Lecture 4 (2025 mar 24) 2019 sep 25 (2023 oct 08) 4001

1) Geometry of universe — well observable universe — or our part of pocket universe a hypothetical in the multi...

2 Robertson-Walker (RW) Metric

4020 3) Geometrical Insight to RW metric

4) Hubble's law from RW metric

5) Cosmological Redshift — other derivations {p. 2...

6) Connection to Observables

7) Summary Pr.

8) Small Δt, z expansions, deceleration parameter

9) Cosmic distance measures

10) Variational calculus — without much connection to anything else in the lecture, but it does find use in cosmology sometimes

1) Geometry of Universe — at least the observable part of our pocket universe (hypothetical)

— salvaged for 2025 mar 24 note p. 4011-4012

of pocket universe

expanding, contracting, or patchy

GR holds everywhere we hope.

expanding infinite false vacuum universe

eternal inflation version of multiverse

#002

1) Geometry of Universe Intro Via Friedmann Equation

a) Getting a_{90}, ~~scale length of universe~~

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G \rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

$k \longrightarrow kc^2$ — Two ways of writing k

here we chooze the kc^2 way.

~~k~~

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = H_0^2\left[\frac{8\pi G}{3H_0^2}\rho - \frac{kc^2}{H_0^2 a^2} + \frac{\Lambda}{3H_0^2}\right]$$

$$\left[\frac{\rho_M}{\rho_c} + \Omega_k + \Omega_\Lambda\right]$$

$\rho_c = \frac{3H_0^2}{8\pi G}$

Ω_M

$\frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3H_0^2}$

0 symbolizes the fiducial time which is usually cosmic present.
At $t = t_0$, $H = H_0$ and $\sum_p \Omega_p = 1$.
Scaled time makes H_0 ~~vanish~~; $d\tau = H_0 dt$

$$\Omega_k \equiv -\frac{kc^2}{H_0^2 a^2} = \frac{-kc^2}{H_0 a_0^2}\frac{1}{x^2} = \Omega_{k,0}\frac{1}{x^2} \quad \text{where } x = \frac{a}{a_0}$$

$\rho_\Lambda = \frac{\Lambda}{8\pi G}$

Which leads to the eternal confusion that

$k > 0$ is +ve curvature (hypersphere) $\Omega_k < 0$

$k < 0$ is −ve curvature (hyperbola) $\Omega_k > 0$

Now previously we defined usually $x_0 = 1$ and also if one likes $a_0 = 1$ and dimensionless

1004

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Then $D = a(t) D_0 = a_0 \times D_0$

proper distance at any cosmic time t

comoving distance at all cosmic time

and this is the good way of defining a and ~~[illegible]~~ a_0 for most purposes $\Rightarrow$ solving the Friedmann equation and it solutions most obviously

But for theoretical discussions of geometry, there is another way

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2 a_0^2} \equiv -\frac{k_g c^2}{H_0^2 a_{0g}^2 (k_g)}$$

+ for +ve
- for -ve curvature

$\Omega_{k,0}$ is found in fit model cosmic expansion solution in Demand

g for geometry

$$k_g = \begin{cases} 1 & \text{+ve curvature} \\ 0 & \text{0 curvature} \\ -1 & \text{-v. curvature} \end{cases}$$

dimensionless

e Friedmann eqn. solution or otherwise (?)

But $\Omega_{k,0}$ is also dimensionless and so a_{0g} now ~~has a dimensionless~~ has dimensions

(2025 mar 4) ('400B

$$a_{0g}^2 = \frac{c^2}{H_0^2 \underbrace{(\cancel{k_g}\,\Omega_{k0})}_{= |\Omega_{k0}|}}$$

since k_g and Ω_{k0} have opposite sign

$$a_{0g} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{c\,t_{H_0}}{\sqrt{\Omega_{k0}}}$$

where t_{H_0} is Hubble time

a_{0g} is a ~~___~~ physical length in units of length, but is undefined for $\Omega_{k0} = 0$, i.e., a flat geometry universe

or $a_{0g} = \infty$ if you prefer

$$R_G \text{ (Gaussian Curvature Radius)} \equiv \frac{a_{g0}}{\sqrt{k_g}}$$

(CL-12)

→ real for hyperspherical, +ve curvature

→ imaginary for hyperbolic, -ve curvature

But we don't use R_G much

$$a_{0g} = \underbrace{\frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{c\,t_{H_0}}{\sqrt{|\Omega_{k0}|}}}_{\text{Hubble distance}} = \frac{(4.2827\ldots)\,h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_{k0}|}}$$

$$= \frac{(13.968\ldots)\,h_{70}^{-1}\ \text{Gly}}{\sqrt{|\Omega_{k0}|}}$$

where $h_{70} = \frac{H_0}{70} \simeq 1$ ~~which is~~ to within $\sim 4\%$

4006

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$\Omega_{k,0} \lesssim 0.02$ (Wik circa 2025)

$\Omega_{k,0} = 0.0019(15)$ ACT Atacama Cos. Tel. (2025)

1σ

so −ve curvature at face value

$\Omega_{k,0} = 0.0022(15)$ Planck (2020)

about the same

~~So $\sqrt{0.0019}$~~

→ So $|\Omega_{k,0}$ fiducial size$| = 0.0009 - 9 \times 10^{-4}$

So $\sqrt{|\Omega_{k,0_{\mathrm{fid}}}|} = 3 \times 10^{-2}$

Current measurements allow that Ω_{k0} could be of order this ~~size~~ size.

$$a_{0g} = \frac{c/H_0}{\sqrt{|\Omega_{k,0}|}} \doteq \frac{1.43 \times 10^2 \, h_{70}^{-1} \text{ Gpc}}{\sqrt{|\Omega_{k\mathrm{fid}}|} \Big/ \sqrt{\Omega_{k,0\,\mathrm{fid}}}}$$

143 Gpc fiducial estimate

(about $10 *$ observable radius)

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Comparison

$D_{\text{observable universe}} = 14.25$ Gpc — scale length of universe — With Base on ΛCDM fit

a_0 mean?

6) But what does a_0 mean?

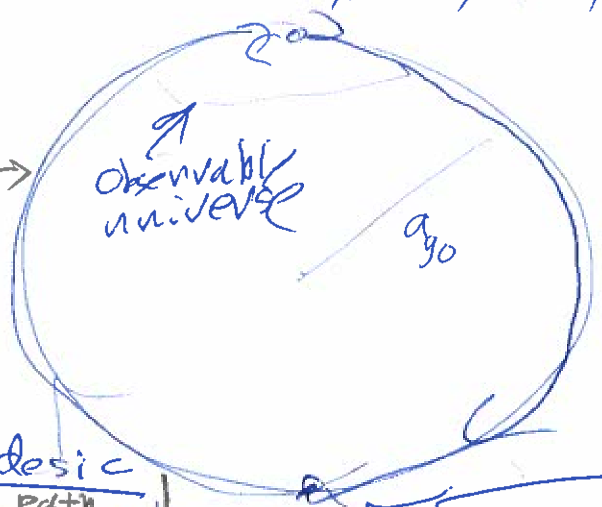
i) For flat (Euclidean space) undefined and of No use.

ii) Easiest for to understand hyperspherical universe (+ve curvature)

iii) For hyperbolic universe (-ve curvature) a_0 length scale over which you would notice hyperbolic effect

It's sort of a parameter.

Milky Way (MW)



2-d analog of 3-d curved space.

D_L (MW to AMW) — No proof Just an assertion

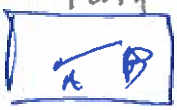
$= \pi a_0 = 3.143$

$= 430$ Gpc

see p. 402 for pro

$\gg D_{\text{observable universe}}$

geodesic path

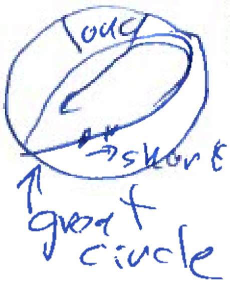


Fills whole ~~whole~~ sky

antipodal point the Milky Way (AMW)

Nothing special ↓ every point has an antipodal point

and so the curvature is very small on ~~obs~~ observable universe scale.



locally shortest distance between points in space

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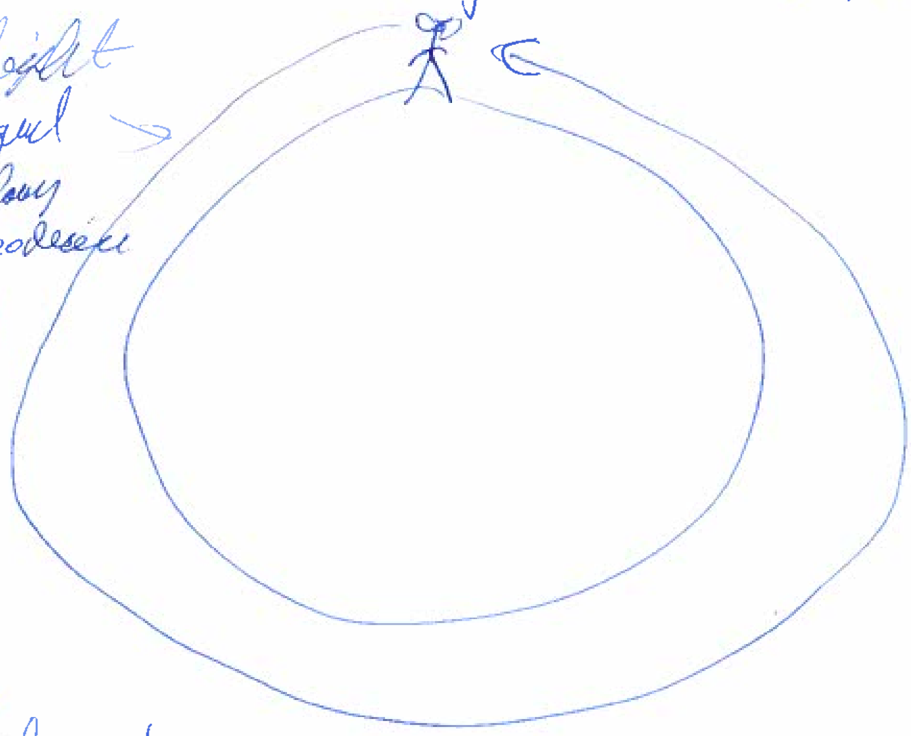
Seeing back of your head

light signal follows a geodesic



$2\pi a_0$

$\simeq 900$ Gpc

$\simeq 2700$ Gly

$\therefore$

$t = 2700$ Gly

to see the back of your head

it just follows a "straight" line

Einstein universe (1917)

Einstein when he posited his static hyperspherical universe didn't worry about this.

every ~~direction~~ direction sees antipodal object

NOT same as increasing Angular Diameter (decreasing angular diameter distance) in a flat universe

→ That is an evolution effect

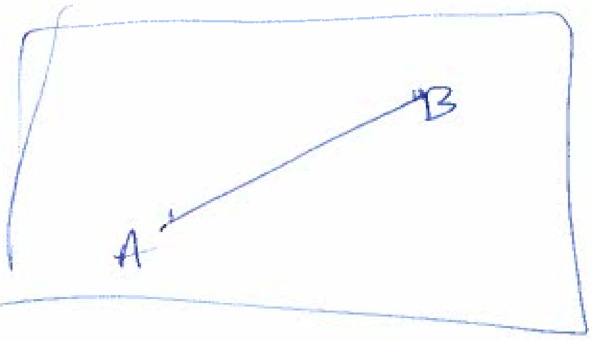
→ The ruler was closer when light started out to us.

(2025 mar 24) 400

c) Geodesics

- stationary paths
- light in GR follows the simplest kind

In flat (Euclidean) space, just traditional straight line



$$\mathcal{L}_{\text{expl}} = \int_A^B [\text{Integrand}]\, d\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0$$

perturbation parameter

invariant to small perturbations

So locally a geodesic is shortest path (~~or longest~~) but maybe not globally

For Example

small circle

great circle - cuts sphere in half

Great Circles are the geodesics on spheres

Need variational calculus
- Not so hard when you know trick

7-010 | 2025mar27

for $\theta < 180°$, the great circle path is locally and globally shortest path.

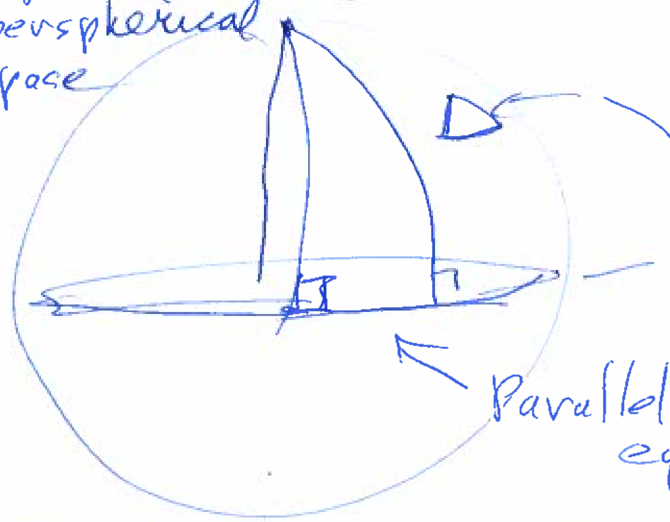
But $\theta = 180°$, all ~~great~~ great circle paths are ~~paths~~ the same length

And $\theta > 180°$ still stationary but not globally shortest path. ~~path~~.

Very small insight into differential geometry ~~geometry~~ — it's hard topic to get into, but if you are going to do general relativity ....

analog to hyperspherical space

Pole

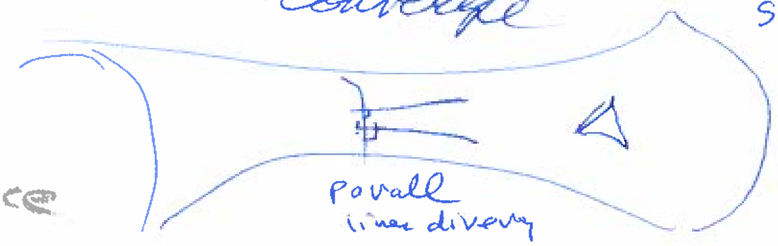


Triangle of great circle paths

Sum of Angles > 180°

Parallel lines at equator meet a poles - converge

Saddle analog ~~to hyperbolic space~~ to hyperbolic space



Sum of Angle < 180°

Parall lines diverg

2029 march | 4 011

~~Making of~~

An online video on hyperbolic space

think saddle analogy poor, but maybe the best you can do in one diagram

It looks right to me.

The maker argues hyperbolic space is easier to understand than hyperspherical space

It's like Euclidean space, but more so.

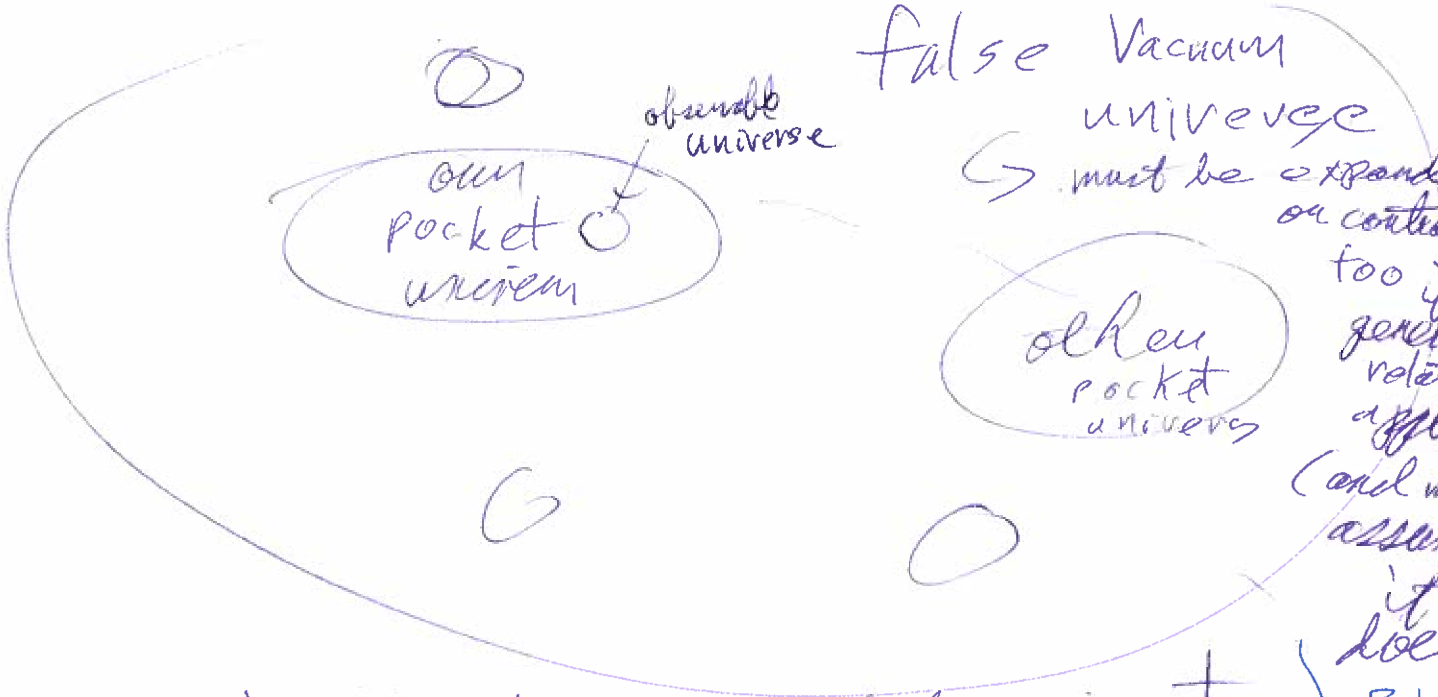
d) How Far Does Friedmann Universes Extend?

4012

(2(12)sep30) 401

We have discussed our FE universes as everywhere without boundary if they extended — but finite if hypersphere

But they may not — in inflation theory they don't probably.



↳ must be expanding or contracting too if general relativity applies (and most assume it does!)

But homogeneous & isotropic except maybe on a superscale

Actually basic inflation just applies ~~to~~ our pocket universe

— It does not require that others exist.

— But a fairly natural extension is ~~that~~ they do

⟹ the Multiverse picture.

(eternal inflation is the inflation version)

(But some believe this is wild speculation and not worth elaboration.)

There are many arguments about the multiverse pro & con.

7012

Pro — The laws of physics may be general (but maybe Not)

We don't really understand it, but could we build a biophilic reality without it, Yes/No and others guess no our lack of imagination

— 2nd law of thermodynamics seems a logical necessity
— Quantum mechanics seems so fundamental
— So does classical limit and General relativity

But the parameters; c, G, ħ, α of physics and the observable universe parameters and particle masses all seem to have no relation other than being restricted by being sufficiently biophilic for life as we know it (carbon based, liquid water based)

...ecided probabilistically ...ut how?

Con —) the multiverse is so unconstrained.
— one can imagine anything.
— and there is no obvious natural way to determine the probability distributions of parameters.
— untestable — except does pass the test of parameters not being greatly fined tuned to special values. (Overly Fine tuned)

Not a Scientific theory?

Let's not bother with "it"

But as Mario Livio says what if a TOE explains all we see and implies it.

2025Jan27

4015

But as our Colleague Mario Livio has opined if we found the fundamental theory of physics and it implied the multiverse, we might concede it's more likely than not.
But maybe it would prove No multiverse.

e) Multiverse: More

passes one falsification test

parameters are tuned well enough for life as we know it, but not over fine tuned,

Huge and controversial topic

Anthropic principle ⇒ universe is as it is or we wouldn't ~~be here~~ have a chance to be here

My view is multiverse is something to put in category whole universe faute de mieux

But it's not really about us ~~but life~~: (as we know it)

Actually K the universe

$P(us|K)$ ~~[illegible]~~ $= 1$

$P(~~K~~|Us)$ = near Zero

exactly us humans now on Earth 13.8 Gyr after big bang

universe as it is every knob has to be right

So many random elements in galaxy formation, planet formation, dinosaur-cidal asteroids, evolution

Some think the anthropic principle is a useful science principle others that it isn't. The subject seems to dissipate into Socratic dialogs.

I get dizzy.

The universe rolled the dice a jillion times and we're one of the lucky winners

4016 [2025mar24]

There once seemed a lot of arguments about ~~things~~ having to well tuned enough for life

→ they all seem to dissipate ~~into~~ others saying, they don't have to that well fine-tuned. (Wiki fine-tuned universe)

But Martin Rees et al. argue that ~~they~~ are Not over-fine-tuned.

Fine-structure constant (Google AI)
— some
~~Some too structu~~
suggest QFT says if

$$\alpha = \frac{1}{137.035\,999\,177} = \frac{e^2}{\hbar c} \text{ (CGS)}$$

~~If α~~ was ~ 2% different proton decay would be too rapid for life to evolve

But do protons decay? ~~not do~~ ~~we know~~ We don't have an undisputed theory

But if it were exactly $\frac{1}{137}$ that is over fine-tuned and some principle other than life fixed it.

after all

But why exactly $\frac{1}{137}$ maybe the principle merely required it to be ~~and~~ integer or a prime and then it isn't over fine-tuned

2025 mar 24

4017

2) General Relativity & Robertson-Walker Metric

a) GR gives Einstein field Equation EFE (2nd order) (Driver)

4x4 set of DE's represented by an element.

$\triangleq 2\times10^{-43}$ m/J

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \left(\frac{8\pi G}{c^4}\right) T_{\mu\nu}$$

stress-energy tensor

mass-energy momentum

Einstein tensor $= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ metric tensor

Rank 2 tensor

→ 2 indices

and μ ~~runs~~ over 0, 1, 2, 3

time dimension — space dimension

Greek letters for spacetime

Roman letters just for 3-d space.

Replacement for Newton law of of Universal grav.

with $g_{\mu\nu}$ being the tensor which describes the geometry of spacetime which tells matter ~~how~~ to move

$g_{\mu\nu}$ is the solution of the 4x4 DE system

Then the geodesic equation (which we won't write down) (GR analog for F = ma) for gravity tells matter what motion to follow

(2nd order! dynamics limit for 1st order equation)

Mass-energy tells spacetime how to curve and curvature tells mass-energy how to move

⇓

Circular but Newtonian physics is the same

9018

EFE can be written a 4x4 matrix differential equation

But GR notation is just to use a representative element.

For example instead of writing vector $\underset{\sim}{v} = \vec{v}$

you just write x_i

one element stands for all.

a very compact notation.

- should we use it only. and teach it intro? to classes.
- probably, ~~different~~ multiple representation give you multiple understanding and so no

GR Tensor equations are invariant under coordinate transformation.

So they can be physical law.

~~Of~~ Of course, just as in ordinary physics, the Newtonian coordinate system should be chosen to be convenient.

(2025mar24)

E.g., spherical or Cartesian [401]

Choice depends on system.
In fact, a lot of GR work
depends on choosing
the good coordinate
system — and often it's ~~hard~~ hard
to choose

Famous Example (Wik: Gwaves)
Einstein thought about Gravitational
waves for a long time,
but he + Rosen 1936 submitted
a paper to Phys Rev
concluding they couldn't
exist.

Howard Robertson (of the metric)
— a referee pointed out
the conclusion was wrong
based on poor choice
of coordinates,
Einstein was miffed and
withdrew the paper, but
eventually made the
correction and published
elsewhere (Wik: Gwaves)

Actually, this paper may have been Einstein's
last important result. and he was
only 57

(people aged faster then — smoking, etc).

1020

What are tensors any way?

→ quantities that depend on directions in space in physics

→ rank 1 are vectors depend on x_1, x_2, x_3 for 3-d

10 independent elements symmetric or antisymmetric

Rank 2 can be written as matrices

→ depend on $x_1 x_1$, $x_1 x_2$, $x_1 x_3$ etc.

You extract information by contraction.

For example, length of vector x_i

→ contract with itself

dot product or inner product

letter letter indices for space 1, 2, 3

Greek letters for spacetime 0, 1, 2, 3

$$\ell^2 = x_i x_i$$

Einstein summation over indices.

Now x_i the element value are coordinate system dependent, but length isn't.

Example of cordinate transformation

coordinate transformation

$$x_i' = A_{ij} x_j$$

primed system

unprimed system

Transpose

Contract on i: $x_i' x_i' = A_{ij} x_j A_{ik} x_k = A_{ij} A^T_{ki} x_j x_k$

2025 mar 24

9021

Kronecker delta

$$= \delta_{jk} x_j x_k$$

$$= x_j x_j$$

So a valid transformation of ~~coordinate~~ tensor must have property

$$A_{ij} A^T_{ki} = \delta_{jk}$$

or $A A^T = \mathbb{I}$

matrix   unit value.

But other tensor can be physically active. → i.e., Not coordinate transformation, physical rotation, etc.

it can all get confusing

b) Curved spacetime ~~doesn't~~ obeys a generalized Pythagorean theorem.

actually so does ordinary flat 3-d space if you used ~~a cartesian~~ Not cartesian coordinates

4022)

take thermodynamics 1st law $dE = TdS - PdV + \mu dN$

the denominator could be time or some other path in S, V, N space

— a great the virtue of physical laws written as tensor equations is the law is invariant under coordinate changes (but not values of elements of tensors)

what are differentials with respect to? path parameter ℓ in spacetime
$x_i = x_i(\ell)$
ℓ could be time or just a parameter measure a path at an instant in time
$\left(\frac{ds}{d\ell}\right)^2 = g_{\mu\nu} \frac{dx^\mu}{d\ell} \frac{dx^\nu}{d\ell}$
connection is not to complicate the formulae ~~the~~ path number

The generalized Pythagorean thm of GR is

differential. and so in general only valid for differential (small) displacements ... to aggregate ... to ...

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Einstein summation

ds^2 is the invariant interval ~~of~~ of spacetime

indices can be lower or upper — and we won't go into why?

→ but when you do raise upper/lower contraction only valid between upper & lower (Carl Hosler's GR course teaches all that)

$g_{\mu\nu}$ is the metric

But often it seems people refer to this equation as the metric but properly metric equation

(2025 mar 24) 4023

The metric of special relativity is "flat" for spacetime

at least for Minkowski (pseudo Euclidean ~~coordinate~~) coordinate

Carroll-8 uses equal sign between element representation and the matrix representation

$$\underbrace{g_{\mu\nu}}_{\text{general}} \Rightarrow \underbrace{\eta_{\mu\nu}}_{\text{Minkowski metric}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Alas, there are two conventions

Carroll-8 takes this one

~~But~~ others use $\eta^{other}_{\mu\nu} = -\eta_{\mu\nu}$

It seems every other persons chooses differently.

with Carrolls convention

Other convention	
	$ds^2 = g_{\mu\nu} dx^\mu d$
$ds^2 < 0$ spacelike	$ds^2 > 0$ is spacelike; ds is a real length if the displacement is at one instant in time
$ds^2 > 0$ is timelike	$ds^2 < 0$ is time-like $\Rightarrow$ ds real time if no change in place e.g. clock at rest in a frame
$ds^2 = 0$ is still lightlike	$ds^2 = 0$ is light like

+ something with negative ~~a~~

(2024) 2023 oct 08

2c) Robertson Walker Metric

In 1935 RW determined their eponymous metric (Wiki) which was known earlier to Friedmann + Lemaître,

but RW proved it is the most general metric that is a manifold that at any point is asymptotically Lorentzian → $g \to \eta = \pm \begin{pmatrix} 1 & \\ & -1 & \\ & & -1 \end{pmatrix}$ (local inertial frame, local Lorentzian frame = LLF)

3 space coord. 1 time coord.

different conventions (Carroll-8 time-like sign)

Minkowski metric. No curvature, coordinates which are allowed coupled

locally Minkowski space

and homogeneous & isotropic in unbounded space (can be finite for hyperspherical), [But infinite for flat or hyperbolical]

It is not specific to general relativity (Wiki: RW history)

But GR provides the dynamics for $a(t)$ the scale factor i.e. the Friedmann Eq. (FE) (which we derived from Newtonian physics) (assuming the perfect fluid)

Note c is the highest velocity relative to a ~~local~~ LLF.

But there is no limit on recession velocities between LLFs and, in fact cosmologically remote space have $v_{rec} > c$.

Since in the classical limit Newtonian physics holds, it would be strange if there were no Newtonian approach to some GR results.

FE derived fairly with extra hypotheses, but the Newtonian derivation tells us nothing about geometry of course

RW metric (or more properly interval since the coefficients are the metric tensor) (4002

a dimensionless
a_g with dimensions — but usually just "a" too
One has to tell difference by context.

CL-9 form
CL-10 $ds^2>0$ is timelike

in most standard form (but not most general)

$$ds^2 = c^2 dt^2 - a^2 \left[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right]$$

time part; space part

get ds by integration: $\int ds = \int \sqrt{\ldots\ldots}\, dl$, l is path parameter

~~space part~~

$= d\tau^2$

(but Carroll 329-338 uses $d\tilde{s}^2 = -ds^2$) Carroll

But $ds^2 > 0$ (CL-10) is timelike and this seems more memorable to me → a clock at rest in its own frame measures this

(r is a dimensionless comoving coordinate — cosmic time and $a(t)$ have units ~~of~~ ~~$a(t)$~~. But now use D for Distance

in spherical coordinates where the origin can be anywhere since space is homogeneous and isotropic (homist?)

but usually we take the Milky Way or Local Group center of mass at origin)

Dimensionless a bit of a misnomer. Dimensionless quantities have a physical nature, they are just in nat...

$$k = \begin{cases} +1 & \text{hyperspherical} \\ 0 & \text{Euclidean or flat space} \\ -1 & \text{hyperbolical} \end{cases}$$

The scaling of k to these values (1, 0, -1) means (as discussed on p. 4004-4005)

~~$a_{g0} = \ldots \dfrac{c/H_0}{\sqrt{|\Omega_{k0}|}}$~~

curvature "radius" has units — Gpc are good.

$$a_{g0} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{4.2827\ldots\, h_{70}^{-1}\,\text{Gpc}}{\sqrt{|\Omega_k|}} = \frac{13.968\ldots\, h_{70}^{-1}\,\text{Gpc}}{\sqrt{|\Omega_k|}}$$

See p. 4004, 4005 / 2025 mar 24 notes

7026

out of place but built into RW GR solution (?)

The fluid equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \quad (Li-26)$$

in GR follows from the energy-momentum conservation equation (see Carroll-117-119)

$\nabla_\mu T^{\mu\nu} = 0$ (Carroll-120)

energy momentum tensor RHS of Einstein field equations

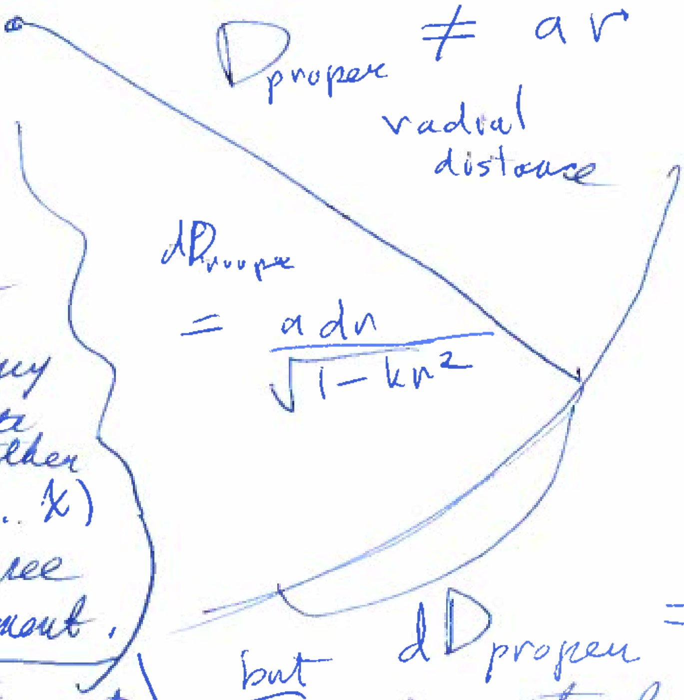
Do you get an singularity / explosion / indefinite / infinity when $\frac{dr^2}{1-kr^2}$ when $kr^2 \to 1$?

No, It turns out that $\frac{dr}{d\chi} = 0$ there (a stationary point) and that stops an singularity

1) Proper Distances

Note Recall proper distance can be measured at one instant in time with a ruler (meaning of distance)

Origin



$D_{proper} \neq ar$ except if $k=0$ and asymptotically as $kr^2 \to 0$.

proper radial distance

$dD_{proper} = \frac{a\,dr}{\sqrt{1-kr^2}}$

This is a choice. One can choose the comoving coordinate in another way (i.e. χ) as we'll see in a moment.

χ makes the radial measurements "normal."

Recall $kr^2 \to 1$ doesn't give an infinity and only happen for hyperspherical space)

but dD_{proper} tangential distance $= ar\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$

$D_{proper}(\phi=0) = ar\,\Delta\theta$

The r comoving coordinate was chosen to make tangential measurements "normal"

as noted $a_0 = a(t_0) = 1$
is not available with
the physical $a(t)$
but one just flips back and forth between
the two $a(t)$'s as needed.
Context tells you

Using $a_g(t)$ to represent ~~physical~~ dimension a but of course "a" is used for both cases

What if $a(t) \to 0$
a point origin (but really a spatial point only for hyperspherical universe)
(Point ~~in time~~ in Time)
but $\rho \to \infty$.

– Big Bang singularity point origin was older term

But we don't think that can happen.
At some point GR must go
to some quantum gravity.
Maybe at the Planck density
or thereabouts

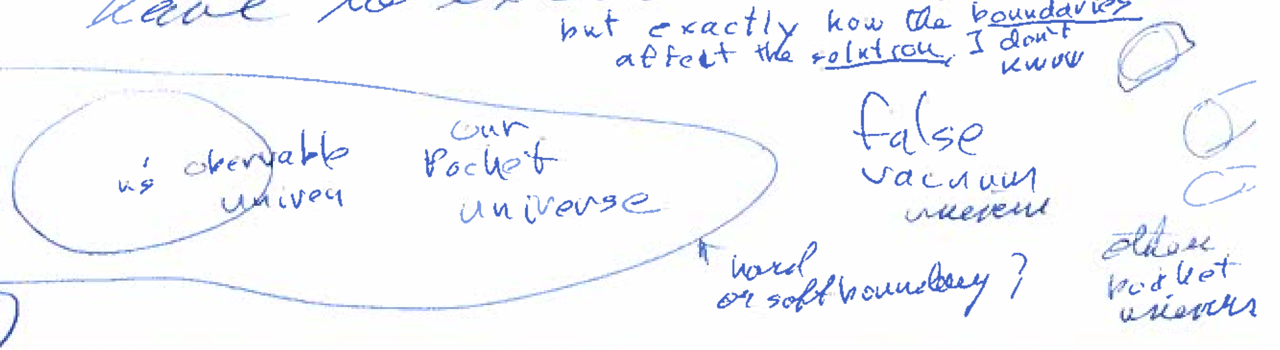
$$\rho_{planck} = \frac{c^5}{\hbar G^2} = 5.1550 \times 10^{96} \frac{kg}{m^3}$$

(Wik: Planck units)

as we run the clock back everywhere
in the observable universe goes
thru reionization, recombination,
big bang nucleosynthesis, the quark era
.... ?

But the FE models don't
have to extend to infinity
but exactly how the boundaries affect the solution, I don't know

FE equation assumed boundless universe.
How do things change ~~with~~ unknown and maybe complex boundaries far away?



1028) Omit (2023 oct 08)

Multiverse

Pro - 2nd law of thermo < logical necessity even
- GR
- quantum mechanics
seem all very elegant
as if universal
But G, ħ, c, M_e and other
constants seems
to have whimsical values
except they allow us
- chosen from a distribution
randomly except
anthropically.

Con - no guidance
- very unconstrained
— what are those distribution
some argue not even a scientific theory
never provable or falsifiable

My view is that it is good a ~~useful~~ fruitful
idea but only to a limited degree.
Maybe a great fundamental theory of everything
will just imply multiverse and then we
~~accept it as the best~~ we can do,
~~or ruled it out by~~ showing all physics
must be as it is indifferent to us.

3) Geometrical Insight

a) k=1 spherical geometry
on a 2-sphere (a sphere
in 3-d Euclidean space; Wiki: n-sphere)

(2021oct03) 4029

Metric is

$$dD^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(CL-1(), last equation

a is radius of sphere and r isn't continuously but origin rotation consistent with everything is tricky

Let $\underbrace{r}_{\text{funny } r} = \sin\theta$ (yes it looks weird but r is a dimensionless comoving coordinate on the sphere)

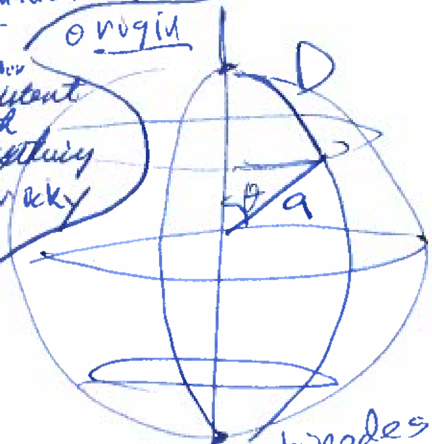
$\theta \in [0, \pi]$ polar coordinate range

$r \approx \theta$ only for $r \ll 1$

$dr = \cos\theta\, d\theta$

$d\theta = \pm \dfrac{dr}{\sqrt{1-r^2}}$ {which implies the limits $\theta \in [0,\pi]$ actual

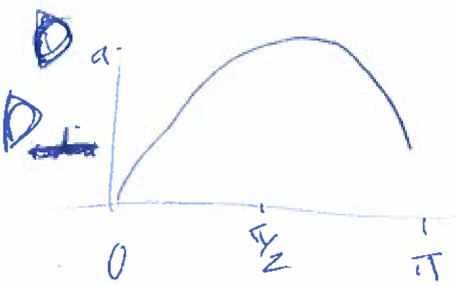
So $dD^2 = a^2\left(\dfrac{dr^2}{1-r^2} + r^2\, d\phi^2\right)$



All geodesics lead to the antipodes

So the sphere surface looks flat only for $r \ll 1$

and $D_{\text{radial}} = \begin{cases} a\theta & \\ ar & \text{for } r \ll 1 \\ a\frac{\pi}{2} & \text{for } \theta = \frac{\pi}{2} \\ a\pi & \text{for } \theta = \pi \end{cases}$



$D_{\text{circumference}} = a r 2\pi = 2\pi a \sin\theta$

$\begin{cases} 2\pi a r & \text{for } r \ll 1 \\ 2\pi a & \text{for } \theta = \frac{\pi}{2} \\ 0 & \text{for } \theta = \pi \end{cases}$

2-sphere

b) $K = 0$, Just flat space, 2-d flat space

Metric $dD^2 = a^2(dr^2 + r^2\, d\phi^2)$ {Just polar coordinate space

$r \in [0, \infty]$

030

(surface of hypersphere in 4-d Euclidean space — but that spatial 4th dimension has no physical meaning)

c) $k=+1$ for 3-sphere which is the RW metric space part, $dt=0$

Metric $dD^2_{proper} = a^2\left[d\chi^2 + \sin^2\chi\,(d\theta^2+\sin^2\theta\, d\phi^2)\right]$

χ makes the radial measurements normal

where χ is one version of comoving coordinate.

Here we define $r=\sin\chi$

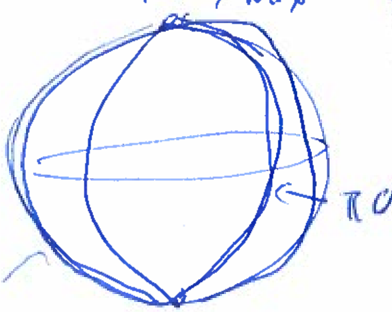
so that $dD_{\substack{proper\\ tangential}} = a\, r\sqrt{d\theta^2+\sin^2\theta\, d\phi^2}$

which means

$dr = \cos\chi\, d\chi$

and $d\chi = \pm\dfrac{dr}{\sqrt{1-r^2}}$ just as for the 2-sphere case

2-sphere analog

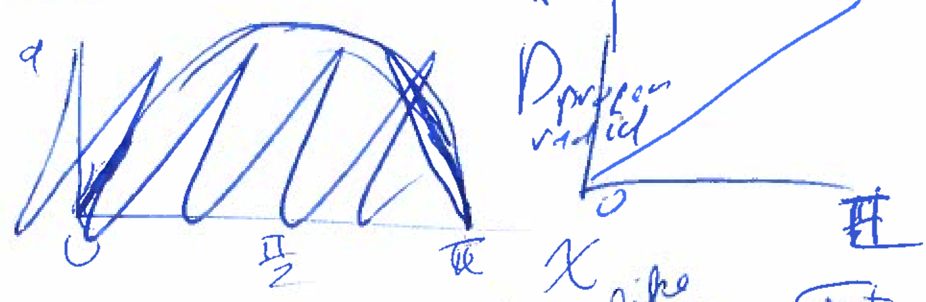


which implies the $\chi\in[0,\pi]$ limits since we assume the RW metric

all good

δ_0

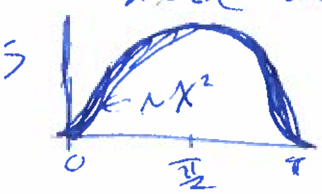
$D_{radial\ proper} < 2\pi a$



What is the area of a 2-sphere-like of comoving frame coordinate r in our RW space.

Not proof Just what is.

Well integral over all θ,ϕ gives 4π just like in flat space.



$\therefore\ S(r) = 4\pi a^2 r^2 = 4\pi a^2\sin^2\chi$

$4\pi a^2\chi^2$ for $\chi\ll 1$
$4\pi a^2$ for $\chi=\pi/2$
0 for $\chi=\pi$

surface area goes to zero!!!

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What is the volume of a 2-sphere-like?

$$V = \int_0^{\chi} S(\chi')\, a\, d\chi' \quad < \begin{array}{l} \text{Chi} \\ \text{no } x \end{array}$$

$$= 4\pi a^3 \int_0^{\chi} \underbrace{\sin^2 \chi}_{\frac{1}{2}(1 - \cos 2\chi)} \, d\chi \;\rightarrow\; 4\pi a^3 \frac{\chi^3}{3} \text{ for } \chi << 1$$

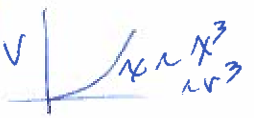
from standard trig identity (Wiki: trig Double angle formulae)

$$= 4\pi a^3 \left[\tfrac{1}{2}\right]\left(\chi - \frac{\sin 2\chi}{2}\right)$$

$$= \frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)\left(\chi - \frac{\sin 2\chi}{2}\right) \text{ in general}$$

$$\chi - \left(\cancel{\chi} - \frac{1}{3!}\frac{(2\chi)^3}{2}\right) \qquad \chi << \frac{\pi}{2}$$

$$\frac{2}{3}\chi^3$$



$$\frac{4\pi}{3} a^3 \chi^3 = \frac{4\pi}{3} a^3 r^3$$

as it should asymptotically flat space.

$$\frac{4\pi a^3}{3}\left(\frac{3}{2}\right)\left(\frac{\pi}{2} - 0\right) \quad \chi = \frac{\pi}{2}$$

$$\pi^2 a^3$$

$$\frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)\pi$$

$$2\pi^2 a^3$$

So a finite space but unbounded.

V_{max} proof

$$\frac{dV}{d\chi} = \frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)\left[1 - \cos 2\chi\right]$$

for $\chi = 0$ which is a min

for $\chi = \pi$ which is max

$$\frac{dV}{d\chi} = 4\pi a^3\, 2 \sin\chi \cos\chi \begin{cases} 0 & \text{for } \chi = 0 \text{ \& } \chi = \pi \\ 0 & \text{for } \chi = \pi/2 \end{cases}$$

7032)

d) $k=0$ RM metric flat space

$$dD^2_{\text{proper}} = a^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$\chi = r$ in this case

Just ordinary spherical coordinates

e) $k=-1$, RW metric hyperbolic space

$$dD^2 = a^2\left[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

might have guessed $\chi \in [0, \infty]$

Wiki: hyperbolic functions

$\sinh\chi = \frac{e^{\chi}-e^{-\chi}}{2}$

$\cosh\chi = \frac{e^{\chi}+e^{-\chi}}{2}$

" let

$r = \sinh\chi$

$dr = \cosh\chi\, d\chi$

$\cosh^2 x - \sinh^2 x = 1$

$$d\chi = \frac{dr}{\cosh\chi} = \frac{dr}{\sqrt{1+r^2}}$$

$$dD^2 = a^2\left[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

again tangential space is "normal"

$$S(r,\chi) = \begin{cases} 4\pi a^2 r^2 \\ 4\pi a^2 \sinh^2\chi \end{cases}$$

$$V = \int_0^{\chi} S\, a\, d\chi' = 4\pi a^3 \int_0^{\chi} \sinh^2\chi\, d\chi$$

can be integrated

$4\pi a^3 \frac{\chi^3}{3} = \frac{4\pi a^3}{3}\chi^3$ for $\chi \ll 1$ asymptotic flat space as it should

4) Hubble's Law

Proof from RW Metric (it's easy)

But there are some subtleties I ignore CL-14 -15

First recall the FE frame as

Hubble parameter $H = \frac{\dot{a}}{a}$

$\dot{a} = Ha$

Can't use v anymore in sheet 4

either interpretation of a leads to

D is proper length — measurable by ruler at one instant in cosmic time

$D = H D$ (recession velocity; proper distance)

for any cosmic time t

or

$v = H D$

D: two point in spacetime at same point in cosmic time

recession velocity not velocity relative to an inertial frame — can be any size and can be greater than c

So NOT an ordinary velocity except asymptotically as $a \to 0$

Not "v" of RW just proper distance

Two cases:

$r = a r_0$, $a(t{=}t_0) = 1$ and a is dimensionless scale factor; r_0 are present proper distance of observable universe

"v" is Not RW metric v here

$r = a_0 \chi$ for $k \neq 0$; $\frac{a}{\sqrt{k}}$ is Gaussian curvature radius in units of length (e.g. Gpc or Gly) and $\chi = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$ is comoving distance in natural units (in units of a)

Proof from RW metric is just (CL-13)
(pretty simple)

$$D = \int_0^\chi a\, d\chi = \int_0^r a \frac{dr'}{\sqrt{1-kr'^2}} = a(t) f(r)$$

$a(t)$ has no comoving frame dependence, $f(r)$ no cosmic time dependence

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$D = a f(r)$

Origin

at cosmic time t.

$\therefore \quad \dot{D} = \dot{a} f(r) = \dot{a}\left(\frac{D}{a}\right)$

or $\quad \dot{D} = H D \qquad$ QED

This is an exact result,
but except asymptotically as $D \to 0$, $\dot{D}$ and D
are not direct observables.

We will show they are in this limit soon

Remarks

a) What's the difference between
a direct and indirect observable?
Probably a matter of taste often.

Maybe! A direct observable is one where you
trust all the theoretical steps from
observation to desired value
(e.g., Volume → Temperature on
an old fashioned thermometer)
or where the steps seem ~~short~~
few to you.

Bertrand Russell → the only truly direct observation
is ~~there~~ is thinking — all ~~other~~
observation are theory laden, to one
degree or another.

(2021oct03) | ~~35~~ 40

b) Maybe with FRBs and GW and quasar lensing ~~cosmologically~~ remote D ~~and~~ and $\dot{D}$ are becoming possible
But those methods still have a lot of uncertainty and are not yet competitive though
Omit Done before in lect. 3 promising

and asymptotic form $zc = Hr_{lum}$

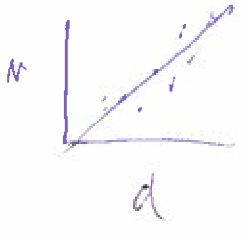
Recapitulate history

1927 Lemaître derived Hubble law explicitly from GR (but Friedmann probably knew it earlier ~~in~~ vaguely) and using literature ~~with~~ values suggested ~~6~~ 575 $\frac{km/s}{Mpc}$ or 670 $\frac{km/s}{Mpc}$ } for H_0

but I think one can say he only found values assuming the theoretical result was true (yes)

1929 Hubble gave an empirical discovery of Hubble's law $v = Hr$
which can be done from ~~$r \to 0$~~ $r \to 0$ asymptotically as we'll prove

He presented a Hubble diagram



His $H_0 \equiv 500 \frac{km/s}{Mpc}$ due to poor Cepheid calibration

403)

In the 1950s, H_0 was ~~brought~~ down a long way.
Which fixed the age problem of that era

$$t_{\text{age of Universe}} \triangleq \frac{1}{H_0} = 13.968\, h_{70}^{-1}$$

age of Universe assuming big bang (i.e. point origin)

of order ~~exact~~

Arthur Holmes (Wiki: Age of Earth) from radioactive dating in 1921 said age of Earth ~ few Gigayears in 1927 1.6–3.0 Gyr

In 1960's — 1990s values between ~50 and 100 (Sandage + Tamman mainly) (de Vaucouleurs mainly) (I exchanged emails with him once)

Since 2010

~68 from Planck (Indirect by fitting ~~to~~ a model but ΛCDM model very robust

~73 from direct local measurements (Riess et al) (but may Cepheids are again giving wrong answers)

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but ~~that~~ we cannot do (except asymptotically → but we can do that

When we look out we look back in cosmic time and light has traveled to us in ~~that~~ lookback time

and the source has receded and it is recession velocity has changed.

But local measurement do establish H_0 (if right!!)

Confusion of source confus mee

Not Really !!, 1927 ✗

Le maitre in 1927 derived it Hubble's law explicitly from GR and estimated $H_0 \simeq 620$ km/s/Mpc ~~625~~ (575+670)/2 Livio 2011

obscure Brussels journal

Friedman eqn Implicit in Friedmann & de Sitter universe but apparently not written explicitly

Lundmark on so had the law Empirically with $H_0 \approx 70$ km/s/Mpc a very good ~~value~~ (Not really

~~all three~~ apparently Not accepted

from data ~~obscure~~

in 1929 Hubble published his discovery and $H_0 = 500$ km/s/Mpc

celebration errors.

$H_0 = 67$ Planck

73 Riess et al. & SNe

~~the tension~~ The Hubble tension

Lemaitre had theory and fitted value H_0

But he did not have ...

So a theoretical discovery and H_0 if true but proof of theory.

~~[illegible]~~ ~~[illegible]~~

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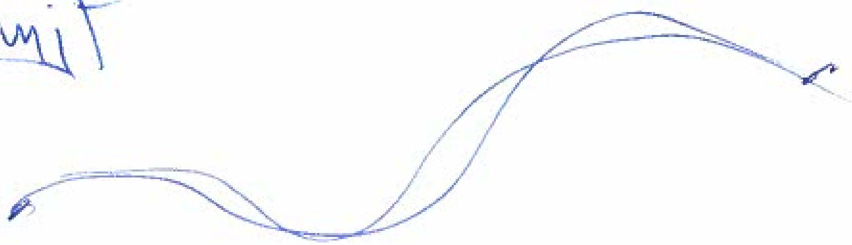
Unit

2017 sep 26



~~I think~~ it's

$$\dot{S}(t) = \cancel{H}S_0\, a(t)$$

true ~~that~~ Hubble's Law ~~follow~~ generally if non vigorously just by saying if space is homogeneous & isotropic and just scales with time by general factor $a(t)$

0
However
possible
exactly
$=0$
1
$=\pm 1$

Then any shape formed by test particles should just scale and one ~~[illegible]~~

$$\dot{S} = S_0\, \dot{a}$$

No freedom to distort

and

$$\frac{\dot{S}}{S} = \frac{\dot{a}}{a}$$

$$\dot{S} = \frac{\dot{a}}{a} S = HS$$

5) Cosmological Redshift

— Multiple derivations (all give same result — which is satisfactory)
— and the analogue non-relavistic velocity + energy loss formula

a) All redshifts $z = \frac{\lambda_o - \lambda_e}{\lambda_e}$ { Doppler, Cosmological, Gravitational + a negative redshift is a blueshift }

λ_o observed wavelength

λ_e emitted wavelength known from recognizing a pattern of lines (emission or absorption)

Thus z is a direct observable.

Now $z+1 = \frac{\lambda_o}{\lambda_e}$

Therefore you can just compound redshifts of any kind

$$(z_{total}+1) = \frac{\lambda_o}{\lambda_n}\frac{\lambda_n}{\lambda_{n-1}} \cdots \frac{\lambda_3}{\lambda_2}\frac{\lambda_2}{\lambda_{1=e}}$$

Most interest

$$z_{total}+1 = (\underset{\text{Doppler}}{z_{local}}+1)(\underset{\text{cosmological unadorned}}{z}+1)(\underset{\text{Doppler}}{z_{remote}}+1)$$

So $z+1 = \frac{z_{total}+1}{(z_{local}+1)(z_{remote}+1)}$ to correct to

Since $z+1$ is a direct observable and independent of theory it is what people cite rather than distance (proper distance) or look back time $(t_0 - t(z))$

These are only direct observables asymptotically as $z \to 0$.

7040

Personally, I do not think the cosmological redshift is a Doppler shift. It has a different formula. It can be derived from the Doppler effect as we will show, and so is kind of a compounded ~~Doppler~~ Doppler effect

~~And the interpretation of the quantit~~

The two formula do agree to lowest order as $z \to 0$

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c},$$ where the distinction between ordinary velocity and recession velocity vanishes.

Classical limit effect which we used in Newtonian derivation of the Fundamental Eqn

b) Proof of Cosmological Redshift Formula from General Relativity (CL-16) via RW metric

$$ds^2 = 0 = c^2 dt^2 - a^2 \frac{dr^2}{1-kr^2}$$

interval between two spacetime events connected by a light beam on a radial path to an observer

$d\theta = d\phi = 0$ for radial

δt_e source e δt_0 observer o

$$\therefore \chi = \int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_{t_e}^{t_0} \frac{c\,dt}{a}$$

independent of time

δt_0 ob

$$= \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{c\,dt}{a}$$

Just a bit later — in fact one period of EMR later

$\frac{dF}{dt} = \frac{c}{a}$

$F(t_0+\delta t_0) - F(t_e+\delta t_e)$

$- F(t_e+\delta t_e) - \delta t_e \frac{dF}{dt}$

$\delta t_0 \frac{dF}{dt_0} - \delta t_e \frac{c}{a}$

$= \frac{\delta t_0 c}{a_0} - \frac{\delta t_e c}{a}$

$$= \cancel{\int_{t_e}^{t_0} \frac{c\,dt}{a}} \quad \int_{t_0}^{t_0+\delta t_0} \frac{c\,dt}{a} \quad \int \frac{c\,dt}{a}$$

$$0 = \frac{\delta t_0}{a_0} - \frac{\delta t_e}{a_e}$$

$= c\delta t_0/a_0 - c\delta t_e/a_e$

$\therefore$ ~~$\frac{\nu_0}{a_0} = \frac{\nu_e}{a_e}$~~

~~$\frac{1}{a_0 \lambda_0} = \frac{1}{a_e \lambda_e}$~~

~~$\frac{\lambda_0}{\lambda_e}$~~

$$\frac{1}{\nu_0 a_0} = \frac{1}{\nu_e a_e}$$

$$\frac{\lambda_0}{a_0} = \frac{\lambda_e}{a_e}$$

$$z+1 = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e}$$

Also $\lambda(t) = \lambda_0 \frac{a(t)}{a_0}$

$\lambda \propto a(t)$

$$\therefore \quad z = \frac{a_0}{a_e} - 1$$

But everyone remembers $\frac{a_0}{a_e} = z + 1 \approx z$ for $z \gg 1$

or $\frac{a_e}{a_0} = \frac{1}{z+1}$

Amazing factoid, z gives ~~us~~ ~~the~~ scaling up of the universe since ~~the~~ time of emission

t of emission time

$$\frac{a_0}{a(t)} = z + 1$$

But we don't know cosmic time directly. [we know $a(t)$ but not t!!!]

It would be great if galaxies had clock faces on them, ~~but they~~ that told cosmic time ~~t~~, but they don't.

There is work on trying to use passively evolving elliptical galaxies (ETGs) as cosmic chronometers, but there are large uncertainties

9.6.2) In Friedmann equation

$$H = \left(\frac{\dot{x}}{x}\right)^2 = \sum_{p=4}^{0} \Omega_{p,0}\, x^{-p}$$

where $x^{-1} = z+1$, $x = \frac{a}{a_0}$ in my notation

$$= \sum_{p=4}^{0} \Omega_{p,0} (1+z)^{p}$$

and so there is a lot of formalism solving in for quantities in terms of z rather than x (i.e. $a(t)$)

Since

$$x = \frac{1}{1+z}$$

$$\dot{x} = \frac{-1}{(1+z)^2}\left(\frac{dz}{dt}\right)$$

evaluated at cosmic time of emission

t of emission

$$\therefore H = \frac{\dot{x}}{x} = -\frac{1}{1+z}\frac{dz}{dt} \hat{=} -\frac{1}{1+z}\frac{\Delta z}{\Delta t}$$

So if you could measure Δz for an object over cosmic time Δt (time of emission) then you could determine $H(z)$

Changing Δz is called redshift drift and it may become measurable in the 2030s. See Roos (2024) in Ast727 reviews (cosmological models)

c) Relativistic Energy loss

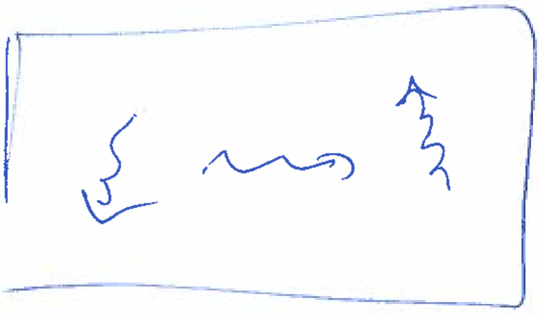
$\lambda \propto a$ but the de Broglie relation $E_{photon} = h\nu = \frac{hc}{\lambda}$

$\therefore E_{photon} \propto \frac{1}{a}$

[2024mar24]

[404.

Where does the energy go?



In an expanding box ~~the~~ photon gas does PdV work on ~~the~~ walls and the energy goes somewhere.

(Ordinary Doppler shifting to lowest order)

But a boundless universe where does it go?

Just vanishes.

GR guarantees itself not ordinary conservation of energy

$$\nabla_\mu T^{\mu\nu} = 0$$

energy momentum conservation equation. (Carroll-117, 120)

∇_μ is covariant derivative (Carroll 93-94, 97)

→ a differential equation true at all points in spacetime

— Not an integral of motion (i.e., a fixed number)

J & the GR replacement

~~4) Non-Relat~~

4044

d) Doppler Shift Derivation of Cosmological Redshift

Plus Hubble law

$\frac{\lambda_o}{\lambda_e} = \sqrt{\frac{1+\beta}{1-\beta}}$ for relativistic Doppler effect

$\approx (1+\frac{1}{2}\beta)(1+\frac{1}{2}\beta)$

$\approx 1+\beta$

$\frac{\Delta\lambda}{\lambda} = d\beta$

1st order $\frac{\Delta\lambda}{\lambda} = \frac{\Delta v}{c}$

where v is ordinary velocity in one inertial frame

ΔD

source — receiver

light signal travels ΔD at speed of light

but in the NR, non-cosmological classical limit, it is valid (for some darn good reason) to conflate v ordinary and v recessional (velocity types degenerate in this classical limit)

We already used this in our Newtonian derivation of the Friedmann eqn.

I suppose, the universe would more weird if it weren't valid.

$\frac{\Delta\lambda}{\lambda} = \frac{\Delta v}{c} = \frac{H\Delta D}{c}$

$= \frac{\Delta a}{a}\frac{1}{c}\frac{\Delta D}{\Delta t}$

Recall $H = \frac{\dot a}{a}$, $H \doteq \frac{\Delta a}{a\Delta t}$, $\frac{\Delta D}{\Delta t} = c$ for light

$\frac{d\lambda}{\lambda} = \frac{\Delta a}{a}$

$d\ln\lambda = d\ln a$

$\lambda \propto a$

e) Non-Relativistic Particle (could be a macroscopic object) Velocity & Energy Loss (almost the same point)

v_p particle $v_p - \Delta v$

recession velocity of 2 relative to 1

$dv_p = (v_p - dv) - v_p$

$= -dv$

$= -H\Delta D$

$= -\frac{da}{a}\frac{dD}{dt} = -\frac{da}{a}v_p$

$\therefore \frac{dv_p}{v_p} = -\frac{da}{a}$

$v_p \propto \frac{1}{a}$

2021 June 24

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But recall

$KE_p = \frac{1}{2} m v_p^2$

$\therefore KE_p \propto \frac{1}{a^2}$

NR energy loss

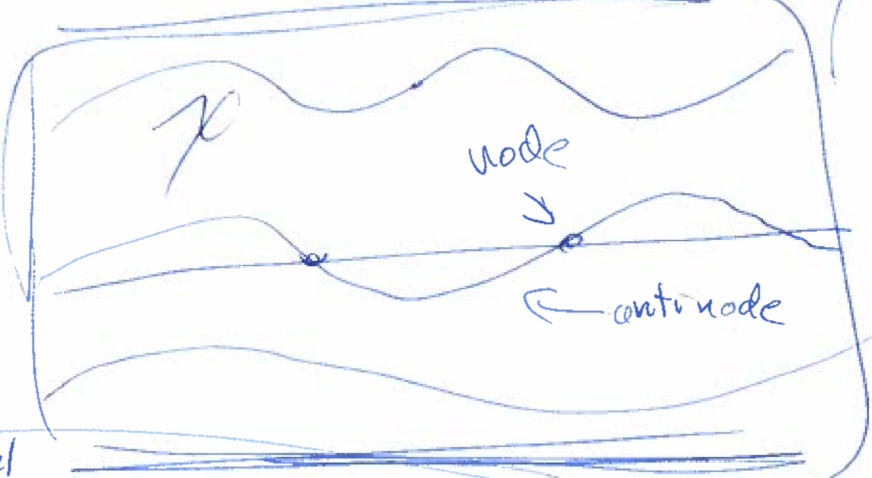
$E \propto \frac{1}{a}$

Extreme Relativistic loss

Maybe BC don't matter if the interior is so homogeneous

Seems like boundary conditions don't matter But matter all the time in all cases

f) Quantum Mechanical Derivation of energy Loss



infinite potential wall

- 1-dimensional

Box quantization

- An ideal limit but close enough to reality to be infinitely useful especially in solid state And White Dwarfs & Neutron Stars
- Even when there seem no Box

Somehow the ideal box gives same result as irregular shapes of infinite size

maybe a proof of this, but it is well hidden

$\psi(BC) = 0$

Boundary conditions

$n(\lambda/2) = L$

number of antinodes

$\lambda = \frac{2L}{n}$

$k = \frac{2\pi}{\lambda}$ wave number of a single-particle state

1046)

Quantum mechanics dictates

$P = \hbar k$ for momentum of single particle state
$= h/\lambda$

where $\hbar = \frac{h}{2\pi}$ Planck constant

For all degrees of relativisticness (non-relativistic (classical limit) to extreme relativistic (photon-like)

NR limit $E \propto p^2 \propto \frac{1}{L^2}$

ER limit $E \propto P \propto \frac{1}{L}$

If you expand box adiabatically, the particles don't change states and the ~~and the~~ single particle energy states change energy with L

$E \propto \frac{1}{L^2}$ NR

$E \propto \frac{1}{L}$ ER

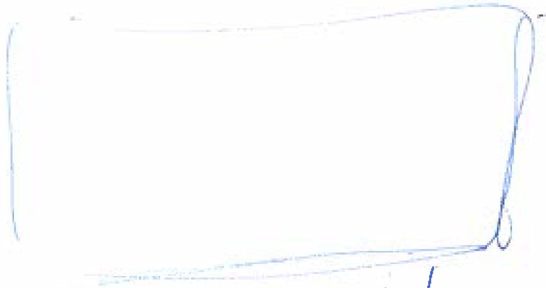
The lost energy goes into PdV work and then somewhere in the external world → except maybe vanishing in cosmology.

But maybe it goes somewhere — But where

2025 mar 24

4047

Universe made of arbitrary boxes that grow with universal expansion.



$L = aL_0$

As box size goes to infinity (or finity for a hyperspherical universe), we can believe results still hold — at least maybe this believable.

$\therefore$

$$p \propto \frac{1}{a}$$

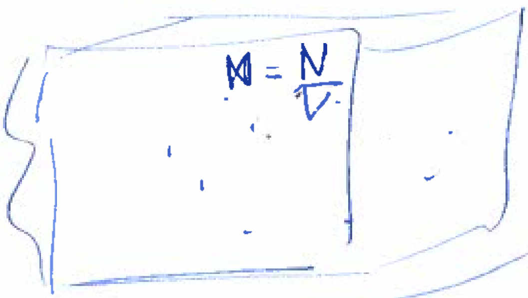
$$E \propto \frac{1}{a^2}$$

NR limit

→ this is for QM particle, but macroscopic object are made of QM particles and it seems it should apply to macroscopic objects and it does — see 4044 particle)

9) Energy Density of Particles

$L = aL_0$



$$E_{total} = N E_{particle}$$

$\mathcal{E}$ = energy density

$$\mathcal{E} = \begin{cases} n E_{rest} \left(\frac{a_0}{a}\right)^3 \\ n E_{avg} \left(\frac{a}{a}\right)^2 \left(\frac{a_0}{a}\right)^3 & NR \\ n E_{part,0} \left(\frac{a}{a}\right)\left(\frac{a_0}{a}\right)^3 & ER \end{cases}$$

$$= \begin{cases} n E_{r0} \left(\frac{a_0}{a}\right)^5 & NR \\ n E_{part} \left(\frac{a_0}{a}\right)^4 & ER \end{cases}$$

Assuming particles conserved and only change ~~lose~~ energy via expansion effect

This turns out to be true for cosmic background

CR[illegible] and th[illegible]

7048

Later we will prove

$E \propto \frac{1}{a^4}$ for Cosmic Background radiation (CBR) which become the CMB (cosmic microwave background)

But also remarkable the radiation field stays Planckian

and this means $E = a_{Rad} T^4$

$\therefore \; T \propto \frac{1}{a}$

The cosmic temperature

There is one.

Before recombination also stuff had this temperature, but now only the CMB

$$T(t) = T_0 \left(\frac{a_0}{a(t)}\right)$$

$T_0 = 2.72548(57)$ Fixsen 2009

If you know $a(t)$, you can just find $T(t)$ to back to some ~~pre~~ pre-Big Bang Nucleosynthesis time. Remarkable.

2025mar24 | 4049

6) Cosmological Distance Measures

a) z — direct observable

$a(t)$ $\downarrow$ $\underbrace{x(\tau)}$ scaled $x = \frac{a}{a_0}$ (* NOT χ)

even though $a_0 = 1$ for flat universe, but in unflat case

$$a_{0g} = \frac{c/H_0}{\sqrt{|\Omega_{k_0}|}} = \frac{4.2827..\, h_{70}^{-1}\ \mathrm{Gpc}}{\sqrt{|\Omega_{k_0}|}} = \frac{13.968\, h_{70}^{-1}\ \mathrm{Gly}}{\sqrt{|\Omega_{k_0}|}}$$

(p. 4005)

t cosmic time

$\tau = H_0 t$ scaled cosmic time

$t_{\mathrm{lookback}} = t_0 - t$

r χ — alternative comoving coordinate

Also $v_{\mathrm{rec}} \equiv H D_{\mathrm{proper}}$ — recession velocity true at any instant in cosmic time

$v_{\mathrm{redshift}} \equiv zc$ — a definition of what it is

$D_{\substack{\mathrm{proper}\\ \mathrm{tangential}}} = a(t)\, r$

$D_{\substack{\mathrm{proper}\\ \mathrm{radial}}} = a(t)\, \chi$

$a, (t)$ you can get from FE, but what is $r(z)$, $\chi(z)$

Also $D_{\mathrm{luminosity}} \equiv \sqrt{\frac{L_{\mathrm{intrinsic}}}{4\pi F}}$ — measured flux (Cepheids, SNe Ia)

Direct observable with a lot of effort. Not easy

D_{angular} — used for BAO and we'll define later

Now Friedmann Eq. solutions give $a(t)$ or $x(\tau)$ in scaled terms.

But direct observable is $z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{a_0}{a} - 1$

or $z + 1 = \frac{a_0}{a} = \frac{a_{0g}}{a_g}$ — ~~scaled~~ Dimensioned or undimensioned a. Most sources simply don't assume you'll know by context

So you know $z = z(a(t)) = z(t)$ if you know $a(t)$

But you still need $r(z)$, $\chi(z)$ to connect direct observable D_{lum}, D_{angular}

9050 | 2025 mar 24

But intuitively obvious? Time travel & curvature effects go to zero, but intuitively obvious is not a proof!

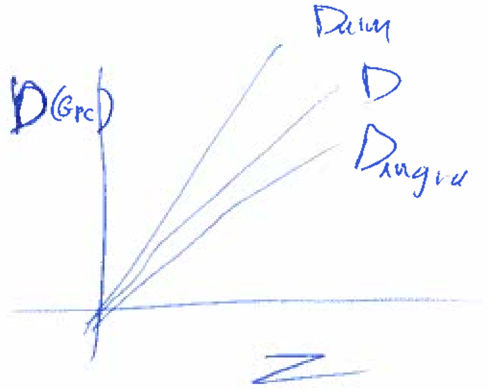
1) As $z \to 0$, it is true in 1st order in small z (but takes a proof) that

$$D^{1st}_{generic} = D^{1st}_{proper} = D^{1st}_{luminosity} = D^{1st}_{angular} = D_{geometric}$$

(– parallax – Masers)

and $v_{rec} \overset{1st}{=} v_{red}$

$$t^{1st}_{look\,back} = \frac{D^{1st}_{generic}}{c}$$



It is ~~important~~ ~~to~~ prove these so ~~that~~ $v_{rec} = H D_{proper}$ can be shown ~~to~~ yield $v_{red} = H_0 D^{1st}_{generic}$

and ~~so~~ we can prove Hubble's law locally ~~and~~ can find H_0 — the current Hubble constant.

2) ~~Deflecting~~ Finding $r(z)$, $\chi(z)$

If $k=0$, $r = \chi$

Distant galaxy — light signal → Milky Way

RW ~~metric~~ connecting light-like spacetime events along a radial direction

$$\underbrace{ds^2 = 0}_{\text{light like}} = c^2 dt^2 - a(t)^2 \frac{dr^2}{1-kr^2}$$

$$= c^2 dt^2 - a(t)^2 d\chi^2$$

Alternative comoving frame coordinates — we need them both in general.

[2025 mar 24]

4051

$$\therefore\ f(r) = \int_0^r \frac{du}{\sqrt{1-kr^2}} = \int_t^{t_0} \frac{c\,dt}{a} = \cancel{\chi}(t)$$

$$= \frac{c}{H_0 a_0} \int_\tau^{\tau_0} \frac{d\tau}{x(t)}$$

x NOT χ

Invert in principle

$r = f^{-1}(\cancel{\chi})$

$r = f^{-1}(z)$

Solve for

$\chi(t) = \chi(t(a)) = \chi[t(a(z))]$

using $\frac{a}{a_0} = \frac{1}{1+z}$ as on p.4049

Note ~~#~~ $k = \begin{cases} 1 & +ve \\ 0 & \text{flat} \\ -1 & -ve \end{cases}$

if $k=0$, then $r = \chi$

and things are simpler

and the universe seems to obey that to good approx.

Recall in general

$\chi = \begin{cases} \sin^{-1} r & k=1 \\ r & k=0 \\ \sinh^{-1} r & k=-1 \end{cases}$

$r = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$

So we do know f and f^{-1}

for flat $a_0 = \cancel{a}.\ 1$ conventionally but bor unflat

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}}$$

$$= \frac{4.2827\ h_{70}^{-1}\ \text{G}}{\sqrt{|\Omega_{k0}|}}$$

$$= \frac{13.968\ h_{70}^{-1}\ \text{G}}{\sqrt{|\Omega_{k0}|}}$$

see p. 4005

7052

d) Can we find $\chi(t) = \chi(t(a)) = \chi(t(a(z))) = \chi(z)$

Yes, but in general not analytically

But analytic cases exist.

i) De Sitter universe

— pure cosmological constant univers

$$\chi = \int_t^{t_0} \frac{c\,dt}{a}$$

$$= \int_a^{a_0} \frac{c\,dt}{H_\Lambda a^2}$$

$$= \frac{c}{H_\Lambda}\left(-\frac{1}{a}\right)\Big|_a^{a_0}$$

$$= \frac{c}{H_\Lambda}\left(\frac{1}{a} - \frac{1}{a_0}\right)$$

$$= \frac{cz}{H_\Lambda a_0}$$ {dimensionless for $a_0 = a_{sc}$

$\therefore D_{proper} = a_0 \chi$

$$D_{proper} = \frac{zc}{H_\Lambda}$$

$v_{rec} = H_\Lambda D_{proper}$

$= zc = v_{red}$

In this special case $v_{rec} = v_{red}$ for all z

More than one way to solve, but in this case

$t = t(a)$

$dt = \frac{\partial t}{\partial a} da$

$= \frac{da}{\dot a}$

$= \frac{da}{H_\Lambda a}$

Recall

$\frac{a}{a_0} = \frac{1}{1+z}$

$\frac{1}{a} = \frac{1+z}{a_0}$

More work to get

D_{lum}

D_{ang}

but we won't do that here

$$\frac{\dot x}{x} = \sqrt{\Omega_\Lambda}$$

≈ 1 for scaled time

$x = x_0 e^{(\tau - \tau_0)}$

$a = a_0 e^{H_0(t - t_0)}$

where $H_0 = H_\Lambda$

$= \sqrt{\Lambda/3}$

Now Λ-CDM

$h = \sqrt{\Omega_{m0} + \Omega_\Lambda}$

in scaled time τ

$H = H_0\sqrt{\Omega_{m0} x^{-3} + \Omega_\Lambda}$

In far future

$H_\Lambda = H_0\sqrt{\Omega_\Lambda}$

$H_\Lambda = 70\, h_{70} \sqrt{\Omega_\Lambda}$

$= \underbrace{70\sqrt{.7}}_{\text{fiducial value}}\, \underbrace{h_{70}\sqrt{\frac{\Omega_\Lambda}{0.7}}}_{\text{fiducial value correction}}$

$= (58.566 \ldots \frac{km/s}{Mpc})\, h_{70}\sqrt{\frac{\Omega_\Lambda}{0.7}}$

far future $H_\Lambda = H$ in Λ-CDM model which currently is challenged by DESI DR2.

iii) Other cases with Analytic Solutions?

Well, the power-law solution

$$x(\tau) = \left[\underbrace{\sqrt{\Omega_{p,0}}}_{\Omega_{p,0}=1} \frac{p}{2}\tau\right]^{2/p}$$ where

$\Omega_{p,0} = 1$ for pure power law universe

$p \neq 0$

$p = 0$ is the de Sitter universe

$$a = a_0\left[\frac{p}{2}H_0 t\right]^{2/p}$$ (Lect 3 p. 328)

$$a = a_0\left[\frac{t}{t_0}\right]^{2/p}$$ where $t_0 = \left(\frac{2/p}{H_0}\right)$ universe age

$$X = \frac{c}{a_0}\int_t^{t_0}\frac{dt}{[t/t_0]^{2/p}} = \frac{c\tau_0}{a_0}\int_y^1\frac{dy}{y^{2/p}} = \frac{c\tau_0}{a_0}\left.\frac{y^{-2/p+1}}{-2/p+1}\right|_y^1$$

$$= \frac{c\tau_0}{a_0}\left[\frac{1}{-2/p+1} - \frac{(a/a_0)^{-1+p/2}}{-\frac{2}{p}+1}\right]$$

$$y = \frac{t}{t_0} = \left(\frac{a}{a_0}\right)^{p/2}$$

A tedious homework problem on one of those days when feel like making up tedious homework problems

$$\frac{a}{a_0} = \frac{1}{z+1}$$

but it's not so neat.

e) Small z Simplification

$$f(r) = \int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_0^r dr\left[1 + \tfrac{1}{2}kr^2 + O(r^3)\right]$$
$$= r + O(r^3)$$

$$\therefore X(z) = X(r) = f(r) = r + O(r^3)$$

We will show eventually that $r \overset{\text{2nd order}}{=} b_1 z + b_2 z^2$

$$\therefore X(z \text{ or } r) \overset{\text{2nd in } z}{=} r = b_1 z + b_2 z^2$$

So $X(z) \overset{\text{2nd}}{=} r \overset{\text{2nd}}{=} b_1 z + b_2 z^2$

Since they are the same to 2nd order in small z conventionally one just uses r and omits the X symbol

7054 | 2025 mar 24

f) Conformal time

- a useful auxillary quantity in some cases.
- you can sometimes find solutions easily in conformal time ~~but unless~~ but it's not real time which coordinates everything else in the universe (e.g., planet orbit frequencies, pendulum frequencies, nuclear reaction rates)

and so ~~unless~~ you ~~can~~ must convert to real time ~~easily~~, ~~somehow, it may not be a big help~~ ~~help, but it does~~ help with observables

for $\chi = \chi(z)$ needed ~~to compare~~ get D_{lum}, D_{ang} and compare to observations

Steiner-3 ~~as~~ Wik, CL-13

$$\chi = \int \frac{c\,dt}{a}$$

eta

$$d\chi = \frac{c\,dt}{a} = \frac{c\,da}{\dot a\, a} = \frac{c\,da}{a^2 h} = \frac{c}{a_0 H_0} \frac{dx}{x^2\sqrt{\sum_p \Omega_{p0} x^{-p}}} = \frac{c}{a_0 H_0}\frac{dz}{\sqrt{\sum_p \Omega_{p0}(1+z)^p}}$$

$t = t(a)$
$dt = \frac{\partial t}{\partial a} da = \frac{da}{\dot a}$

There don't include c in definition, but it doesn't matter since just a constant

$x = \frac{1}{1+z}$
$dx = -\frac{1}{(1+z)^2} dz = -x^2 dz$

The minus just flip the integration direction

(2024 Mar 24) 4055

~~desired for~~ computing observable quantities D_{lum}, $D_{angular}$

$$\chi(z) = \frac{c}{a_0 H_0}\int_0^z \frac{dz}{\sqrt{\sum_p \Omega_{p0}(1+z)^p}} = \frac{c}{a_0 H_0}\frac{dy}{\sqrt{\sum_p y^p}}$$

if $a = a_0 g$ dimensionless
but if $a = a_0 = 1$ has units of length
— either way
$D_{proper} = a_0 \chi(z)$
and the a_0 vanishes

$$D_{proper} = \frac{c}{H_0}\int_0^z \frac{dz}{\sqrt{\sum_p \Omega_{p0}(1+z)^p}}$$

widely considered

$p =$
- 4 radiation
- 3 matter
- 2 curvature $R_k = ct$ univ
- 1 quintessence in some theory
- 0 the Λ term,

Sometimes can be solved analytically

Say ~~for Λ~~ ~~Alber~~ and any other for $p = 0, 1, 2$.

~~$p = 0, 2, 3$~~

May be other cases

In any case, you can integrate numerically to get $\chi(z)$.

~~but dt ... $\frac{a}{c}d\chi$ ... $= \frac{1}{H_0}\frac{d(t)\ dz}{\sqrt{\sum \Omega_{r,0}(1+z)^p}}$~~

~~You need to know $a(t) =$~~ $a(z)$

But if you want $t(z)$

$$dt = \frac{a}{c}d\chi = \frac{1}{a_0 H_0}\frac{\frac{a_0}{1+z}\,dz}{\sqrt{\sum_r \Omega_0(1+z)^p}}$$

056

2025 march

$$dt = \frac{d_0}{H_0} \frac{dz}{\sqrt{\sum_p (1+z)^{p+2}}}$$

– another integration has to be done.

~~4049~~ 4049

Since $k=0$ $\quad r = f(r)$

$\therefore \quad r = A\left[t\left(\frac{a_0}{1+z}\right)\right]$

and recall ~~the~~ de Sitter universe has exponential growth

t_0

$a = a_0 e^{H_0(t-t_0)} = a_0 e^{H_0 \Delta t}$ and $\Delta t = t - t_0$, $d\Delta t = dt$

$$A(t) = \int_t^{t_0} \frac{c\,dt}{a} = \int_{\Delta t}^{0} \frac{c}{a_0} e^{H_0 \Delta t'} \, d\Delta t'$$

$$= -\frac{c}{a_0 H_0} e^{-H_0 \Delta t'}\Big|_{\Delta t}^{0} = \frac{c}{a_0 H_0}\left(e^{-H_0 \Delta t} - 1\right)$$

$$= \frac{c}{a_0 H_0}\left(\frac{a_0}{a} - 1\right) = \frac{zc}{a_0 H_0}$$

$\therefore \quad r = \frac{zc}{a_0 H_0}$ ~~with~~ $\quad L = a_0 r$ $\Bigg\}$ Proper distance at t_0

~~whether a_0~~ whatever the choice of a_0 ~~or~~ r ~~to~~ have dimensions

$\therefore \quad zc = H_0 L$ $\quad$ Hubble's law recovered

zc is called redshift ~~velocity~~

since $z = \left(\frac{a_0}{a} - 1\right)$ and ~~in this special~~ case redshift velocity = recession velocity

4050 (3b)

7) Summary of Distance Measures and small z formulae

i) $a(t) = a_0\left[1 + H_0\Delta t - \frac{1}{2}q_0 H_0^2 \Delta t^2 + \ldots\right]$ CL-17

lookback $= t_0 - t = -\Delta t$

$\Delta t = t - t_0 < 0$ for usual cases

$q_0 \equiv \frac{\ddot{a}_0 a_0}{\dot{a}_0^2}$ is the deceleration parameter which for ΛCDM is -ve since the universe is accelerating

$q_0 H_0^2 = \frac{\ddot{a}\, a_0}{\dot{a}^2}\frac{\dot{a}_0^2}{a_0^2} = \frac{\ddot{a}_0}{a_0}$

ii) For ΛCDM (with k=0 of course)

$$q_0 = \frac{1}{2}\left[\Omega_{m_0}(1+3w_{matter}) + \Omega_\Lambda(1+3w_\Lambda)\right]$$

with $w_{matter} = 0$, $w_\Lambda = -1$

$$= \frac{1}{2}\left[\Omega_{m_0} - 2\Omega_\Lambda\right]$$

See p. 4053 — 4055

Ω_{m_0}	Ω_Λ	
0.3	0.7	fiducial values
0.3153(73)	0.6847(73)	Planck 2018 p. 15

$q_0 = -\frac{\ddot{a}}{a_0}\frac{1}{H_0^2}$ Li-5, Li-27

$= -\frac{\left[\left(-\frac{4\pi G}{3}\right)\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}\right]}{\left[\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}\right]}$ Li-27

$\doteq -0.55$ fiducial ΛCDM value

iii) $z = -H_0\Delta t + \left(1 + \frac{1}{2}q_0\right)H_0^2\Delta t^2 + \ldots$

(CL-17) Recall $\Delta t < 0$, good to 2nd order in Δt for all k see p. 4055

iv) $t_{lookback} = -\Delta t = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2}q_0\right)z + \ldots\right]$

(CL-17)

see 4056

$\doteq \frac{z}{H_0} = \frac{zc}{c\, H_0} = \frac{v_{redshift}}{c\, H_0} = \frac{D_{1st}}{c}$ to 1st order

$= \frac{D_{1st}}{c}$ to 1st order in z

$= \frac{D_{angular,1st}}{c}$

4051

v) $r = \frac{zc}{a_0 H}\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right]$ (4049, 4059 (see p.)

χ (to 2nd order in z as on p. 4046) (CL-18)

comoving frame coordinate to 2nd order in z

So to 1st order $zc = H_0 a_0 r = HD$

$\underbrace{zc}_{v_{redshift}}$ — proper distance at t_0 for any k.

(see p. 4048)

First derived by Lemaitre 1927

This is the observable low redshift Hubble law for measurable L

vi) $D_{proper} \stackrel{= a_0 r}{=} \frac{zc}{H_0}\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right]$

(see p. 4058)

$= \frac{zc}{H_0}$ to 1st order

D_p — model, data — z

vii) Recession velocity since $v_{res} = H_0 D_{proper}$ in general — ideal Hubble's law

$v_{recession} = zc\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right]$

from $v_{res} = H_0 D_{proper}$

Model gives $D(z) = a_0 r$ proper tangential

$= zc$ to 1st order in z

So fit D_L observed to model giving D_L from FE universe

$= v_{redshift}$

viii) $D_{Luminosity} = \left(\frac{\mathcal{L}_{intrinsic}}{4\pi f_{observed}}\right)^{\frac{1}{2}}$ ← observable definition (see p. 4063)

$= a_0 r(1+z)$ for universe — Linear — all k (see Weinberg p. 421)

$= \frac{zc}{H_0}\left[1 + \frac{1}{2}(1-q_0)z + \ldots\right]$ (CL-19)

$k=0$, $a_0=1$, $r = D_{proper}$

$k=\pm$, a_0 from Ω_{k0}

and $r = \begin{cases} \sin\chi, & k=1 \\ \sinh\chi, & k=-1 \end{cases}$

D_p — $= D$ — $= \frac{z}{H}$ — $= \frac{v_r}{H}$

∴ Obs. Hubble law

$v_{reds} = H_0 \ldots$ which makes it observable, possible measure

~~1052~~ ~~08~~

ix) $D_{Angular} \equiv \frac{S}{\Delta\theta} = \frac{L_{ruler}}{\Delta\theta}$ (observable)

1st $D^{1st}_{proper} = D^{1st}_{Lum} = D^{1st}_{Ang} = \frac{zc}{H_0} = \frac{N_{vel}}{H_0}$ = asymptotic Hubble law

$= \frac{a_0 r(z)}{1+z}$ (among ~~Flat space~~ ? (p. 4007) ~~i.e.~~ result for all k)

Weinberg 422

CL-19

$= \frac{zc}{H_0}\left[1 - \underbrace{\left(\frac{3}{2}+\frac{q_0}{2}\right)}_{-\frac{1}{2}(3+q_0)} z + \dots\right]$

x) Distance Duality Relation (Etherington's Reciprocity theorem 1933 Wik)

(see p. 4070 for repeat)

$\frac{D_L}{1+z} = a_0 r = D_{angular}(1+z)$

$\therefore D_L = D_{ang}(1+z)^2$

general for all k Weinberg 423

easier to test since independent of scale of D_L and $D_{angular}$

— It has been verified within error (Wik)

— a violation would mean exotic physics of some kind

eg. photons not conserved, GR wrong

But actually Richard C. Tolman 1881-1948 gave the correct formula and suggested it's use in cosmology (Wik)

~~Wik: Etherington Reciprocity theorem implies true~~ for all FE models NOT just flat. Yes Weinberg - 423

~~Is this just for flat space and ... ? ...~~

7a) Proving the Small z approximations

(don't depend on curvature to z2)

These allow us to set ~~model~~ parameters from ~~fom~~ the local small z universe

That are independent of models

Of course, if the ~~models are~~ Friedmann Equation is wrong the parameters may not mean what they mean in ~~the model~~ the FE.

If you can do straight forward modeling from a model but some parameters the higher ones can

Without a full model specific and so are generous of you have fit them correctly and the FE is true!!

a) Expand $a(t)$ in a Taylor series about t_0 (cosmic present)

$$a(t) = a_0 + \dot{a}_0 \Delta t + \frac{1}{2}\ddot{a}_0 \Delta t^2 + \ldots$$

The next term is jerk, but too far for us

$$\Delta t = t - t_0 < 0$$

since we usually think of signals from the past.

$t_{lookback} = -\Delta t$, the look back time

Some argue that Taylor series is not best for expansion, Padé approximants have been used for faster convergence ?

$$a(t) = a_0\left[1 + \frac{\dot{a}_0}{a_0}\Delta t + \frac{1}{2}\frac{\ddot{a}_0}{a_0}\Delta t^2 + \ldots\right]$$

$$= a_0\left[1 + H_0\Delta t + \frac{1}{2}(-q_0)H_0^2\Delta t^2 + \ldots\right]$$

where $H_0 = \frac{\dot{a}_0}{a_0}$ is the Hubble constant (unit of inverse time)

$= \sqrt{\frac{8\pi G}{3}\rho_{crit \text{ or Hubble density}}}$ (Li-51)

and

$q_0 = -\frac{\ddot{a}_0}{a_0 H_0^2} = -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2}$ is the deceleration parameter (Li-53)

It's dimensionless and is positive for negative acceleration because people wanted a positive parameter and thought acceleration would be negative when they defined it in ~~the~~ before 1970 sometime

1054

Li-53

$$q_0 = -\frac{\ddot{a}_0}{a_0}\frac{1}{H_0^2} = -\left[\frac{\left(-\frac{4\pi G}{3}\right)\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}}{\frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}}\right]_0$$

(see Li-55) $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$

$$= -\left[-\frac{1}{2}\frac{1}{\rho_{crit}}\left(\rho + \frac{3P}{c^2}\right) + \Omega_\Lambda\right]$$

Li-51 Li-56

Let $\rho = \sum_i \rho_i$ and $P_i = c^2 w_i \rho_i$ (EOS)

counting dark energy which is in Ω_Λ

$$= \frac{1}{2}\sum_{i \neq \Lambda} \Omega_i(1+3w_i) - \Omega_\Lambda$$

$$= \frac{1}{2}\sum_{i \neq \Omega} \Omega_i(1+3w_i) + \frac{1}{2}\Omega_\Lambda(1-3)$$

$$= \frac{1}{2}\sum_i \Omega_i(1+3w_i)$$

if we parameterize $w_\Lambda = -1$ then for the cosmological constant

Li-57 If a constant $P_\Lambda = -\rho_\Lambda c^2$ to obey fluid equation

$$= \frac{1}{2}\sum_i \Omega_i(1+3w_i)$$

in general for w parameter equation of state (EOS)

$$= \frac{1}{2}[\Omega_M - 2\Omega_\Lambda]$$

$$\frac{1}{2}[.3 + .7(-2)]$$

$$\frac{1}{2}[-1.1]$$

-0.55

$\Omega_M = .3$, $w_M = 0$

$\Omega_\Lambda = .7$, $w_\Lambda = -1$

the fiducial ΛCDM values

Before 1998 when the flat EdS model was still favored

$$q_0 = \frac{1}{2}\Omega_M = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

We know to some degree $H_0 \approx 70$, $q_0 = -.55$

In 1970s Sandage described cosmology as the search for two numbers H_0 and q_0 (somewhat humorously)

Nowadays q_0 has lost a bit of luster if ΛCDM is true since Planck 2018 data fits the model ~~seeing~~ other data.

However, if ΛCDM needs replacement local values of q_0 become important again but hard to measure

b) From $a(\Delta t)$ to $z(\Delta t)$

$$\frac{a}{a_0} = \underbrace{\frac{1}{1+z}}_{1 - z + z^2 + \ldots} = 1 + H_0\Delta t + \tfrac{1}{2}(-q_0)(H_0\Delta t)^2 + \ldots$$

to 2nd order
geometric series
(Arf-279)

$$\therefore \quad z = -H_0\Delta t + \tfrac{1}{2}q_0(H_0\Delta t)^2 + \ldots$$
$$+ z^2 + \ldots$$

assume $z = \sum_{\ell=1}^{\infty} b_\ell \Delta t^\ell = b_1\Delta t + b_2\Delta t^2 + \ldots$

$$\therefore \quad z^2 = b_1^2\Delta t^2 + 2b_1b_2\Delta t^3 + \ldots$$

$\therefore$ To 1st order in Δt, then $\Big($ ~~$z^2 = (H_0\Delta t)^2 + 2b_1b_2\Delta t$~~ $\Big)$

dropping $2b_1b_2\Delta t^3$

$$z^{1st} = -H_0\Delta t$$

$$\therefore \quad z^{2nd} = -H_0\Delta t + (H_0\Delta t)^2 + \tfrac{1}{2}q_0(H_0\Delta t)^2$$

$$z = -H_0\Delta t + (\tfrac{1}{2}q_0 + 1)(H_0\Delta t)^2 + \ldots$$

$\therefore$ Recall z is from a source observed at lookback time $t_{lookback} = -\Delta t$

But Δt is not an observable in general (though it is asymptotically as $z \to 0$ as we'll show) unless someone puts clocks on galaxies,

and so to determine Δt from z we need to invert the series

4056)

From Arf-316-317

$$\Delta y = \sum_{\ell=1}^{\infty} a_\ell \Delta x^\ell$$

$$\Delta x = \sum_{\ell=1}^{\infty} b_\ell \Delta y^\ell$$

$$b_1 = \frac{1}{a_1}$$

$$b_2 = -\frac{a_2}{a_1^3} \quad \text{etc}$$

$a_1 = -H_0$

$a_2 = (1+\tfrac{1}{2}q_0)H_0^2$

$$\therefore \Delta t = -\frac{1}{H_0} z - \frac{(1+\frac{1}{2}q_0)H_0^2}{(-H_0)^3} z^2 + \dots$$

$$= -\frac{z}{H_0}\left[1 - (1+\tfrac{1}{2}q_0) z + \dots\right]$$

$$t_{\text{lookback}} = \frac{z}{H_0}\left[1 - (1+\tfrac{1}{2}q_0) z + \dots\right] \quad \Big\{ \text{CL-17}$$

$$t_{\text{lookback}}^{\text{1st ord}} = \frac{z}{H_0} = \frac{z}{N_{\text{red}}/D^{\text{1st}}} = \frac{D^{\text{1st}}}{c} \quad \Big\{ \text{to 1st order in small } z$$

$$= z\, t_{\text{Hubble}}$$

~~not~~ proved formally on p. 4058 + 4067 ~~proven yet~~ in most formal way

since $N_{\text{red}} = zc$

But $N_{\text{rec}} = H D_{\text{proper}}$ exi

$zc = H_0 D_{\text{lumonosity}}$ (static limit) is vaguely proven

c) From $a(t)$ to $r(z)$

Recall for a signal $\displaystyle\int_t^{t_0} \frac{c\,dt}{a} = f(r) = r + O(r^3)$

see p. 4046

use $\dfrac{1}{1+x} = 1 - x + x^2$ Arf-279 and see p. 4053

see p. 4044 – 4045

$t - t_0 = \Delta t$

$$\int_t^{t_0} \frac{c\,dt}{a_0}\left[1 - H_0\Delta t - \tfrac{1}{2}(-q_0)H_0^2\Delta t^2 + H_0^2\Delta t^2 + \dots\right]$$

$$\int_{\Delta t}^{0} \frac{c\,d\Delta t}{a_0}\Big[\underset{(0-3)}{1} - \underset{-(0-3)}{H_0\Delta t} + \underset{+(0-3)}{\tfrac{1}{2}(q_0+1)H_0^2\Delta t^2} + \dots\Big] = r + O(r^3)$$

$$\frac{c}{a_0}\left[-\Delta t + \tfrac{1}{2}H_0\Delta t^2 - \tfrac{1}{2}(q_0+1)H_0^2\frac{\Delta t^3}{3} + \dots\right] = r + O(r^3)$$

static limit or small t_{lookback}

Validity of Doppler effect proof of cosmological redshift implies $\frac{\Delta\lambda}{\lambda} = \beta = \frac{N_{\text{rec}}}{c}$ it's z $\therefore N_{\text{rec}} = zc$

$$\frac{c}{a_0 H_0}\left[-H_0 \Delta t + \tfrac{1}{2} H_0^2 \Delta t^2 + \dots \right] = r + O(r^3)$$

see p. 4056   see p. 4056

Assume $r = \sum_{\ell=1}^{\infty} a_\ell z^\ell$

$$z\left[1 - (1 + \tfrac{1}{2} q_0) z\right] + \tfrac{1}{2} z^2 + \dots \Big] = r + O(r^3)$$

this term contributes $O(z^3)$

$$\frac{c}{a_0 H_0}\left[z\left[1 - (\tfrac{1}{2} + \tfrac{1}{2} q_0) z\right] + \dots \right] - O(z^3) = r$$

$$\therefore\ r^{\text{2nd order in } z} = \frac{z\,c}{a_0 H_0}\left[1 - \tfrac{1}{2}(1 + q_0) z\right]$$

$$r(z) = \frac{z\,c}{a_0 H_0}\left[1 - \tfrac{1}{2}(1 + q_0) z + \dots\right] \qquad \text{(C1-18)}$$

8) Cosmological Distance measures

And making use of results p. 4053-70

a) z the easiest direct observable.

b) Cosmic time and Lookback time

Not a direct observable.

If you know $a(t)$, then $t = t(a)$ the inverse

$$t_{\text{lookback}} = \begin{cases} t_0 - t & \text{in general} \\ \frac{z}{H_0}\left[1 - (1 + \tfrac{1}{2} q_0) z + \dots\right] & \text{2nd order in } z \quad \text{(C1-17, p. 4056)} \\ \frac{z}{H_0} & \text{1st order in } z \end{cases}$$

So if you know H_0, q_0 as well as easy z, you know $t_{\text{lookback}}^{\text{2nd}}$ but how do you know H_0, q_0?

4058

c) $D_{proper} = a_0 X(z)$ in general

Recall $D_{proper}(z)$ exact available in special cases; see p. 4047, 4048
desitter universe, power law universe

$a_0 r(z)$ to 2nd order in small r (see p. 4046) and small z

$$\frac{zc}{H_0}\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right]$$

to 2nd order in small z (see p. 4047) & CL-18

$D^{1st}_{proper} = \frac{zc}{H_0}$

$$= z\left\{\begin{array}{l} \frac{4.2827\ldots}{h_{70}}\ \text{Gpc} \\ \frac{13.968\ldots}{h_{70}}\ \text{Gly} \end{array}\right.$$

If you knew z, H_0, q_0 you'd know D_{proper} 2nd, but D_{proper} not a direct observable

Note a_0 has cancelled. Physical scale comes from c, H_0, q_0 information
$\frac{zc}{H_0}$ to 1st order in z.

d) Recession Velocity

~~$D_p(t) = a(t)X(z)$~~

~~$v_{rec}(t) = \dot{D}_{prop} = \dot{a}_0 X = \ldots$~~

From general Hubble's law $v_{rec} = H D_{proper}$

$v_{rec\,0} = H_0 D_{proper\,0}$

$v_{rec} = H_0 D_{proper}$

$= \cancel{H_0}\, zc\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right]$

Dropping 0 on v_{rec} and D_{proper} for simplicity

To 1st order in z, v^{1st}_{rec} is a direct observable which we already know as formally (see p. 4056)

$\therefore v^{1st}_{rec} = \cancel{H_0} zc = H_0 D^{1st\ order\ in\ z}$

confirms the $z \to 0$ asymptotic Hubble law first derived by Lemaître 1927

and

e) $v_{redshift} \equiv v^{1st}_{rec} = zc$ is a definition.

So we have z, $\nu_{redshift}$,

and $\nu_{rec}^{1^{st} \text{ in } z}$ as direct observables,

but we still don't ~~dete~~ have

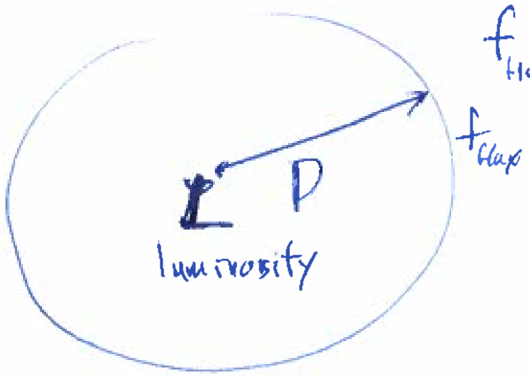
$t_{lookback}$ and D_{proper} as direct observables to 1^{st} order in z $\Big\{$ need to fit H_0

to 2^{nd} order in z ~~need~~ $\Big\{$ need to fit H_0 and q_0

or exactly.

f) Luminosity Distance

$$f = f_{flux} = F_{des} = \frac{L}{4\pi D^2}$$ in a static flat universe assuming zero extinction



hereafter we assume extinction ~~can~~ negligible or can be corrected for

$$\therefore D = \sqrt{\frac{L}{4\pi f_{flux}}}$$

Now if the universe is not static (i.e., moving and additionally time dependent)

4060 We define luminosity distance

$$D_L \equiv \sqrt{\frac{L}{4\pi f_{flux}}}$$

which is a direct observable if L is known.

(which it ~~sort~~ ~~of~~ is for SNe Ia as standardizable candles to good approximation. (Other ~~cosmological~~ bright sources too more approximately))

But to use this to measure parameters independent of models for z small or dependent on models for all z, we need to connect D_L to theoretical quantities connected to z

i.e. $r(z)$ and $\chi(z)$.

Now for any k ($k = \{ \begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix} \}$), we have

$$S = 4\pi (a_0 r)^2$$ (see p. ~~4024~~ 4022)

In p. 4022 only proved for a k=+1 case, but sort of obviously true for k=0 and k=-1 too

recall r is the comoving coordinate designed to give tangetial lengths correctly (p. 4017)

4061

$S = S(t_0, r) = 4\pi(a_0 r)^2$

the spherical surface surrounding the source at t_0

Milky way at t_0

Say at t_e a light pulse was emitted over dt_e

from p. 4032, $\frac{dt_e}{a} = \frac{dt_0}{a_0}$ which we used on p. 4032 to find the cosmological redshift formula

$\frac{dt}{dt_0} = \frac{a}{a_0}$

From p. 4032 $E a = E_0 a_0$

photon energy

$\frac{E_0}{E} = \frac{a}{a_0}$

cosmological redshift itself

$\frac{a}{\lambda} = \frac{a_0}{\lambda_0}$

$$\therefore f_{obs} = \frac{L}{4\pi(a_0 r)^2} \left(\frac{dt}{dt_0}\right)\left(\frac{E_0}{E}\right) = \frac{L}{4\pi(a_0 r)^2}$$

photon energy reduction factor

time dilation factor

photon energy reduction factor

$L \propto \frac{NE}{dt}$

multiply by time of emission

$L \propto \left(\frac{NE}{dt}\right)\frac{dt}{dt_0} \quad \left(\frac{E_0}{E}\right)$

divide by time of observ

to get observer photon energy

Recall

$\frac{a_0}{a} = 1 + z$

$$\therefore f = f_{obs} = \frac{L}{4\pi(a_0 r)^2}\frac{1}{(1+z)^2}$$

4062

$$D_L = \sqrt{\frac{L}{4\pi f}} = \sqrt{(a_0 r)^2 (1+z)^2}$$

observable quantities

$$D_L = a_0 r\,(1+z) \quad \langle \text{ Theoretical connection to } z$$

$$= a_0\, r(z)\,(1+z)$$

~~where it is~~ which is the way it is usually written.

Now ~~$r = f^{-1}\left\{A\left[t\left(\frac{a_0}{1+z}\right)\right]\right\}$ (see p 4046)~~

~~$f(r) = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$, $A(t) = \int_t^{t_0} \frac{c\,dt}{a(t)}$~~

Not for ~~or r~~ ~~and~~ normal r and X ...ut r and X connecting ...nts ~~ob~~ separated by a light-like interval

$$r = f^{-1}[X(t)] = f^{-1}[X(z)]$$

f^{-1} analytic exist see p. 4046

~~$X(t)$~~ $X(z)$ ~~analytic exist~~

$$f(r) = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$$

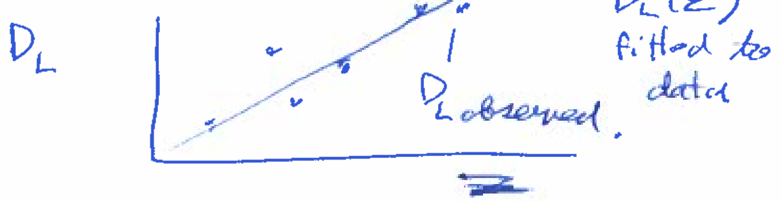
$r = \sin X$, $k=1$

$r = \sinh X$, $k=-1$

$k=0$ is easy $f(r) = r$

$X(t)$ analytic and $X(z)$ analytic exist ~~only~~ in special cases.

In general you'd have to grind out $D_L(z)$ numerically



$$X(t) = \int_t^{t_0} \frac{c\,dt}{a} = \int_a^{a_0} \frac{c\,da/\dot a}{a} = \int_a^{a_0} \frac{c\,da}{a\,\dot a(a)}$$

exists only in special cases.. see p. 4047 de Sitter U. p. 4048 power law U.

How does χ depend on a_0? 4063

$$\frac{a_0}{a} = z + 1$$

Say we have a solution $a(t) = a_0\, y(t)$,

then $z = \frac{1}{y(t)} - 1$

and $z(t)$ has no a_0 dependence.

$$\chi(t) = \int_t^{t_0} \frac{c\,dt}{a} = \frac{c}{a_0}\int_y^{y_0=1} \frac{dt}{y} = \frac{c}{a_0}\int_z^{0} (z+1)\,dt$$

$$= \frac{c}{a_0}\int_z^{0} \frac{(z+1)}{\dot{z}(t)}\,dz = \frac{c}{a_0}\int_z^{0} \frac{(z+1)}{\dot{z}(z)}\,dz$$

but $z = z(t)$

implies $t = t(z)$

Analytic solutions rarely. Usually a numerical solution

and so $\chi = \frac{c}{a_0}\cancel{\chi}_R(z)$

$\therefore$ $D_L = a_0 r (1+z) = c\,\cancel{\chi}_R(z)(1+z)$

if $K = 0$ and a_0 is eliminated and of course can be set to be 1

but if $K \neq 0$, then

then $r = \begin{cases} \sin\chi & \Rightarrow K = 1 \\ \sinh\chi & \Rightarrow K = -1 \end{cases}$

and a_0 cannot be eliminated, and of course the RW metric with K scaled to 1 or -1

does have $a = a_{og} = \frac{c/H_0}{\sqrt{|\Omega_{K0}|}}$

a physical scale.

$$= \frac{4.2827\ldots\, h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_{K0}|}}$$

$$= \frac{13.968\ldots\, h_{70}^{-1}\ \text{Gly}}{\sqrt{|\Omega_{K0}|}}$$

see p. 4006

So a_{og} becomes part of the model to be fitted to data.

But what of small z limit?

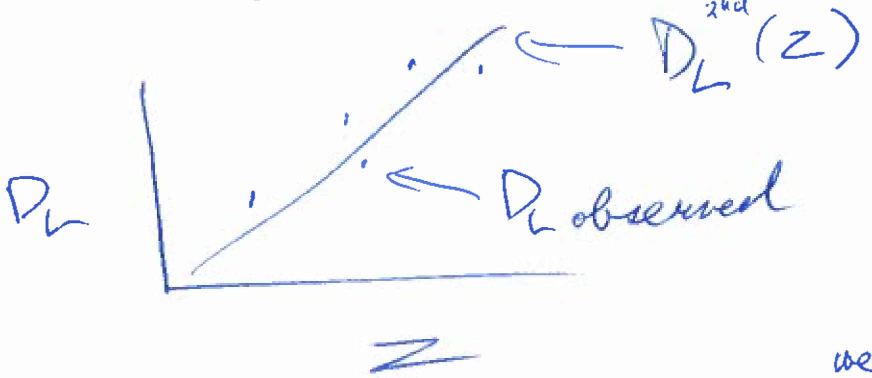
$$D_L = a_0 r(z)(1+z)$$

$$= a_0 \frac{zc}{a_0 H_0}\underbrace{\left[1 - \tfrac{1}{2}(1+q_0)z + \ldots\right]}_{\text{see p. 4057}}(1+z)$$

CL-19

$$D_L = \frac{zc}{H_0}\left[1 + \left(\tfrac{1}{2} - q_0\right)z + \ldots\right]$$

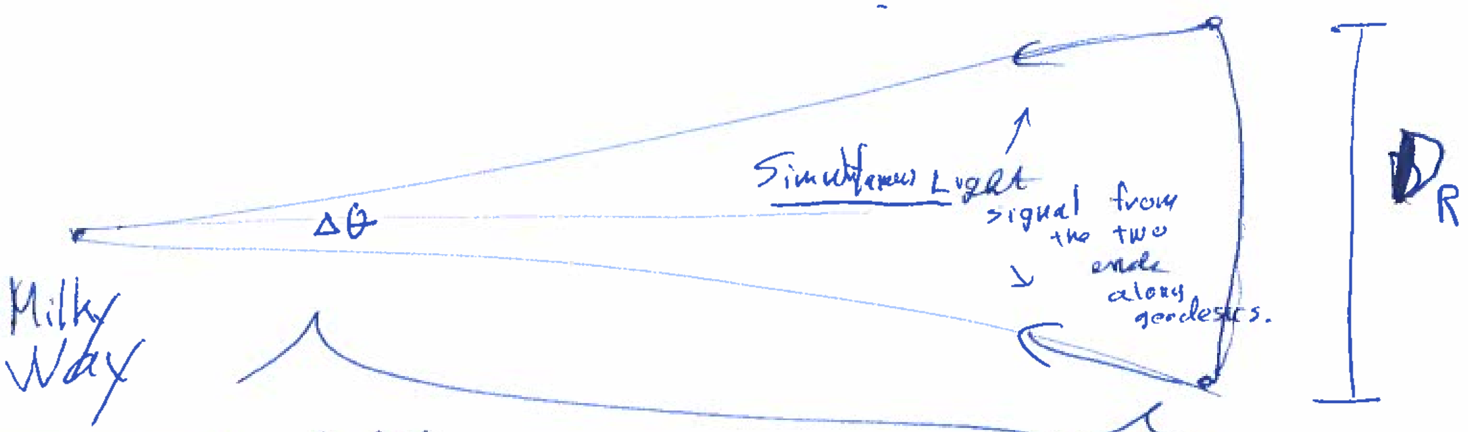
So



So finally we can solve for H_0 and q_0 for small z from a fit to observations but we have no ability to fit a_{og} and find curvature information

9) Angular Diameter Distance

- a direct observable if you have a standard ruler of proper length D_R



Curvature is general $K = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$

We assume $\Delta\theta$ is small enough (small angle approximation) that arc length and geodesic ("straight") ruler are the same to negligible error — some one has looked into this, but not Weinberg - 421-423.

Observable $D_A \equiv \dfrac{D_R}{\Delta\theta}$

define this

we know D_R somehow e.g., size scale of BAO,

we observe $\Delta\theta$

But we have to connect it to theory.

D_R is a length at one instant in time along a tangential curve relative to a ~~the~~ comoving distance $\underline{\underline{r}}$ from us.

4066

from RW metric

$$dD_R = a\,dr\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$$

set to zero by choice of our polar axis

$$D_R = a r \Delta\theta$$

$$= a_0 r \Delta\theta \left(\frac{a}{a_0}\right)$$

$$= \frac{a_0 r \Delta\theta}{1+z} \quad \text{or} \quad \Delta\theta = \frac{D_R}{a_0 r}(1+z)$$

$$\therefore D_A = \frac{D_R}{\Delta\theta} = \frac{a_0 r(z)}{1+z}$$

$r(z)$ for light signal connection

General $k = \{+1, 0, -1\}$

CL-198, Weinberg 422
$ar = \frac{a_0}{1+z} r$

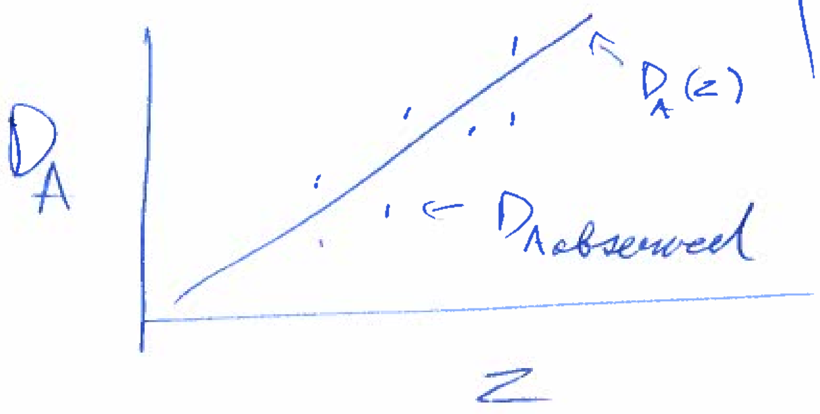
$$\frac{a_0 \chi(z)}{1+z} \quad \text{if } k = 0$$

$$= \frac{c F(z)}{1+z}$$

If $k \neq 0$, then $a = a_{0g}$ (see p. 4063 - 4064) is part of fit and a model parameter

and a_0 is not part of the fit — not a model parameter

by the same argument as on p. 4063)



Small z limit (see p.4057) (4067)

$$D_A = \frac{a_0 r(z)}{1+z} = \frac{a_0 \frac{zc}{a_0 H_0}}{1+z}\left[1 - \tfrac{1}{2}(1+q_0)z + \dots\right]$$

$$= \frac{zc}{H_0}\left[1 - (\tfrac{3}{2} + q_0)z + \dots\right]$$ to 2nd order

So

$$D^{2nd}_{proper} = \frac{zc}{H_0}\left[1 - \tfrac{1}{2}(1+q_0)z + \dots\right] \quad (p.4059)$$

$$D^{2nd}_{L} = \frac{zc}{H_0}\left[1 + (\tfrac{1}{2} - q_0)z + \dots\right] \quad (p.4064)$$

$$D^{2n}_{A} = \frac{zc}{H_0}\left[1 - (\tfrac{3}{2} + q_0)z + \dots\right]$$

$$D^{1st\,z} = \frac{zc}{H_0} \text{ in all cases.}$$

We identify $v^{1st}_{rec} = zc = v_{redshift}$

since $v_{rec} = H_0 D_{proper}$ in general

see p.4058 $v^{1st\,z}_{rec} = H_0 D^{1st\,z}$

$v_{red} = H_0 D^{1st\,z}$

The asymptotic Hubble's law proven by Lemaître 1927 and we less formally show it on p.4056

4068)

Curiosity of $D_A = \dfrac{a_0 r(z)}{1+z}$

is ~~that~~ it doesn't have to grow monotically with z!

↪ it can shrink!

and it does for ΛCDM

As an exact example consider the flat de Sitter universe

so $r = \chi$

$$\chi = \frac{zc}{H_0 a_0} = r(z)$$

$$\therefore \; D_A = \frac{zc}{H_0}\,\frac{1}{1+z}$$

Plateaus.

What does this mean?

From p. 4066

$$D_R = \frac{a_0 r(z)\Delta\theta}{1+z}$$

(held fixed)

$$\Delta\theta = \frac{(z+1)}{r(z)}\,\frac{D_R}{a_0}$$

so $\Delta\theta(z)$ can grow.

Standard ruler can start looking bigger farther away

it is ~~at cosmic present~~.

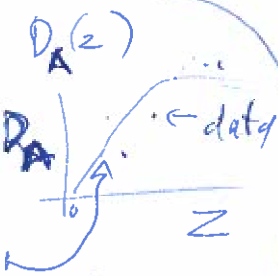
recall light started on cosmic past.

Sort of seeing it when closer.

Two points:

a) It still gets fainter with z and so fades from view

b) The small angle approximation breaks down if ~~the~~ object is too big on sky.



Model, so far ΛCDM fits data but much more data to come from Euclid (2023 – 2029 ~ wik) RST (2027 – 2032) (wik)

(Standard Ruler we have BAOs

baryonic acoustic oscillations

The dense regions emitted spherical waves. ~~The~~ acoustic waves stopped at recombination $z = 1090$ when photons flowed freely. The shells all overlap but impose a ~~scale~~ scale — small excess in galaxy pairs set

imposed ~~oscillation~~ ~~etc~~ structure on matter.

Since 2-point correlation excess at ~~150 Mpc~~ at ~~cosmic present~~

$$D_R = D_{R_0} \frac{1}{(z+1)}$$

150 Mpc to 0.15 Mpc at recombination

So standard ruler at at z but they shrink to the past

D_R fiducial value $D_{R_0} = 150$ Mpc

4070

(2023 Oct 08)

Etherington Reciprocity Theorem (Wiki)

AKA Distance Duality Relation

$$D_L = a_0 r(z)(1+z)$$ (p. 4062)

$$D_A = \frac{a_0 r(z)}{1+z}$$ (p. 4066)

Etherington 1933
but Richard C. Tolman
1881–1948
gave the correct formula
and suggested its
use in cosmology

$$\frac{D_L}{1+z} = a_0 r(z) = D_A(1+z)$$

or $$\frac{D_L}{D_A} = (1+z)^2$$

(Applies to general curvature $k = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$)

Independent of overall scale and cosmological model
has been verified within error.
Something bad wrong if it failed.
– exotic physics
e.g. photon not conserved
GR wrong etc.

~~2~~
~~50~~

40~~7~~

9 ~~10~~) Variational Calculus

In QM, and for geodesics in GR

min not just

But there is a Hull question.

Just stationary

Great circle

$$I = \int_a^b f(x_i, \dot{x}_i, \ldots, t)\, dt$$

set of coordinate and derivative

path param. Not necessarily time but could be

little blip anywhere

find f that makes I stationary

→ max, min, inflexion

~~func~~ f stationary

Won't get all global necessary

Do all possibilities get all: global & locals

→ ~~global~~ ~~just~~ ~~local~~

No variation at end points

$$x_{i\,\text{varied}}(t) = x_i(t) + \alpha\, \eta(t)$$

stationary

variation parameter

general function of the coordinate

except η_i =

Let's consider

$$f(x_i, \dot{x}_i, t)$$

coord. 1st derivative

This form is good for geodesics + Hamilton principle in classical mech

(4073) (455)

$$\frac{dI}{d\alpha} = \int_a^b \left[\frac{\partial f}{\partial x_i}\underbrace{\frac{\partial x_{iv}}{\partial \alpha}}_{\eta_i} + \frac{\partial f}{\partial \dot{x}_i}\underbrace{\frac{\partial \dot{x}_{vi}}{\partial \alpha}}_{\dot{\eta}_i}\right] dt$$

use integration by parts

$$\left.\frac{\partial f}{\partial \dot{x}_i}\eta_i\right|_a^b - \int \left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)\right]\eta_i$$

zero at endpoints.

for stationarity

$$0 = \frac{dI}{d\alpha} = \int_a^b \left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)\right]\eta_i\, dt$$

Notice total Not partial derivative

must be zero for stationary ~~path~~ since η_i is general

$\Rightarrow$ any little blip except zero at end point

Not a constrained stationary path

$$0 = \frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)$$

(a differential equation for each coordinate x_i

Called Euler's equation Art-928

40 | 201/Sep 21

2 fixed ends
2 ⇒ BCs at two time
but same as 2 BCs at start for 2nd order DE

Uses –
1) Eqn of motion
1 endpoints
2) Geodesics, GR
3) Path Integral formulation of QM
4) Fermat's Principle to Reflection/refraction

Path integrals Formulation of QM → particle can be thought of as following all paths but only along stationary ones, do waves add coherent – other paths lead to incoherence and cancellation.

(Feynman etc.)

But is this just an emergent principle? true if all paths really were followed but other paths collapse – decoherence – long arguments
which
– Any way a ~2000 atom molecule has been diffracted
Feinetal, 2019 Nature physics