Lecture 4 | 2026jan24 | 4001 L

2025 mar 24

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4.1) Geometry of Universe Intro via Friedmann Equation

a) Getting a_0, ~~scale length of universe~~

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G \rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

$k \longrightarrow kc^2$

Two ways of writing k

here we choose the kc^2 way.

~~k~~

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = H_0^2\left[\frac{8\pi G}{3H_0^2}\rho - \frac{kc^2}{H_0^2 a^2} + \frac{\Lambda}{3H_0^2}\right]$$

$$\left[\frac{\rho_M}{\rho_c} + \Omega_k + \Omega_\Lambda\right]$$

$\rho_c = \frac{3H_0^2}{8\pi G}$

Ω_M

$$\frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3H_0^2}$$

0 symbolizes the fiducial time which is usually cosmic present.

At $t = t_0$, $H = H_0$ and $\sum_p \Omega_p = 1$.

Scaled time makes H_0 ~~invisible~~; $d\tau = H_0 dt$

$$\Omega_k \equiv -\frac{kc^2}{H_0^2 a^2} = \frac{-kc^2}{H_0 a_0^2}\frac{1}{x^2} \qquad \rho_\Lambda = \frac{\Lambda}{8\pi G}$$

$= \Omega_{k,0}\frac{1}{x^2}$ where $x = \frac{a}{a_0}$

Which leads to the eternal confusion that

$k > 0$ is +ve curvature hypersphere $\quad \Omega_k < 0$

$k < 0$ is −ve curvature hyperbolic $\quad \Omega_k > 0$

$k = 0$ for flat (Euclidean) space

Now previously we defined usually

$x_0 = 1$ and also if one likes $a_0 = 1$ and dimensionless

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~~then~~ $D = a(t) D_0 = a_0 \times D_0$

proper distance at any cosmic time t

comoving distance at all cosmic time

and this is the good way of defining a and ~~~~ a_0 for most purposes $\Longrightarrow$ solving the Friedmann equation and it solutions most obviously

But for theoretical discussions of geometry, there is another way

$\Omega_{k,0}$ is determined by fit of model to cosmic expansion evolution via the Friedmann eqn. solution or otherwise (?)

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2 a_0^2} \equiv -\frac{k_g c^2}{H_0^2 a_{0g}^2 (\pm k_g)}$$

+ for +ve, − for −ve curvature

g for geometry

Demand

$$k_g = \begin{cases} 1 & \text{+ve curvature} \\ 0 & \text{0 curvature} \\ -1 & \text{−ve curvature} \end{cases}$$

dimensionless

But $\Omega_{k,0}$ is also dimensionless and so a_{0g} now ~~has a dimensions~~ has dimensions

[2025 mar 4] ['4005

$$a_{0g}^2 = \frac{c^2}{H_0^2 \underbrace{(-k_g \Omega_{k0})}_{= |\Omega_{k0}|}}$$

since k_g and Ω_{k0} have opposite signs

$$a_{0g} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{c\, t_{H_0}}{\sqrt{|\Omega_{k0}|}}$$

where t_{H_0} is Hubble time

a_{0g} is a ~~...~~ physical length in units of length, but is undefined for $\Omega_{k0} = 0$, ie, a flat geometry universe (or $a_{0g} = \infty$ if you prefer)

$$R_G \text{ (Gaussian Curvature Radius)} \equiv \frac{a_{g0}}{\sqrt{k_g}}$$

(CL-12)

→ real for hyperspherical, +ve curvature

→ imaginary for hyperbolic, -ve curvature

But we don't use R_G much

$$a_{0g} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{\overbrace{c\, t_{H_0}}^{\text{Hubble distance}}}{\sqrt{|\Omega_{k0}|}} = \frac{(4.2827\ldots)\, h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_{k,0}|}}$$

$$= \frac{(13.968\ldots)\, h_{70}^{-1}\ \text{Gly}}{\sqrt{|\Omega_{k0}|}}$$

where $h_{70} = \frac{H_0}{70} \simeq 1$ ~~which is~~ to within $\approx 4\%$

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$|\Omega_{k,0}| \lesssim 0.02$ (Wik circa 2025)

$\Omega_{k,0} = 0.0019(15)$ ACT Atacama Cos. Tel. (2025)

$\underbrace{(15)}_{1\sigma}$

so -ve curvature at face value

$\Omega_{k0} > 0$ so $K < 0$ and -ve curvature

$\Omega_{k,0} = 0.0022(15)$ Planck (2020)

about the same

~~So~~

→ So $|\Omega_{k,0\,\text{fiducial size}}| = 0.0009 = 9 \times 10^{-4}$

So $\sqrt{|\Omega_{k,0_{\text{fid}}}|} = 3 \times 10^{-2}$

Current measurements allow that Ω_{k0} could be of order this size.

$$a_{0g} = \frac{c/H_0}{\sqrt{|\Omega_{k,0}|}} \doteq \frac{1.43 \times 10^{2}\, h_{70}^{-1}\ \text{Gpc}}{\sqrt{\Omega_{k\,\text{fid}}}\ /\sqrt{\Omega_{k0\,\text{fid}}}}$$

143 Gpc fiducial estimate

(about $10 * D_{\text{observable radius}}$)

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Comparison

$D_{\text{observable universe}} = 14.25$ Gpc { With Base on ΛCDM fit

scale length of universe ?

a_0 mean?

b) ~~But what does~~ undefined and of No use.

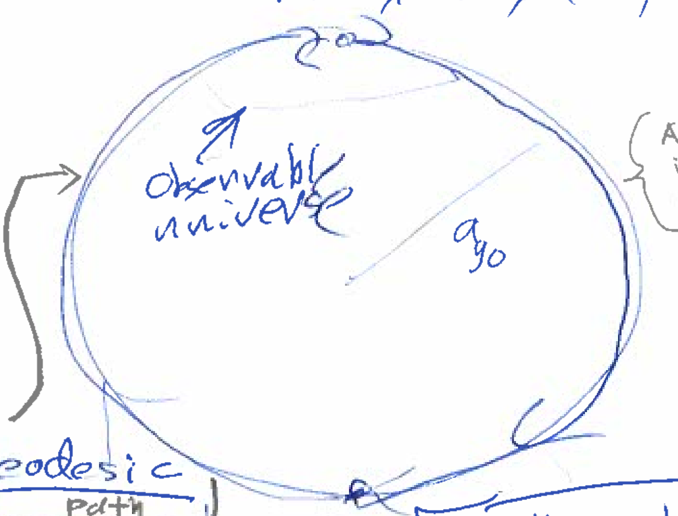
i) For Flat (Euclidean space)

ii) Easiest then to understand: hyperspherical universe (+ve curvature)

iii) ~~Boom~~ hyperbolic universe a — length scale over which you would notice hyperbolic effect (−ve curvature)

It's sort of a parameter.

Milky Way (MW)



2-d analog of 3-d curved space.

At an instant in cosmic time

D_{proper} (MW to AMW) — No proof. Just an assertion.

$= \pi a_{90} = 3 \times 143$

$= 430$ Gpc

$\gg D_{\text{observable universe}}$

See p. 4029 for proof

geodesic path

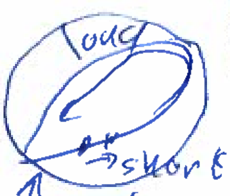
$A \quad B$

Fills whole ~~whole~~ sky

antipodal point to the Milky Way (AMW)

Nothing special → every point has an antipodal point

and so the curvature is very small on ~~our~~ observable universe scale.



great circle

locally shortest distance between points in space

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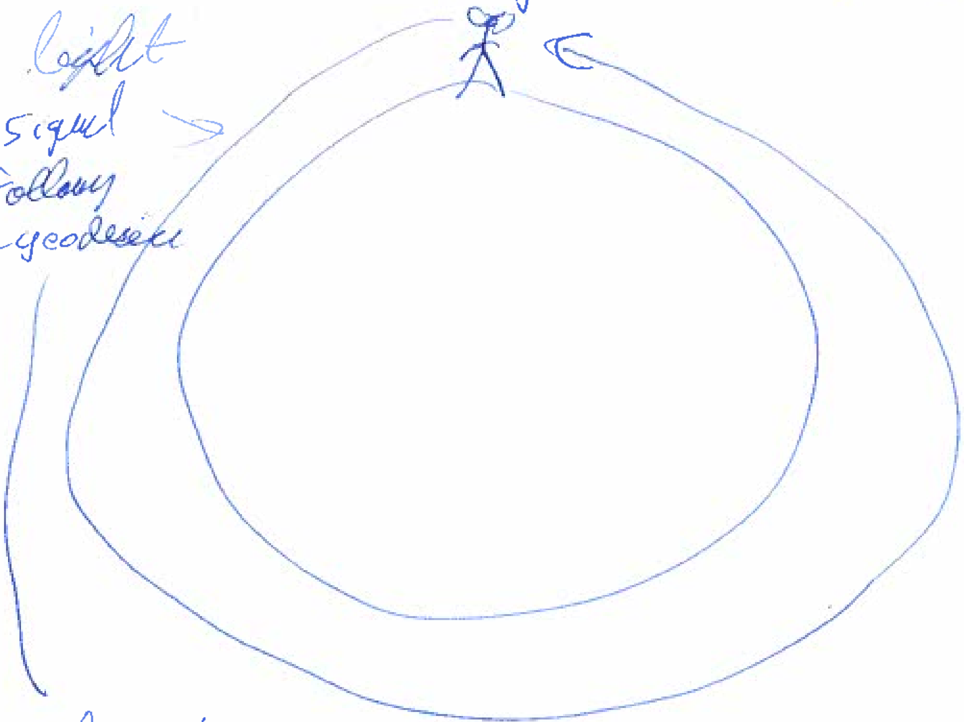
Seeing back of your head

light signal follows a geodesic



$2\pi a_0$

≤ 900 Gpc

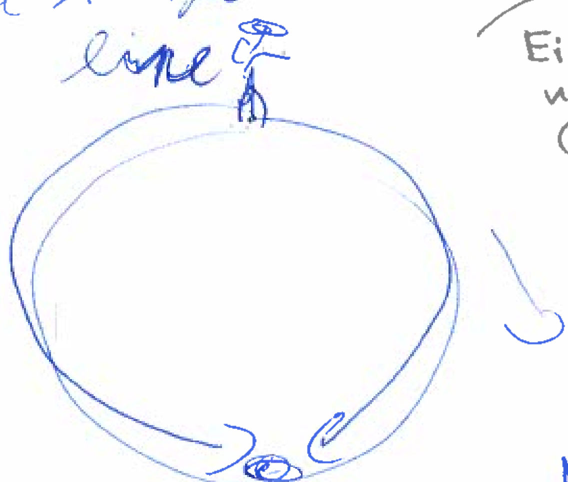
$\cong 2700$ Gly

$\therefore$

$t = 2700$ Gly

to see the back of your head

it just follows a "straight" line



Antipoda

Einstein universe (1917)

Einstein when he posited his static hyperspherical universe didn't worry about this.

every direction sees antipodal object

NOT same as increasing Angular Diameter (decreasing angular diameter distance) in a flat universe

→ That is an evolution effect

→ The ruler was closer when light started out to us.

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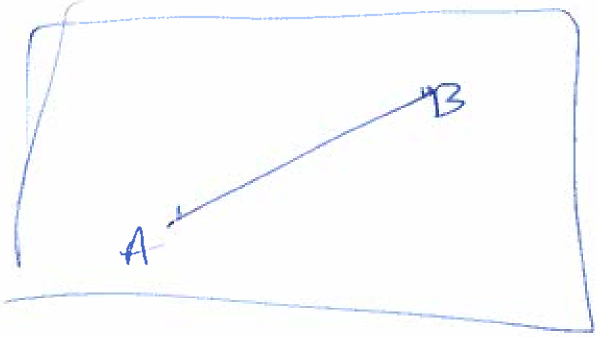
c) Geodesics

- stationary paths
- light in GR follows the simplest kind



In flat (Euclidean) space, just traditional straight line

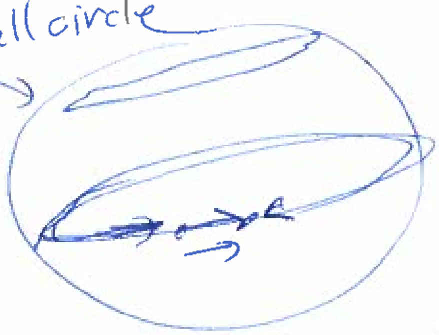
$$\mathcal{L} = \int_A^B [\text{Integrand}]\, d\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0$$

perturbation parameter

So locally a geodesic is shortest path (~~or longest~~) but maybe not globally

For Example



Great Circles are the geodesics on spheres

Need variational calculus
- Not so hard when you know trick.

4010 | 2025mar24

for $\theta < 180°$, the great circle path is locally and globally shortest path.

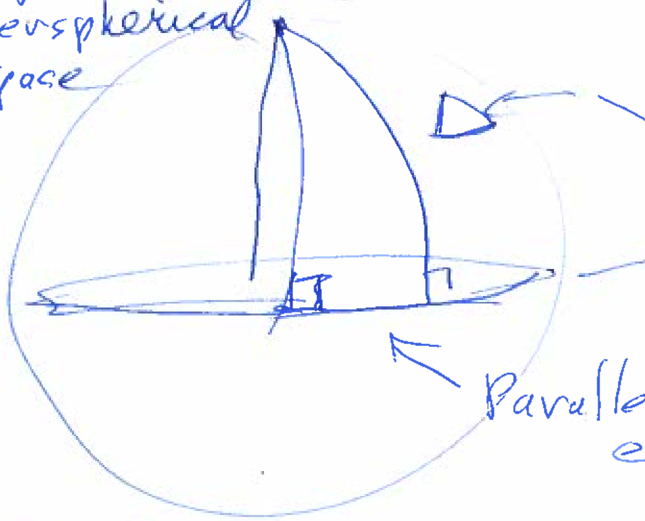
But $\theta \doteq 180°$, all great circle paths are ~~path~~ the same length

And $\theta > 180°$ still stationary but not globally shortest path. ~~path~~.

Very small insight into differential geometry ~~geometry~~ — it's hard topic to get into, but if you are going to do general relativity ....

Analog to hyperspherical space

Pole

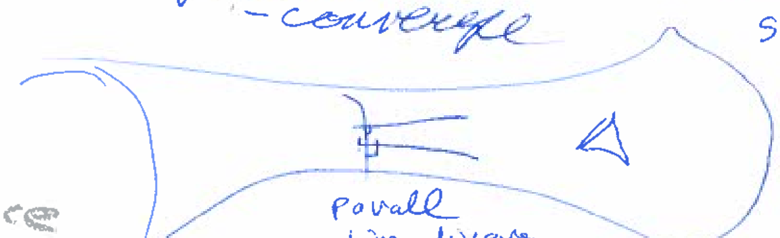


triangle of great circle paths

Sum of Angles > 180°

Parallel lines at equator meet a poles - converge

Saddle analog ~~to hyperbolic space~~ to hyperbolic space



Sum of Angle < 180°

~~Making of~~ 2025 march 11 | 4 011

An online video on hyperbolic space think saddle analogy poor, but maybe the best you can do in one diagram

It looks right to me. The maker argues hyperbolic space is easier to understand than hyperspherical space. It's like Euclidean space, but more so.

d) How Far ~~Do~~ Do Friedmann Universes Extend?

(2026 jan 24)

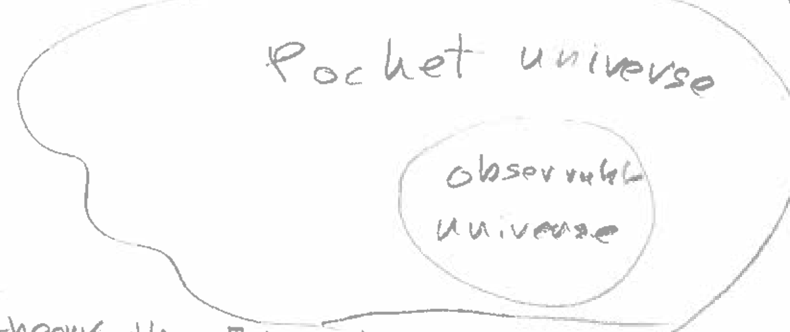
The cosmic scale factor $x(t)$ or $x(\tau)$ is so suppose to apply everywhere in a homogeneous isotropic universe, infinite or finite unbounded

Curvature
- $k>0$ finite unbounded hyperspherical
- $k=0$ infinite Euclidean (flat)
- $k<0$ infinite hyperbolical

Part of the assumption at derivation in Newtonian derivation and GR derivation.

But perhaps the extent is finite

Although there is some evidence for some inhomogeneity and/or anisotropy in the observable universe, it's not completely accepted, but some slight variation is NOT impossible.



Background universe

in Inflation theory the False Vacuum universe

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(2026jan24)

But are the boundaries sharp or soft or are there none with a gradual variation?
And how does the pocket universe boundary conditions map into the background universe if there are any?

If none of physical laws apply beyond our pocket universe, then we have no guidance and can't say anything.
I think people generally believe GR and quantum mechanics and quantum field theory holds, and, of course thermodynamics (in particular the 2nd law of thermo).
Maybe our lack of imagination, but these seem indispensable.
But it may be the fundamental constants (e.g., G, c, ħ, Λ(?), masses, charges, fundamental particles, etc.) vary, but no accepted guidance on the distribution of variation.

the 2nd law qualitatively at least seems to be a logical necessity. Maybe all physical law will be that way at high enough level, but maybe only AI could understand it.

e) Inflation and the Multiverse

Actually basic inflation, only our pocket universe is posited, Nothing beyond that is needed, but it seems to many that it is natural that there is a false vacuum universe beyond the pocket universe.

The false vacuum universe of inflation obviates $a(t) \to 0$, $\rho(t) \to \infty$ as $t \to 0$, the point origin or mythical Big Bang singularity. Somehow prevented by inflation or some other quantum gravity effect by or before Planck density

$R_{\text{observable universe}}$ (ΛCDM) $= 14.26$ Gpc $= 46.5$ Gly at cosmic present what you can measure with a ruler if you froze expansion.

$\rho_{pl} = \frac{c^5}{\hbar G^2} = 5.1550... \times 10^{9}$ kg/m3 (Wik)

Edge = Particle Horizon

Of course, we cannot see to the edge of the observable universe since the universe was opaque to photons before Recombination. Some we might see closer to edge in cosmic neutrinos or primordial gravitational waves

The multi-pocket universe inflation theory is often called eternal inflation, but there are probably many versions of that with all kinds of mathematical detail.

Eternal inflation is itself a version of multiverse theory and I tend to use the words as synonyms.

Many people think speculating about the multiverse is pointless since even if it were true there is so little guidance to say anything about it. Perhaps if we found TOE (theory of everything that explains all we observe in single unified elegant theory), it would imply the multiverse and tell us about it and we accept that as the best we can do OR it might imply there is No multiverse OR it might say Nothing.

f) The Anthropic Principle Approach

Anthropic Principle as an aphorism

Some things must be the way there are or we would NOT have a chance of being here: i.e., we'd have zero probability of being here.

Note $P(\text{observable universe} \mid \text{us just as we are}) \doteq 1$

every detail of us depends on some feature of universe

but

$P(\text{us just as we are} \mid \text{observable universe}) \doteq 0$

So random events went into making us just as we are (e.g., ~~the~~ Cretaceous–Paleogene impactor event wiping out the non-avian dinosaurs)

The observable universe rolled the dice a jillion times and we just as we are are the lucky winners

Some (e.g., Martin Rees) argue that the multiverse has passed a significant falsification test:

Fundamental constants, etc., are fine-tuned enough for life (as-we-know-it), but not overly fined tuned,

If ~~there~~ they were overly fined for life, then some principle other than a probability distribution decided them and life would be an ineluctable consequence of that principle (or principle)

E.g. Fine-structure constant. It's $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.035\,999\,177(21)}$ and Not exactly $\alpha = \frac{1}{137}$ which is overly fine-tuned and even spooky

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But it still seems that all Anthropic principle arguments dissipate into discussions of counterfactuals and philosophical arguments.

Yours truly finds eternal inflation (multiverse) useful to put in the slot theory of whole universe, but has lost patience with discussions until some great discovery illuminates the issue. (Maybe the true theory of inflation, quantum gravity dark energy, dark matter, etc.)

discussions seem so indecisive, unconstrained and arcane. Discussion at Wiki make me dizzy.

Summary

pro Multiverse	con multiverse
Seems no logical reason for fundamental constants, except consistent with life-as-we-know-it	So unconstrained that progress seemingly impossible

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4.2) General Relativity & Robertson-Walker Metric

True at every point in spacetime above quantum gravity - whatever that is,

a) GR gives Einstein field Equation EFE (2nd order)

4x4 set of DE's represented by an element,

$\hat{=} 2 \times 10^{-43}$ m/J

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \left(\frac{8\pi G}{c^4}\right) T_{\mu\nu}$$

Driver

stress-energy tensor ⇑ mass-energy, momentum

Einstein tensor $= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

metric tensor

Rank 2 ~~tensor~~

→ 2 indices

and μ ~~...~~ over 0, 1, 2, 3

time dimension

space dimension

- Greek letters for spacetime
- Roman letters just for 3-d space.

Mass-energy tells spacetime how to curve and curvature tells mass-energy how to move ⇓ ~~Newton~~ Circular but Newtonian physics is the same,

Replacement for Newton law of of Universal grav

with $g_{\mu\nu}$ being the tensor which describes the geometry of spacetime which tells matter ~~...~~ to move

$g_{\mu\nu}$ is the solution of the 4x4 DE system

Then the geodesic equation (which we won't write down (GR analog for F=ma) for gravity tells matter what motion to follow)

(2nd order dynamics eqn. ~~... 1st order ...~~)

9018

EFE can be written a 4x4 matrix differential equation

But GR notation is just to use a representative element.

For example instead of writing vector $\underset{\sim}{r} = \vec{r}$

you just write x_i

one element stands for all,

a very compact notation.

- Should we ~~use~~ it only, and teach it to intro? classes.
- probably, ~~different~~ multiple representations give you multiple understanding and so no

GR Tensor equations are invariant under coordinate transformation.

So they can be physical law.

Of course, just as in ordinary physics, the Newtonian coordinate system should be chosen to be convenient.

(2025mar24)

E.g., spherical or Cartesian [4019]

Choice depends on system of application

In fact, a lot of GR work depends on choosing the good coordinate system — and often it's ~~hard~~ hard to choose

Famous Example (Wik: Gwaves)

Einstein thought about Gravitational waves for a long time, but he & Rosen 1936 submitted a paper to Phys Rev concluding they couldn't exist.

Howard Robertson (of the metric) the referee pointed out the conclusion was wrong based on poor choice of coordinates,

Einstein was miffed and withdrew the paper, but eventually made the correction and published elsewhere (Wik: Gwaves)

Actually, this paper may have been Einstein's last important result. and he was only 57

(people aged faster then — smoking, etc).

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What are tensors anyway?

→ quantities that depend on directions in space in physics

→ rank 1 are vectors depend on x_1, x_2, x_3 for 3-d

Rank 2 can be written as matrices → depend on $x_1 x_1$, $x_1 x_2$, $x_1 x_3$ etc.

- 10 independent elements
- symmetric or antisymmetry

You extract information by contraction.

For example, length of vector x_i

→ contract with itself

dot product or inner product

$$l^2 = x_i x_i$$

Einstein summation over indices.

- latter letters indices for space 1, 2, 3
- Greek letters for spacetime 0, 1, 2, 3

Now x_i the element values are coordinate system dependent, but ~~is not~~ length is NOT

Example of coordinate transformation

coordinate transformation

$$x_i' = A_{ij} x_j$$

primed system

unprimed system

Contract on i: $x_i' x_i' = A_{ij} x_j A_{ik} x_k = A_{ij} A^T_{ki} x_j x_k$

transpose

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Kronecker delta

$$= \delta_{jk} x_j x_k$$

$$= x_j x_j$$

So a valid transformation of ~~coordinate~~ tensor must have property

$$A_{ij} A^T_{ki} = \delta_{jk}$$

or $A A^T = \mathbb{I}$

matrices   unit matrix.

But other tensor can be physically active, i.e., Not coordinate transformation, physical rotation, etc.

it can all get confusing

b) Curved spacetime ~~doesn't~~ obeys a generalized Pythagorean theorem.

actually so does ordinary flat 3-d space if you ~~use a curvilinear~~ do NOT use ~~that~~ cartesian coordinates

4022) take thermodynamics 1st law $dE = TdS - PdV + \mu dN$

the "denominator" could be time or some other path in S, V, N space

again the virtue of physical laws written as tensor equations is the law is invariant under coordinate changes (but not values of elements of tensors)

What are differentials with respect to? A path parameter through spacetime $x_i = x_i(l)$

l could be time or just a parameter measuring a path at an instant in time

$\left(\frac{ds}{dl}\right)^2 = g_{\mu\nu}\frac{dx^\mu}{dl}\frac{dx^\nu}{dl}$

- convention is not to complicate the formulae with path parameter

The generalized ~~Pythagorean~~ Pythagorean theorem of GR is

- differential and so in general only valid for differential (local) displacements and need to integrate for finite intervals

Einstein summation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

ds^2 is the invariant interval of spacetime

indices can be lower or upper and we won't go into why?

but when you do have upper/lower contraction only valid between upper & lower

(Carl Hartle's GR course teaches all that)

$g_{\mu\nu}$ is the metric (tensor)

But often it seems people refer to this equation as the metric, but it is properly metric equation

The metric of special relativity for flat spacetime is the Minkowski metric

$$g_{\mu\nu} \rightarrow \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \eta_{\mu\nu} = -\eta_{\mu\nu}$$

(CL-24)

The alternative convention of Carroll-8

It seems the two conventions are about equally used. A nuisance

Returning to the general GR metric equation

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$d\tau^2 = ds^2$

proper time measured by said clock

Interval characterizations

$ds^2 > 0$ time-like

In this case, the interval can be measured by a single clock moving between two points in spacetime.

$ds^2 = 0$ light-like

which gets a lot of use in our developments

A light signal can connect the two points in spacetime with $ds^2 = 0$

$ds^2 < 0$ space-like

Carroll-7,9 chooses the opposite convention

The interval can be measured by a ruler at rest with with respect to an observer,

Length = a spatial interval whose endpoints can be determined simultaneously by an observer

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c) Robertson-Walker Metric: Preliminary

(AKA RW Metric, Friedmann-Lemaître RW metric FLRW metric)

In 1935, Robertson & Walker determined their eponymous metric known earlier to Friedmann & Lemaître,

but RW proved it is the most general metric that is a manifold: i.e., any local region is asymptotically Minkowskiian,
— as you approach its defining point $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$
(i.e., There is always a local (Minkowskiian) comoving frame) (Carroll-54: see both above and below section heading)

And this metric is for a homogeneous, isotropic unbounded space.

Exact homogeneity and isotropy is equivalent to the cosmological principle holding exactly.

Unbounded implies infinite Euclidean (flat) space
infinite hyperbolic (-ve curvature) space
finite hyperspherical (+ve curvature) space

Now the RW metric is geometry,

If you impose the RW metric on the Einstein field equations,

you get the time evolution equation for a cosmic scale factor $a(t)$
(i.e., the Friedmann eqn. which we derived from Newtonian physics with extra hypothesis without using or getting any information on curved spaces).

Since Newtonian physics is the classical limit of GR, it is reasonable there is a Newtonian to some GR results (e.g., a classical derivation of the Schwarzschild radius),

but, of course, a priori such approaches
could've been misleading (even if interesting)
but at least for the Friedmann eqn
the approach gave the exactly right result.

The fact that all local regions are
comoving frames means that
the vacuum light speed is *the* (invariant)
fastest speed relative to them.

<u>But</u> recession velocities between
comoving frames can be arbitrarily large.

<u>Exact Hubble law</u> $v_{rec} = H D_{proper}$
holds for proper distance
at any instant in cosmic time.

$$v = Hr \doteq \left(\frac{c}{3 \times 10^5\,\mathrm{km/s}} \cdot 70\,\frac{\mathrm{km/s}}{\mathrm{Mpc}} \right) D \, h_{70}\, c$$

$$\doteq \left(2.3 \times 10^{-4} D_{\mathrm{Mpc}} \right) h_{70}\, c$$

$\therefore$ $v > c$ for $D_{\mathrm{Mpc}} \gtrsim 4000\,\mathrm{Mpc}$

corresponds in Λ-CDM
to $z \gtrsim 1.3$

d) RW Metric (CL-10)

Ordinary spherical coordinates for an origin that is anywhere

spacetime interval

$$ds^2 = c^2 dt^2 - a^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$d\tau^2 = ds^2$

where $d\tau$ is proper time (Carroll-9), Time for a clock moving between endpoints

r is a dimensionless comoving radius-like coordinate

Metric or metric interval or metric equation

The Metric is really the coefficients

$a = a_g$

cosmic scale factor, with dimensions of length. Some book use R (Carroll-331-332)

As I labeled it on p.4004 to distinguish it from the dimensionless cosmic scale factor of the Friedmann eqn. Note dimensionless quantities have a physical nature. They are just in natural units

R Gaussian curvature radius

$$= \frac{a_g}{\sqrt{K}} \quad \text{(CL-12)},$$

and the dimensioned form is the magnitude of the Gaussian curvature radius.

$$a_{g0} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{c\, t_{\text{Hubble}\,0}}{\sqrt{|\Omega_{k0}|}} = \frac{(4.2827\ldots)\, h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_{k0}|}} = \frac{13.968\, h_{70}^{-1}\ \text{Gly}}{\sqrt{|\Omega_{k0}|}}$$

see p.4005

We will NOT do a derivation (which I do not know) but the equation is scaled so again

$k = 1$ is +ve curvature unbound finite hyperspherical universe

$k = 0$ is Euclidean (flat) infinite universe

$k = -1$ is the −ve curvature infinite hyperbolical universe

We note the ~~FR~~W metric is an equation of differentials like, e.g., the 1^{st} law of Thermodynamics

$$dE = TdS - PdV + \mu dN.$$

The independent variable is any path parameter between the endpoint of the spacetime interval, e.g., ℓ

You may wonder does $\frac{d^2 r}{1-kr^2}$ have a singularity as $kr^2 \to 1$. No, $\frac{dr}{d\ell} = 0$ at that point (i.e., there is a stationary point) and the comoving coordinate smooth (differentiable there)

e) Proper Distance $= D_p \equiv D_{proper}$

Recall, a proper distance is a spatial extent measured at one instant in time

Here cosmic time and in the Metric $dt = 0$

Consider radial D from an origin anywhere

$$dD_{p\,radial} = \frac{a\,dr}{\sqrt{1-kr^2}}$$

The proper distances scale with $a(t)$

Origin

$$dD_{p\,tangential} = a\,dr\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$$

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So the radius-like comoving coordinate r
was chosen to be "normal"
for tangential proper distance
and "non-normal" for radial distance

In fact, as we will show there is also
a comoving coordinate χ (chi not α)
that is "normal" for radial distance.
Both r and χ are needed.

4.3 Geometrical Insight Into the Curved Space of Cosmology

a) First some insight into the surface of a 2-sphere

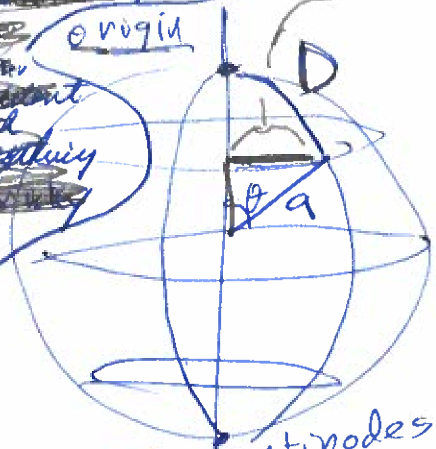
A 2-sphere is just an ordinary sphere in 3-d Euclidean space
(Wiki n-sphere)
but its surface is an unbounded finite curved space
with $k = 1$

Metric is

$$dD^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$ (CL-10, last equation)

(20140130) 4029

a is radius of sphere and r is [illegible] origin



Let "funny r"

$r = \sin\theta$ (yes it looks weird but

$\theta \in [0, \pi]$ polar coordinate range; $r \simeq \theta$ only for $r \ll 1$; r is a dimensionless comoving coordinate on the sphere)

$dr = \cos\theta\, d\theta$

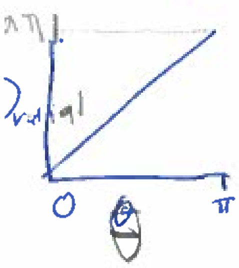
$d\theta = \pm \dfrac{dr}{\sqrt{1-r^2}}$ {which implies the limits $\theta \in [0, \pi]$ actually

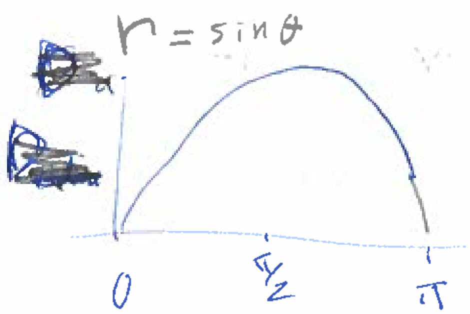
So $dD^2 = a^2\left(\dfrac{dr^2}{1-r^2} + r^2\, d\phi^2\right)$

All geodesics lead to the antipodes

So the sphere surface looks flat only for $r \ll 1$

and $D_{radial} = \begin{cases} a\theta \\ ar & \text{for } r \ll 1 \text{ where } \sin\theta \simeq \theta \\ a\frac{\pi}{2} & \text{for } \theta = \frac{\pi}{2} \\ a\pi & \text{for } \theta = \pi \end{cases}$





$D_{circumference} = ar\,2\pi = 2\pi a \sin\theta$

$\begin{cases} 2\pi a r & \text{for } r \ll 1 \\ 2\pi a & \text{for } \theta = \frac{\pi}{2} \\ 0 & \text{for } \theta = \pi \end{cases}$

Infinite Plane Case

b) $k = 0$, Just flat space, 2-d flat space

Metric $dD^2 = a^2(dr^2 + \cancel{\sin^2\theta}\, d\phi^2)$ {Just polar coordinate space

$r \in [0, \infty]$

Surface of hypersphere in 4-d Euclidean space – but the spatial 4th dimension has no physical meaning

c) $k = +1$ for 3-sphere which is the RW metric space part. $dt = 0$

Metric $dD^2_{proper} = a^2\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$ (CL-11)

χ makes the radial measurements normal!

where χ is one version of comoving coordinate.

chi NOT x

Here we define $r = \sin\chi$

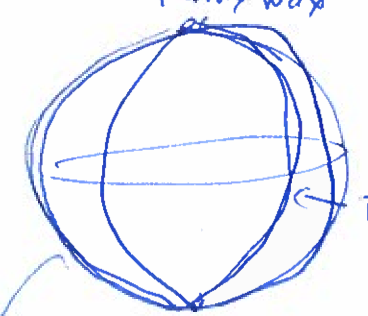
so that $dD_{\substack{proper\\ tangential}} = a\, r\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$

and which means

$dr = \cos\chi\, d\chi$ &

and $d\chi = \pm \dfrac{dr}{\sqrt{1-r^2}}$ just as for the 2-sphere case

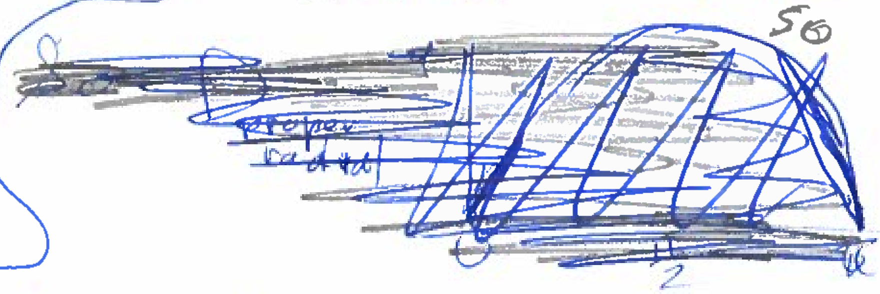
2-sphere analog

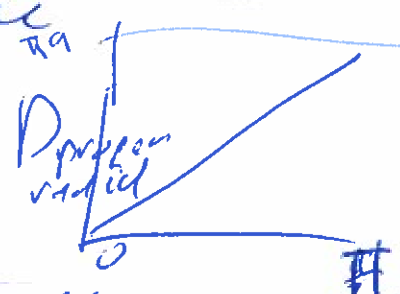


all geodesics lead to the antipodes

D_{radial} $_{proper}$ $\le 2\pi a$

which implies the $\chi \in [0, \pi]$ limits since we assume the RW metric

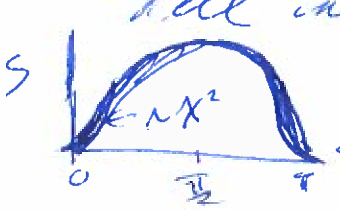




What is the area of a 2-sphere-like region in the comoving frame coordinate r in our RW space.

Not proof Just what is from the RW metric CL-11

Well integral over all θ, ϕ gives 4π just like in flat space.



$\therefore\ S(r) = 4\pi a^2 r^2 = \left\{ 4\pi a^2 \sin^2\chi \right.$

$4\pi a^2 \chi^2$ for $\chi \ll 1$; $4\pi a^2$ for $\chi = \pi/2$

surface area does go to zero!!!

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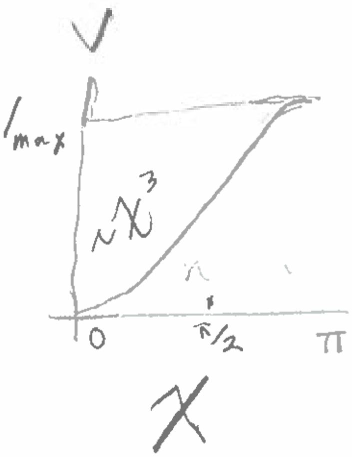
What is the volume of a 2-sphere-like region? ← a 2-sphere or "ordinary sphere" in the 3-sphere space

$$V = \int_0^{\chi} S(\chi')\, d\chi' \quad \leftarrow \text{Chi, no } x$$

$$= 4\pi a^3 \int_0^{\chi} \underbrace{\sin^2\chi}_{\frac{1}{2}(1-\cos 2\chi)}\, d\chi \quad \Big\{ 4\pi a^3 \frac{\chi^3}{3} \text{ for } \chi << 1$$

from standard trig identity (Wiki: trig Double angle formulae)



$$= 4\pi a^3 \left[\tfrac{1}{2}\right]\left(\chi - \frac{\sin 2\chi}{2}\right)$$

$$V = \frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)\left(\chi - \frac{\sin 2\chi}{2}\right) \text{ in general}$$

$\chi - \left(\cancel{\chi} - \frac{1}{3!}\frac{(2\chi)^3}{2}\right)$ ($\chi << \frac{\pi}{2}$

$\frac{2}{3}\chi^3$

$V \sim \chi^3 \sim r^3$

$\frac{4\pi a^3 \chi^3}{3} = \frac{4\pi}{3} a^3 r^3$ as it should for asymptotically flat space.

$\frac{4\pi a^3}{3}\left(\frac{3}{2}\right)\left(\frac{\pi}{2} - 0\right)$ For $\chi = \frac{\pi}{2}$

$\pi^2 a^3$

For $\chi = \pi$ and this is a maximum

$\frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)\pi$

$2\pi^2 a^3$

So a finite space but unbounded.

V_{max} proof

$\frac{dV}{d\chi} = \frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)[1 - \cos 2\chi] = ($

for $\chi = 0$ which is a min.

for $\chi = \pi$ which is a max

$\frac{d^2V}{d\chi^2} \{ 4\pi a^3\, 2 \sin\chi \cos\chi$

0 for $\chi = 0$ & $\chi = \pi$

Show video of hyperspherical universe (4:00) I won't guarantee accuracy, but it seems OK.

40~~3~~2)

d) $k=0$ RM metric flat space

$$dD^2_{\text{proper}} = a^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$\chi = r$ in this case

Just ordinary spherical coordinates

e) $k=-1$, RW metric hyperbolic space

$$dD^2 = a^2\left[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

might have guessed $\chi \in [0, \infty]$

with: hyperbolic functions

$\sinh\chi = \frac{e^{\chi}-e^{-\chi}}{2}$

$\cosh\chi = \frac{e^{\chi}+e^{-\chi}}{2}$

let

$\cosh^2\chi - \sinh^2\chi = 1$

$$\begin{cases} r = \sinh\chi \\ dr = \cosh\chi\, d\chi \\ d\chi = \dfrac{dr}{\cosh\chi} = \dfrac{dr}{\sqrt{1+r^2}} \end{cases}$$

$$dD^2 = a^2\left[\frac{dr^2}{1+r^2} + \cancel{\ }r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

again tangential space is "normal"

surface of 2-sphere-like region

$$S(r,\chi) = \begin{cases} 4\pi a^2 r^2 \\ 4\pi a^2 \sinh^2\chi \end{cases}$$

Volume of 2-sphere-like region

$$V = \int_0^{\chi} S\, a\, d\chi' = 4\pi a^3 \int_0^{\chi} \sinh^2\chi\, d\chi$$

$= 4\pi a^3 \int_0^{\chi} \frac{1}{2}(\cosh(2\chi) - 1)\, d\chi$ ← can be integrated

$4\pi a^3 \frac{\chi^3}{3} = \frac{4\pi a^3}{3}\chi^3$ for $\chi \ll 1$ asymptotic flat space as it should be.

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4) Hubble's Law: Proof from Friedmann Equation and Proof from RW Metric (it's easy)

But there are some subtleties I ignore CL-14 -15

a) First recall the FE gave us (recapitulation)

Hubble parameter

$$H = \frac{\dot{a}}{a}$$

$$\dot{a} = Ha$$

Can't use v anymore in Sect 4

either interpretation of a leads to

D_p proper length — measurable by a ruler at the instant in cosmic time

$$\dot{D}_p = H D_p$$

recession velocity; proper distance

for any cosmic time t

D two point in spacetime at same point in cosmic time

or

$$v = H D_p$$

recession velocity not velocity relative to an inertial frame — can be any size and can be greater than c

So NOT an ordinary velocity except asymptotically as $a \to 0$

proper distance

(see p. 4025)

Two cases:

$D_p = a D_{p0}$

$a(t=t_0)=1$

and a is dimensionless scale factor

D_0 are present proper distance of observable universe

$D_p = a_g \chi$

for $k \neq 0$

$\frac{a}{\sqrt{k}}$ is Gaussian curvature radius in units of length (e.g. Gpc or Gly)

and

$$\chi = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}}$$

is comoving distance in natural units (in units of a)

b) Proof from RW metric is just (CL-13) (pretty simple)

$$D = \int_0^\chi a\, d\chi = \int_0^r a \frac{dr'}{\sqrt{1-kr'^2}} = a(t)\chi(r)$$

$a(t)$ has no comoving frame dependence; $f(r)$ no cosmic time dependence

$$D = a\cancel{\chi}(r)$$

origin

at cosmic time t,

$$\therefore \quad \dot{D} = \dot{a}\cancel{\chi}(r) = \dot{a}\left(\frac{D}{a}\right)$$

$$\text{or} \quad \dot{D} = H D \qquad \text{QED}$$

This is an exact result, (but except asymptotically as $D \to 0$), $\dot{D}$ and D are not direct observables.

We will show they are in this limit soon

c) ~~a)~~ Remarks on What's the difference between a direct and indirect observable?

see p.

Probably a matter of taste often.

Maybe! A direct observable is one where you trust all the theoretical steps from observation to desired value (e.g., Volume → Temperature on an old fashioned thermometer)

and/or where the steps seem ~~short~~ few to you.

An indirect observable is where you don't trust all the steps to a high level and/or there a lot of them: e.g., determining paleo climate from an ice core

Bertrand Russell → the only truly direct observation ~~is there~~ is thinking — all other observation are theory laden, to one degree or another.

b) May be with FRBs
~~and GW and quasar lensing~~
~~cosmologically remote D and~~
~~and D are becoming possible~~
~~But those methods still have a~~
~~lot of uncertainty and~~
~~are not yet competitive though~~
~~promising~~
Omit Done before in sect. 3

and asymptotic form $cz = HD$ luminosity of Hubble's law

d) Recapitulate history

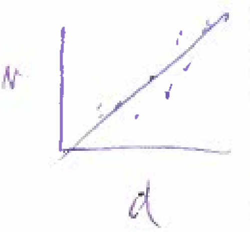
1927 Lemaître derived Hubble law explicitly from GR (but Friedmann probably knew it earlier at least vaguely) and using literature ~~valu~~ values suggested 575 $\frac{km/s}{Mpc}$ or 670 $\frac{km/s}{Mpc}$ for H_0

luminosity Distance

but I ~~think~~ one can say he only found values assuming the theoretical result was true (yes)

1929 Hubble gave an empirical discovery of Hubble's law $v = Hr$
which can be done ~~from D → 0~~ $D \to 0$ asymptotically as we'll prove (see p.
His $H_0 \cong 500 \frac{km/s}{Mpc}$ due to poor Cepheid calibration

He presented a Hubble diagram



403)

In the 1950s, H_0 was brought down a long way. ← thanks to realizing there were two types of Cepheids mainly

Which fixed the age problem of that era

$$t_{age} \triangleq \frac{1}{H_0} = 13.968\, h_{70}^{-1}\ \text{Gyr}$$

of Universe assuming big bang (i.e., point origin

→ of order ~~-exact~~

Arthur Holmes (Wiki: Age of Earth) from radioactive dating in 1921 said age of Earth ~ few Gigayears

in 1927 1.6 – 3.0 Gyr.

In 1960's — 1990s values between ~50 (Sandage + Tamman mainly) and 100 (de Vaucouleurs mainly) (I exchanged emails with him once)

Since 2010, the Hubble Tension

~68 from Planck (Indirect by fitting ~~to~~ a model but ΛCDM model very robust

~73 from direct local measurements (Riess et al many papers) (but may Cepheids are again giving wrong answers) seems very unlikely

→ Maybe Λ-CDM needs revision but it still could be we are unluckily just in large (~300 Mpc) low density or maybe there is Non standard Reionization)

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but ~~that~~ we cannot do (except asymptotically → but we can do that)

When we look out we look back in cosmic time and light has traveled to us in ~~that~~ lookback time and the source has receded and it is recession velocity has changed.

But local measurement do establish H_0 if right!!

Confusion of sources confused me

Not Really!! 1927 X

Le maitre in 1927 derived it explicitly from GR and estimated $H_0 \cong 625 \frac{km/s}{Mpc}$ from data

Hubble's law obscure Brussels journal

$=625$
$[575 + 670]/2$
Livio 2011

Lundmark on so had the law empirically with $H_0 \simeq 70 \frac{km/s}{Mpc}$? and apparently Not accepted a very good scale (Not really)

Friedmann eqn Implicit in Friedmann & de Sitter universe but apparently not written explicitly

in 1929 Hubble published his discovery and $H_0 = 500 \frac{km/s}{Mpc}$

celebrated errors.

$H_0 = 67$ Planck

73 Riess et al. of SNe

~~the tension~~ The Hubble tension

Lemaitre had theory and fitted value H_0

But he did not WH

So a theoretical

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2017sep26

e) Another Remark

I think it's true that $S(t) = S_0\, a(t)$

true that Hubble's Law follow generally if non vigorously just by saying if space is homogeneous & isotropic and just scales with time by general factor $a(t)$

of whatever possible geometry $k=0$ or $k=\pm 1$

Then any shape formed by comoving test particles should just scale and one has

$$\dot{S} = S_0\, \dot{a}$$

No freedom to distort

and

$$\frac{\dot{S}}{S} = \frac{\dot{a}}{a}$$

$$\dot{S} = \frac{\dot{a}}{a} S = HS$$

4.5) Cosmological Redshift

— Multiple derivations (all give same result — which is satisfactory)

— and the analogue non-relavistic velocity & energy loss formula

a) All redshifts $z = \dfrac{\lambda_o - \lambda_e}{\lambda_e}$ { Doppler, Cosmological, Gravitational + a negative redshift is a blueshift.

λ_o observed wavelength

λ_e emitted wavelength known from recognizing a pattern of lines (emission or absorption)

Thus, z is a direct observable.

Now $z + 1 = \dfrac{\lambda_o}{\lambda_e}$

Therefore you can just compound redshifts of any kind: 0 ← n ⋯ 3 2 e=1

$$(z_{total} + 1) = \frac{\lambda_o}{\lambda_n} \frac{\lambda_n}{\lambda_{n-1}} \cdots \frac{\lambda_3}{\lambda_2} \frac{\lambda_2}{\lambda_{1=e}}$$

Our Most interest is for

$$z_{total} + 1 = \underbrace{(z_{local} + 1)}_{\text{Doppler}}\,(z + 1)\,\underbrace{(z_{remote} + 1)}_{\text{Doppler}}$$

cosmological unadorned

So $z + 1 = \dfrac{z_{total} + 1}{(z_{local} + 1)(z_{remote} + 1)}$ to correct to get

$\hat{=}\ z_{total} + 1$ if z_{local} and z_{remote} negligible.

Since $z + 1$ is a direct observable and independent of theory, it is what people cite rather than distance (proper distance) or look back time $(t_0 - t(z))$

These are only direct observables asymptotically as $z \to 0$.

4040

Personally, I do not think the cosmological redshift is a Doppler shift. It has a different formula. It can be derived from the Doppler effect as we will show, and ~~—~~ is a kind of a compounded ~~D~~oppler effect.

~~And the interpretation of the quantit~~

The two formulae do agree to lowest order as $z \to 0$

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c},$$ where the distinction between ordinary velocity and recession velocity vanishes.

Recall in NR limit only do velocities just add.

Classical limit effect which we used in Newtonian derivation of the Friedmann Eqn.

~~Formula~~

b) Proof of the Cosmological Redshift Formula From the RW Metric

observer ⟵⟶ emission

The interval is light-like for a light signal. Thus $ds^2 = 0$

$$\therefore\ ds^2 = 0 = c^2 dt^2 - a\frac{dr^2}{1-kr^2}$$ for a radial path from emitter to observer

$$\therefore\ \chi = \underbrace{\int_0^r \frac{dr}{\sqrt{1-kr^2}}}_{\text{time independent}} = \int_{t_e}^{t_0} \frac{c\,dt}{a} = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{c\,dt}{a}$$

One light pulse and then another just a bit later,

$$= F(t_0) - F(t_e) = F(t_0+\delta_0) - F(t_e+\delta t_e)$$

$$= F(t_0) + \delta t_0 \left.\frac{\partial F}{\partial t}\right|_0 - F(t_e) - \delta t_e \left.\frac{\partial F}{\partial t}\right|_e$$

where $\frac{\partial F}{\partial t} = \frac{c}{a}$

One wave emission later. and so a 1st order approximation is very well justified

$$\therefore\ \delta t_0 \frac{c}{a_0} = \delta t_e \frac{c}{a_e}$$

$$\frac{1}{\nu_0}\frac{c}{a_0} = \frac{1}{\nu_e}\frac{c}{a_e},$$ and so

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$\cancel{\frac{\nu_0}{a_0}} \; \cancel{\frac{\nu_e}{a_e}}$

$\cancel{\frac{1}{a_0 \lambda_0} = \frac{1}{a_e \lambda_e}}$

$\cancel{\frac{\lambda_0}{\lambda_e}}$

$\cancel{\frac{1}{\lambda_0 a_0} \quad \frac{1}{\lambda_e a_e}}$

$$\frac{\lambda_0}{a_0} = \frac{\lambda_e}{a_e}$$

$$z+1 = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e}$$

Also $\lambda(t) = \lambda_0 \frac{a(t)}{a_0}$

$\lambda \propto a(t)$

Note time dilation

$\frac{\delta_0 t}{\delta_e t} = \frac{a_0}{a_e}$

So something emitting this time interval is observed in time interva δt_0

$$\therefore \quad z = \frac{a_0}{a_e} - 1$$

But everyone remembers $\frac{a_0}{a_e} = z + 1 \approx z$ for $z \gg 1$

or $\frac{a_e}{a_0} = \frac{1}{z+1}$

Amazing factoid: z gives ~~us the~~ scaling up of ~~the~~ universe since ~~the~~ time of emission of a light signal

$$\frac{a_0}{a(t)} = z + 1$$

(t is ~~t_e~~ emission time

But we don't know cosmic time t directly. [we know $a(t)$ but not t!!!]

It would be great if galaxies had clock faces on them, ~~but they~~ that told cosmic time ~~t~~, but ~~they~~ don't.

There is work ~~on~~ trying to use passively evolving elliptical galaxies (ETGs) as cosmic chronometers, but there are large uncertainties (model dependence among other things).

There is a lot progress and they are a great consistency check, but not yet competitive in deciding ~~on~~ models. Consistent with Λ-CDM ...

9042) Note, Friedmann equation

$$H = \left(\frac{\dot{x}}{x}\right)^2 = \sum_{p=4}^{0} \Omega_{p,0}\, x^{-p}$$

where $x^{-1} = z+1$, $x = \frac{a}{a_0}$ in my notation

$$= \sum_{p=4}^{0} \Omega_{p,0}\,(1+z)^p$$

and so there is a lot of formalism solving in for quantities in terms of z or $(1+z)$ rather than x (i.e. $a(t)$); Maybe much the same thing, except maybe for $z \ll 1$ cases.

Note also, since

$$x = \frac{1}{1+z}$$

$$\dot{x} = \frac{-1}{(1+z)^2}\left(\frac{dz}{dt}\right)$$

evaluated at cosmic time of emission

t of emission

$$\therefore\ H = \frac{\dot{x}}{x} = -\frac{1}{1+z}\frac{dz}{dt} \triangleq -\frac{1}{1+z}\frac{\Delta z}{\Delta t}$$

So if you could measure Δz for an object over cosmic time Δt (time of emission)

then you could determine $H(z)$ and get more constraints on $\Omega_{p,0}$'s

changing Δz is called redshift drift and it may become measurable in the 2030s. See Roos (2024) in Ast727 reviews (cosmological models)

c) Relativistic Energy loss

$\lambda \propto a$ but the de Broglie relation $E_{photon} = h\nu = \frac{hc}{\lambda}$

$\therefore\ E_{photon} \propto \frac{1}{a}$

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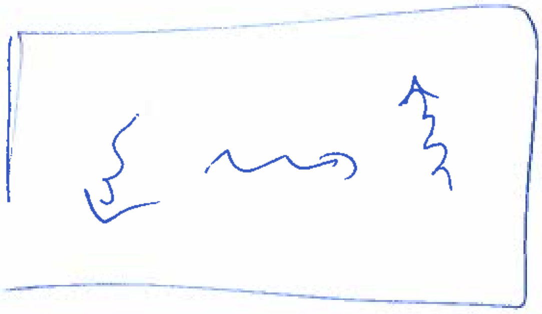
Where does the energy go?



In an expanding box ~~the~~ photon ~~gases~~ does PdV work on ~~the~~ walls where the photon inertial frame energy is deposited and so the energy goes somewhere. (Ordinary Doppler shifting to lowest order)

But in a boundless GR expanding universe where does it go? Just vanishes it seems.

GR guarantees itself, not ordinary conservation of energy.

But there is $\nabla_\mu T^{\mu\nu} = 0$ energy momentum conservation equation. (Carroll-117, 120)

∇_μ is covariant derivative (Carroll 93-94, 97)

a differential equation true at all points in spacetime above quantum gravity level

— ~~Not an integral of motion (i.e., a fixed number)~~

~~#) Non Relat~~

There are 4 equations to balance and that seems unuseful in giving information on motion unlike the classical $E_{mech} = KE + PE$ conservation of mechanical energy equation

Is the GR replacement for ordinary conservation of energy. But it seems much less useful.

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d) A Doppler Shift Derivation of the Cosmological Redshift

From Special Relativity in one inertial fram

$$\mathcal{R} = \frac{\lambda_o}{\lambda_e} = \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{where } \beta = \frac{v}{c}$$

$$\frac{d\mathcal{R}}{d\beta} = \frac{1}{2}\frac{1}{\sqrt{1+\beta}}\frac{1}{\sqrt{1-\beta}} - \frac{1}{2}\frac{\sqrt{1+\beta}}{(1-\beta)^{3/2}}(-1)$$ which is a sort of elementary differential eqn (DE) that can be solved for the original formula.

But take 1st order approximation in small β

$$\mathcal{R}\Big|_{1st} = \frac{\lambda_o}{\lambda_e}\Big|_{1st} = \left(1+\frac{1}{2}\beta\right)\left(1+\frac{1}{2}\beta\right)\Big|_{1st} = 1+\beta$$

$$\therefore \quad \frac{\Delta\lambda_e}{\lambda_o} = \frac{\lambda_o-\lambda_e}{\lambda_e} = \beta \quad \text{or} \quad \frac{d\lambda}{\lambda} = d\beta = \frac{dv}{c}$$

where we have gone to the limit of differentials, but this NOT a DE for the original formula.

However,

We argue (cogently or not) that a photon traveling from one to another must obey differentially the classical differential Doppler shift

dD = differential proper distance in an expanding universe

gives $dv = H\,dD$ by exact Hubble law

The differential recession velocity between differential close comoving frames (i.e., inertial frames)

since recession velocity and ordinary velocity are the same thing in the classical limit

$\therefore$ To differential order accuracy,

$$\frac{d\lambda}{\lambda} = \frac{dv}{c} = \frac{H\,dD}{c} = \frac{(\dot{a}/a)\,dD}{c}$$

$$= \frac{da}{ca}\frac{dD}{dt} = \frac{da}{ca}c = \frac{da}{a}$$

$$\therefore \frac{d\lambda}{\lambda} = \frac{da}{a}, \quad \ln(\lambda/\lambda_0) = \ln(a/a_0)$$

$$\therefore \lambda/\lambda_0 = a/a_0$$

$$\text{or} \quad \lambda \propto a$$

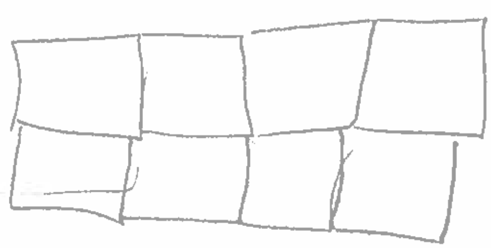
Note the differential Doppler shift is to the wavelength the observer at rest in the differentially displaced comoving frame observes, and so it is (is not) cogent that one can integrate up the shifts

f) Quantum Mechanical Derivation of Cosmological Redshift

I do NOT think this is a cogent derivation, but it does show consistency of cosmological redshift with an extreme 'alternate perspective' with delocalized particle states in boxes in an expanding universe with boundary conditions which take more justification than I know.

Box Quantization (i.e., cubic infinite square wells)

Tile all space with the boxes,



$$\therefore \lambda = \frac{2L}{n}$$

$$k = \frac{2\pi}{\lambda}$$

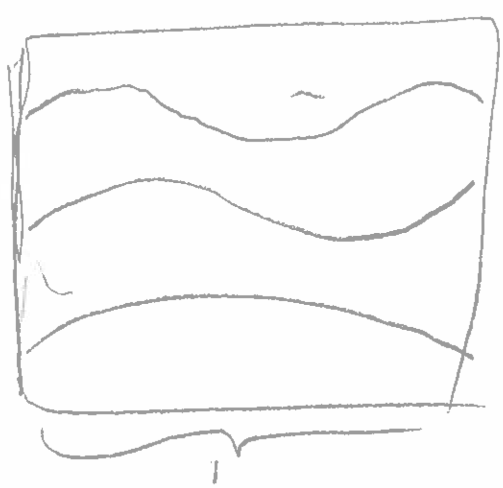
Wavenumber

$$p = \hbar k = h/\lambda$$

In a box

$$\psi(\text{boundaries}) = 0$$

$$\therefore \underbrace{n}_{\text{number of antinodes}}(\lambda/2) = L$$



NR limit (nonrelativistic), $E \propto \sum_i p_i^2 \propto \frac{1}{L^2} \propto \frac{1}{a^2}$

ER limit (extreme relativistic), $E = \sum_i p \propto \frac{1}{L} \propto \frac{1}{a}$

Expand boxes adiabatically and only PdV work into walls.

So particles in superpositions of states including localized wave packets should see their ~~energies~~ scale as $\begin{cases} \frac{1}{a^2} & NR \\ \frac{1}{a} & ER \text{ as for photons.} \end{cases}$

But here to preserve homogeneity, the energy vanishes. Is this an over-idealization? Or a real "discovery" ~~of~~ non-conservation of energy?

The "mean" wavelengths of the packets should scale as $\begin{cases} a^2 & NR \\ a & ER \text{ as for photons} \end{cases}$

Thus $\lambda \propto a$ as the proof of the cosmological redshift from the RW metric on p. 4040

Note a macroscopic object has a sort of Dirac Delta wave packet wave function

Note, $E_{NR} \propto \frac{1}{a^2}$ is the same as for the proof for a classical object moving in the expanding universe given in Lecture 3 p. 3101.

But does phoney box quantization make any sense in a boundless and maybe infinite universe? It's used all the time in solid state and for other cases which have no well defined boundaries to count states (e.g., white dwarfs and neutron stars).

Recapitulate

$dv = v_2 - v_1 = -H\,dD = -\frac{\dot a}{a}\,dD$

$= -\frac{da}{a}\frac{dD}{dt} = -\frac{\dot a}{a}v$

$\therefore \frac{dv}{v} = -\frac{\dot a}{a}$

$\therefore \ln v \propto -\ln a$

$v \propto a^{-1}$

$\therefore KE \propto \frac{1}{a^2}$

And what vacuum groundstate? Maybe we need quantum field theory to elucidate.

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Universe made of arbitrary boxes that grow with universal expansion.

$L = aL_0$

As box size goes to infinity (or finity for a hyperspherical universe) we can believe results still hold — at least maybe this believably

$$p \propto \frac{1}{a}$$

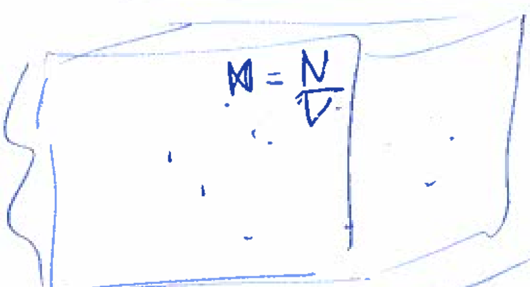
$$E \propto \frac{1}{a^2}$$

NR limit

→ this is for QM particle, but macroscopic objects are made of QM particles and it seems it should apply to macroscopic objects and it does — see 4044 part(e)

9) Energy Density of particles

$L = aL_0$



Assuming particles conserved and only change ~~lose~~ energy via adiabatic expansion effect

$$E_{total} = N E_{particle}$$

$$\underset{\text{energy density}}{\mathcal{E}} = \begin{cases} n E_{rest} \left(\frac{a_0}{a}\right)^3 & \text{CBR and then CNB} \\ n E_{any} \left(\frac{a}{a}\right)^2 \left(\frac{a_0}{a}\right)^3 & \text{NR} \\ n E_{particle} \left(\frac{a}{a}\right)\left(\frac{a_0}{a}\right)^3 & \text{ER} \end{cases}$$

$$= \begin{cases} n E_{r0} \left(\frac{a_0}{a}\right)^5 & \text{NR} \\ n E_{part} \left(\frac{a_0}{a}\right)^4 & \text{ER} \end{cases}$$

This turns out to be cosmic background radiation

4048 So for adiabatic expansion for ER particles

~~Later, we will prove~~

$E \propto \frac{1}{a^4}$ ~~for~~

But also remarkable, the CMB ~~[illegible]~~ radiation field stays Planckian

$E = a_{Rad} T^4$

and this means $\therefore$

$T \propto \frac{1}{a}$

The Cosmic Background radiation (CBR) which becomes the CMB (cosmic microwave background) has energy scale like this.

The cosmic temperature

There is one Before baryon-temp. decoupling which happened after recombination all ~~[illegible]~~ stuff had the ~~temperature~~, but now only the CMB has it: i.e.,

All stuff no matter how relativistic. So radiation and matter

See Hergt & Scott (2024, p.6)

$t_{recombination} = 372.6(10)$

$t_{baryon\text{-}Temperature\ decoupling} = 11.86(4)$ Myr

using Λ-CDM

$$T(t) = T_0 \left(\frac{a_0}{a(t)}\right)$$

$T_0 = 2.72548(57)$ Fixsen 2009

If you know $a(t)$, you can just find $T(t)$ to back to some ~~pre~~ pre-Big Bang Nucleosynthesis time. Remarkable.

back to end of inflation era $t \hat{=} 10^{-36}$ — 10^{-32} s

4.6) Cosmological Distance Measures Introduced

a) z — direct observable

$a(t)$ $\downarrow$ $\alpha(\tau)$ — scaled $\alpha = \frac{a}{a_0}$ ⟨ α NOT chi χ ⟩

even though $a_0 = 1$ for flat universe, but in unflat ~~can~~ use

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_{k_0}|}}$$

$$= \frac{4.2827... \; h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_{k_0}|}}$$

$$= \frac{13.968 \; h_{70}^{-1}\ \text{Gly}}{\sqrt{|\Omega_{k_0}|}}$$

(p. 4025)

t cosmic time — $\tau = H_0 t$ scaled cosmic time

$t_{\text{lookback}} = t_0 - t$

r, χ {chi Not α} — alternative comoving coordinates

Also

$v_{\text{rec}} \equiv H D_{\text{prime}}$ — recession velocity true at any instant in cosmic time

$v_{\text{redshift}} \equiv zc$ — a definition of what it is

$v_{\text{rec}} \to v_{\text{red}}$ asymptotically as $z \to 0$

Also

$D_{\text{proper tangential}} = a(t) r$

$D_{\text{proper radial}} = a(t) \chi$

{ $a(t)$ you can get from FE, but what is $r(z)$, $\chi(z)$? } Those have to be determined

Also

$D_{\text{luminosity}} = \sqrt{\dfrac{L_{\text{intrinsic}}}{4\pi F}}$ — Cepheids, SNe Ia; measured flux

Direct observable with a lot of effort. Not easy ~~Also~~

D_{angular} — used for BAO and we'll define later

Now Friedmann Eq. Solutions give $a(t)$ or $\alpha(\tau)$ in scaled form. ~~term~~

But direct observable is $z = \dfrac{\lambda_0 - \lambda_e}{\lambda_e} = \dfrac{a_0}{a} - 1$

or $z + 1 = \dfrac{a_0}{a} = \dfrac{a_0}{a_g}$ { ~~scaled~~ Dimensioned or undimensioned a. Most sources simply don't assume you'll know by context without a subscript. }

So you know $z = z(a(t)) = z(t)$ if you know $a(t)$

But you still need $r(z)$, $\chi(z)$ to connect direct observable D_{lum}, D_{angular}

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... intuitively obvious. Time travel & curvature effects go to zero, but intuitively obvious is not a proof!

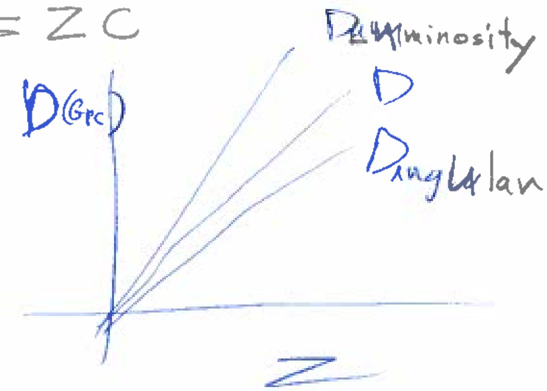
b) As $z \to 0$, it is true in 1st order in small z (but takes a proof) (see p. 4092) that

$$D^{1st}_{generic} = D^{1st}_{proper} = D^{1st}_{luminosity} = D^{1st}_{angular} = D_{geometric}$$

(- parallax
- Masers)

and $v^{1st}_{rec} = v_{red} = zc$

and $t^{1st}_{look\,back} = \dfrac{D^{1st}_{generic}}{c}$



It is important to prove these so that $v_{rec} = H\, D_{proper}$ can be shown to yield $v_{red} = H_0 D^{1st}_{generic}$

and so we can prove Hubble's law locally and can find H_0 — the current Hubble constant.

c) ~~Deriving~~ Finding $r(z)$, $\chi(z)$

If $k=0$ $r = \chi$

Distant galaxy — light signal → Milky Way

RW metric connecting light-like spacetime events along a radial direction

$$ds^2 = 0 = c^2 dt^2 - a(t)^2 \frac{dr^2}{1-kr^2}$$

light like

$$= c^2 dt^2 - a(t)^2 d\chi^2$$

Alternative comoving frame coordinates — we need them both in general. r and χ!

c) Finding $r(z)$ and $\chi(z)$

These are the comoving coordinates, but to theoretically predict key direct observables, you need to find them as functions of z for particular objects at cosmological redshifts z.

The key direct observables are

i) $D_{\text{luminosity distance}} \equiv \sqrt{\dfrac{L_{\text{intrinsic}}}{4\pi F}}$ ← known somehow from calibration or theory. A standard or standardizable candle

$D_{\text{luminosity distance}}$: Only a real distance in limit that the object doesn't move in travel time and there is correction if needed for extinction, so cosmologically $D_L \to D_{\text{proper}}$ only as $z \to 0$.

F: Flux measured for object

It's theoretical value for some cosmological model requires computing $r(z)$ or $\chi(z)$ for that model.

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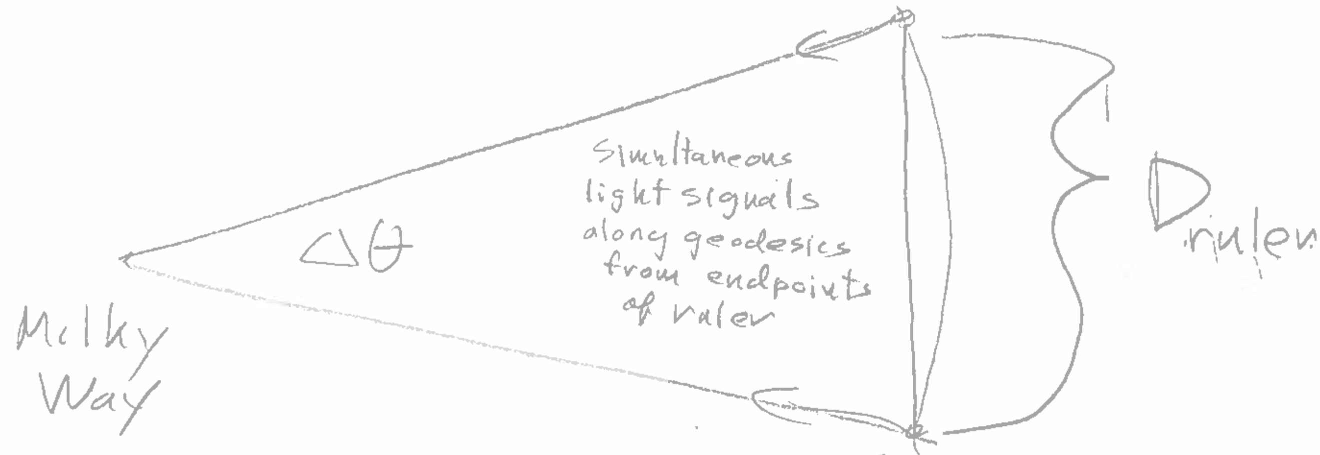
ii) $D_{\text{angular diameter distance}} = \dfrac{D_{\text{ruler}}}{\Delta\theta}$ ← known somehow / ← direct observable



The lack of curvature of ruler is assumed negligible or can be corrected for.

So D_L and D_A are considered direct observables if you know the standard candle and standard ruler. Getting the standards and observations are their own problems.

Now $r(z)$ and $\chi(z)$ are connected via the RW metric applied to light signal from object at redshift z.

For a light signal

$$\cancel{0} = c^2 dt^2 - a\left(\frac{dr^2}{1 - kr^2}\right)$$

for $ds^2 = 0$ for a light signal on a radial path to us

Milky Way $\longleftarrow$ Emitter

$$\therefore \; f(r) = \int_0^r \frac{dr}{\sqrt{1 - kr^2}} = \int_t^{t_0} \frac{c\,dt}{a(t)} = \chi(t)$$

where

$$r = f^{-1}(\chi)$$

We know the formulae relating the two comoving coordinates!

$$r = f^{-1}(\chi) = \begin{cases} \sin\chi & \text{for } k = 1 \\ \chi & \text{for } k = 0 \text{ (the easy case)} \\ \sinh\chi & \text{for } k = -1 \end{cases}$$

(see p. 4030 – 4032)

see p. 4030 – 4032 for chi χ

χ is a comoving coordinate but evaluating it for light signal travel time gives it the same formula as for conformal time η (see Lecture 3 p. 3262)

and the inverses are

$$\chi = f(r) = \begin{cases} \sin^{-1} r, & k = 1 \\ r, & k = 0 \\ \sinh^{-1} r, & k = -1 \end{cases}$$

$d\tau = H_0\,dt$ is scaled time

$$\text{Now } \chi = \int_t^{t_0} \frac{c\,dt}{a(t)} = \frac{c}{H_0 a_0} \int_\tau^{\tau_0} \frac{d\tau}{x(\tau)}$$

x not chi χ

$x(\tau)$ is a Friedmann eqn solution.

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and $a_0 = \begin{cases} 1 & \text{conventionally for flat space and is dimensionless or for Friedmann eqn solutions not worrying about curvature} \\ a_{0g} = \dfrac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \dfrac{4.2827\ldots\, h_{70}^{-1}\,\text{Gpc}}{\sqrt{|\Omega_{k0}|}} = \dfrac{13.968\ldots\, h_{70}^{-1}\,\text{Gly}}{\sqrt{|\Omega_{k0}|}} \\ \text{for curved space when you want to give it dimensions} \\ a_{0g} = \underbrace{|R_G|}_{\text{Gaussian curvature radius (CL-12)}} = \left|\dfrac{a_{0g}}{\sqrt{k}}\right| \end{cases}$

Recall $\dfrac{a_0}{a} = z + 1$

and so solving for chi $\chi(t)$

gives $\chi(t) = \chi[t(a)] = \chi[t(a(z))] = \chi(z)$,

and so $\chi(z)$ and $r(z)$ (see p. 4053)

4.7 Solving for $\chi(z)$

) Can we solve for $\chi(t) = \chi(z)$?

Yes, but analytically only for relatively simple Friedmann solutions $a(t)$

a) de Sitter Universe: A Pure Cosmological Constant universe

$k = 0$ and $r = \chi$ (see p 4053)

$$a = a_0 e^{H_0(t - t_0)}$$ where $H_0 = \sqrt{\Lambda/3} = H_\Lambda$

(see Lecture 3 p. 3240)

which is a constant

and recall $H = \frac{\dot{a}}{a}$

$$\frac{\Delta a}{\Delta t} = H_0 a$$

$$\frac{\Delta t}{\Delta a} = \frac{1}{H_0 a}$$

And $z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{a_0}{a} - 1$

$$a = \frac{a_0}{1+z}$$

$$\chi = \int_t^{t_0} \frac{c\,dt}{a} = \int_a^{a_0} \frac{c \frac{dt}{da} da}{a}$$

$$= \frac{c}{H_0} \int_a^{a_0} \frac{da}{a^2} = \frac{c}{H_0} \left(-\frac{1}{a}\right)\Big|_a^{a_0}$$

$$= \frac{c}{H_0}\left(\frac{1}{a} - \frac{1}{a_0}\right)$$

$$= \frac{c}{H_0}\left[\frac{z+1}{a_0} - \frac{1}{a_0}\right]$$

$$\chi = \frac{zc}{H_0 a_0}$$ which is a very simple result

$$\therefore D_{proper} = \frac{zc}{H_0}$$

$$H = H_0 \sqrt{\Omega_M x^3 + \Omega_\Lambda}$$

$$H_\Lambda = 70 h_{70} \sqrt{\Omega_\Lambda} = 70\sqrt{.7}\, h_{70} \sqrt{\frac{\Omega_\Lambda}{0.7}}$$
$$= \left(58.566\ldots \frac{\text{km/s}}{\text{Mpc}}\right) h_{70} \sqrt{\frac{\Omega_\Lambda}{0.7}}$$

and $v_{recession} = H_0 D_{proper} = \underbrace{zc \equiv v_{redshift}}_{\text{definition of the redshift velocity}}$

This is the only case where $v_{rec} = v_{red}$ for z, and NOT just asymptotically as $z \to 0$

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(2026 jan 24)

(i) A general Approach to $\chi_1(z)$

$\left\{ d\eta = \frac{d\tau}{x(t)} \right.$ is conformal time

See p. 4063

$$\chi(t) = \int_t^{t_0} \frac{c\,dt}{a(t)} = \frac{c}{H_0 a_{g0}} \int_\tau^{\tau_0} \frac{d\tau}{x(t)} = \frac{c}{H_0 a_{g0}} \int_x^{x_0} \frac{dx}{x^2\sqrt{\cdots}} = \frac{c}{H_0 a_{g0}} \int_x^{x_0} d\eta$$

(see p. 4053)

But recall the Friedmann equation form

$$\left(\frac{\dot{x}}{x}\right)^2 = \sum_p \Omega_{p0} x^{-p} \quad \text{for } x_0 = 1$$

If you have a numerical or exact solution you can solve for $\chi(t)$, $\chi(a)$, and $\chi(z)$.

$$\pm \frac{dx}{x\sqrt{\cdots}} = d\tau$$

$$d\eta = \frac{d\tau}{x} = \frac{dx}{x^2\sqrt{\cdots}} = \frac{dx}{x\sqrt{\sum_p \Omega_{p,0}\, x^{-(p-2)}}}$$

$p_\eta = p - 2$

$$\eta(x, \{p\}, \{\Omega_{p0}\}) = \tau(x, \{p_\tau = p-2\}, \{\Omega_{p0}\})$$

set of actual power parameters for a Friedmann eqn. solution.

The set of p_τ power parameters in $\tau(x, p_\tau = p-2, \{\Omega_{p,0}\})$

p NOT p_τ

$$z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{x_0 - x}{x} = \frac{x_0}{x} - 1$$

$$x = \frac{x_0}{1+z} = \frac{1}{1+z} \quad \text{if } x_0 = 1$$

$$\text{So } \chi(z) = \frac{c}{H_0 a_{g0}}\left[\eta(x=1) - \eta\left(x = \frac{1}{1+z}\right)\right] = \frac{c}{H_0 a_{g0}} \Delta\eta(z)$$

To recapitulate the elementary solutions (where $\Omega_{p0} = 1$ actually)

For $p \neq 0$ $\left\{ d\tau = \frac{dx}{\sqrt{\Omega_{p0}}\, x^{-p/2+1}} = \frac{x^{p/2-1}\,dx}{\sqrt{\Omega_{p0}}}, \quad \tau = \frac{x^{p/2}}{\sqrt{\Omega_{p0}}\,(p/2)} \right.$

for a Point origin

I) Elementary solution χ's

$$\chi(z, \underbrace{p}_{p \neq 2}, p_\gamma = p-2) = \frac{c}{H_0 a_{g0} \sqrt{\Omega_{p0}}} \left[\frac{1}{\frac{p-2}{2}}\right]\left[1 - \left(\frac{1}{1+z}\right)^{\frac{p-2}{2}}\right]$$

Always $\chi \geq 0$ for $p > 2$ and for $p < 2$

$\Omega_{p0} = 1$

0	$z = 0$
$\left[\frac{1}{\frac{p-2}{2}}\right]\left[+\left(\frac{p-2}{2}\right)z\right] = z$	z small
$\left(\frac{1}{\frac{p-2}{2}}\right)\left[1 - \left(\frac{1}{z}\right)^{\frac{p-2}{2}}\right]$	$p > 2$, z large
$\left(\frac{1}{\frac{p-2}{2}}\right)$	$p \geq 2$ asymptotic value
$\left(\frac{1}{\frac{2-p}{2}}\right)\left[(1+z)^{\frac{2-p}{2}} - 1\right]$	$p < 2$
$\left(\frac{1}{\frac{2-p}{2}}\right)\left[z^{\frac{2-p}{2}}\right]$	$p < 2$, z large
∞	$p < 2$, $z \to 0$
z	$p = 0$, just as on p. 4055

But what of $P = 2$ giving $P_\gamma = 0$?

Recall $P = 2$ is the power for curvature, the $R_h = ct$ universe, and some cosmic string density components (Steiner-6)

Well, $d\tau = \frac{dx}{\sqrt{\Omega_{20}}\, x^{-1}}$

$\tau = \frac{1}{\sqrt{\Omega_{20}}} \ln x + C$

$= \ln[Cx]$

setting $\Omega_{20} = 1$

$$\therefore \chi(z, p = 2, p_\gamma = p - 2 = 0) = \frac{c}{H_0 a_{g0}}\left[\ln 1 - \ln\left(\frac{1}{1+z}\right)\right]$$

$$= \frac{c}{H_0 a_{g0}}\left[\ln(1+z)\right]$$ / wik

z	$z \ll 1$
$\ln z$	$z \gg 1$

4058 ('2026 jan 24)

d) What of interesting multi density component X's?

Radiation matter universe } $P = 4$, $P = 3$ } $P_z = 2$, $P_z = 1$

{ parameters from J26
$V = \frac{2}{2-1} = 2$
$U = \frac{1}{2-1} = 1$
$W = \frac{V}{P} = \frac{1}{2}$
so an exact solution exists

Matter curvature universe } $P = 3$, $P = 2$ } $P_z = 1$, $P_z = 0$

{ $V = \frac{2}{2-0} = 2$
$U = 0$
$W = 0$
and so an exact solution exists

Matter-Λ universe } $P = 3$, $P = 0$ } $P_z = 1$, $P_z = -2$

J26 didn't consider power parameters < 0, but this would be

$$d\eta = \frac{d\tau}{x} = \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_{30} x^{-3}}} = \frac{dx}{\sqrt{\Omega_\Lambda x^4 + \Omega_{30} x}}$$

(see p. 4056)

Google AI says No exact solution in terms of elementary functions, but

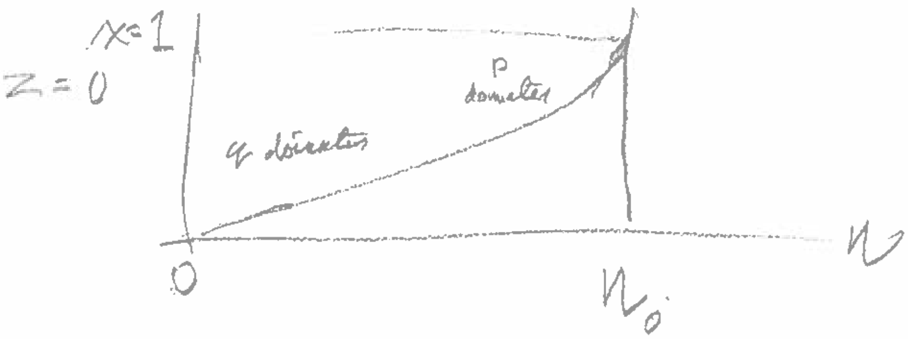
$$y = \int \frac{dx}{\sqrt{ax^4 + bx}} = \frac{2\sqrt{x}}{\sqrt{b}} \cdot \underbrace{{}_2F_1\left(\frac{1}{6}, \frac{1}{2}; \frac{7}{6}; -\frac{ax^3}{b}\right)}_{\text{Gauss hypergeometric function}} + C$$

I don't consider this an exact solution. It's a pity since the matter-Λ universe is the ΛCDM solution after the short radiation era (from inflation to radiation-mass equality at $t \approx 50$ kyr).

Omit in class. It's worthless

f) An Idea for Solving $d\eta = \dfrac{dx}{\sqrt{ax^{p} + bx^{q}}}$

where $p > q \ge 1$ and $p = -(p_{real} - 4)$
$q = -(q_{real} - 4)$



for example $q_{real} = 3$
and $q = 1$
$p_{real} = 0$
$p = 4$
As for Matter - Λ universe

Ansatz

$$\eta_{apr} = \begin{cases} \dfrac{2}{qb}\sqrt{ax^{p} + bx^{q}} + Cx^{n} + Kx^{m} & \text{where } C \text{ and } n \text{ need to be fitted} \\ 0 & \text{for } x = 0 \end{cases}$$

$$\frac{\partial \eta_{apr}}{\partial x} \begin{cases} \dfrac{1}{qb}\,\dfrac{pax^{p-1} + qbx^{q-1}}{\sqrt{\ldots\ldots}} + nCx^{n} + mKx^{m-1} & \text{general} \\ \dfrac{1}{qb}(pa + qb) + nC + mK = 1 \\ K = \dfrac{1 - nC - \frac{1}{qb}(pa + qb)}{m} \end{cases}$$

at $x = 1$ $\dfrac{\partial \eta}{\partial x} = 1$ by scaling

Specialize $q = 1$ since general q too hard.

$$\frac{\partial \eta_{apr}}{\partial x} = \frac{1}{\sqrt{ax^{p} + bx}} + \frac{pax^{p-1}}{\sqrt{bx}} + nCx^{n} + mKx^{m-1}$$

as $x \to 0$

$$= \frac{1}{\sqrt{bx}}\left[1 - \frac{1}{2}\frac{a}{b}x^{p-1} + \frac{3}{8}\left(\frac{a}{b}x^{p-1}\right)^{2}\right] + \frac{pax^{p-1}}{\sqrt{bx}} + nCx^{n} + mKx^{m-1}$$

recall $\dfrac{\partial \eta}{\partial x}$ asymptotic

$n = p - 3/2$

$$C = -\frac{pa}{\sqrt{b}}\,\frac{1}{(p - 3/2)}$$

4060

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Seems hopeless. With no k, C is overconstrained. With k, k is overconstrained.

Maybe Set $k = 0$ and $n = p - 3/2$

and $C = \dfrac{1 - \frac{1}{b}(p_a + b)}{n}$

and forget the 2nd order small x fit.

If $p = 4$, $b = 0.3$, $a = 0.7$ and $n = 4 - 3/2 = 5/2$

then $C = \dfrac{1 - \frac{1}{.3}(2.8 + .3)}{5/2}$

$\doteq \dfrac{1 - 10}{5/2} = -\frac{18}{5} \doteq -3.6$

But for Λ-CDM $\eta(x=1) = ?$ I should find out.

$\eta(x=1) \doteq \frac{2}{b} \cdot 1 - 3.6 \doteq 6 - 3.6 = 2.4$

which at least is positive

But the whole rigamarole is too complex for a crude uncontrolled approximation

9) So say we have $\chi(z) = \frac{c}{H_0 a_{g0}} \Delta\eta(z)$

First, if $k = 0$ (i.e., zero curvature)

$a_{g0} = a_g(t_0)$ the scale factor at cosmic present.

$$D_{\text{proper radial}} = a_g(t_0) \int_0^{r} \frac{dr}{\sqrt{1-kr^2}}$$

{see p. 4026 for RW Metric

$$= a_{g0} \int_{z}^{0} d\chi$$

{(see p. 4056) & p. 4030 - 4032 for relations of r and χ

for $k = 0$,
$dr = d\chi$
(see p. 4032)

$$= \frac{c}{H_0} \Delta\eta(z)$$

If you don't have an exact $\eta(z)$ or adequate approximation, then you'll need to integrate numerically, see p. 4056 for the integration form,

Note in this case a_{g0} cancels out which is good since it is undefined for $\Omega_{k0} = 0$. See p. 4005.

In fact for $k = 1$, ~~you~~ you can just use the dimensionless $a_0 = 1$.

4062

What if $k \neq 0$, i.e., $k = \pm 1$

$$D_{\text{proper radial}} = a_{g0} \int_0^r \frac{dr}{\sqrt{1 - kr^2}}$$

(see p. 4026 for the RW metric

$$= a_{g0} \begin{cases} \sin \chi = \sin\left[\dfrac{c}{H_0 a_{g0}} \Delta\eta(z)\right] \\ \sinh \chi = \sinh\left[\dfrac{c}{H_0 a_{g0}} \Delta\eta(z)\right] \end{cases}$$

see p. 4030 or p. 4053

see p. 4032 or p. 4053)

Except to lowest order in $\frac{c}{H_0 a_{g0}} \Delta\eta(z)$, the absolute value of the Gaussian curvature radius does NOT cancel out.

(see p. 4005

Fortunately, so far the observable universe is flat to within $\sim 1\sigma$

(see p. 4006)

Recall
$$a_{g0} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{4.2827\ldots\, h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_{k0}|}}$$
$$= \frac{13.968\ldots\, h_{70}^{-1}\ \text{Gly}}{\sqrt{|\Omega_{k0}|}}$$

(p. 4005)

and $\Omega_{k0} \doteq 0.0019(15)$

(ACT $\equiv$ Atacama Cos. Tel., 2025)

There is a small z approximation that we will use later in using comoving coordinates $r(z)$ and $\chi(z)$

$$\chi(r) = \int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_0^r dr\left[1 + \tfrac{1}{2}kr^2 + O(r^4)\right]$$

$$\therefore \quad \chi(r) = r + O(r^3)$$

On p. 4077, we will show that

$$r_{2nd} = b_1 z + b_2 z^2$$

to 2nd. order in small z (see p. 4077)

$$\therefore \quad \chi_{2nd}(r \text{ or } z) = r = b_1 z + b_2 z$$

4.8 More on Conformal Time

Should I have done this before all the work on p. 4056?

Recall conformal time is an auxillary quantity defined $\eta = \int \frac{c\,dt}{a(t)} = \frac{c}{H_0}\int \frac{dz}{X(z)}$

Of course, everything runs on real time: e.g., pendulums, nuclear reactions, planet orbits, etc.

where η = eta

and definitions vary slightly by, e.g., setting $c = 1$

4064

$$\chi = \int_{\tau}^{\tau_0} d\eta = \frac{c}{H_0 a_{g0}} \int_{\tau}^{\tau_0} \frac{d\tau}{x}$$ where

$a(t) = a_{g0}\, x(\tau)$

and $a_{g0} = 1$ for flat space and derived on p. ~~4004~~ on p. 4062

$$= \frac{c}{H_0} \int_{x}^{x_0} \frac{dx}{x^2 \sqrt{\sum_p \Omega_{p0} x^{-p}}}$$

where recall the Friedmann eqn

$$\left(\frac{\dot{x}}{x}\right)^2 = \sqrt{\sum_p \Omega_{p0}\, x^{-p}}$$

for (inverse) power parameters

$$d\tau = \frac{dx}{x\sqrt{\sum_p \Omega_{p0}\, x^{-p}}}$$

Now $z = \frac{\lambda_0 - \lambda}{\lambda}$

$= \frac{x_0}{x} - 1$

$x = \frac{x_0}{1+z}$

$= \frac{1}{1+z}$ if $x_0 = 1$

$dx = \frac{-dz}{(1+z)^2} = -x^2 dz$

$$\therefore \chi = \frac{c}{H_0 a_{g0}} \int_0^z \frac{dz}{\sqrt{\sum_p \Omega_{p0} (1+z)^p}}$$

Recall
- $p = 5$ classical kinetic energy
- $p = 4$ "radiation" = extreme relativistic particles
- $p = 3$ "matter" = rest mass
- $p = 2$ curvature, $K_n \neq 0$ universe, some cosmic strings
- $p = 1$ quintessence (some theories)
- $p = 0$ cosmological constant or constant dark energy

On p. 4056, we already considered how exact solution for $\chi(z)$ and don't think anymore progress can be made with the explicit integral in terms of z for exact solutions, but for numerical $\chi(z)$ the formula on p. 4064 is probably the cleanest to use.

If you want $\tau(z)$

$$d\tau = a(\tau)\, d\chi = \frac{\chi c}{H_0 a_{g0}} \frac{dz}{\sqrt{\sum_p \Omega_{p0}(1+z)^p}}$$

$$= \frac{c}{H_0 a_{g0}} \frac{1}{(1+z)} \frac{dz}{\sqrt{\sum_p \Omega_{0p}(1+z)^p}}$$

which will have to be solved numerically unless an exact solution $\tau(\chi) = \tau(z)$ is available.

4066

4.9) Proving the Small z Approximations (or an exercise in todium)

The small z approximations apply to the local observable universe ($z << 1$) and allow us to determine cosmological parameters without specifying a Friedmann Equ. model, but still assuming the Friedmann Equ. is correct because without that our "parameters" may NOT mean what we think they do.

a) Note, nowadays fitting whole data sets to Friedmann Equ models or other cosmological models, may have made such expansions less important than in the past.

Also we are just doing Taylor series expansions, but Padé approximant expansions are probably better — need fewer terms (?) and large radius of convergence (?) (Wiki: Pade Ap.)

Hower, just as with Taylor series, Padé approximants may be NOT of such use given fits to overall cosmological models

(

4068

a) <u>Cosmic Scale factor $a(t)$ Expansion in small $\Delta t = t - t_0 \leq 0$</u>

$$a(t) = a_0 + \dot{a}_0 \Delta t + \tfrac{1}{2} \ddot{a}_0 \Delta t^2 + \ldots$$

where $\Delta t = t - t_0 \leq 0$

and $t_{\text{lookback}} = -\Delta t \geq 0$

Rewriting in terms of Hubble constant H_0 and deceleration parameter q_0

$$a(\Delta t) = a_0 \left[1 + \frac{\dot{a}_0}{a_0} \Delta t + \frac{1}{2} \frac{\ddot{a}}{a_0} \Delta t^2 + \ldots \right]$$

$$= a_0 \left[1 + H_0 \Delta t + \tfrac{1}{2}(-q_0) H_0^2 \Delta t^2 + \ldots \right]$$

Recall $H_0 \equiv \frac{\dot{a}_0}{a}$ and $q \equiv -\frac{\ddot{a} a}{(\dot{a})^2} = -\frac{\ddot{a}}{a} \frac{1}{H^2}$ (Li-53)

$$= -\left[\frac{(-\frac{4\pi}{3} G)(\rho + \frac{3P}{c^2}) + \frac{\Lambda}{3}}{\frac{8\pi G}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3}} \right] \quad \begin{matrix}\text{(Li-27)} \\ \text{(Li-28)}\end{matrix}$$

in general.

$$= \frac{\sum_p \Omega_{p0} x^{-p} (p/2 - 1)}{\sum_p \Omega_{p0} x^{-p}}$$

for density parameters depending only on (inverse) powers p (Lect. 3, p. 3235)

$$q_0(x=1) = \sum_p \Omega_{p0} (p/2 - 1)$$

$$= \left[0.3(3/2 - 1) + 0.7(-1) \right]$$

$$= [0.15 - 0.7]$$

$= -.55$ fiducial Λ-CDM value

q_0 is still of some interest, but $q(t)$ or $q(z)$ as a diagnostic for an overall fit to observations may be more important nowadays.

Note

H_0 is the 'relative rate' of expansion of the universe

$$H_0 = 70 \,\frac{km/s}{Mpc}\, h_{70}$$

$\rho_{crit} = \frac{3H_0^2}{8\pi G}$

where $h_{70} = \frac{H_0}{(70 \frac{km/s}{Mpc})}$

if $\rho_{M\Lambda} = \rho_c$, the universe is flat

CMB gives $H_0 \doteq 68 \,\frac{km/s}{Mpc}$

Local universe $H_0 = 73 \,\frac{km/s}{Mpc}$

They disagree by ~ 4 or 5σ.

This is the famous Hubble tension.

Endless cosmological fixes have been suggested

Is this a failure of Λ-CDM, or are one (or both) affected by some systematic error.

Astrophysical Fixes

It may be we live large low density region

Banik et al. (2026) suggest out to $z \doteq 0.2$ (or $D \doteq 800$ cMpc).

But some suggest the Planck reionization calculation is NOT right. A non standard treatment needed.

b) From $a(\Delta t)$ expansion to $z(\Delta t)$ expansion

From p. 4072, $\frac{a(\Delta t)}{a_0} = \left[1 + H_0 \Delta t + \frac{1}{2}(-q_0) H_0^2 \Delta t^2 + \dots\right]$

where recall $t_{lookback} = -\Delta t$

but recall

$z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{a_0}{a} - 1$

$z + 1 = \frac{a_0}{a}$

or $\frac{a}{a_0} = \frac{1}{1+z}$

$= \frac{1}{1+z} = 1 - z + z^2 - \dots$

2nd order geometric series (Arf -279)

$\therefore \; z - z^2 + \dots = -H_0 \Delta t + \frac{1}{2} q_0 H_0^2 \Delta t^2 + \dots$

Expand $z = a_1 \Delta t + a_2 \Delta t^2 + \dots$

and substitute for z and determine the coefficients b_i

$(a_1 \Delta t + a_2 \Delta t^2 + \dots) - (b_1 \Delta t + \dots)^2 = -H_0 \Delta t + \frac{1}{2} q_0 H_0^2 \Delta t^2 + \dots$

$\therefore \; a_1 = -H_0$

and $a_2 - a_1 = \frac{1}{2} q_0 H_0^2$

$a_2 = \frac{1}{2} q_0 H_0^2 + H_0^2 = (\frac{1}{2} q_0 + 1) H_0^2$

$\therefore \; z = -H_0 \Delta t + (\frac{1}{2} q_0 + 1) H_0^2 \Delta t + \dots$

Which a true result, but we really want $\Delta t(z)$.

So let $\Delta t = b_1 z + b_2 z^2 + \dots$

and try again.

Or from Arf-316-317 use the power series inversion formula

If $\Delta y = \sum_{\ell=1}^{\infty} a_\ell \Delta x^\ell$, then $\Delta x = \sum_{\ell=1}^{\infty} b_\ell \Delta y^\ell$

$$\begin{cases} a_1 = -H_0 \\ a_2 = (1+\tfrac{1}{2}q_0)H_0^2 \end{cases}$$

where $b_1 = \frac{1}{a_1} = \frac{1}{-H_0}$

$b_2 = -\frac{a_2}{a_1^3} = -\frac{(1+\frac{1}{2}q_0)H_0^2}{(-H_0)^3} = (1+\tfrac{1}{2}q_0)/H_0$

$\therefore \Delta t = -\frac{1}{H_0}z + \left[(1+\tfrac{1}{2}q_0)/H_0\right]z + \ldots$

$= -\frac{z}{H_0}\left[1 - (1+\tfrac{1}{2}q_0)z + \ldots\right]$

to 2^{nd} order in z CL-17

$\therefore t_{lookback} = \frac{z}{H_0}\left[1 - (1+\tfrac{1}{2}q_0)z + \ldots\right]$

to 2^{nd} order in z

$= \frac{z}{H_0} = z\,t_{H_0}$ to 1^{st} order in z

So $t_{lookback}$ is an observable to 1^{st} order if you know H_0 and to 2^{nd} order if you know H_0 and q_0

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c) From $a(\Delta t)$ to $\chi(z)$ and $r(z)$ to 2nd order in z

Recall for a light signal from the RW metric
$$ds^2 = 0 = c^2 dt^2 - a^2 d\chi^2$$ (see p. 4026 and 4032) where chi χ is a comoving coordinate

$$\therefore \chi = \int_t^{t_0} \frac{c\,dt'}{a(t')} = \int_{\Delta t}^{0} \frac{c\,d\Delta t'}{a_0\left[1 + H_0\Delta t' + \frac{1}{2}(-q_0)H_0^2\Delta t'^2 + \dots\right]}$$

Note $\Delta t = t - t_0$
$\therefore d\Delta t = dt$ where $\Delta t < 0$

from p. 4072

$$\chi = \frac{c}{a_0}\int_{\Delta t}^{0} d\Delta t\left[1 - H_0\Delta t' - \frac{1}{2}(-q_0)H_0^2\Delta t'^2 + (H_0\Delta t')^2 + \dots\right]$$

Using the geometric series (Arf-279)

$$= \frac{c}{a_0}\int_{\Delta t}^{0} d\Delta t\left[1 - H_0\Delta t' + \left[(\tfrac{1}{2}q_0 + 1)H_0^2\right]\Delta t'^2 + \dots\right]$$

$$= \frac{c}{a_0}\left[\Delta t' - \frac{1}{2}H_0\Delta t'^2 + \frac{1}{3}\left[(\tfrac{1}{2}q_0+1)H_0^2\right]\Delta t'^3 + \dots\right]_{\Delta t}^{0}$$

$$= \frac{c}{a_0}\left[-\Delta t + \frac{1}{2}H_0\Delta t^2 - \frac{1}{3}(\tfrac{1}{2}q_0+1)H_0^2\Delta t^3 + \dots\right]$$

and recall $\Delta t < 0$

$$\chi = \frac{c}{a_0}\left[t_{lookback} + \frac{1}{2}H_0 t_{lookback}^2 + \dots\right]$$

to 2nd order in $t_{lookback} = -\Delta t$

$$\chi = \frac{c}{a_0}\left[\frac{z}{H_0} - \underbrace{\left(1+\tfrac{1}{2}q_0\right)\frac{z^2}{H_0} + \tfrac{1}{2}H_0\left(\frac{z}{H_0}\right)^2}_{} + \dots\right]$$

substituting for t_{lookback} from p. 4075

$$\chi = \frac{c}{a_0}\left[\frac{z}{H_0} - \tfrac{1}{2}(1+q_0)\frac{z^2}{H_0} + \dots\right]$$

$$= \frac{zc}{a_0 H_0}\left[1 - \tfrac{1}{2}(1+q_0)z + \dots\right]$$

to 2nd order in z

$$= \frac{zc}{a_0 H_0}$$ to 1st order in z

From p. 4063, we have $\chi_{\substack{\text{2nd} \\ \text{in } r}} = r$

$$\therefore \quad \chi_{\substack{\text{2nd} \\ \text{in } z \\ \text{and } r}} = r = r_{\substack{\text{2nd} \\ \text{in} \\ z}} = \frac{zc}{a_0 H_0}\left[1 - \tfrac{1}{2}(1+q_0)z + \dots\right]$$

(CL-18)

4.10 More Elementary Cosmological Distance Measures Formulae

Observational and Theoretical in General and for small z

a) Cosmological Redshift z $a = \frac{a_0}{1+z}$

$$z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{\lambda_0}{\lambda} - 1 = \frac{a_0}{a} - 1$$

is an easily obtained direct observable

Of course, you can calculate it theoretically if you know $a(t)$ from an exact or numerical solution of the Friedmann equation for which you would need all cosmic present density parameters and how they evolve with cosmic time

If they were all inverse power density parameters you need all $\Omega_{p,0}$'s or all Ω_{p0}'s less 1 and Hubble constant H_0

b) Lookback Time t_{lookback} and Cosmic Time

Observationally, $t_{\text{lookback}} = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2}q_0\right)z\right]$ to 2nd order in small z

$= \frac{z}{H_0}$ to 1st order in small z

(see p. 4075 & CL-17) But only a direct observable if you know H_0 for 1st order and H_0 and q_0 for 2nd order

If you know $t(a)$ from numerical or exact solution of the Friedmann eqn.,

then $t_{\text{lookback}} = t_0 - t(a)$

$$= t_0 - t\left(\frac{a_0}{1+z}\right).$$

c) Comoving Coordinates χ and r

These are really theoretical quantities in first understanding.

You can calculate $\chi(t) = \int_{t_0}^{t} \frac{c\,dt}{a(t)}$

$$= \frac{c}{H_0 a_0} \Delta\eta(z)$$

$$= \chi(z)$$

if you you a solution $a(t)$ from a numerical or exact solution of the Friedmann eqn, see p. 4056 for a general approach to solutions. Then you can solve

for $r(\chi) = r(t) = r(z) = \begin{cases} \sin\chi, & k=+1 \\ \chi, & k=0 \\ \sinh\chi & k=-1 \end{cases}$

see p 4030-4032

4076

For small z,

$$r(z) = \chi(z) = \frac{zc}{H_0 a_0}\left[1 - \tfrac{1}{2}(1+q_0)z\right]$$

to 2nd order in small z (see p. 4077 & CL-18)

d) D_{proper}

see p. 4027

$dD_{proper\ tangential} = \underbrace{a_g(t)}_{\text{So you need the solution } a_g(t) \text{ which for } k\neq 0 \text{ means you need } \Omega_{k,0}} r\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$ where $r = \begin{cases} \sin\chi, & k=1 \\ \chi, & k=1 \\ \sinh\chi, & k=-1 \end{cases}$

$D_{proper\ radial} = a_{g0}\chi$ (see p. 4027)

For small z at cosmic present

$$D_{proper\ radial} = \frac{zc}{H_0}\left[1 - \tfrac{1}{2}(1+q_0)z\right] \quad \text{(CL-18)}$$

to 2nd order

$$= \frac{zc}{H_0} \quad \text{to 1st order}$$

(see just above)

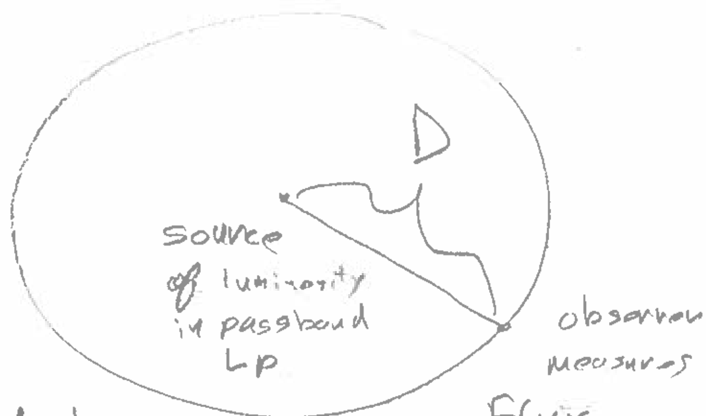
4.11 D_L = $D_{\text{Luminosity}}$ Distance

a) To recapitulate from p. 4051

In ordinary unexpanding flat space with source and receiver at rest with respect to each other and negligible (or corrected for extinction)

$$F_p = \frac{L_p}{4\pi D^2}$$

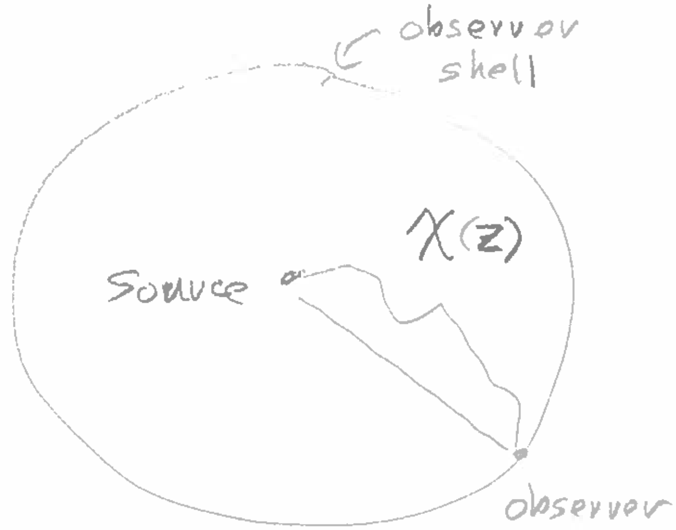
$$\therefore \quad D = \sqrt{\frac{L_p}{4\pi F_p}}$$



flux F_p in passpand P.

and D is the true physical distance (i.e., proper distance, a spacetime interval measured at one instant in time)

In a curved expanding spacetime, we define luminosity distance



$$D_L = \sqrt{\frac{L_{pK}}{4\pi F_{p_0}}} = \sqrt{\frac{L}{4\pi F_0}}$$

which is length quantity but is a direct observable <u>if</u>

4078

The second definition uses total intrinsic luminosity and total Flux; ie, quantities integrated over all frequency (or energy or wavelength),

You measure F_{po} (the Flux at observer at cosmic present measured in passband p)

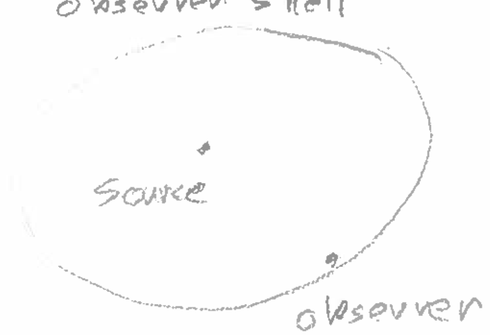
and know L_{pk} (luminosity the intrinsic luminosity in (inverse) K-correction in passband p

which we can know (ie, calculate) if we know enough about the emission of a standard (or standardizable) candle: e.g., Cepheids, SNe Ia (which are calibrated largely using Cepheids)

There are a lot of random and systematic errors to beat down but that has been done to high degree — but with caveats

b) How do we calculate L_{pk} and total intrinsic luminosity L given the intrinsic L_E (intrinsic luminosity as a function of energy)?

observer shell



The source releases a burst of photons n in energy interval dE in time dt

In the ideal limit of no extinction, etc.,

$$\underbrace{n_0}_{\text{photon burst at observer shell in energy interval } dE_0 \text{ and time } dt_0} = n \quad \left\{ \text{photons as discrete entities are conserved} \right.$$

∴ from $n_0 = n$,

$$\frac{L_{0E_0}\, dt_0\, dE_0}{E_0} = \frac{L_E\, dt\, dE}{E}$$

E_0 is the observed photon energy and E is the source photon energy

$$L_{0E}\, dE_0 = L_E\, dE \left(\frac{dt}{dt_0}\right)\left(\frac{E_0}{E}\right)$$

Now on p. 4040 – 4041, we found the cosmological redshift formula.

$$\nu \propto \frac{1}{a}$$

but $\nu \propto \frac{1}{\text{Period of a wave}} \propto \frac{1}{dt}$

a differential small time

∴ $dt \propto a$

$$\frac{dt}{dt_0} = \frac{a}{a_0}$$

$$E \propto \frac{1}{a}$$

$$\frac{E_0}{E} = \frac{a}{a_0}$$

∴ two factors of $\left(\frac{a}{a_0}\right)$ $= \left(\frac{dt}{dt_0}\right)\left(\frac{E_0}{E}\right) = \left(\frac{a}{a_0}\right)^2 = \frac{1}{(1+z)^2}$

$$∴ \quad L_{0E_0}\, dE_0 = L_E\, dE \left(\frac{a}{a_0}\right)^2 = L_E\, dE\, (1+z)^{-2}$$

$$L_{P0} = \int \underbrace{P(E_0) L_{0E_0}\, dE_0}_{\text{Passband } P} = \int \underbrace{P\left[E_0 = E\left(\tfrac{a}{a_0}\right)\right] L_E\, dE}_{\text{An (inverse) K-correction passband } P} (1+z)^{-2}$$

$$= L_{PK} (1+z)^{-2}$$

4080

A K-correction turns an observed redshifted F_{po} into an intrinsic F_p (i.e., rest frame flux in passband p)

The idea of K-correction goes back to Wirtz (1918) and maybe independently to Hubble at bit later (Wik)

But here we want calculate the redshifted L_{po} from the intrinsic luminosity L_E, and so need an inverse K-correction

We find $L_{pK} = L_{po}(1+z)^2$

If $p(E) = 1$,

$$\underbrace{L}_{\text{total intrinsic luminosity}} = \underbrace{L_0(1+z)^2}_{\text{total luminosity at the observer shell}}$$

Note $D_L = \sqrt{\dfrac{L_{pK}}{4\pi F_{po}}}$ uses L_{pK} and NOT L_{po} which would've an OK definition

so that $D_L = \sqrt{\dfrac{L}{4\pi F}}$ is also true, and this seemed the more natural general definition of luminosity distance

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c) How do we calculate D_L theoretically in order to compare to observations?

From p. 4030–4032, the surface area of a sphere in general space at cosmic present is

$$S_0 = 4\pi a_{g0}^2 r^2 = 4\pi a_{g0}^2 \begin{Bmatrix} \sin^2\chi \ (k=1) \\ \chi^2 \ (k=0) \\ \sinh^2\chi \ (k=-1) \end{Bmatrix}$$

Recall from p. 4053

$$\underset{\text{chi}}{\chi} = \int_t^{t_0} \frac{c\,dt}{a(t)} = \left(\frac{c/H_0}{a_{g0}}\right)\int_\tau^{\tau_0} \frac{d\tau}{\alpha(\tau)}$$

where α is $\alpha(\tau_0) = 1$ and $a_{g0} = 1$ (if one likes) for $k=0$

and

$$a_{g0} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} \quad \text{for } k \neq 0$$

$$= \frac{4.2827\ldots h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_{k0}|}}$$

$$= \frac{13.968\ldots h_{70}^{-1}\ \text{Gly}}{\sqrt{|\Omega_{k0}|}}$$

see p. 4005

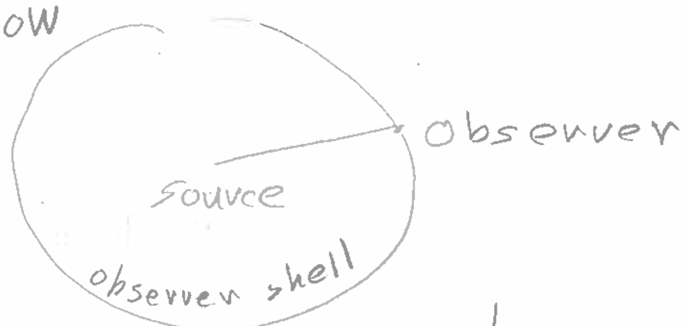
Note for $k=0$,

$r = \chi$ (chi), and

$$a_{g0} r = a_{g0}\chi = \frac{c}{H_0}\int_\tau^{\tau_0} \frac{d\tau}{\alpha(\tau)}$$

and we never needed to specify a_{g0} at all.

Now



and so $$F_{P0} = \frac{L_{P0}}{S_0} = \frac{L_{PK}(1+z)^{-2}}{S_0}$$ (see p. 4083)

And so

$$D_L = \sqrt{\frac{L_{pk}}{4\pi F_{po}}} = \sqrt{\frac{L_{pk}}{4\pi\left[L_{pk}(1+z)^{-2}/(4\pi a_{go}^2 r^2)\right]}}$$

observational

$$= a_{go}\, r\,(1+z)$$ (Theoretical

a_{go} calculated as on p. 4085

r calculate as on p 4085, You need the cosmic scale factor $\cancel{\chi}(z)$ an ex NOT a chi

$\therefore D_L = a_{go}\, r\,(1+z)$ in general

$$= a_{go}\begin{Bmatrix}\cancel{\chi}\ \text{2nd order in } z \\ r\ \text{2nd order in } z\end{Bmatrix}(1+z)$$

see p. 4077

To 2nd order in Z

$$= \frac{zc}{H_0}\left[1 - \tfrac{1}{2}(1+q_0)z\right](1+z)$$

$$= \frac{zc}{H_0}\left[1 + \tfrac{1}{2}(1-q_0)z\right]$$

See p. 4069 and CL-19

4.12 Angular-Diameter Distance

To recapitulate from p. 4052

$$D_{\text{angular diameter distance}} = D_A = \frac{D_{\text{ruler}}}{\Delta\theta}$$

Observational definition

direct observable

Known somehow. The most important rulers are BAOs (baryonic acoustic oscillations) which we may cover later

$D_{BAO} = 147.09(26)$ cMpc

Planck 2018

and unfortunately usually referred to as the drag scale (r_d) but is best called the BAO scale

Simultaneously emitted light signals from the ruler endpoints that follow geodesics to the observer.

$D_{\substack{BAO\\ \text{fiducial}}} = 150$ cMpc (wik)

c = comoving and means

$D(\text{Mpc}) = a(t)\,D(\text{cMpc})$

Note for BAOs, the ruler actually scales with a

$$D_{\text{ruler}}(z) = D_{\substack{BAO\\ \text{fiducial}}}\, a(z)$$

$$= D_{BAO\,\text{fiducial}}\left(\frac{1}{1+z}\right)$$

But for the theoretical formula for D_A, the convention is to consider the ruler D_R fixed,

From RW metric (p. 4026)
for $dt = 0$ (implying a spatial distance)
and $dr = 0$ (implying a tangential distance)

$$D_{ruler} = a_g(t)\, r \sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$$

$$= a_g r \Delta\theta$$

$$= \frac{a_{g0}\, r \Delta\theta}{1+z}$$

set $d\phi = 0$ for a polar angle. We are free to make this choice

$$\therefore D_A = \frac{D_{ruler}}{\Delta\theta} = \frac{a_{g0}\, r(z)}{1+z}$$

Theoretical formula $\Bigg\}$ $D_A = \dfrac{a_{g0}\, r(z)}{1+z} = \dfrac{a_{g0}}{1+z}\begin{cases} \sin\chi, & k=1 \\ \chi, & k=0 \\ \sinh\chi, & k=-1 \end{cases}$

Just as on p. 4085.
Also just as on p. 4085, a_{g0} cancels out for $k=0$.

The small z approximation

$$D_A = \frac{a_{g0}\, r^{\text{zip}}(z)}{1+z} = \frac{a_{g0}\, \chi^{\text{zup}}(z)}{1+z}$$

$$= \frac{zc}{H_0}\, \frac{\left[1 - \frac{1}{2}(1+q_0)z\right]}{1+z}$$

$$= \frac{zc}{H_0}\left[1 - \frac{1}{2}(3+q_0)z\right]$$

4.13 Example Theoretical D_P, D_L D_A Formulae for Particular Friedmann Equation Solutions

a) de Sitter Universe: $\Omega_\Lambda = 1$, $\chi = \frac{zc}{H_0 a_{g0}}$ (p. 4055), $k_\Lambda = 0$

D_P (p. 4080)

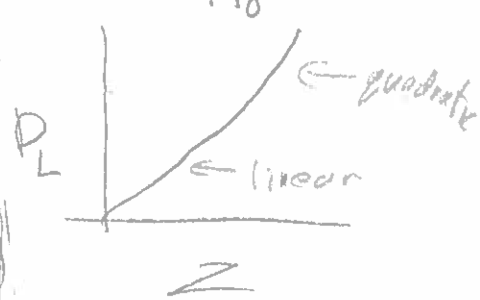
$$D_P = a_{g0}\,\chi$$

$$= \frac{zc}{H_0}$$

D_L (p. 4086)

$$D_L = a_{g0}\,\chi(1+z)$$

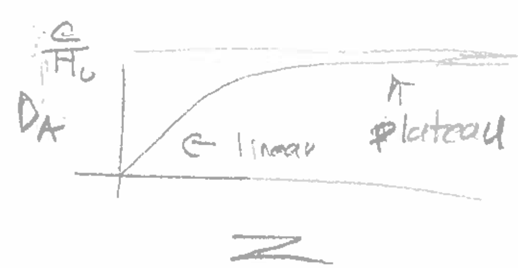
$$= \frac{cz}{H_0}(1+z)$$



D_A (p. 4088)

$$D_A = \frac{a_{g0}\,\chi}{(1+z)}$$

$$= \frac{zc}{H_0}\,\frac{1}{1+z}$$



Note $\frac{c}{H_0}$ = Hubble length

4086

see p. 4057

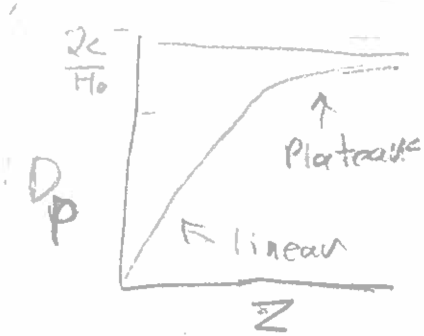
b) EdS universe, $P_{ower} = 3$, $\Omega_{03} = 1$, $k = 0$

$$D_p = a_{g0}\chi$$

$$= \frac{c}{H_0}(2)\left[1 - \left(\frac{1}{1+z}\right)^{\frac{1}{2}}\right]$$

$$D_p^{1st} = \frac{c}{H_0}(2)\left[1 - (1 - \tfrac{1}{2}z)\right]$$

$$= \frac{2c}{H_0}$$



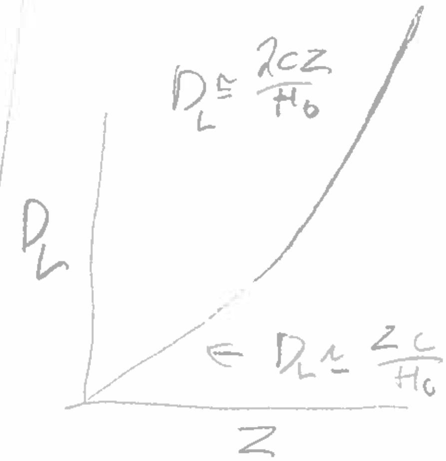
$$\frac{2c}{H_0}$$

= 2 times Hubble length

The proper distance saturates as $Z \to \infty$ i.e., as one approaches the Big Bang singularity

$$D_L = a_{g0}\chi(1+z)$$

$$= \frac{c}{H_0}(2)\left[1 - \left(\frac{1}{1+z}\right)^{\frac{1}{2}}\right](1+z)$$



$D_L \propto Z$

as $Z \to \infty$

Recall from p. 4086

$$D_L = a_{g0}\chi(1+z)$$

$$= \sqrt{\frac{L_{pk}}{4\pi F_{p0}}}$$

$$\because F_{p0} = \frac{L_{pk}}{4\pi D_L^2},$$

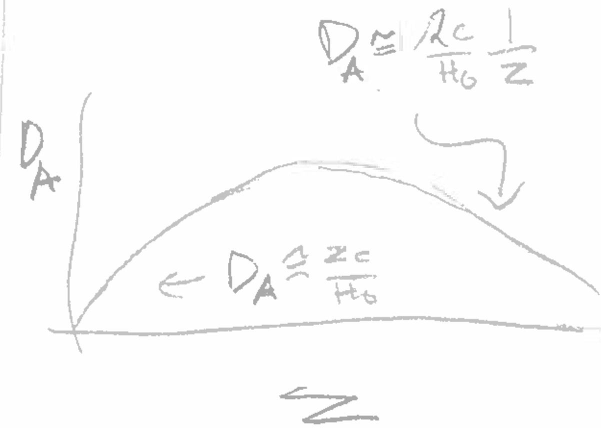
$$\propto \frac{1}{Z^2}$$

as $Z \to \infty$

The object gets fainter

$$D_A = a_{g0}\chi/(1+z)$$

$$= \frac{c}{H_0}(2)\frac{\left[1 - \left(\frac{1}{1+z}\right)^{\frac{1}{2}}\right]}{1+z}$$



$D_A \propto \frac{1}{Z}$ for large Z

From p. 4087

$$D_A = \frac{D_{ruler}}{\Delta\theta}$$

$$\Delta\theta = \frac{D_{ruler}}{D_A}$$

$$\propto Z$$

as $Z \to \infty$

The angular size of the ruler ~~starts~~ increasing.

2026 jan 24

4087

A remarkable fact, ~~the~~ ruler can get bigger on the sky as $z \to \infty$, but still fainter, and so you would loose track of it.

Before the Λ-dominated era, the observable universe was approximately a EdS universe, and so this result ~~to~~ some approximation applies to the observable universe.

Being a bit more exact

see p. 4087

BAO angular scale

$$\Delta\theta = \frac{D_{ruler}}{\frac{c}{H_0}(2)\frac{\left[1-\left(\frac{1}{1+z}\right)^{1/2}\right]}{1+z}} = \frac{D_{BAO}/(1+z)}{\frac{2c}{H_0}\frac{\left[1-\left(\frac{1}{1+z}\right)^{1/2}\right]}{1+z}}$$

$$= \frac{D_{BAO}}{2c/H_0}\frac{1}{\left[1-\left(\frac{1}{1+z}\right)^{1/2}\right]}$$

$$\triangleq \frac{150\,\mathrm{Mpc}}{2(4282.7\,\mathrm{Mpc})}\frac{1}{\left[1-\left(\frac{1}{1+z}\right)^{1/2}\right]}$$

see p. 4005

$$\triangleq \frac{1}{2}(0.03\,\mathrm{rad})\frac{1}{\left[1-\left(\frac{1}{1+z}\right)^{1/2}\right]}$$

$$\cong 1^\circ\,\frac{1}{\left[1-\left(\frac{1}{1+z}\right)^{1/2}\right]}$$

The asymptotic value Θ_* ?

4.088

According to Google AI

$$\Theta_* = 0.8 - 1°$$

from most recent CMB measurements,

but Google AI fails to give an exact reference and relevant papers seem to avoid giving this number explicitly

In any case, our rough calculation based on the EdS universe (NOT the Λ-CDM model) is in fair agreement.

4.14 Cosmic Distance Duality Relation (CDDR

(AKA Etherington reciprocity theorem though only Wikipedia seems to call it this. Richard Tolman (1881–1948) first proposed it as Etherington reported in 1933.)

It's very simple

$$D_L = a_{g0}\, r(z)\,(1+z)$$ (p. 4082)

$$D_A = a_{g0}\, r(z)/(1+z)$$ p. 4088)

$$\therefore \frac{D_L}{D_A} = (1+z)^2$$ the cosmic distance-duality relation (CDDR)

eq $$\eta = \frac{D_L}{(1+z)^2 D_A}$$ Fortunato et al. 2026 CDDR parameter

Not conformal time here

They use SNe Ia for D_L's and FRBs for D_A's which may NOT be the best objects, but their analysis is model independent (or so they say).

Fortunato et al. find $\eta(z) = 1.0 \pm 1.0$ for median for $z = 0$ to 2.5 from some sort of data modeling

4090)

As far as I know,
no significant deviation from
CDDR (i.e., $\eta(z) = 1$)
has ever been found.
However, recent papers seem
to be coy about given
obvious explicit comparisons.

The old
if-they-read-the-whole-paper-carefully
thing,
they'll know what
key numbers are

Some basic theory of cosmology
(e.g., the RW metric, the Friedmann eqn,
the cosmolgical redshift,
general relativity)
would be wrong if the
cosmic duality relation were wrong.

4.15 Summary of Small z Approximations and Cosmological Distance Measures

a) $a(\Delta t)$: Expansion in small $\Delta t = t - t_0 = -t_{lookback} < 0$

$$a(\Delta t) = a_0\left[1 + H_0 \Delta t + \tfrac{1}{2}(-q_0)H_0^2 \Delta t + \dots\right]$$

(see p. 4068, CL-17)

b) $z(\Delta t)$ Expansion in small Δt

$$z = -H_0 \Delta t + \left(\tfrac{1}{2}q_0 + 1\right)H_0^2 \Delta t^2 + \dots$$

$= -H_0 \Delta t$ to 1^{st} order (see p. 4070, CL-17)

c) $t_{lookback}$ Expansion in small z

$$t_{lookback} = \frac{z}{H_0}\left[1 - \left(1 + \tfrac{1}{2}q_0\right)z + \dots\right]$$

$= \frac{z}{H_0}$ to 1^{st} order

(see p. 4071)

d) chi $\chi(z)$ and $r(z)$ expansion in small z

$$\chi = \frac{zc}{H_0 a_0}\left[1 - \tfrac{1}{2}(1 + q_0)z + \dots\right]$$

$= \frac{zc}{H_0 a_0}$ to 1^{st} order (see p. 4073)

4092

Since $\chi^{\text{2nd order in } r} = r$

and $r = b_1 z + b_2 z$

(see p. 4063)

$$r(z) = \chi(z) = \frac{zc}{H_0 a_0}\left[1 - \tfrac{1}{2}(1+q_0)z + \dots\right]$$

$$= \frac{zc}{H_0 a_0} \text{ to } 1^{st} \text{ order}$$

(see p. 4073, CL-18)

e) D_p, D_L, D_A Formulae

$$D_p = a_{g0}\,\chi(z)$$

$$= \frac{zc}{H_0}\left[1 - \tfrac{1}{2}(1+q_0)z + \dots\right]$$

$$= \frac{zc}{H_0} \text{ to } 1^{st} \text{ order}$$

(see p. 4076)

$D_{parallax}$ || { length measurable at one instant in time

$$D_L = \sqrt{\frac{L_{pk}}{4\pi F_{p0}}} = \sqrt{\frac{L}{4\pi F_0}}$$

(observational p. 4079 CL-19)

$$D_L = a_{g0}\,r(z)\,(1+z)$$

$$= \frac{zc}{H_0}\left[1 + \tfrac{1}{2}(1-q_0)z\right] \text{ to 2nd order in } z$$

$$= \frac{zc}{H_0} \text{ to } 1^{st} \text{ order}$$

(Theoretical p. 4082)

$$D_A = \frac{D_{rulen}}{\Delta\theta_0}$$ (observational p. 4083)

$$D_A = \frac{a_{g0}\,r(z)}{1+z}$$ (theoretical (see p. 4084))

$$= \frac{zc}{H_0}\left[1 - \tfrac{1}{2}(3+q_0)z\right] \text{ to 2nd order in small } z$$

$$= \frac{zc}{H_0} \text{ to } 1^{st} \text{ order in small } z$$

$$D^{1st}_{generic} = D_p^{1st} = D_L^{1st} = D_A^{1st} = \frac{zc}{H_0}$$ { Intuitively suggested but now proven.

f) Recession Velocity & Redshift Velocity

$$v_{recession} = \dot{D}_p$$

but since

$$v_{recession} = H_0 D_p$$ (see p. 4033 - 4034)

$$\therefore v_{recession} = H_0 \, a_0 \, \chi(z)$$ (see p. 4092)

$$= zc\left[1 - \tfrac{1}{2}(1+q_0)z + \ldots\right]$$

$$\simeq zc \quad \text{to } 1^{st} \text{ order in } z$$

$$v_{redshift} \equiv zc$$

$\therefore v_{recession} = v_{redshift}$ to 1^{st} order in z.

Also, $v_{redshift} = H_0\left(\frac{zc}{H_0}\right) = H_0 D^{1st}_{generic}$

(see p. 4092)

4094

~~Q2~~ 9 4.16 2012sep27 4095

Vareational Calculus

In QM, and for geodesics in GR

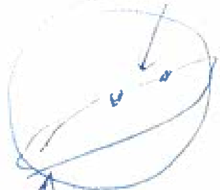
Not min

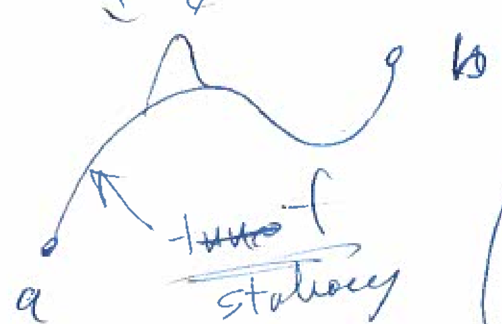
But there is a Hull question

$$I = \int_a^b \underbrace{f(x_i, \dot{x}_i, \ldots, t)}\, dt$$

set of coordinate and derivative

path parameter Not necessarily time but could be





find f that makes I stationary → max, min, inflexion

What I get global necessarily

Do all possibilities get all: global & locals

~~global or~~ just ~~local~~

No variation at end-points

$$x_{i\,\text{varied}}(t) = \underbrace{x_i(t)}_{\text{stationary}} + \alpha\, \eta_i(t)$$

α: variation parameter

$\eta_i(t)$: general function of the coordinate

except $\eta_i(a) = \eta_i(b) = 0$

Let's consider

$$f(x_i, \dot{x}_i, t)$$

x_i: coord.

$\dot{x}_i$: 1st derivative

This form is good for geodesics + Hamilton principle in classical mech.

4096

4017

$$\frac{dI}{d\alpha} = \int_a^b \left[\underbrace{\frac{\partial f}{\partial x_i}\frac{\partial x_{iv}}{\partial \alpha}}_{\eta_i} + \underbrace{\frac{\partial f}{\partial \dot{x}_i}\frac{\partial \dot{x}_{vi}}{\partial \alpha}}_{\dot{\eta}_i}\right] dt$$

for stationarity

zero at endpoints

use integration by parts

$$\left.\frac{\partial f}{\partial \dot{x}_i}\eta_i\right|_a^b - \int \left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)\right]\eta_i dt$$

$$0 = \frac{dI}{d\alpha} = \int_a^b \underbrace{\left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)\right]}_{\text{must be zero for stationary}} \eta_i dt$$

Notice total Not partial derivative

Not a constrained stationary path

~~Must~~ be zero for stationary ~~path~~ since η_i is general
→ any little blip except zero at end point

$$0 = \frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)$$

(a differential equation for each coordinate x_i

→ Called Euler's equation Art-928

4097

fixed ends

2 → BCs at two times

but same as 2 BCs at start for 2nd order DE

Uses –
1) Eqn of motion
2) Geodesics, GR
3) Path Integral formation of QM
4) Fermat's Principle to Reflection/refraction

1 endpoints

Path integrals Formulation of QM → particle can be thought of as following all paths but only along stationary ones do waves add coherent — other paths lead to incoherence and cancellation

(Feynman etc.)

But is this just an emergent principle? true if all paths really were followed

but then paths collapse — decoherence — long argument

which

Any way a ~ 2000 atom molecule has been diffracted

Fein et al, 2019 Nature physics