[\frac{1}{\sqrt{2}} z_2 + B - 1 + \sqrt{\frac{1}{2}}\right] \to \frac{\ln(1/\Delta x_0)}{\ln\left(\frac{B}{\ldots}\right) + B - 1 + \sqrt{\frac{1}{2}}}$

$$\tau_2 = z_2 \tau_a \quad \frac{\ln\left(\frac{B}{1-\sqrt{\frac{1}{2}}}\right) \ln\left(\frac{1}{\Delta x_0}\right)}{\left[\sqrt{\frac{1}{2}} + B(1 - e^{-z})\right]} = \begin{cases} 0.8 \ln\left(\frac{1}{\Delta x_0}\right) \\ 0.9076\ldots \text{ fiducial} \end{cases}$$

$$\Delta x_a = \Delta x_0 e^{B\tau_a} = \Delta x_0 e^{c \ln\left(\frac{1}{\Delta x_0}\right)} = \Delta x_0^{1-c} = \begin{cases} 1 \\ 0.8250\ldots \text{ fiducial} \end{cases}$$

$1 - c \simeq \begin{cases} 0 \\ 0.17497\ldots \end{cases}$

So a rather weak power law

$x_a = 1 + \Delta x_a \doteq 2$

$\Delta x_0 \ll \Delta x \leq 1$

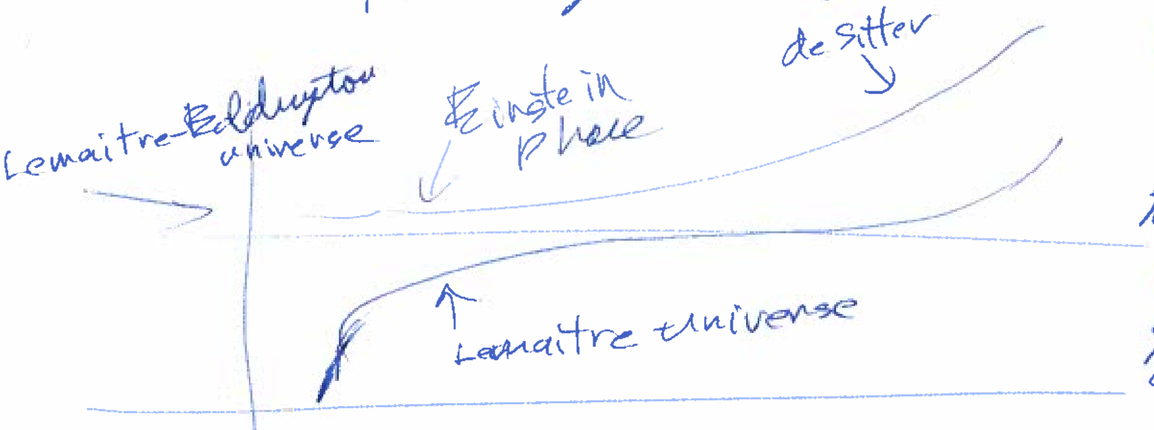
$\Rightarrow \Delta x_0 \ll 1$ by hypothesis.

reversed factor 5053

~~Actually~~, ~~the~~ the Lemaitre-Eddington universe (with Λ approximated by $\Lambda(\tau)$ on p. 5056) is of historical interest

Lemaître presented it in 1925 but did not favor it. Eddington did

1925 – 1935 it had a bit of vogue — since there were only about 10 cosmologists then, a vogue means probably Eddington and his best friend



Why did Eddington like it? Well the long slow growth phase allowed collapses to make galaxies and avoided the age problem of that time (see p. 5043)

It avoided any point origin, and ~~therefore~~ left the primordial universe, if there was one, an avoided problem

Of course, Lemaître wanted ~~the~~ ~~point out~~ primordial universe of high density

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to create the elements including the radioactive elements that he believed powered stars (wrongly) and geologically heat (rightly). {About source of ~ $\frac{1}{2}$ the heat, but the radioisotopes come from stars & supernovae, not a primeval atom)

See Bondi 84, 85, 117-119, 121, 175

for the Lemaitre-Eddington universe.

autonomous, no explicit dependence on independent variable

8) (1st Order DEs which are autonomous | 5063

But not required to be linear

and stationary Points and static solutions

Rule's stationary points only at ±∞ and those correspond to static solutions but there exceptions

assume x is infinitely differentiable too

and one at least is very important and turns up late in physics including the Friedmann equation

Say we have

$$x' = f(x) \quad \text{where } x = x(t)$$

and f is just a function x not derivatives of x, so the DE is 1st order

a 1st order (ordinary not partial) DE

but it can be non linear

Say x_s is a stationary point of x and t_s, $x_s = x(t_s)$ ∴ $x'(x_s) = f(x_s) = 0$

d) n=2 unit l=0 of Leibniz Rule $x'' = \frac{df}{dx} x'$ chain rule $= 0$ at x_s

$$x''' = \frac{d^2f}{dx^2} x'^2 + \frac{df}{dx} x''$$ (step 1 of proof by induction) since $x'(x_s) = 0$ usually. But there are exceptions as we've said.

$= 0$ at x_s

and if you do a formal proof by induction, you always generate $x^{(i)}$ $i < n$ new x' or $x^{(i)}$ for derivative $x^{(n)}$ and no $x^{(n)}$ ...

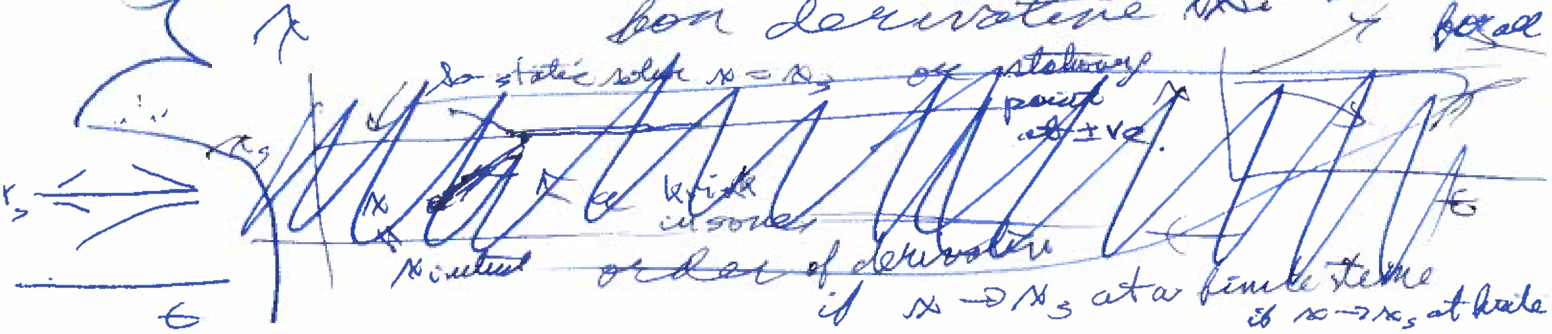
if all $x^{(n)}(t_s) = 0$, then $x(t)$ is flat or asymptotically flat at $t = \pm\infty$

So static soln $x = x_s$ or stationary point at ±ve

write as order of derivative

if $x \to x_s$ at a finite time

if $x \to x_s$ at finite



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expand $f(x) = f(x_s) + (x - x_s)\left(\frac{df}{dx}\right)_s + \ldots$

$f(x_s) = 0$

to 1st order in small perturbation Δx_0

$\frac{d\Delta x}{dt} = \Delta x \left(\frac{df}{dx}\right)_{x_s}$

assume non-zero

$\frac{d\Delta x}{\pm \Delta x} = \left(\frac{df}{dx}\right)_{x_s} dt$ { +ve for $\Delta x_0 > 0$; −ve for $\Delta x_0 < 0$

$\ln|\Delta x/\Delta x_0| = \pm \left(\frac{df}{dx}\right)_{x_s} t$

$\Delta x = \Delta x_0 e^{\pm \left(\frac{df}{dx}\right)_{x_s} t}$

There none same overall +ve

$d\ln|x| = \frac{dx}{\pm x}$

Say $\left(\frac{df}{dx}\right)_{x_s} > 0$, then Δx_0 +ve leads to divergence, Δx −ve lead to converge

Say $\left(\frac{df}{dx}\right)_{x_s} < 0$, then Δx_0 +ve lead to convergence, Δx_0 −ve leads to divergence

slope always positive

~~What if $\left(\frac{df}{dx}\right)_s$ trickier cases arise will divergence to 0 at finite t,~~

Proof by induction.

$x^{(n)} = \frac{d^{n-1}}{dx^{n-1}}(x')^n + \frac{d}{dx}(x')^{n-2}$ — 2 order leibniz rule

$x^{(n-1)}$ ← highest order on RHS

all other terms are less than $x^{(i \le n-1)}$ clearly. assume all are zero at $x = x_s$ step 2 proof by induction.

then $x^{(n)}(x_s) = 0$ for all n QED of proof by induction.

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To understand the behavior for small perturbations from the static solution, expand $f(x)$ around x_s

$$x' = f(x_s) + \Delta x \left(\frac{\partial f}{\partial x}\right)_{x_s} + \ldots$$

O and truncate to the 1st order assuming

$$\left(\frac{\partial f}{\partial x}\right)_{x_s} \neq 0$$

then the DE becomes

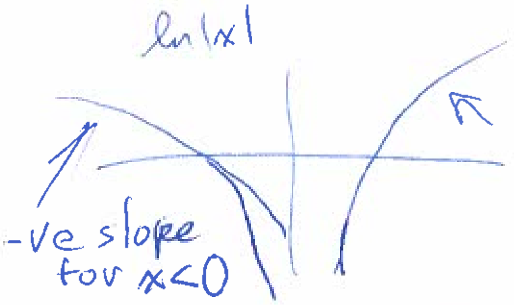
$$\Delta x' = \Delta x R \quad \text{where } R = \left(\frac{\partial f}{\partial x}\right)_{x_s}$$

$$\frac{d\Delta x}{\Delta x} = R\,dt$$

$$\ln\left|\frac{\Delta x}{\Delta x_0}\right| = Rt, \text{ where}$$

$\Delta x(t=0) = \Delta x_0$ the initial perturbation

Note $\dfrac{d \ln|x|}{dx} = \dfrac{d \ln|x|}{d|x|}\dfrac{d|x|}{dx} = \dfrac{1}{|x|}(\pm 1) = \dfrac{1}{x}$



↖ positive slope for $x > 0$

So $\int \frac{dx}{x} = \ln|x|$ is general and correct.

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$\therefore\ |\Delta x| = |\Delta x_0| e^{Rt}$

$\pm \Delta x = \pm \Delta x_0 e^{Rt}$

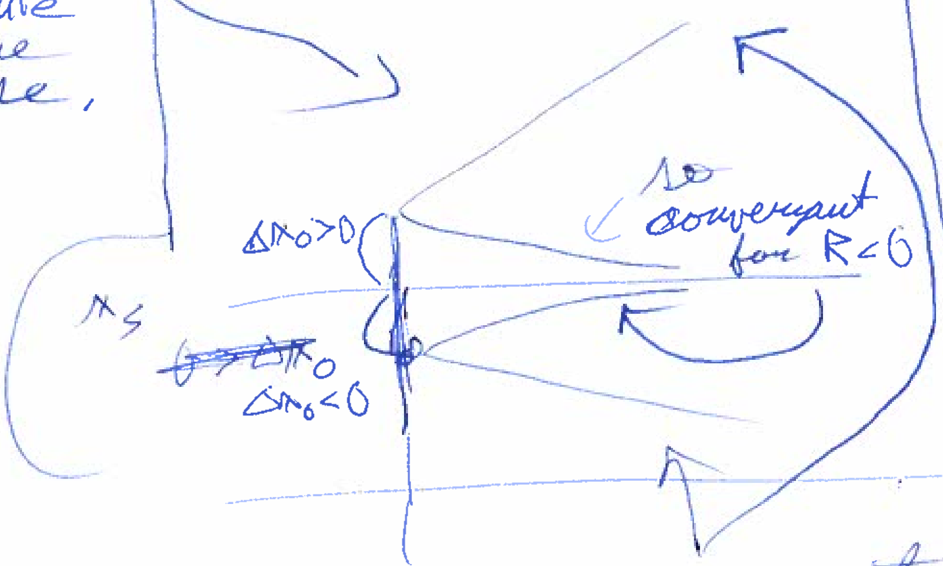
$\Delta x = \Delta x_0 e^{Rt}$

$\Delta \dot{x} =$ ~~Δx_0~~ $R \Delta x$

Absolute value decreases for $R<0$ and increases for $R>0$ with time

But it is simplest to understand using the absolute value case,

so slope is +ve if they have the same sign
-ve if the signs are different



to the
Part the other way:
convergent for $R>0$,
divergent for $R<0$.

What if $\left(\frac{df}{dx}\right)_{x_s} = 0$? Then you have to go to ~~higher or~~ the lowest nonzero derivative of $f \rightarrow \frac{d^{(n)}x}{dt^{(n)}} \neq 0$ (which is sort of messy) to decide on convergence/divergence

h) The Most Important Exception to the Rule

~~Note g(x) can go negative,~~ but we assume g[x(t)] doesn't at the stationary point

Let $x' = \sqrt{g(x)}$ where $g(x)$ is infinitely differentiable — so no singularities at least where $g(x)=0$ and t is independent variable

$x'(x_s) = 0$ since $g(x_s)=0$ and $x_s = x(t_s)$

a) $x'' = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{dg}{dx}x'$

$= \frac{1}{2}\frac{1}{\sqrt{g}}\frac{dg}{dx}\sqrt{g}$

(Minimum if $\frac{dg}{dx} > 0$; maximum if $\frac{dg}{dx} < 0$)

$= \frac{1}{2}\frac{dg}{dx}$, and so if $\left(\frac{dg}{dx}\right)_{x_s} \neq 0$ there is a stationary point at x_s $\left\{ \begin{array}{l} x(x_s)' = 0 \\ x(x_s)'' \neq 0 \end{array} \right.$

A zero over zero cancellation. You can think of it as a limiting process at $t \to t_s$ and $x \to x_s$

What about $\frac{1}{\sqrt{g}}$ singularities arising as you go to higher derivatives of x?

Every time you take a derivative of $x' = \sqrt{g}$ that appears you get a factor $\frac{1}{2}\frac{1}{\sqrt{g}}\frac{dg}{dx}x' = \frac{1}{2}\frac{dg}{dx}$

and so no $\frac{1}{0}$ cases ever arise. This is true whether $\left(\frac{dg}{dx}\right)_{x_s}$ is zero or not.

b) But can you get a static solution if $\left(\frac{dg}{dx}\right)_{x_s} = 0$?

Yes, if g contains no fractional powers of x or functions of x.

<u>First</u> $x''/_{x_s} = \frac{1}{2}\left(\frac{dg}{dx}\right)_{x_s} = 0$

so $x''(t_s) = 0$

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~~What of higher order derivative of x?~~ ~~Can you generate a nonzero term at x_s?~~ A priori, it seems unlikely

~~Well again every time you take a derivative of x' you get a factor of~~

$$x'' = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{dg}{dx}x' = \frac{1}{2}\frac{dg}{dx}$$

and $\left.\frac{dg}{dx}\right|_{x_s} = 0$

So all terms in $x'(x_s) = 0$

and $x''(x_s) = 0$

~~What of~~

Consider

$$x''' = \frac{1}{2}\frac{d^2g}{dx^2}(x')^2 + \frac{1}{2}\frac{dg}{dx}x''$$

$$x^{(n-1)} = \ldots\ldots$$

$$x^{(n)} = \ldots\ldots + \frac{1}{2}\frac{dg}{dx}x^{(n-1)}$$

if all $x^{(i)}(x_s) = 0$

$i \le n-1$

~~this last term~~ will always be zero at x_s.

~~In fact all terms will be zero~~ if they contain any $x^{(i)}$ factor

What of higher order derivatives of x: i.e. $x^{(n \geq 3)}(x_s)$ will they all equal zero and so give a static solution? Yes.

We assume $g(x)$ and $x(t)$ are both infinitely differentiable in themselves to avoid pathological points

Proof by induction

Step 1 ($n=3$)

$$\bar{x}^{(3)} = \frac{1}{2}\frac{d^2 g}{dx^2}(x')^{1}$$

($n-1$, $n-2$)

$$x^{(4)} = \frac{1}{2}\frac{d^3 g}{dx^3}(x')^2 + \frac{1}{2}\frac{d^2 g}{dx^2}x^{(2)}$$

This must be step since $x^{(4)}$ not zero if $x''(x_s) \neq$

all higher derivatives $x^{(n)}$ will have terms with factors of $x^{(i \leq n-2)}$

The trick of getting rid of them by fractional powers of x on functions of x is not available.

Step 2 assume for n ~~that~~ all $x^{(n \leq n-2)}(x_s) = 0$

Step 3

$$x^{(n)} = \frac{1}{2}\left[\frac{d^{n-1}}{dx^{n-1}}(x')^{n-2} + \dots + \frac{d^2 g}{dx^2}x^{(n-2)}\right]$$

all zero at x_s by assumption

highest order derivative of x in RHS

$\therefore$ $x^{(n)}(x_s) = 0$

QED

n is general and so that completes the proof by induction.

So static solutions occur for this case of $\frac{dg}{dx}(x_s) = 0$

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~~where dg contains no fractional powers of x or functions of x~~

Both $\frac{dg}{dx}(x_s) = \begin{cases} 0 & \text{static solution} \\ \neq 0 & \text{stationary point exist for the Friedmann equation.} \end{cases}$

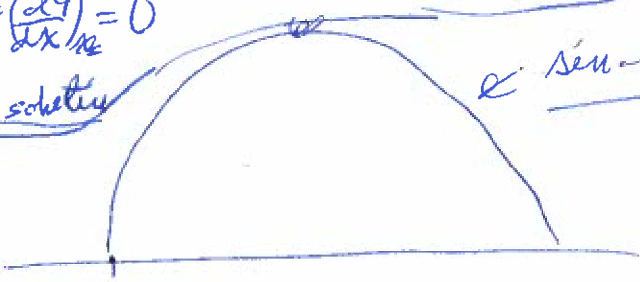
Einstein universe

and Radiation analogue Einstein universe

$g = g(x_s) + \Delta x \left(\frac{dg}{dx}\right)_s^{\nearrow 0} + \Delta x^2 \left(\frac{dg}{dx}\right)_{x_s} + \ldots$

∴ only 5036 $\frac{1}{x}\frac{dx}{dt} = \pm\sqrt{C\Delta x^2}$ ← when Δx^2

since $\left(\frac{dg}{dx}\right)_{x_s} = 0$ for static solution



← sin-like

matter +ve curvature universe

Bounce universe cosh-like

stationary point solutions

There other cases of exception to ordinary rule for 1st order (ordinary autonomous) DEs and I ~~but~~ do explicate in my never ending FE paper, but ~~they~~ are tedious and probably never physically relevant

Another example of the important exception to the rule is the orbital motion 1st order DE for orbits in the Schwarzschild metric that comes up in the General relativity course:

$$\left(\frac{dr}{d\tau}\right) = \pm\sqrt{\hat{E}^2 - \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{\hat{L}}{r^2}\right)}$$

(Not in Carroll, not obvious)

Because of square root again you can have stationary points and oscillatory solutions ⇒ physically reals as must be for orbits

The Friedmann Equation allows sine-like solutions

Note you cannot add the oscillating solution to the static solution and get a solution since FE is nonlinear

but you only get halfacycle since $x < 0$ is unphysical

⇒ the orbital equation allows physical real oscillatory solutions (r > 0 solutions) (at least so Carl tells me) unlike the FE.