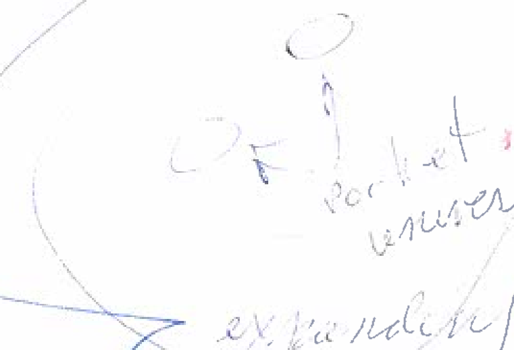
Lecture 4 2019 sep 23 2023 oct 08 400 1a

1) Geometry of universe — well observable universe or our part of Pocket universe a hypothetical in the multiverse
2 Robertson-Walker (RW) Metric
3) Geometrical Insight to RW metric
4) Hubble's law from RW metric
5) Cosmological Redshift - other derivations
6) Connection to Observables
7) Small Δt, z expansions, deceleration parameter
8) Cosmic distance measures
9) Variational calculus — without much connection to anything else in the lecture, but it does find use in cosmology sometimes

{p. 2}

1) Geometry of Universe

— at least the observable part of our pocket universe (hypothetical)



expanding (?) infinite False vacuum universe

expanding, contracting, or patchy

GR holds everywhere we hope.

eternal inflation version of multiverse

4001b

2021sep30 | 4001

1) Geometry of Universe intro via the Friedmann Eq (FE)

Recall most standard form — two versions, the kc^2 version is the useful form for geometry; $k' = kc^2$

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

Hubble parameter. H_0 is Hubble constant value at present in cosmic time $= t_0$

$\frac{\dot a}{a}$

$\rho \propto \left(\frac{a_0}{a}\right)^p$

Only this term depends on size of "a" not a ratio. This is a clue if really smart that k is related to a size scale of the universe

No a dependence

DE true for spacetime where our assumption holds. Note our sphere was small but $dr = a\,dr_0$, $r = a r_0$ so integrate to all sizes, but what do UE & FE mean then ... maybe nothing clear cut

$[k] = L^{-2}$

$[k'] = T^{-2}$

when defined this way

Recall $\frac{kc^2}{a^2} = \frac{-2E}{m a^2 r_0^2}$ Explicit

We first defined

$r = a(t)\,r_0$

proper distance as a function of cosmic time t

unitless

proper distance at our cosmic time t_0

E and m and r_0 for our test particle — but a general particle and so we argue $\frac{E}{m r_0^2}$ had to be a universal constant in spacetime — but it could differ between different universe models — but we can put

But not particular test particle.

But we have partitioned the proper distance to the test particle a different way

$r = a_g(t)\,\chi$

Chose $\chi = 1$ for present

Now $a_g(t)$ is a real distance

$a_g(t)$ is that natural unit in terms of ruler measurable distance so we could put in Gigaparsecs.

comoving frame distance in natural units a_g or dimensionless content in time

χ can still be as small as you like

Peculiarities of test particle cancel

Now $k = \frac{-2E_0}{mc^2\chi^2}$ must be a universal constant since our particle is general.

Still a universal k but with a different value

$$\frac{kc^2}{a_g^2} = \left(\frac{-2E_0}{m\chi^2}\right)\frac{1}{a_g^2}$$

I use a_g to distinguish from a — but a is used for both usually and context decides.

9002 (2021 Sep 30)

k we already called the curvature but that's anticipating our discussion in this chapter.

The other standard form of FE is derived from

$$\left(\frac{\dot a_g}{a_g}\right) = H_0^2 \left[\frac{\rho}{\left(\frac{3H_0^2}{8\pi G}\right)} - \underbrace{\frac{kc^2}{H_0^2 a_g^2}}_{\Omega_k = \Omega_{k0}\left(\frac{a_{g0}}{a_g}\right)^2} + \frac{\Lambda}{3H_0^2} \right]$$

Just pull out H_0^2 not critical density

Endless confusion $k>0$ has $\Omega_k<0$, $k<0$ has $\Omega_k>0$

$$\Omega_k = -\frac{kc^2}{H_0^2 a_g^2}$$

$$\Omega_{k_0} = -\frac{kc^2}{H_0^2 a_{g0}^2}$$

present time value.

where $\rho_{critical} = \frac{3H_0^2}{8\pi G}$ (Hubble density)

Li-56

$$\frac{\Lambda}{\left(\frac{3H_0^2}{8\pi G}\right)} \frac{(8\pi G)}{} = \frac{\rho_\Lambda}{\rho_{crit}}$$

ρ_Λ is either Dark energy, or a cosmological constant parameterized as a fictitious density

put these together and

$$\frac{\rho_{total}}{\rho_{crit}} = \sum_p \Omega_{p_0} \left(\frac{a_0}{a}\right)^p$$

a or a_g same here since a ratio

$p = 0, 1, 2, 3, 4$ { Other possibilities are allowed

Λ-stuff

Radiation "ER stuff"

Matter "NR stuff" (Steiner-6, -7)

cosmic strings (also curvature parameterized as a density)

Some version of quintessence

At $t = t_0$ the FE becomes

$$H_0^2 = H_0^2 \left[\underbrace{\sum_p \Omega_{p_0}}_{=\Omega_0} + \Omega_{k_0} \right]$$

$= \Omega_0$ but excluding the curvature omega

∴ $1 - \Omega_{k_0} = \Omega_0$

So if $\Omega_0 = 1$

$\Omega_{k0} = 0$ and $k = 0$

and there is no way to define a special length a_g ~~weather~~ Anticipating this is the flat geometry case

2021sep30

4003

But say $k \neq 0$, then

$$-\Omega_{k_0} = \Omega_0 - 1$$

$$+\frac{kc^2}{H_0^2 a_{g0}^2} = \Omega_0 - 1$$

a_g at time t_0

$\Rightarrow \rho_0/\rho_{crit}$

Ω_0 all density except curvature, Matter, Radiation, dark matter (cosmological constant)

H_0 is fixed, c is fixed as fundamental constant.

Observed value

But how do we ~~partition~~ factorize $\frac{k}{a_{g0}^2}$ into two values?

Fixes a_{g0} to relevant physical size in RW metric

The natural choice — plus clairvoyance — is $k = 1$ if $\Omega_0 > 1$

hyperspherical space

$k = -1$ if $\Omega_0 < 1$

hyperbolical space

$k = 0$ if $\Omega_0 = 1$

flat space

$K = 0$ for $k = 0$ $\Omega_0 = 1$

$$\cancel{\frac{kc^2}{H_0^2 a_{g0}^2} = \Omega_0 - 1}$$

$$\frac{c^2}{H_0^2 a_{g0}^2} = \frac{\Omega_0 - 1}{k} = |\Omega_0 - 1|$$

$$= |\Omega_{k_0}|$$

and $a_{g0} = \frac{c/H_0}{\sqrt{|\Omega_0 - 1|}} = \frac{c/H_0}{\sqrt{|\Omega_{k_0}|}}$

either way

Better

$= \cancel{\ldots}$

4004

2021 sep 30

Not that we will make a ~~lot~~ much use of the definition but

$K \equiv$ curvature

Gaussian curvature radius of universe $\equiv R_G \equiv \frac{a_{g0}}{\sqrt{k}}$ (CL-12)

→ real for $k > 0$

→ imaginary for $k < 0$

→ $k = 0$ undefined as is a_{g0}

We show this latter

Now if we take the Robertson-Walker metric for an isotropic homogeneous relativistic universe and substitute it into the Einstein field equations (+ geodesic equation for the like) you derive the Friedmann Eq (as the very Friedmann & independently Lemaitre did in the 1920s)

Got dynamics of all (eg $a_g(t)$)

in standard form with $k = 1$ +ve curvature

$k = -1$ -ve curvature

$k = 0$ flat Euclidean

and you get also

$a_g(t)$ is cosmic time

$$a_{g0} \equiv \frac{c/H_0}{\sqrt{|\Omega_0 - 1|}} = \frac{c/H_0}{\sqrt{|\Omega_k|}}$$

for $k \neq 0$. Same for $k = 1$ and -1.

$a_{g0} =$ undefined for $\Omega_k = 0$ or $k = 0$

No physical scale for universe

The same formula we arrived at for natural length scale in e.g., Gigaparsecs and X is the length in natural units

So ℓ (proper length to particle at X from origin at ...) $= a_g(t) X$

So our handwavy procedure gives ~~$a(t)$~~ a_{g0} and the Friedmann Eq. itself gives the dynamics $a_g(t)$
(But rely on FRW ~~metric for the~~ geometric meaning which we don't prove)

But, of course, in the standard form $a_g(t)$ is just written $a(t)$

For description of our universe

$r = a(t) r_0$

is fine for most purposes when $a(t=t_0) = a_0 = 1$ and has no units

but for studying the geometry we need

$r = a_g(t) \chi$

of course $a(t) r_0 = a_g(t) \chi$

$a(t) \propto a_g$

since r_0 and χ are both comoving coordinates

r_0 has units e.g. gigaparsecs and χ is in natural units: i.e. units of a_{g0}

$a_g(t)$ is the value of that unit in gigaparsecs

400 6)

What is Ω_{K0} and a_{g0}?

Well Planck-2018 p.68

The last Planck collaboration data and analysis release — others can re-analyze their data but they themselves have said their last word.

Handley (2019) got $\Omega_{K0} = -.045(15)$ or $-.012(6)$ by his Bayesian analysis

But Planck 2018 and other data a poll among Bayesian evidence disfavors his results

Now they primarily analyzed in terms of ΛCDM model which is flat exactly $\Omega_K = 0$ but they considered curved extension

$$\Omega_{K0} = \begin{cases} -.058(51) \\ -.037(34) \\ -.011(13) \\ +.0005(40) \\ -.012(10) \text{ Tristram et al (2023)} \end{cases}$$

including more data sets but within 1 sigma or 2 all consistent with zero

Recall $\Omega_{K0} < 0$ gives $k = 1$ or +ve curvature.

Let's consider what a_{g0} values you get

$$a_{g0} = \frac{c/H_0}{\sqrt{|\Omega_K|}} = \frac{4.2827... \, h_{70}^{-1} \text{ Gpc}}{\sqrt{|\Omega_K|}} = \frac{13.968... \, h_{70}^{-1} \text{ Gly}}{\sqrt{|\Omega_K|}}$$

where c/H_0 is the Hubble length and $h_{70} = H_0/(70 \text{ km/s})$

4007

So $|\Omega_k| << 1$ implies

$$a_{g0} >> l_{Hubble} = c/H_0$$

Just as an example let

$$|\Omega_k| = \frac{1}{100} = \frac{1}{10^2} \approx .1$$

close to Tristram 2023 $\Omega_k = -.012(10)$

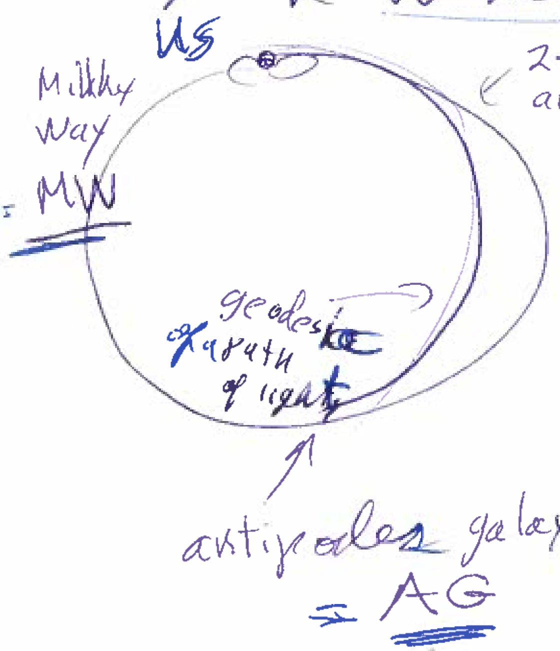
$\frac{1}{8} = .125$ ✓

So $k > 0$, $\Omega_k < 0$ and +ve curvature

Then $a_{g0} = \frac{4.3 \text{ Gpc}}{\frac{1}{10}} \approx 43 \text{ Gpc}$

Now what does this mean for a curved 3-d space (a curved spacetime too of course)

Anticipating our analysis of RW metric of hypersphere



D Proper distance to antipodes (MW to AG) $= \pi a_{g0} \approx 130 \text{ Gpc}$ [3.14, 43]

proper distance at our instant in cosmic time.

$>> 14.3$ Gpc Radius of observable universe assuming flat ΛCDM

So any direction we look out we see the AG but as it was in past due to finite travel speed of light.

4008

Actually observable universe

define as the greatest distance to the ~~point~~ comoving frame (The comoving frame ~~point~~ was much closer when signal started out) that a signal at light speed could have reached us since the Big Bang

observable universe — an ordinary sphere centered us

MW

R = Particle Horizon = 14.3 Gpc proper distance at an instant in cosmic time in ΛCDM model. (which is flat by assumpt but about the same for low curv i.e. big a_{g0})

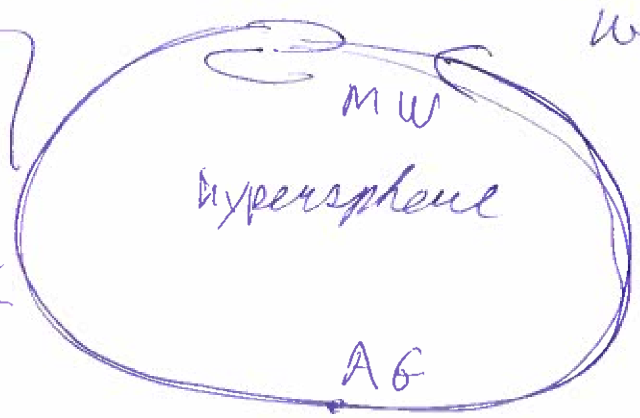
We can't see ~~that far~~ in light because Universe is opaque before recombination (decoupling)

$t \sim 377{,}700$ years in ΛCDM model, — which is a flat model — but using it anyway.

So we can't see AG (antipodal Galaxy) if $a_{g0} \gtrsim 14.3$ Gpc

If we could see ~~past the~~ AG we'd see the opposite side ~~of~~ of MW in any direction

e.g. $a_{g0} \simeq 100$ Gpc



← at t_0 all the way around the proper distance is $2\pi a_{g0}$ just as for a circle

2021 Sep 20

4009

Why is $D_{circ\ now} = 2\pi a_{g0}$ / e.g. $D_{circ} = 100$ Gp

~~I think~~ Robertson & Walker had a choice. D_{circ} is ~~the~~ physical distance around, but they could have factorized it differently between k and a_{g0}

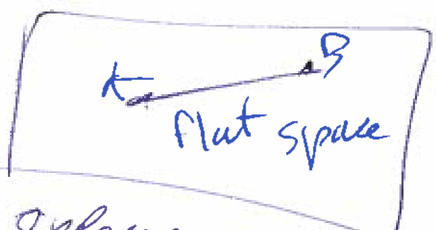
But the natural factorization is one consistent with an ordinary sphere.

choosing $k = \pm 1$ and $D_{circ} = 2\pi a_g$

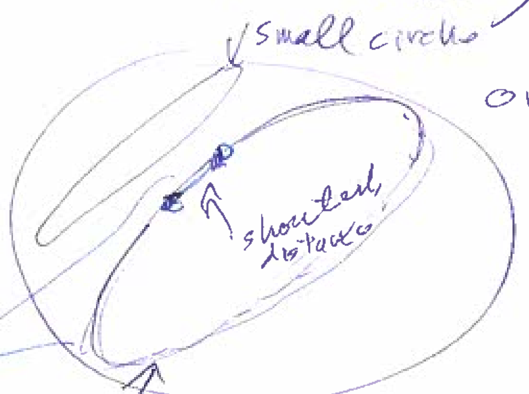
Comment — Other ~~Points to~~ Remarks

a) Geodesic → a stationary path in ~~some~~ a geometry

in flat space a ordinary line



- only one geodesic path in flat space between A + B (that does go to infinity,



ordinary sphere (2-sphere) — a hypersphere is a 3-sphere (Wiki: n-sphere)

- a great circle divides a sphere in half and is the geodesic

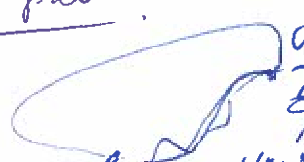
both stationary to blip perturbations

Maximum geodesic path

global maximum is infinity

In general in variational calculus

But necessarily global



variation

$\frac{dA_{blip}}{ds} = \int_1^2 d\,[\text{integral}]$ dl path

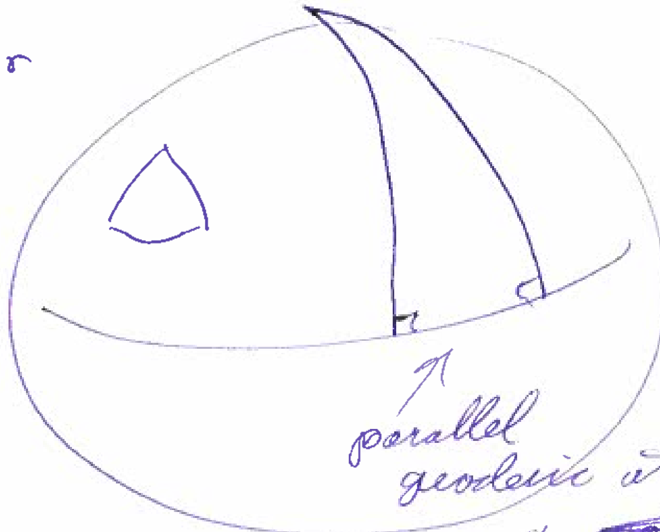
→ DB = 0 to be stationary with respect to all blip perturbations

2016

[2021sep30]

4) 2-d analogue for k = 1 space

$\sum \theta_i > 180°$ triangle



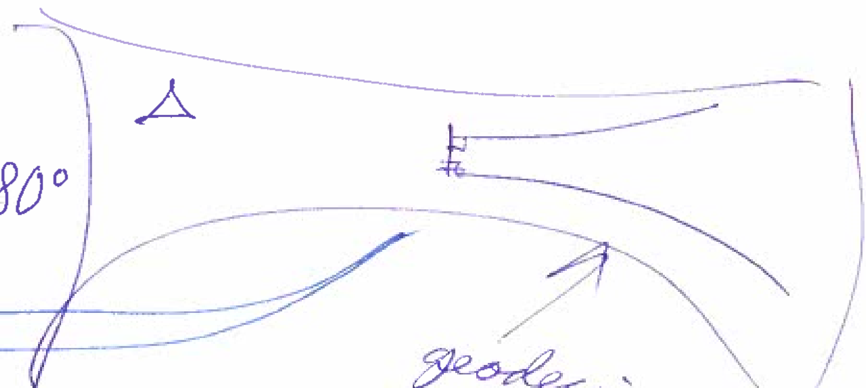
parallel geodesic at a ~~point~~ location meet at a pole

Actually the video on hyperbolic space thinks this is a very poor way to represent hyperbolic space — still OK for just following geodesics from start

omit

k = -1 hyperbolic space

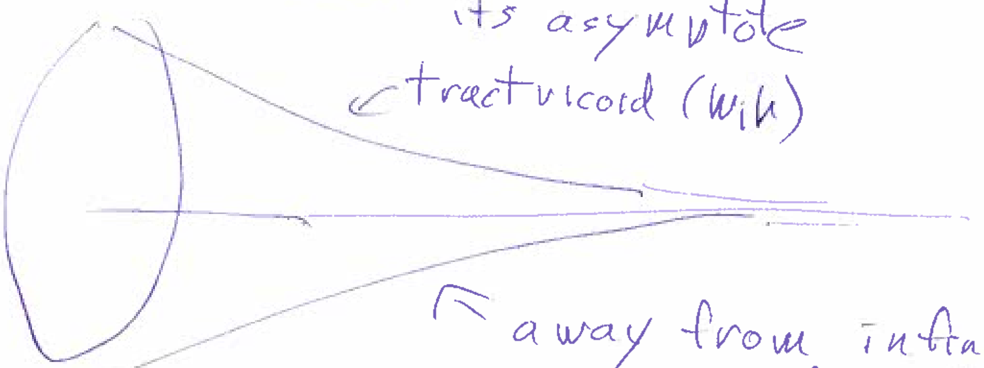
$\sum \theta_i < 180°$ triangle



geodesic parallel a location ~~meet at the pol~~ diverge

Another analog is tractrix rotated around its asymptote

← tractricoid (wiki)

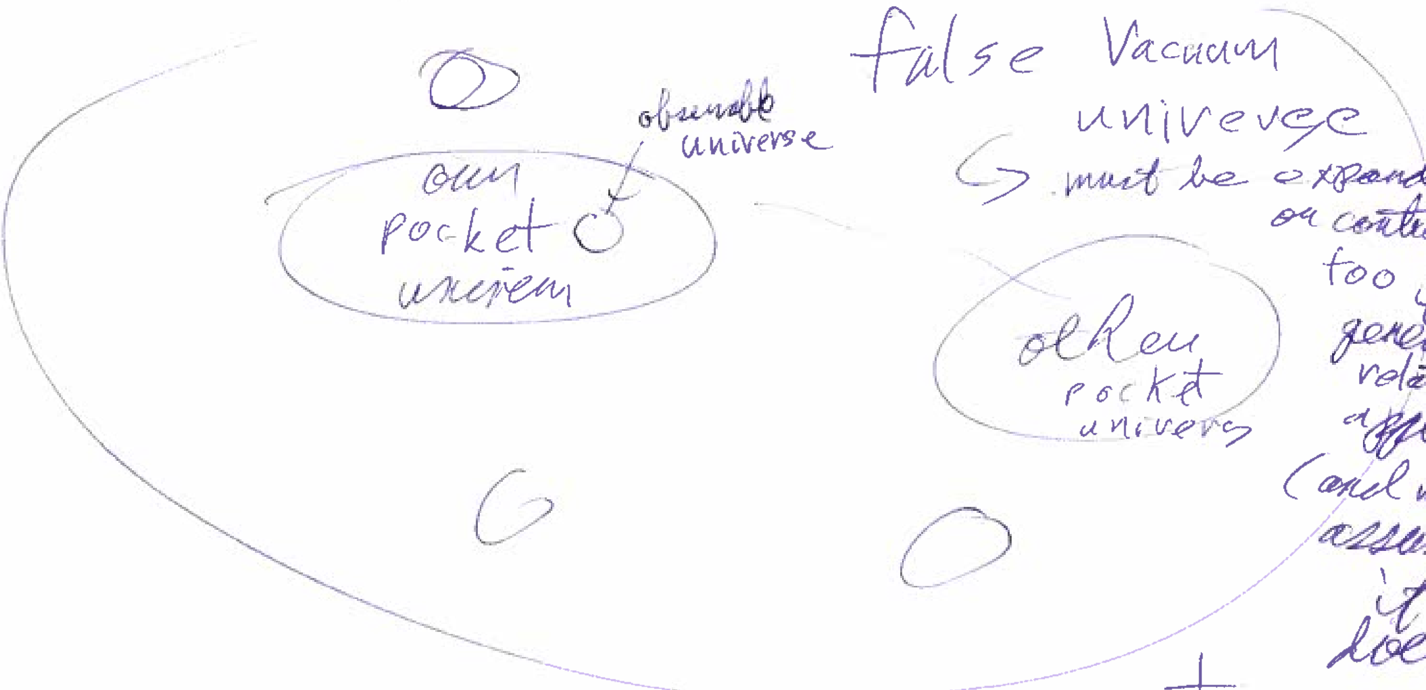


meet asymptote at infinity

← away from infinity and equator has constant -ve Gaussian curvature as defined by geometers

c) We have discussed our / if they extended FE universes as everywhere

But they may not — in inflation theory they don't



↳ must be expanding or contracting too if general relativity applies (and most assume it does)

Actually basic inflation just applies ~~to~~ our pocket universe
- It does not require that others exist.
- But a fairly natural extension is ~~that they do~~

⟹ the Multiverse picture. (~~But some~~ believe this is wild speculation and not worth elaborating on)

There are many arguments about the multiverse pro & con.

4012)

Pro The laws of physics may be general:

- 2nd law of thermodynamics seems a logical necessity
- Quantum mechanics seems so fundamental
- So does classical limit and General relativity

But the parameters; c, G, ħ, e of physics and the observable universe and particle masses all seem to have no relation other than being restricted by being sufficiently biophilic for life as we know it (carbon based, liquid water based)

Decided probabilistically but how

Con → the multiverse is so unconstrained.

- one can imagine anything.
- and there is no obvious natural way to determine the probability distributions of parameters.
- untestable — except does pass the test of parameters not being fined tuned to special values.

"Let's not bother with it"

But as Mario Livio says what if a TOE explains all we see and implies it.

2) Metric & the Robertson–Walker Metric

In GR all physical law should be formulatable as tensor equations to be explicitly coordinate system invariant

A space with a metric of the Pythagorean sort anyway (there are more general spaces) has the metric ($\binom{0}{2}$ — two form tensor / vectors)

$$ds^2 = g_{\kappa\nu}\, dx^\kappa dx^\nu$$

Einstein summation implied

|| elements change value

Lorentzian metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

or $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

ds is the differential interval

Unlike metric which has both geometry and coordinate dependence.

Invariant under all coordinate changes — so it represents geometry or physics ~~physics~~ not coordinate system

In relativistic (SR or GR) use

$ds^2 >$ time-like "can" (opposite from Carroll-9) $d\tau^2$ proper time

$ds^2 <$ space-like "cannot" $d\tau^2$

→ so proper time in Carroll-9

events in spacetime ~~can~~ be causally connected by a light signal

$ds^2 = 0$ is also ~~time-like~~ light-like or null (CL-10 (Ideally ~~causally~~ can causally connected))

The Convention is to use ~~refer to~~ a general component as the symbol for the whole. We should teach this in intro physics but $x = x_i$ is still nonsense $d\tau^2$

$g_{\kappa\nu}$ is metric tensor or often just metric (g is ~~a metric~~ ambiguous — $g = 9.8$ m/s² ?)

⟹ in contains both physics and coordinate choice

In GR choosing the good coordinate system for the physical system ~~of~~ interest seems essential.

but of all honest

4084 (2021 sep 26)

A {ironic / incongruous} episode from history of GR including Robertson

Wik — in 1936 Einstein & Rosen submitted a paper to Phys Rev concluding gravitational waves could not exist by GR, but the referee Howard Robertson pointed out that the conclusion depended on coordinate system being wrongly interpreted.

Einstein was peeved, but his assistant L. Infield convinced him the criticism was correct and the paper was published elsewhere with opposite conclusion.

Requirement that good is physics invariant under transformation ~~I~~ right order

New The tensor formulation of the metric and GR (and SR) makes the physical formulae invariant for different coordinate choices, but the price you pay is the components are coord. system dependent.

Analogous to vectors (~~tensors~~ tensors of rank 1)

vector pictured

→ a quantity with a direction and magnitude — sort of visual

but x_i is a component

But higher rank tensors etc.

But we map them into vectors and scalars & analyze their b...

Anyway we can't visual tensors of rank > 1

— So we might as well use components

2015

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

What are the differentials with respect to,

Oh ~~some~~ a general path in ~~$l(x_i)$~~ $x_i(l)$

in spacetime (or in jargon a world line)

Could be an actual motion

$$\left(\frac{ds}{dl}\right)^2 = g_{\mu\nu} \frac{dx^\mu}{dl} \frac{dx^\nu}{dl}$$

but the convention ~~that in~~ ~~the~~ is not to complicate the formulae by making ~~that~~ explicit → like in thermo where $dE = TdS - PdV + \mu dN$

→ with respect to time or anything

e.g., geodesic path — stationary with respect to small local variations — but they don't have to be.

Using path is useful in other areas of math e.g. moving Lagrange multipliers

Actual Metrics

Minkowski metric for flat spacetime of SR

Wik of Wolfram Carroll-8 →

$$g = \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Wik Minkowski space)

$g = \eta \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ colors-blurry

but this convention occurs too Carroll-8: or $g' = \eta' = -\eta$

or if one uses the complex number form

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

but this formulation of SR is disfavored nowadays since it doesn't generalize to GR

Another one of those 2-convention class things.

~~Wik has it both ways~~

4016 (2023 oct 08)

27) Robertson Walker Metric

In 1935 RW determined their eponymous metric (Wiki) which was known earlier to Friedmann + Lemaitre,

but RW proved it is the most general metric that is a manifold that at any point is asymptotically Lorentzian $\rightarrow g \rightarrow \eta = \pm \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ {3 space coord. 1 time coord}

(local inertial frame
local Lorentzian frame = LLF

different conventions

and homogeneous & isotropic in unbounded space (can be finite for hyperspherical),

It is not specific to general relativity (Wiki: RW history)

But GR provides the dynamics for $a(t)$ the scale factor i.e., the Friedmann Eq. (FE) (which we derived from Newtonian physics fairly with extra hypotheses, but the Newtonian derivation tells us nothing about geometry of course)

Note c is the highest velocity relative to a ~~local~~ LLF.

But there is no limit on recession velocities between LLF and, in fact cosmologically remote space has $v_{rec} > c$.

Since in the classical limit Newtonian physics, it would be strange if there was no Newtonian approach to some GR results.

RW metric (or more properly interval since the coefficients are the metric tensor)

in most standard form (but not most general)

$$ds^2 = c^2 dt^2 - a^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$= d\tau^2$ (but Carroll 329-333 uses $ds^2_{\text{Carroll}} = -ds^2$)

But $ds^2 > 0$ is timelike and this seems more memorable to me

time part

space part

in spherical coordinates where the origin can be anywhere since space is homogeneous and isotropic (homist?) — but usually we take the Milky Way or Local Group center of mass as origin)

Dimensionless a bit of a misnomer. Dimensional quantities have a physical nature, they are just in natural units

(r is a dimensionless comoving coordinate — time and units are in $a(t)$.)

$$k = \begin{cases} +1 & \text{hyperspherical} \\ 0 & \text{Euclidean or flat space} \\ -1 & \text{hyperbolically} \end{cases}$$

The scaling of k to these values means

~~$a_0 = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}}$~~

curvature radius

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{4.2827\ldots\ h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_k|}}$$

$$= \frac{13.968\ldots\ h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_k|}}$$

See p. 4003, 4006

4018

The Fluid equation

$$\dot{\rho} = 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \quad \text{(Li-26)}$$

in GR follows from the energy-momentum conservation equation $\nabla_\mu T^{\mu\nu} = 0$ (Carroll-120)

(see Carroll-117-119)

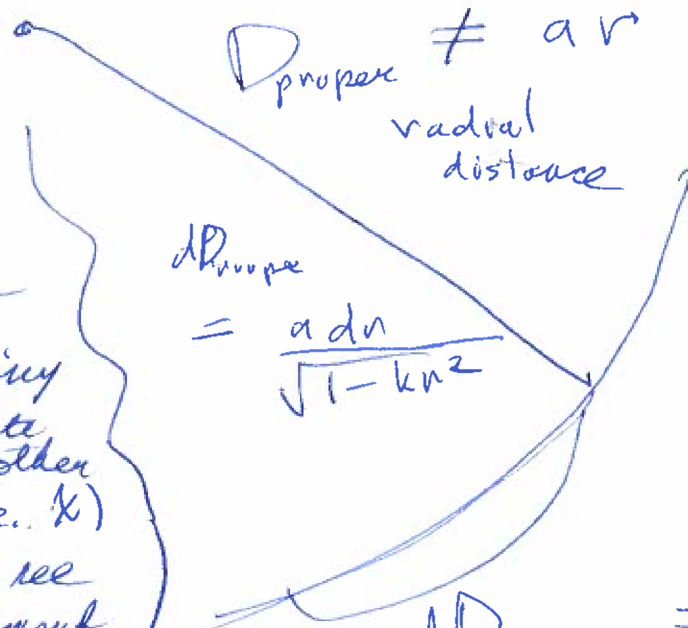
energy momentum tensor
RHS of Einstein field equations

Do you get an explosion/indefinite when for $\frac{dr^2}{1-kr^2}$ when $kr^2 \to 1$?

No, It turns out that $\frac{dr}{d\lambda} = 0$ there
→ a stationary point.

Note Recall proper distance can be measured at one instant in time with a ruler (meaning of distance)

Origin



$D_{proper} \neq ar$ except if $k=0$ and asymptotically as $kr^2 \to 0$.

radial distance

$$dD_{proper} = \frac{a\,dr}{\sqrt{1-kr^2}}$$

This is a choice. One can choose the comoving coordinate in another way (i.e. χ) as we'll see in a moment

Recall $kr^2 \to 1$ doesn't give an infinity and only happen for hyperspherical space,

but $dD_{proper} = ar\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$

tangential distance.

as note $a_0 = a(t_0) = 1$
is not available with
the physical $a(t)$
but one just flips back and forth between
the two $a(t)$'s as needed.

What if $a(t) \to 0$
a point origin, but really only for hyperspherical universe

but $\rho \to \infty$.

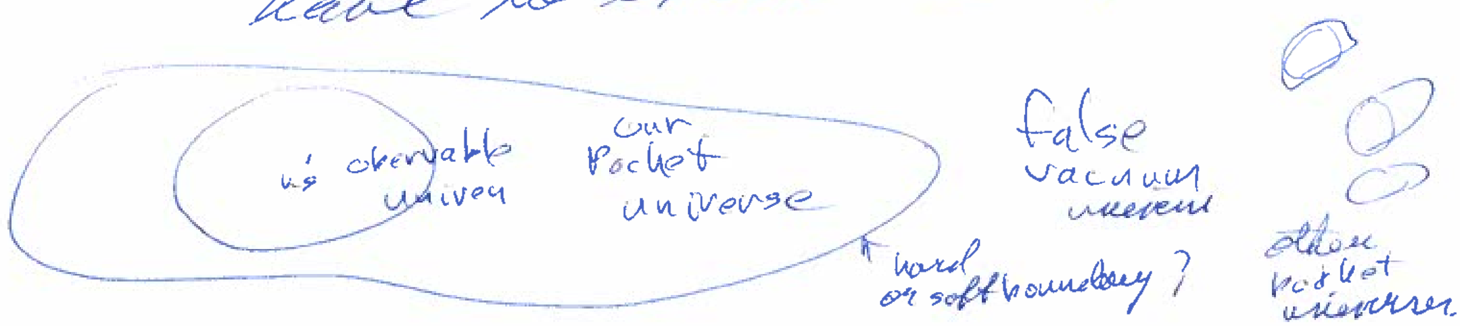
But we don't think that can happen.
At some point GR must go
to some quantum gravity.
Maybe at the Planck density
or thereabouts

$$\rho_{\text{planck}} = \frac{c^5}{\hbar G^2} = 5.1550.. \times 10^{96} \frac{\text{kg}}{\text{m}^3}$$

(Wik: Planck units)

as we run the clock back everywhere
in the observable universe goes
thru reionization, recombination,
big bang nucleosynthesis, the quark era
.... ?

But the FB models don't
have to extend to infinity



4020) Omit

Multiverse

Pro - 2nd law of thermo
- GR
- quantum mechanics

logical necessity even

seem all very elegant as if universal

But G, ħ, c, i.e. and other constants seems to have whimsical values except they allow us - chosen from a distribution randomly except anthropically.

Con - no guidance
- very unconstrained
- what are those distribution

never provable or falsifiable

some argue not even a scientific theory

My view is that it is and a ~~useful~~ fruitfull idea but only to a limited degree.

Maybe a great fundamental theory of everything will just imply multiverse and then we accept it as the best we can do,

— or ruled it out by showing all physics must be as it is indifferent to us.

3) Geometrical Insight

a) $k=1$ spherical geometry on a 2-sphere (a sphere in 3-d Euclidean space; Wik: n-sphere)

Metric is

$$dD^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

4021

(CL-10, last equation)

a is radius of sphere and r isn't confusingly but notation consistent with everything is tricky

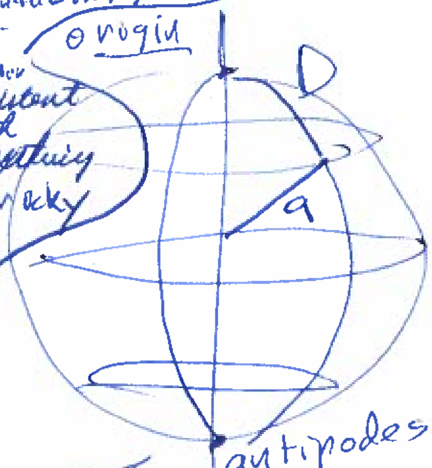
Let $r = \sin\theta$ (yes it looks weird but r is a dimensionless comoving coordinate on the sphere)

$\theta \in [0, \pi]$

$r \approx \theta$ only for $r \ll 1$

$dr = \cos\theta \, d\theta$

$d\theta = \pm \frac{dr}{\sqrt{1-r^2}}$ {which implies the limits $\theta \in [0,\pi]$ actually

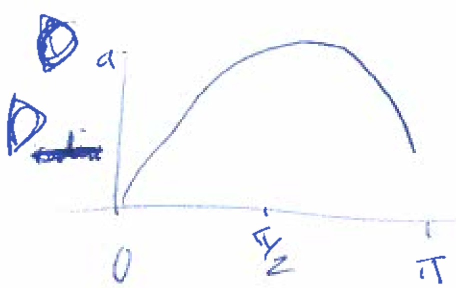


So $dD^2 = a^2\left(\frac{dr^2}{1-r^2} + r^2 \, d\phi^2\right)$

All geodesics lead to the antipodes

So the sphere surface looks flat only for $r \ll 1$

and $D_{radial} = \begin{cases} a\theta \\ ar & \text{for } r \ll 1 \\ a\frac{\pi}{2} & \text{for } \theta = \frac{\pi}{2} \\ a\pi & \text{for } \theta = \pi \end{cases}$



$D_{circum} = a r 2\pi = 2\pi a \sin\theta$

$\begin{cases} 2\pi a r & \text{for } r \ll 1 \\ 2\pi a & \text{for } \theta = \frac{\pi}{2} \\ 0 & \text{for } \theta = \pi \end{cases}$

2-sphere

b) $K = 0$, Just flat space

Metric $dD^2 = a^2(dr^2 + \cancel{\sin^2\theta} \, d\phi^2)$ { Just polar coordinate space

$r \in [0, \infty]$

4 022

c) $k=+1$ for 3-sphere which is the RW metric space part.

Metric $dD_{proper}^2 = a^2\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$

where χ is one version of comoving coordinate.

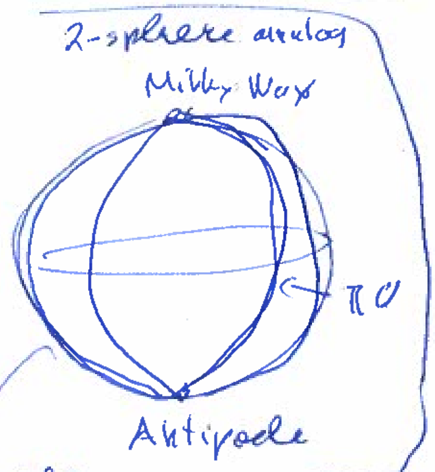
Here we define $r = \sin\chi$

so that $dD_{proper\ tangential} = a\, r\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$

which means

$dr = \cos\chi\, d\chi$ &

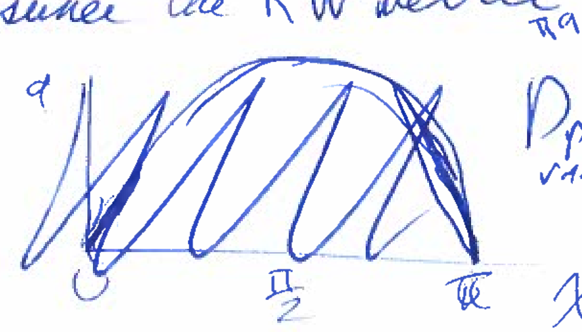
and $d\chi = \pm \dfrac{dr}{\sqrt{1-r^2}}$ just as for the 2-sphere case

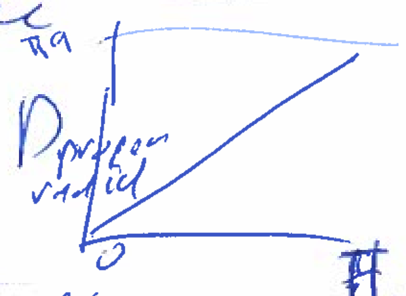


which implies the $\chi \in [0, \pi]$ limits since we assume the RW metric

all geod

So $D_{proper\ radial}$

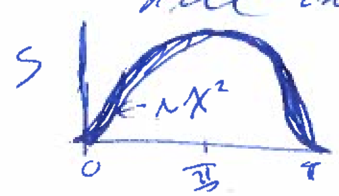




D_{radial} radium $\leq 2\pi a$

What is the area of a 2-sphere-like of comoving frame coordinate r in our RW space.

Full integral over all θ, ϕ gives 4π just like in flat space.



$\therefore\ S(r) = 4\pi a^2 r^2 = \begin{cases} 4\pi a^2 \sin^2\chi \\ 4\pi a^2 \chi^2 \text{ for } r \ll 1 \\ 4\pi a^2 \text{ for } \chi = \pi/2 \end{cases}$

surface area goes to small!!

4023

What is the volume of a 2-sphere-like ?

$$V = \int_0^{\chi} S(\chi')\, a\, d\chi'$$

$$= 4\pi a^3 \int_0^{\chi} \underbrace{\sin^2 \chi}_{\frac{1}{2}(1-\cos 2\chi)}\, d\chi \rightarrow 4\pi a^3 \frac{\chi^3}{3} \text{ for } \chi << 1$$

from standard trig identity (Wiki: trig Double angle formulae)

$$= 4\pi a^3 \left[\frac{1}{2}\right]\left(\chi - \frac{\sin 2\chi}{2}\right)$$

$$= \frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)\left(\chi - \frac{\sin 2\chi}{2}\right) \text{ in general}$$

$\chi << \frac{\pi}{2}$:

$$\chi - \left(\frac{2\chi}{2} - \frac{1}{3!}\frac{(2\chi)^3}{2}\right)$$

$$\frac{2}{3}\chi^3$$

$$\frac{4\pi}{3} a^3 \chi^3 = \frac{4\pi}{3} a^3 r^3$$ as it should asymptotically flat space.

$\chi = \frac{\pi}{2}$:

$$\frac{4\pi a^3}{3}\left(\frac{3}{2}\right)\left(\frac{\pi}{2} - 0\right)$$

$$\pi^2 a^3$$

V_{max}

$$\frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)\pi$$

$$2\pi^2 a^3$$

$$\frac{dV}{d\chi} = \frac{4\pi}{3} a^3 \left(\frac{3}{2}\right)\left[1 - \cos 2\chi\right] = 0$$

for $\chi = 0$ which is a min.

for $\chi = \pi$ which is a max

So a finite space but unbounded.

4024)

d) $k=0$ RM metric flat space

$$dD^2_{\text{proper}} = a^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$\chi = r$ in this case

Just ordinary spherical coordinates

e) $k=-1$, RW metric hyperbolic space

$$dD^2 = a^2\left[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

might have guessed $\chi \in [0, \infty]$

N.B.: hyperbolic functions

$\sinh\chi = \frac{e^{\chi}-e^{-\chi}}{2}$

$\cosh\chi = \frac{e^{\chi}+e^{-\chi}}{2}$

$\cosh^2 x - \sinh^2 x = 1$

$\therefore$ let

$$r = \sinh\chi$$
$$dr = \cosh\chi\, d\chi$$
$$d\chi = \frac{dr}{\cosh\chi} = \frac{dr}{\sqrt{1+r^2}}$$

$$dD^2 = a^2\left[\frac{dr^2}{1+r^2} + \cancel{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

again tangential space is "normal"

$$S(r,\chi) = \begin{cases} 4\pi a^2 r^2 \\ 4\pi a^2 \sinh^2\chi \end{cases}$$

$$V = \int_0^{\chi} S\, a\, d\chi' = 4\pi a^3 \int_0^{\chi} \sinh^2\chi\, d\chi$$

can be integrated

$4\pi a^3 \frac{\chi^3}{3} = \frac{4\pi a^3}{3}\chi^3$ for $\chi \ll 1$ asymptotic flat space as it should

(2021oct03) 40215

Hubble's Law

Proof from RW Metric (it's easy)

First recall the FE game or

Hubble parameter $\rightarrow$ $H = \frac{\dot{a}}{a}$

$\dot{a} = Ha$

Two cases:

$r = a r_0$

$a(t = t_0) = 1$

and a is dimensionless scale factor

r_0 are present proper distance of observable universe

"r" is Not RW metric r here

$r = a_g \chi$ for $k \neq 0$

$\frac{a}{\sqrt{k}}$ is ~~curvature~~ Gaussian curvature radius in units of length (e.g. Gpc or Gly)

and $\chi = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}}$ is comoving distance in natural units (in units of a)

either interpretation of a leads to

$\dot{D} = HD$

from any cosmic time t

or

$v = HD$

– is proper length – measurable by a ruler at the instant in cosmic time

recession velocity not velocity relative to an inertial frame – can be any size and can be greater than c

Not "r" of RW just proper distance

Proof from RW metric is just (CL-13)
(pretty simple)

$$D = \int_0^{\chi} a\, d\chi = \int_0^{r} a \frac{dr'}{\sqrt{1-kr'^2}} = a(t) f(r)$$

$a(t)$ has no comoving frame dependence, $f(r)$ no time: comoving ... dependence

4028

$$\overbrace{\text{Origin} \longrightarrow \cdot}^{D \,=\, a f(r)}$$

at cosmic time t.

$$\therefore \quad \dot{D} = \dot{a} f(r) = \dot{a}\left(\frac{D}{a}\right)$$

$$\text{or} \quad \dot{D} = H D \qquad \text{QED}$$

This is an exact result, but except asymptotically as $D \to 0$, $\dot{D}$ and D are not direct observables. We will show they are in this limit soon

Remarks

a) What's the difference between a direct and indirect observable?
Probably a matter of taste often.

Maybe! A direct observable is one where you trust all the theoretical steps from observation to desired value (e.g., Volume $\Rightarrow$ Temperature on an old fashioned thermometer) or where the steps seem ~~short~~ few to you.

Bertrand Russell $\rightarrow$ the only truly direct observation is ~~there~~ is thinking — all other observations are theory laden, to one degree or another.

b) Maybe with FRBs
and GW and quasar lensing
~~cosmologically~~ remote D ~~and~~
and $\dot{D}$ are becoming possible
But ~~those methods~~ still leave a
lot of uncertainty and
are not yet competitive though
Omit Done before in lect. 3 promising

and asymptotic form $cz = H r_{\text{lumin dist}}$

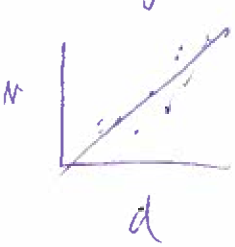
Recapitulate history

1927 Lemaître derived Hubble law explicitly from GR (but Friedmann probably knew it earlier ~~+~~ vaguely) and using literature ~~vals~~ values suggested 575 $\frac{km/s}{Mpc}$ or 670 $\frac{km/s}{Mpc}$ for H_0

but I ~~think~~ one can say he only found values assuming the theoretical result was true (yes)

1929 Hubble gave an empirical discovery of Hubble's law $v = Hr$ which can be done from ~~$r \to 0$~~ $r \to 0$ asymptotically as we'll prove

He presented a Hubble diagram



His $H_0 \cong 500 \frac{km/s}{Mpc}$ due to poor Cepheid calibration

4028)

In the 1950s, H_0 was ~~brought~~ down a long way.
Which fixed the age problem of that era

$$t_{age} \triangleq \frac{1}{H_0} = 13.968\, h_{70}^{-1}$$

of Universe assuming big bang (i.e., point origin

of order ~~- exact~~

Arthur Holmes (Wiki: Age of Earth) from radioactive dating in 1921 said age of Earth ~ few Gigayears
in 1927 1.6 - 3.0 Gyr.

In 1960's — 1990s values between ~50 and 100 (Sandage + Tamman mainly) (de VaucouIeurs mainly) (& I exchanged emails with him once)

Since 2010

~68 for Planck (Indirect by fitting ~~to~~ a model but ΛCDM model very robust

~73 from direct local measurements (Riess et al) (but may Cepheids are again giving wrong answers)

1902a

~~but that we cannot do~~ (except asymptotically → but no one does that)

When we look out we look back in cosmic time and light has traveled to us in ~~that~~ lookback time and ~~the~~ source has receded and it is recession velocity has changed.

But local measurement do establish H_0 if right!!

Confusion of sources confused me

Lemaitre in 1927 derived it explicitly from GR and estimated $H_0 \simeq 625 \frac{km/s}{Mpc}$ from data

Hubble's law Obscure Brussels journal

~~~~

$\frac{625}{670}$ $\frac{+575}{+670}/2$ Livio 2011

Friedmann equation Implicit in Friedmann & de Sitter universe but apparently not written explicitly

Not Really!! 1927

Lundmark on so had the law empirically with $H_0 \simeq 70 \frac{km/s}{Mpc}$

apparently a very good Not accepted ~~value~~ (Not really)

in 1929 Hubble published his discovery and $H_0 = 500 \frac{km/s}{Mpc}$

calibration errors.

$H_0 = 67$ Planck

73 Riess et al. of SNe

~~The tension~~ The Hubble tension

Lemaitre had theory and fitted value H_0

But needed not known

So a theoretical
~~~~

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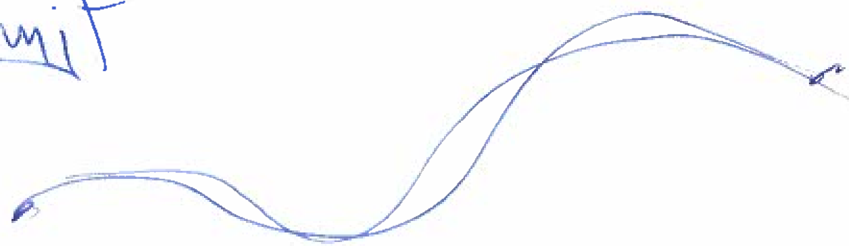
Unit



I think it's true ~~that~~ Hubble's Law ~~follow~~ generally if non vigorously just by saying if space is homogeneous & isotropic and just scales with time by general factor $a(t)$

$$S(t) = \cancel{H}S_0\, a(t)$$

of whatever possible geometry $k=0$ or $k=\pm 1$

Then any shape formed by test particles should just scale and one has

$$\dot S = S_0\, \dot a$$

No freedom to distort

and

$$\frac{\dot S}{S} = \frac{\dot a}{a}$$

$$\dot S = \frac{\dot a}{a} S = HS$$

2021oct03

4033

5) Cosmological Redshift

Proofs for RW metric models

In general redshift is defined

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}$$

- cosmological
- Doppler Redshift
- gravitational

λ_o observed

emitted known by recognizing the pattern of shifted lines from lab measurements, very hard to know λ_e without line spectra, i.e., just from continuum spectra

$z < 0$ is −ve redshift or blueshift

Note $\quad z + 1 = \dfrac{\lambda_o}{\lambda_e}$

Of course there are compounded redshifts

$$(z_{total} + 1) = \frac{\lambda_o}{\lambda_n} \cdots\cdots \frac{\lambda_3}{\lambda_2}\frac{\lambda_2}{\lambda_1}$$

$$\left\{ \frac{\lambda_{\text{pLocal}}}{\lambda_{\text{coslocal}}}\ \frac{\lambda_{\text{cosLocal}}}{\lambda_{\text{cosRemote}}}\ \frac{\lambda_{\text{Doppler}}}{\lambda_{\text{Doppler}}} \right.$$

$$= \frac{\lambda_o}{\lambda_2}\ \ \lambda_{\text{local CMB}}$$

Most of interest

$$(z_{total} + 1) = (z_{\text{us}} + 1)(z + 1)(z_r + 1)$$

us Doppler — This we know from CMB

cosmological Redshift.

peculiar motion Doppler shift of remote galaxy may be estimated or just neglected as small

$$z + 1 = \frac{(z_{total} + 1)}{(z_{us} + 1)(z_{pec} + 1)}$$

Personally I argue that the cosmological redshift is NOT a Doppler shift though there is a derivation from the Doppler shift — you might argue it is a compounded Doppler shift, but its formula is not a Doppler shift formula except asymptotically as $z \to 0$.

4032

a) Cosmological Redshift Proven from Robertson-Walker Metric — a direct GR proof.

$d\tau^2$ (Carroll-9)

$$ds^2 = 0 = c^2 dt^2 - a^2 \frac{dr^2}{1-kr^2}$$

t_0 MW $t_0 + dt_0$ t_e $t_e + dt_e$ CU

light-like interval along a radius to some point

$d\phi = d\theta = 0$

$$\therefore \chi = \int_0^r \frac{dr^2}{1-kr^2} = f(r) = \int_{t_e}^{t_0} \frac{c\,dt}{a} = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{c\,dt}{a} = \int_{t_e}^{t_0} \frac{c\,dt}{a} + \frac{dt_0}{a(t_0)} - \frac{dt_e}{a(t_e)}$$

$a(t)$ evolves in time but we need an FE solution to know how

independent of time so equal

differential time later

MW guide

$$= \int_{t_e}^{t_0} \frac{c\,dt}{a} + \int_{t_0}^{t_0+\delta t_0} \frac{c\,dt}{a} - \int_{t_e}^{t_e+\delta t_e} \frac{dt}{a}$$

starting earlier at t_e

add a bit

so must subtract off this bit

$$0 = \frac{c\,dt_0}{a(t_0)} - \frac{c\,dt_e}{a(t_e)}$$

Let $dt = \frac{1}{\nu}$

the time interval between crests of a wave of electromagnetic radiation

$$\therefore \nu_0 a_0 = \nu_e a_e$$

$a_0/\lambda_0 = a_e/\lambda_e$

$\nu \propto \frac{1}{a}$

$\lambda \propto a$

$$\frac{a_0}{a_e} = \frac{\lambda_0}{\lambda_e}$$

$$\frac{a_0}{a_e} - 1 = z$$

$$z = \frac{a_0}{a} - 1$$

or

$$z + 1 = \frac{a_0}{a}$$

suppress e

QED.

Remarks

Proof on p. 4034

$$\frac{a_0}{a} = z + 1 \hat{=} z \text{ for } z >> 1$$

or $\frac{a}{a_0} = \frac{1}{1+z}$

So any measurement of z gives a at the time t of emission.

But cosmic time t is not a direct observable.

So z gives $a(z)$ but not $a(t)$.

If only galaxies had clock faces on them to read off cosmic time.

Some try to use passively evolving early-type galaxies as cosmic chronometers

Maybe other cosmic chronometers?

(Wiki: Dark Energy; Observational Hubble constant data)

ellipticals

- I don't know quite how
- it can't be so accurate

~~Redshift Drift~~ so far

z_1

z_2

$$\frac{\dot{a}}{a_0} = \frac{-1}{(1+z)^2} \frac{dz}{dt}$$

{ Cosmic time but nearly our time too.

$H > 0$

$\dot{z} < 0$

$$H = \frac{\dot{a}}{a} = \frac{-1}{(1+z)} \frac{dz}{dt} \hat{=} \frac{-1}{1+z} \frac{\Delta z}{\Delta t}$$

must model age difference somehow.

Gives direct knowledge of Hubble parameter.

Redshift Drift is something else.

But $\frac{\dot a}{a_0} \simeq \frac{-1}{(1+z)^2}\frac{\delta z}{\delta t}$

could allow direct empirical integration of $a(t)$ if you could trust your Δt's well enough.

Another one on p.4041-4042

b) $\lambda \propto a$ a Quantum Mechanical Proof of Cosmological Redshift

but the de Broglie relation gives

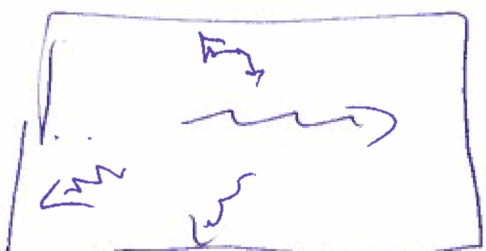
$$E_{photon} = h\nu = hc/\lambda$$ hand wavin proof

$\therefore$ $E_{photon} \propto \frac{1}{\lambda} \propto \frac{1}{a}$ UBD space grows under light waves

So photon energy redshifts away.

Where does it go?

In an expanding box it goes into walls doing PdV work



But what of in a boundless universe?

Recall GR doesn't guarantee conservation of energy in an ordinary sense.

It guarantees $\nabla_\mu T^{\mu\nu}=0$ a energy-momentum conservation equation Carroll-117, 120

Recall Einstein field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein demanded $\nabla^\mu T_{\mu\nu} = 0$ and that guided him to the LHS. ∇^μ LHS $= 0$ too!

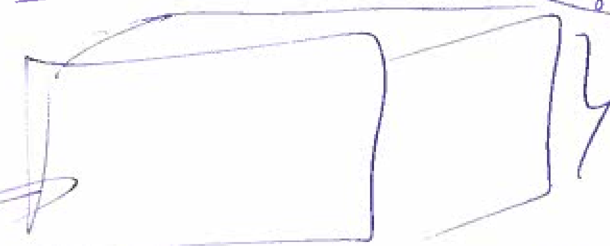
4035

Carroll-120 says ~~the poor~~ ~~redshift energy~~ the photon energy goes nowhere. It's gone. Maybe it goes somewhere but not clear where

In ideal GR derivation of boundless space

I used to say into the expansion of universe which is sort of true, but rather meaningless. Especially in early universe when radiation is dominant up to ~~t=50ky~~ t=50 kyr. Photon energy ~~[illegible]~~ without need to put it anywhere

Consider a box in the expanding universe



$r_0 = a_0 \chi$

or box like if geometry odd

A box too small to worry about curvature, but maybe there is a tiny curvature correction. But our observable universe is nearly flat anyway

box of N photons of energy E_0 each

then $E_{total} = NE_0$

$$\mathcal{E}_0 = \frac{NE_0}{r_0^3} = \frac{NE_0}{a_0^3\chi^3}$$

$N = \mathcal{E}_0 a_0^3 \chi^3 / E_0$

~~at a later time~~

$= E_0\left(\frac{a_0}{a}\right)$

Later in cosmic time

~~but~~ $E = E_0\left(\frac{a_0}{a}\right)$ and ~~$r = a\chi$~~ $= a\left(\frac{r_0}{a_0}\right)$ since $\left(\frac{r_0}{a_0}\right) = \chi$

$$\mathcal{E} = \frac{NE}{(a\chi)^3} = \frac{NE_0\left(\frac{a_0}{a}\right)}{\left(\frac{a}{a_0}\right)^3 \chi r_0^3}$$

$= \frac{NE}{a_0^3\chi^3}\left(\frac{a_0}{a}\right)^3\left(\frac{a_0}{a}\right)$

$$= \mathcal{E}_0\left(\frac{a_0}{a}\right)^4$$

Not dependent on being a Blackbody spectrum but holds for that too

$= \frac{NE_0\left(\frac{a_0}{a}\right)}{\left(\frac{a}{a_0}\right)^3 (a_0\chi)^3}$

$\frac{\mathcal{E}}{\mathcal{E}_0} = \left(\frac{a_0}{a}\right)^3\left(\frac{a_0}{a}\right)$

So the energy of photons and all extreme relativistic particles scales as $\propto \frac{1}{a^4}$

Where ~~[illegible]~~ energy go?

Later on we'll see how the CMB behaves under cosmic expansion.

4036

c) ~~A derivation from the Doppler~~ Formula

$$\cancel{\frac{f_o}{f_e} = \frac{(1-\beta_o\cos\theta_o)}{(1-\beta_e\cos\theta_e)}\sqrt{\frac{1-\beta_e^2}{1-\beta_o^2}}}$$

~~Relativistic Doppler Shift formula for emitter and observer with arbitrary velocities in one~~ inertia frame.

c) A derivation of the Cosmological Redshift From the Relativistic Doppler Formula

In one inertial frame the point can have any separation in space including 0 — the velocities measured in one inertial frame

$e \longrightarrow o$

$\beta = \frac{v}{c}$ where v is relative velocity

$v > 0$ for increasing separation

$v < 0$ for decreasing separation

$$\frac{\lambda_o}{\lambda_e} = \frac{f_e}{f_o} = \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{(Wik)}$$

Now this is one inertial frame but ~~we can say it must be the same~~ if we go to nonrelativistic limit; $\beta \ll 1$

$$\frac{\lambda_o}{\lambda_e} = (1 + \tfrac{1}{2}\beta)(1 + \tfrac{1}{2}\beta) = 1+\beta$$

and so $\frac{\lambda_0}{\lambda} - 1 = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \beta$

or $\frac{\Delta\lambda}{\lambda} = \beta$

Note $\frac{d\lambda}{\lambda} = \beta =$ is not a ~~useful~~ meaningful differential equation since $\ln\lambda \Big| = \int \beta$ no obvious interpretation ~~makes no sense~~ ~~other~~ no useful sense ~~since β is not differential~~ ~~(But I still wonder why the~~

maybe ~~β just~~ ~~is~~ a ~~~~

~~we know what to do with~~ change ~~is~~ ~~~~

Can one make β a differential? $d\beta$ dt ...

But what does that mean?

~~$\beta = \frac{v}{c}$~~

If you make β a change over some differential quantity that has meaning. implied steps

Say $\beta = \frac{v}{c} = \frac{H\,dD}{c}$ where H is the Hubble parameter,

Now the $d\beta \propto dD$ have meaning I think

But we've changed our meaning a bit here. Instead of v being a velocity in one inertial frame, it is now velocity between two infinitesimally close inertial frames.

But this seems to be allowed in the classical limit which should hold in differential limit

$\dot{1}$ $\dot{2}$ in one ~~frame~~ inertial frame

$\dot{1} \longrightarrow \dot{2}$ in two ~~frames~~ ~~with no outside acceleration~~ inertial frames

$\therefore \frac{d\lambda}{\lambda} = \frac{H\,dD}{c}$

but traversing dD takes time $dt = \frac{dD}{c}$

comoving frame time.

~~Ex~~ 4038

$$\therefore \frac{d\lambda}{\lambda} = \frac{\dot{a}}{a}\, dt = \frac{da}{a}$$

$$\therefore \ln\lambda = \ln a$$

$\lambda = \lambda_i \left(\frac{a}{a_i}\right)$ where i is initial wavelength.

$\therefore \lambda \propto a$ again the cosmological redshift.

I think this derivation is valid. ~~only for one but only with the valid assumptions specified — and I'm not sure I have.~~

Actually, the case is "isomorphic" to classical expansion in a container. Consider a ~~reflective~~ cavity and scale up with $a(t)$ and $l = a l_0$

$$\frac{d\lambda}{\lambda} = \frac{v}{c} = \frac{\dot{a} l_0}{c}$$

$$= \frac{(\dot{a}/a) l}{c}$$

$$= \frac{\dot{a}}{a} dt = \frac{da}{a}$$

$$\therefore \lambda \propto a$$

Same if cavity is full of ~~hollow scatters~~ pure scatterers

But where does the energy go in this case? Into PdV work on the walls and hence the exterior (but I can't think of a proof now)

4039

But in FE models, there is no outside — they are boundless — So the energy just vanishes, allowed by GR.

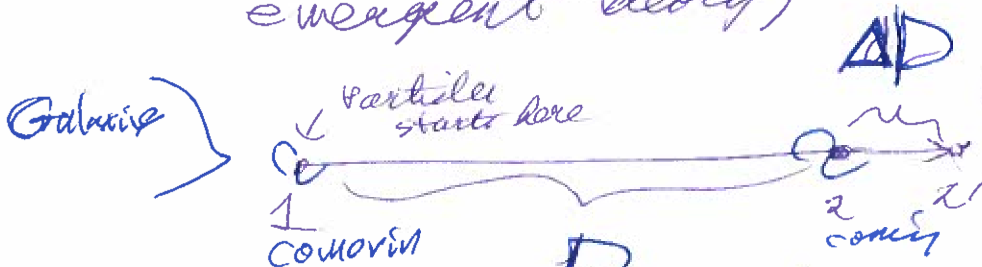
But what if there is really an outside? (outside our pocket universe)

Maybe the photon energy does go there.

Maybe the dark energy into the darkenergy stuff comes from there.

d) What of NR particles and objects? (Non relativistic.)

On the small scale we expand assume classical physics because we know it does hold in the classical limit (and believe exactly so — it's a true emergent theory)



arrives here while frame 2 is moving from frame 1 at velocity v_2

$\Delta L = v_2 \Delta t$

True things ... $v_2 = \Delta L / \Delta t$

30 4(4D)

and

~~similarly~~ $\Delta t = \dfrac{D+\Delta D}{v}$ ~~$= d\left(\frac{\Delta L}{\Delta t}\right)$~~

Same times as constraint

particle velocity ~~to~~ relative to frame 1.

So the change in velocity relative to frame is $\Delta v = (v - v_R) - v$

What we want to know is how ~~at~~ changes

$$= -v_R$$

$$= -\frac{\Delta D}{\Delta t} = -\frac{\Delta D}{(D+\Delta D)/v}$$

$$= -\frac{\Delta D}{D+\Delta D}\, v$$

Going to the differential limit where everything is assume exact

$$dv = -\frac{dD}{L}\, v$$

$$\frac{dv}{v} = -\frac{dD}{D} = -\frac{da}{a}$$

$$\ln v = -\ln a$$

$$\therefore \quad v \propto \frac{1}{a}$$

$$\text{or} \quad v = v_i\left(\frac{a_i}{a}\right)$$ where i is the initial condition

In FE universes the kinetic energy just vanishes

Note $KE \propto v^2 \propto \frac{1}{a^2}$

So unbound objects from Galaxy clusters down to kicked hypervelocity pulsars will perpetually slow down relative to their local comoving frame. $\rightarrow$ i.e., the peculiar velocities will always decrease

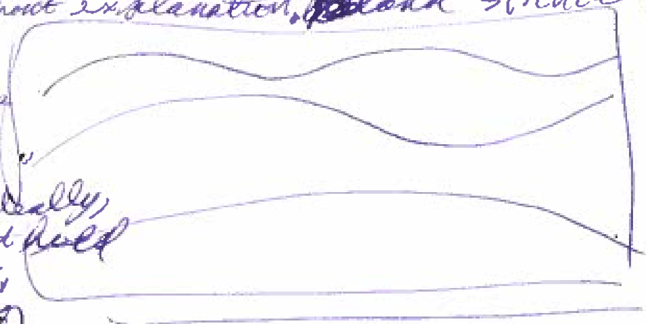
e) Derivation from Quantum Mechanics

4041

Peacock-72

Actually used for ill-defined boxes in medium like neutron stars without explanation, bland silence

I suppose since it holds ideally, it must hold really,



L

Box quantization — an ideal limit, but ~~must~~. it seems to be close enough to real cases ~~that~~ it is infinitely useful, especially in solid state physics

$\psi(BCs) = 0$

since the wavefunction cannot penetrate an infinite ~~potential~~ wall

$\therefore n(\lambda/2) = L$

$\lambda = \frac{2L}{n}$

we just consider 1-dimension of the box

$\therefore k = \frac{2\pi}{\lambda}$ is wavenumber of a single particle state

$= \frac{2\pi}{\left(\frac{2L}{n}\right)} = \frac{\pi}{L} n$ where n is the number of waves.

Somehow in many cases, ideal boxes must asymptotically yield the same or messy ideal real boxes. There may be a proof of this, but QM textbooks are blandly silent

~~Imagine a box the~~

~~NR b~~

$k \propto \frac{1}{L}$ for every single particle state

$p = \hbar k$ general in QM

NR limit $E \propto p^2 \propto \frac{1}{L^2}$

ER limit $E \propto p \propto \frac{1}{L}$ as for photons

In actual box the adiabatic expansion causes no particles to change state and all energy goes into PdV work

4042

If we imagine a box in space expanding with ~~the~~ universe, ~~then~~

Note macroscopic particles are made of microscopic particles, so if then result do extrapolate to nonideal cases, then.... Not sure

$$p \propto \frac{1}{a} \quad \text{in NR limit}$$
$$E \propto \frac{1}{a^2}$$

$$\left.\begin{array}{l} p \propto \frac{1}{a} \propto N \text{ (as on p.4040)} \\ E \propto \frac{1}{a} \end{array}\right\} \text{ in ER limit}$$

There are no boxes in space, but maybe the FE universes ~~are~~ sort of ~~a~~ boxes

— and if there were particles in ideal single particle states (~~tota~~ → Nonlocalized particles)

— they would have to lose energy like this.

— So a congruent result ~~from~~ from the QM perspective

~~The~~ In boundless FE universes the energy again vanishes. But if there is an outside, maybe somehow energy comes or goes from there.

6) Connecting $z, a_0, a, t, t_{\text{lookback}} = t_0 - t$

Cosmological Distance Measures

4043

a) We will look at some general results, but

$a_0 r$, $a_0 \chi$

D_{proper}, D_{lum}, D_{angular},

$v_{\text{regession}}$, $v_{\text{redshift}} \equiv zc$

(This is a definition of v_{red}, not a result.)

also focus on the asymptotic $z \to 0$ case.

Recall a basic result $\frac{a_0}{a} = 1 + z \doteq z$ for $z \gg 1$

and $\frac{a}{a_0} = \frac{1}{1+z}$

and $\frac{a_0}{a} - 1 = z$

So we need to connect cosmological distance measures as functions of z not t including $t(z)$.

Since z is the easiest direct observable, we know $a(z)$ really well but what we want is $a(t)$. Alas galaxies do not have clock faces on them to tell us cosmic time. Some work in that direction with cosmic chronometers.

Of course one could always do forward modeling. All $f(t)$ ~~[illegible]~~ to data or not.

We need to prove, among ~~other~~ things ~~that~~

~~[illegible]~~ as $z \to 0$, ~~or~~ to 1^{st} order in small z (as we'll show)

$$D^{1\text{st}}_{\text{proper}} = D^{1\text{st}}_{\text{luminosity}} = D^{1\text{st}}_{\text{angular}} = D^{1\text{st}}$$

Not always expedient

$A = 4\pi D_L^2$

$D_L \leftarrow f_{\text{obs}}$

$f_{\text{obs}} = \frac{L}{4\pi D_L^2}$

D_L is a direct observable if you know $L_{\text{luminosity}}$

$$= \sqrt{\frac{L_{\text{luminosity}}}{4\pi f_{\text{obs}}}}$$

— only used distances for a static stationary linear case

known if you have a ~~[illegible]~~ standard candle

4044)

$$v^{1st}_{\text{recession}} = v_{\text{red}} \equiv zc$$

and $t_{\text{lookback}} = \dfrac{D^{1st}}{c}$

b) What we need to do first is connect

z and $a_0 \chi$ / $a_0 r$

For $k=0$: $\chi = r$, and in general $\chi = r$ as $z \to 0$.

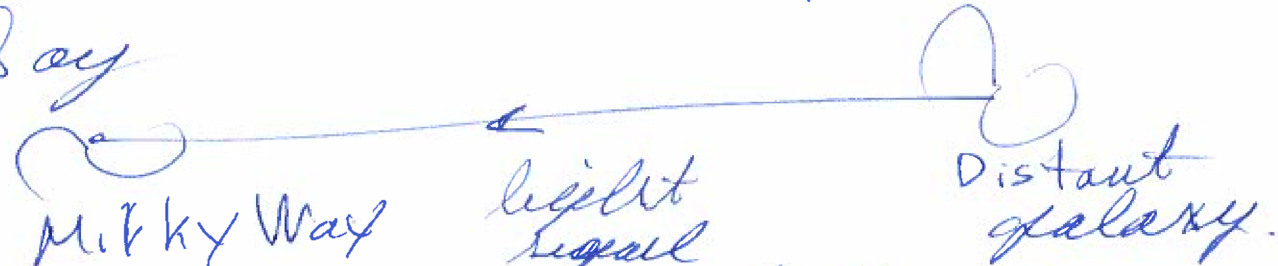
where χ and r are alternative comoving coordinates $\Rightarrow$ they are time independent.

If you can do ~~that, then~~ other results follow.

Analytically you can do it as a power series in small z.

But in general, only in special cases analytically.

c) Say



connects the two

$$\therefore \underbrace{d\tau^2 = ds^2 = 0}_{\text{a lightlike interval}} = c^2 dt^2 - a^2 \underbrace{\frac{dr^2}{(1-kr^2)}}_{\text{or } d\chi^2}$$

If $k=0$, $D_{\text{proper}} = a_0 \chi(z)$ and the partition between a_0 and χ is arbitrary

Note if $k = \pm 1$ we must use $a_g(t)$ where a_{g0} is radius of curvature at cosmic present.

r case	χ case
$\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_t^{t_0} \frac{c\,dt}{a} \equiv A(t)$	$\int_0^\chi d\chi = \int_t^{t_0} \frac{c\,dt}{a} = \chi(t)$

A model gives $a(t)$ and $t(a)$

and $t(a) = t\left(\frac{a_0}{1+z}\right) = t(z)$ and so $\chi(t) = \chi(z)$

$= \chi(z)$

Always in principle

but only in special cases

is $A(z)$ analytic.

$f(r) = \chi(z)$	$\begin{cases} \chi = \chi(z) \\ \chi = f(r) \end{cases}$
$r = f^{-1}[\chi(z)]$	
$dD_{\substack{\text{proper}\\ \text{tangential}}} = a_0 r(z) \sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$	$D_{\substack{\text{proper}\\ \text{radial}}} = a_0 \chi(z)$

In physical connection do disappear if flat. If curved space does ~~do~~ is a parameter

If only to keep track of soln

Now it may be useful to write

$$\int_t^{t_0} \frac{c\,dt}{a(t)} = \frac{1}{a_{s0}} \int_t^{t} \frac{c\,dt}{y(t)} = \frac{1}{a_{s0}} A(z)$$

where $a_{s0} = \begin{cases} a_{s0} \\ 1 \end{cases}$ or whatever

$r = f^{-1}\left[\frac{1}{a_{s0}} A(z)\right]$	$\chi = \frac{1}{a_{s0}} A(z)$
	$D_{\substack{\text{proper}\\ \text{radial}}} = \frac{a_0}{a_{s0}} A(z)$

If $k = 0$, $\cancel{\chi} = f(r)$ and no a_{s0} is defined

$dD_{\substack{\text{proper}\\ \text{tanential}}} = a_0 r(z)$	$D_{\substack{\text{proper}\\ \text{radial}}} = a_0 r(z)$

4046

In the $k=0$ (i.e., flat universe), the partitioning between a_0 and r is arbitrary, and so $a_0 = 1$ is probably best ~~and~~ unless there is some advantage in choosing it to be a fiducial value (e.g. 1Gpc)

The factoring of $a_0 r(z)$ is arbitrary in this case. In fact in many of our results a_0 cancels out.

~~Note~~ Recall $k = \pm 1$, $a_{0} = \dfrac{c/H_0}{\sqrt{|\Omega_{k0}|}}$ (p.4003 4006)

and $\chi = \begin{cases} \sin^{-1} r & k=1 \\ r & k=0 \\ \sinh^{-1} r & k=-1 \end{cases}$ CL-11 $= \dfrac{4.2827\ldots\, h_{70}^{-1}\,\mathrm{Gpc}}{\sqrt{|\Omega_{k0}|}} = \dfrac{13.968\ldots\, h_{70}^{-1}\,\mathrm{Gly}}{\sqrt{|\Omega_{k0}|}}$

$r = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$

Note $f(r) = \int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_0^r dr\left[1 + \frac{1}{2}kr^2 + \ldots\right]$

$= r + O(r^3) + \ldots$

~~It turns out that $r^{1st} = z$ as we~~ will see.

Now $\chi(z) = f(r)$ ~~=~~ $r + O(r^3)$

So χ ~~and~~ has no dependence of curvature to 2nd order in z

and so ~~$\chi(z)^{\text{2nd order in } r}$ $r \Rightarrow b_1 z + b_2 z^2$~~ ~~to 2nd order in z~~

and as we'll show $r = b_1 z + b_2 z^2$ to 2nd order in z

$\therefore \chi^{\text{2nd ord in } r \text{ and } z} = r^{\text{2nd in } z} = b_1 z + b_2 z^2$

So to 2nd order in z one customarily uses r rather than χ

But as we see, our choice doesn't ~~matter too much since one can easily change it later if one leaves a_0 explicit~~

$$\left[\ dt = \frac{da}{\dot a}\ \right.$$

d) Now $\chi = \cancel{\chi}[t(z)] = \int_t^{t_0} \frac{c\,dt}{a(t)} = c\int_a^{a_0} \frac{da}{a\,\dot a}$

We want $\chi(a) = \chi(z)$ not $\chi(t)$ ~~to~~ connect to observations

~~How? You need to be able to change variable $da = \dot a\, dt$ $\therefore \frac{dt}{a} = \frac{da}{a\dot a} =$~~

can be solved analytically in some cases if $a(t)$ is known analytically ~~or at least $\dot a(t)$ and~~ and can be inverted to $t(a)$ so that $\dot a(t) = \dot a[t(a)] = \dot a(a)$ but the integration $\frac{da}{a\,\dot a(a)}$ is usually hard and not exactly analytic

~~E)~~ The simplest example case ~~where~~

~~e)~~ an analytic result exists is the de Sitter universe

where $a(t) = a_0 e^{H_0(t - t_0)}$

$\therefore \dot a = H_0 a.$ ~~and $da = a\,dt$~~

The geometry in general $k = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$

$$\therefore \cancel{\chi} = \int_t^{t_0} \frac{c\,dt}{a} = \int_a^{a_0} \frac{c\,da}{a\,\dot a} = \frac{c}{H_0}\int_a^{a_0} \frac{da}{a^2}$$

$$= \frac{c}{H_0}\left(\frac{-1}{a}\right)\Bigg|_a^{a_0} = \frac{1}{H_0}\left(\frac{-1}{a_0} + \frac{1}{a}\right)$$

$$= \frac{c}{H_0 a_0}\left(-1 + \frac{a_0}{a}\right)$$

$$z = \frac{a_0}{a} - 1 = \frac{1}{e^{H_0(t - t_0)}} - 1 = \frac{1}{e^{-H_0 t_{\rm look}}} - 1$$

$$z = e^{H_0 t_{\rm look}} - 1$$

$$t_{\rm look} = \frac{\ln(z+1)}{H_0}$$

exactly lookback time

$$\cancel{\chi} = \frac{zc}{H_0 a_0}$$

~~where $H_0 a_0$~~

Getting $D_{\rm Lum}$ and $D_{\rm A\,angular}$ takes some more results.

$$D_p(z) = a_0 \frac{zc}{H_0 a_0} = \frac{zc}{H_0}$$

So a_0 disappeared no matter whether you gave it units or not

4048

Note $zc = H_0 D_p$

{ To get D_L and D_A you need to get a connection to z or x which is p.

and since $v_{rec} = H_0 D_p$ by the already know Hubble law (req. 4025–4026)

we recognize $v_{rec} = zc \equiv v_{redshift}$

This is a definition of what we mean by v_{red}

in this special case the recession velocity equals the redshift velocity for all z (not just to first order in small z)

f) Another exact case is the Power-Law case

$$a = a_0 x = a_0 \left[\sqrt{\Omega_{p0}} \frac{p}{2} \tau \right]^{2/p}$$

where $\tau = t/t_{H_0}$

and $p > 0$ and $\Omega_{p0} = 1$

except at early times for dark term, but we are assuming not early times

$$\dot{a} = \frac{2}{p} \frac{a}{\tau} \frac{d\tau}{dt} = \frac{2}{p} \frac{a}{t}$$

$$\tau = \frac{(a/a_0)^{p/2}}{\sqrt{\Omega_p} \frac{p}{2}}$$

$$t = \frac{(a/a_0)^{p/2}}{\sqrt{\Omega_{p0}} \frac{p}{2}} \frac{1}{H_0}$$

$$x = \frac{a}{a_0} = \frac{1}{a_0/a} = \frac{1}{1+z}$$

$$\chi = \int_a^{a_0} \frac{c\, da}{a \dot{a}} = C \int_a^{a_0} \frac{da}{a^2 \left(\frac{a_0}{a}\right)^{p/2}} = C \frac{a_0}{a_0^2} \int_x^1 \frac{dx}{x^{2-p/2}}$$

$$= C \frac{1}{a_0} \left. \frac{x^{p/2-1}}{p/2-1} \right|_x^1$$

not beautiful, but not so bad either.

(Maybe a homework problem.)

g) ~~Maybe exact results~~ ~~from~~ ~~Matt~~ Radiation – Matter early universe and Matter – Λ ΛCDM after early universe

since $t(a)$ exists in both cases, $a(t)$ and so $\dot{a}[t(a)]$ may be analytic but $\int \frac{da}{\dot{a}}$ probably not

Since $k = 0$ $\quad r = f(r)$ 4049

$\therefore \quad r = A\left[t\left(\frac{a_0}{1+z}\right)\right]$

and recall ~~the~~ de Sitter universe has exponential growth

$a = a_0 e^{H_0(t-t_0)} = a_0 e^{H_0 \Delta t}$ and $\Delta t = t - t_0$

$d\Delta t = dt$

t_0

$$A(t) = \int_t^{t_0} \frac{c\,dt}{a} = \int_{\Delta t}^{0} \frac{c}{a_0} e^{-H_0 \Delta t'}\, d\Delta t'$$

$$= -\frac{c}{a_0 H_0} e^{-H_0 \Delta t'}\Big|_{\Delta t}^{0} = \frac{c}{a_0 H_0}\left(e^{-H_0 \Delta t} - 1\right)$$

$$= \frac{c}{a_0 H_0}\left(\frac{a_0}{a} - 1\right) = \frac{zc}{a_0 H_0}$$

$\therefore \quad r = \frac{zc}{a_0 H_0}$ ~~with~~ $L = a_0 r$ Proper distance at t_0

~~whether a_0~~

whatever the choice of a_0 ~~or~~ r ~~to~~ have dimensions

$\therefore \quad zc = H_0 L$

Hubble's law recovered

zc is called redshift ~~velocity~~

since $z = \left(\frac{a_0 - a}{a}\right)$ and in this special case redshift velocity = recession velocity

4050 (32)

7) Summary of Distance Measures and small z formulae

i) $a(t) = a_0\left[1 + H_0\Delta t - \frac{1}{2}q_0 H_0^2 \Delta t^2 + \dots\right]$ CL-17

$t_{lookback} = t_0 - t = -\Delta t$

$\Delta t = t - t_0 < 0$ for usual cases

$q_0 \equiv -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2}$ is the deceleration parameter which for ΛCDM is −ve since the universe is accelerating

$q_0 H_0^2 = \frac{\ddot{a}\, a_0}{\dot{a}^2}\frac{\dot{a}_0^2}{a_0^2} = \frac{\ddot{a}_0}{a_0}$

ii) For ΛCDM (with k = 0 of course)

$$q_0 = \frac{1}{2}\left[\Omega_{m_0}(1+3\underbrace{w_{matter}}_{0}) + \Omega_\Lambda(1+3\underbrace{w_\Lambda}_{-1})\right]$$

See p. 4053 — 4055

$$= \frac{1}{2}\left[\Omega_{m_0} - 2\Omega_\Lambda\right]$$

Ω_{m_0}	Ω_Λ	
0.3	0.7	fiducial values
0.3153(73)	0.6847(73)	Planck 2018 p. 15

$q_0 = -\frac{\ddot{a}}{a_0}\frac{1}{H_0^2}$ Li-5 3, Li-27

$= -\frac{\left[\left(-\frac{4\pi G}{3}\right)\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}\right]}{\left[\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}\right]}$ Li-27

$= -0.55$ fiducial ΛCDM value

iii) $z = -H_0\Delta t + \left(1 + \frac{1}{2}q_0\right)H_0^2\Delta t^2 + \dots$

(CL-17) Recall $\Delta t < 0$, good to 2nd order in Δt for all k see p. 4055

iv) $t_{lookback} = -\Delta t = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2}q_0\right)z + \dots\right]$ (CL-17)

See p. 4056

$\doteq \frac{z}{H_0} = \frac{zc}{cH_0} = \frac{v_{redshift}}{c\,H_0} = \frac{D_{1st}}{c}$ to 1st order

$= D_{1st}$ to 1st order in z

$= \frac{D_{1st}}{c}$

< 4051

v) $r = \frac{zc}{a_0 H}\left[1 - \frac{1}{2}(1+q_0)z + \dots\right]$ (CL-18)

X (to 2nd order in z as on p. 4046)

(40 46, 4058 (see p

comoving frame coordinate to 2nd order in z

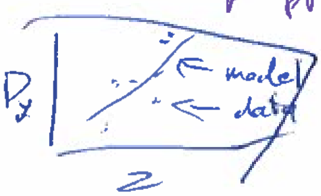
see p. 4028

So to 1st order $\underbrace{zc}_{v_{redshift}} = H_0 a_0 r = H\underbrace{D}_{\text{proper distance at } t_0 \text{ for any } k}$

First derived by Lemaitre 1927

This is the observable low redshift Hubble law for measurable L

vi) $D_{proper} = a_0 r = \frac{zc}{H_0}\left[1 - \frac{1}{2}(1+q_0)z + \dots\right]$ (see p. 4058)



$= \frac{zc}{H_0}$ to 1st order

$D_{prop}^{1st} = D_L^{1st} = \frac{zc}{H_0} = \frac{v_{rec}}{H_0}$

vii) Recession velocity since $v_{res} = H_0 D_{proper}$ in general (ideal Hubble's law)

Model gives $D(z) = a_0 r$ ~~transgential~~

$v_{recession} = zc\left[1 - \frac{1}{2}(1+q_0)z + \dots\right]$ from $v_{res} = H_0 D_{proper}$

$= zc$ to 1st order in z

$= v_{redshift}$

So fit D_L observed to model giving D_L from FE universe

∴ Observed Hubble law $v_{redshift} = H_0 D_L$ which makes it observable possible to measure H_0

viii) $D_{Luminosity} = \left(\frac{L_{luminous}}{4\pi f_{observed}}\right)^{\frac{1}{2}}$ ← observable definition (see p. 4063)

$= a_0 r(1+z)$ ← ? for universe all k (see Weinberg) p. 421

$= \frac{zc}{H_0}\left[1 + \frac{1}{2}(1-q_0)z + \dots\right]$ (CL-19)

k=0, $a_0=1$, $r = $ ~~D proper~~

k=±, a_0 from Ω_{k0}

and $r = \begin{cases}\sin\chi, & k=1\\ \sinh\chi, & k=-1\end{cases}$

407 ~~8~~ 8

ix) $D_{Angular} \equiv \frac{S}{\Delta\theta} = \frac{L_{ruler}}{\Delta\theta}$ (observable)

~~Flat space~~ ? p. 4069

D^{1st}
$= D^{1st}_{proper}$
$= D^{1st}_{Lum}$
$= D^{1st}_{Ang}$
$= \frac{zc}{H_0} = \frac{N_{red}}{H_0}$
The asymptotic Hubble law

$= \frac{a_0 r(z)}{1+z}$ ← theory

is ~~FE~~ recall for all k

Weinberg 422

CL-19

$= \frac{zc}{H_0}\left[1 - \underbrace{\left(\frac{3}{2} + \frac{q_0}{2}\right)}_{-\frac{1}{2}(3+q_0)} z + \ldots\right]$

x) Distance Duality Relation (Etherington's Reciprocity 1933 theorem Wik)

(see p. 4070 for repeat)

$$\frac{D_L}{1+z} = a_0 r = D_{angular}(1+z)$$

$$\therefore \quad D_L = D_{ang}(1+z)^2$$

general for all k Weinberg - 423

easier to test since independent of scale of D_L and $D_{angular}$

- It has been verified within error (Wik)

But actually Richard C. Tolman 1881-1948 gave the correct formula and suggested it's use in cosmology (Wik)

- a violation would mean exotic physics of some kind

e.g. photons not conserved, GR wrong

Wik: Etherington Reciprocity theorem implies true for all FE models Not just flat. Yes Weinberg - 423

Is this just for flat space and only ~~for~~ ... in z ...

7) Proving the small z approximations

(don't denoudou cunanduee to z2)

[If you can do straightforward modeling from a model, but some parameters H_0, higher ones can be determined without a full model specified, and so are general if you have fitted them correctly and the FE is true!!]

These allow us to set ~~model~~ parameters from ~~for~~ the local small z universe [That are independent of models]

Of course, if the ~~model's one~~ Friedmann Equation is wrong the parameters may not mean what they mean in ~~the model~~ the FE.

a) Expand $a(t)$ in a Taylor series about t_0 (cosmic present)

$$a(t) = a_0 + \dot{a}_0 \Delta t + \frac{1}{2}\ddot{a}_0 \Delta t^2 + \ldots$$

The next term is jerk, but too far for us

$$\Delta t = t - t_0 < 0$$

since ~~we~~ we usually think of signals from the past.

$t_{\text{lookback}} = -\Delta t$, the look back time

[Some argue that Taylor series is not best for expansion, Padé approximants have been used. Faster convergence?]

$$a(t) = a_0\left[1 + \frac{\dot{a}_0}{a_0}\Delta t + \frac{1}{2}\frac{\ddot{a}_0}{a_0}\Delta t^2 + \ldots\right]$$

$$= a_0\left[1 + H_0 \Delta t + \frac{1}{2}(-q_0) H_0^2 \Delta t^2 + \ldots\right]$$

where $H_0 = \frac{\dot{a}_0}{a_0}$ is the Hubble constant (unit of inverse time)

$= \sqrt{\frac{8\pi G}{3}\rho_{crit}}$ or Hubble density (Li-51)

and

$$q_0 = -\frac{\ddot{a}_0}{a_0 H_0^2} = -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2}$$ is the deceleration parameter (Li-53)

It's dimensionless and is positive for negative acceleration because people wanted a positive parameter and thought acceleration would be negative when they defined it in ~~the~~ before 1970 sometime

4054

$\llcorner$ Li-53

$$q_0 = -\frac{\ddot{a}_0}{a_0}\frac{1}{H_0^2} = -\left[\frac{\left(-\frac{4\pi G}{3}\right)\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}}{\frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}}\right]_0$$

(see Li-55)

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$$

$$= -\left[\underbrace{-\frac{1}{2}\frac{1}{\rho_{crit}}\left(\rho + \frac{3P}{c^2}\right)}_{\text{Li-51}} + \underbrace{\Omega_\Lambda}_{\text{Li-56}}\right]$$

counting dark energy which is in Ω_Λ

Let $\rho = \sum_i \rho_i$ and $P_i = c^2 w_i \rho_i$ (EOS)

$$= \frac{1}{2}\sum_{i \neq \Lambda}\Omega_i(1+3w_i) - \Omega_\Lambda$$

$$= \frac{1}{2}\sum_{i \neq \Omega}\Omega_i(1+3w_i) + \frac{1}{2}\Omega_\Lambda(1-3)$$

$$= \frac{1}{2}\sum_i \Omega_i(1+3w_i)$$

if we parameterize $w_\Lambda = -1$ even for true cosmological constant

Li-57

If a constant $P_\Lambda = -\rho_\Lambda c^2$ to obey fluid equation

$$= \frac{1}{2}\sum_i \Omega_i(1+3w_i)$$

in general for w parameter equation of state (EOS)

Before ~1998 when the flat EdS model was still favored

$$q_0 = \frac{1}{2}\Omega_M = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$= \frac{1}{2}[\Omega_M - 2\Omega_\Lambda]$$

$$\frac{1}{2}\underbrace{[.3 + .7(-2)]}$$

$$\frac{1}{2}\underbrace{[-1.1]}$$

-0.55

$\Omega_M = .3$, $w_M = 0$

$\Omega_\Lambda = .7$, $w_\Lambda = -1$

the fiducial ΛCDM values

We know to some degree $H_0 \approx 70$, $q_0 = -.55$

In ~~the~~ 1970s Sandage described cosmology as the search for two numbers H_0 and q_0 (somewhat humorously)

Nowadays q_0 has lost a bit of luster if ΛCDM is true since Planck 2018 data fits the model using other data.

However, if ΛCDM needs replacement local values of q_0 become important again but hard to measure

b) From $a(\Delta t)$ to $z(\Delta t)$

4055

$$\frac{a}{a_0} = \underbrace{\frac{1}{1+z}}_{= 1 - z + z^2 + \dots} = 1 + H_0 \Delta t + \tfrac{1}{2}(-q_0)(H_0 \Delta t)^2 + \dots$$

to 2nd order
geometric series
(Arf-279)

$$\therefore \quad z = -H_0 \Delta t + \tfrac{1}{2} q_0 (H_0 \Delta t)^2 + \dots$$

$$+ z^2 + \dots$$

assume $z = \sum_{\ell=1}^{\infty} b_\ell \Delta t^\ell = b_1 \Delta t + b_2 \Delta t^2 + \dots$

$$\therefore \quad z^2 = b_1^2 \Delta t^2 + 2 b_1 b_2 \Delta t^3 + \dots$$

$\therefore$ To ~~1st~~ order in Δt, then $\quad \cancel{z^2 = (H_0 \Delta t)^2 + 2 b_1 b_2 \Delta t^3}$

dropping $2 b_1 b_2 \Delta t^3$

$$z^{1st} = -H_0 \Delta t$$

$$\therefore \quad z^{2nd} = -H_0 \Delta t + (H_0 \Delta t)^2 + \tfrac{1}{2} q_0 (H_0 \Delta t)^2$$

$$z = -H_0 \Delta t + (\tfrac{1}{2} q_0 + 1)(H_0 \Delta t)^2 + \dots$$

$\therefore$ Recall z is from a source observed at lookback time $t_{lookback} = -\Delta t$

But Δt is not an observable in general (though it is asymptotically as $z \to 0$ as we'll show) unless someone puts clocks on galaxies, and so to determine Δt from z we need to invert the series

4056)

From Arf-316-317

$$\Delta y = \sum_{\ell=1}^{\infty} a_\ell \Delta x^\ell$$

$$\Delta x = \sum_{\ell=1}^{\infty} b_\ell \Delta y^\ell$$

$$b_1 = \frac{1}{a_1}$$

$$b_2 = -\frac{a_2}{a_1^3} \quad \text{etc}$$

$a_1 = -H_0$

$a_2 = (1+\frac{1}{2}q_0)H_0^2$

$$\therefore \Delta t = -\frac{1}{H_0} z - \frac{(1+\frac{1}{2}q_0)H_0^2}{(-H_0)^3} z^2 + \ldots$$

$$= -\frac{z}{H_0}\left[1 - (1+\frac{1}{2}q_0) z + \ldots\right]$$

$$t_{\text{lookback}} = \frac{z}{H_0}\left[1 - ((1+\frac{1}{2}q_0) z + \ldots\right] \quad \Big\{ \text{CL-17}$$

$$t_{\text{lookback}}^{\text{1st ord}} = \frac{z}{H_0} = \frac{z}{N_{\text{red}}/D^{\text{1st}}} = \frac{D^{\text{1st}}}{c}$$

to 1st order in small z

$$= z\, t_{\text{Hubble}}$$

since $N_{\text{red}} = zc$ ~~not~~ proved formally on p.4058 + 4067 ~~proven yet~~ in most formal way

But $N_{\text{rec}} = H D_{\text{proper}}$ ex

$zc = H_0 D_{\text{magically}}$ (static limit) is vaguely proven

c) From $a(t)$ to $r(z)$

Recall for a signal $\int_t^{t_0} \frac{c\,dt}{a} = f(r) = r + O(r^3)$

see p.4046

use $\frac{1}{1+x} = 1 - x + x^2$ Arf-279 and see p.4053

see p.4044–4048

$\Delta t = t - t_0$

$d\Delta t = dt$

differential

↓

actual value

↓

$\Delta t < 0$

$$\int_t^{t_0} \frac{c\,dt}{a_0}\left[1 - H_0\Delta t - \frac{1}{2}(q_0)H_0^2\Delta t^2 + \ldots \right. \\ \left. + H_0^2 \Delta t^2 \right.$$

$$\int_{\Delta t}^{0} \frac{c\,d\Delta t}{a_0}\left[\underset{(0-3)}{1} \underset{-(0-3)}{- H_0\Delta t} \underset{+(0-3)}{+ \frac{1}{2}(q_0+1)H_0^2\Delta t^2} + \ldots\right] = r + O(r^3)$$

$$\frac{c}{a_0}\left[-\Delta t + \frac{1}{2}H_0\Delta t^2 - \frac{1}{2}(q_0+1)H_0^2\frac{\Delta t^3}{3} + \ldots\right] = r + O(r^3)$$

static limit or small t_{lookback}

Validity of Doppler effect proof of cosmological velocities implies $\frac{\Delta\lambda}{\lambda} = \beta = \frac{N_r}{c}$ itself z

$\therefore N_{\text{rec}} = zc$

$$\frac{c}{a_0 H_0}\left[-H_0 \Delta t + \frac{1}{2} H_0^2 \Delta t^2 + \dots \right] = r + O(r^3)$$

see 4056   see p. 4056

Assume $r = \sum_{l=1}^{\infty} a_l z^l$

$$z\left[1-\left(1+\tfrac{1}{2}q_0\right)z\right] + \dots \quad + \dots \Big] = r + O(r^3)$$

this term contribute $O(z^3)$

$$\frac{c}{a_0 H_0}\left[z\left[1 - \left(\tfrac{1}{2} + \tfrac{1}{2} q_0\right) z\right] + \dots \right] \cancel{= O(z) + O(z^2) + O(z^3)} - O(z^3) = r$$

$$\therefore\ r^{\text{2nd order in } z} = \frac{z\,c}{a_0 H_0}\left[1 - \tfrac{1}{2}(1+q_0) z\right]$$

$$r(z) = \frac{z\,c}{a_0 H_0}\left[1 - \tfrac{1}{2}(1+q_0) z + \dots\right]$$

(C1-18)

8) Cosmological Distance measures

And making use of results p. 4053-4057

a) z the easiest direct observable.

b) Cosmic time and Lookback time

Not a direct observable.

If you know $a(t)$, then $t = t(a)$ the inverse

$$t_{\text{lookback}} \equiv \begin{cases} t_0 - t & \text{in general} \\[1ex] \dfrac{z}{H_0}\left[1 - \left(1+\tfrac{1}{2}q_0\right)z + \right] & \text{2nd order in } z \\ & \text{(C1-17, p. 4056)} \\ \dfrac{z}{H_0} & \text{1st order in } z \end{cases}$$

So if you know H_0, q_0 as well as easy z, you know t_{lookback} 2nd but how do you know H_0, q_0?

4048

c) $D_{proper} = a_0 \chi(z)$ in general

Recall $D_{proper}(z)$ exact available in special cases, see p. 4047, 4048 (deSitter universe, power law universe)

$a_0 r(z)$ to 2nd order in small r (see p. 4046) and small z

$D^{1st}_{proper} = \frac{zc}{H_0}$

$= z \begin{cases} \frac{4.2827\ldots}{h_{70}} \text{ Gpc} \\ \frac{13.968\ldots}{h_{70}} \text{ Gly} \end{cases}$

$\frac{zc}{H_0}\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right]$

to 2nd order in small z (see p. 4047) & CL-18

So if you knew z, H_0, q_0 you'd know D^{2nd}_{proper}, but D_{proper} not a direct observable

Note a_0 has cancelled. Physical scale comes from c, H_0, q_0 information $\frac{zc}{H_0}$ to 1st order in z.

d) Recession Velocity

~~$D_P(t) = a(t)\chi(z)$~~

t independent

From general Hubble's law $v_{rec} = H D_{proper}$

$v_{rec0} = H_0 D_{proper0}$ } Dropping 0 on v_{rec} and D_{proper} for simplicity

$v_{rec} = H_0 D_{proper}$

$= \cancel{H_0} zc\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right]$

To 1st order in z, v^{1st}_{rec} is a direct observable which we already know less formally (see p. 4056)

$\therefore v^{1st}_{rec} = \cancel{H_0} zc = H_0 D^{1st \text{ order in } z}$

and ~~v^{1st}_{rec}~~ confirms the $z \to 0$ asymptotic Hubble law first derived by Lemaître 1927

e) $v_{redshift} \equiv v^{1st}_{rec} = zc$ ...

So we have z, ν_{redshift},

and $\nu_{\text{rec}}^{1^{\text{st}} \text{ in } z}$ as direct observables,

but we still don't ~~don't~~ have

t_{lookback} and D_{proper} as direct observables to 1^{st} order in z { need to fit H_0

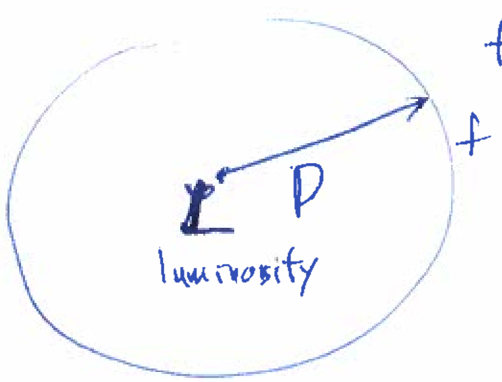
to 2^{nd} order in z ~~need~~ } need to fit H_0 and q_0

or exactly.

f) Luminosity Distance

$$f = f_{\text{obs}} = \frac{L}{4\pi D^2}$$ in a static flat universe assuming zero extinction

hereafter we assume extinction ~~can~~ negligible or can be corrected for.



$$\therefore D = \sqrt{\frac{L}{4\pi f}}$$

Now if the universe is not static (i.e., moving and additionally time dependent)

We define luminosity distance

$$D_L \equiv \sqrt{\frac{L}{4\pi f}}$$

which is a direct observable if L is known.

(which it ~~sort~~ ~~of~~ is for SNe Ia as standardizable candles to good approximation. Other ~~candidates~~ bright sources too more approximately;

But to use this to measure parameters independent of models for z small or dependent on models for all z, we need to connect D_L to theoretical quantities connected to z

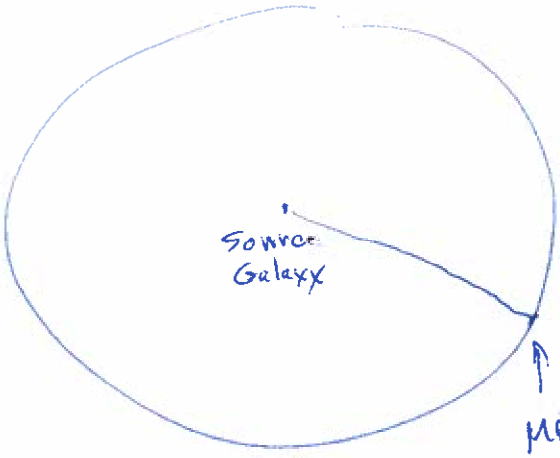
i.e. $r(z)$ and $X(z)$.

Now for any k ($k = \{ \begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix} \}$), we have

$$S = 4\pi (a_0 r)^2 \quad \text{(see p. } \cancel{4024}\ 4022\text{)}$$

On p. 4022 only proved for a k=+1 case, but sort of obviously true for k=0 and k=−1 too

recall r is the comoving coordinate designed to give tangetial lengths correctly (p. 4017



$S = S(t_0, r) = 4\pi(a_0 r)^2$ the spherical surface surrounding the source at t_0

Milky way at t_0

Say at t_e a light pulse was emitted over dt_e

from p. 4032, $\frac{dt}{a} = \frac{dt_0}{a_0}$ which we used on p. 4032 to find the cosmological redshift formula

$\frac{dt}{dt_0} = \frac{a}{a_0}$

From p. 4032 $E\,a = E_0 a_0$

photon energy

$\frac{E_0}{E} = \frac{a}{a_0}$

cosmological redshift itself

$\frac{a}{\lambda} = \frac{a_0}{\lambda_0}$

$$\therefore f_{obs} = \frac{L}{4\pi(a_0 r)^2}\left(\frac{dt}{dt_0}\right)\left(\frac{E_0}{E}\right) = \frac{L}{4\pi(a_0 r)^2}\left(\frac{a}{a_0}\right.$$

~~photon energy reduction factor~~

time dilation factor

photon energy reduction factor

$L = \frac{NE}{dt}$

multiply by time of emission

$L^{eff} = \left(\frac{NE}{dt}\right)\frac{dt}{dt_0} \quad \left(\frac{E_0}{E}\right)$

divide by time of observation

to get observation photon energy

Recall $\frac{a_0}{a} = 1 + z$

$$\therefore f = f_{obs} = \frac{L}{4\pi(a_0 r)^2}\frac{1}{(1+z)^2}$$

4062

$$D_L = \sqrt{\frac{L}{4\pi f}} = \sqrt{(a_0 r)^2 (1+z)^2}$$

observable quantities

$$D_L = a_0 r (1+z)$$ Theoretical connection to z

$$= a_0 r(z)(1+z)$$

~~which is~~ which is the way it is usually written.

Now ~~$r = f^{-1}\{A[t(\frac{a_0}{1+z})]\}$ (see p. 4046)~~

~~$f(r) = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$, $A(t) = \int_t^{t_0} \frac{c\,dt}{a(t)}$~~

Not for ~~for r~~ ~~and~~ formal r and χ but r and χ connecting points separated by a light-like interval

$$r = f^{-1}[\chi(t)] = f^{-1}[\chi(z)]$$

f^{-1} analytic exist see p. 4046

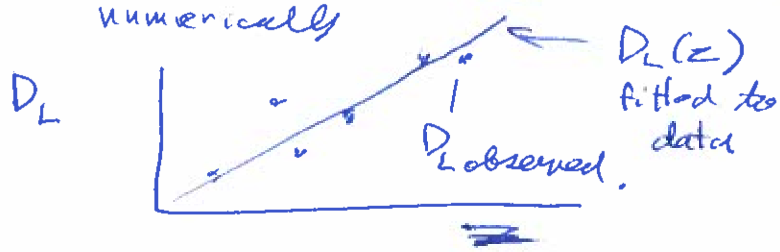
~~$\chi(t)$, $\chi(z)$ analytic exist~~

$$f(r) = \int_0^r \frac{dr}{\sqrt{1-kr^2}}$$ $r = \sin\chi$, $k=1$; $r = \sinh\chi$, $k=-1$

$k=0$ is easy $f(r) = r$

$\chi(t)$ analytic and $\chi(z)$ analytic exist ~~only~~ in special cases.

In generall you'd have to grind out $D_L(z)$ numerically



$$\chi(t) = \int_t^{t_0} \frac{c\,dt}{a} = \int_a^{a_0} \frac{c\,da/\dot{a}}{a}$$

exists only in special cases.

$$= \int_a^{a_0} \frac{c\,da}{a\,\dot{a}(a)}$$

See p. 4047 de Sitter U. p. 4048 power law U.

How does χ depend on a_0? 4063

$$\frac{a_0}{a} = z + 1$$

Say we have a solution $a(t) = a_0\, y(t)$,

then $z = \frac{1}{y(t)} - 1$

and $z(t)$ has no a_0 dependence.

$$\chi(t) = \int_t^{t_0} \frac{c\,dt}{a} = \frac{c}{a_0}\int_y^{y_0=1} \frac{dt}{y} = \frac{c}{a_0}\int_z^0 (z+1)\,dt$$

$$= \frac{c}{a_0}\int_z^0 \frac{(z+1)}{\dot{z}(t)}\,dz = \frac{c}{a_0}\int_z^0 \frac{(z+1)}{\dot{z}(z)}\,dz$$

but $z = z(t)$

implies $t = t(z)$

and so $\chi = \frac{c}{a_0} F(z)$

$$\therefore \quad D_L = a_0 r (1+z) = c F(z)(1+z)$$

if $K = 0$ and a_0 is eliminated and of course can be set to be 1.

but if $K \neq 0$, then

then $r = \begin{cases} \sin\chi & \Rightarrow K = 1 \\ \sinh\chi & \Rightarrow K = -1 \end{cases}$

and a_0 cannot be eliminated, and of course the RW metric with K scaled to 1 or -1

does have a physical scale.

$$a = a_{og} = \frac{c/H_0}{\sqrt{|\Omega_{k0}|}} = \frac{4.2827\ldots\, h_{70}^{-1}\ \text{Gpc}}{\sqrt{|\Omega_{k0}|}} = \frac{13.968\ldots\, h_{70}^{-1}\ \text{Gly}}{\sqrt{|\Omega_{k0}|}}$$

see p. 4006

So a_{og} becomes part of the model to be fitted to data.

But what of small z limit?

$$D_L = a_0 r(z)(1+z)$$

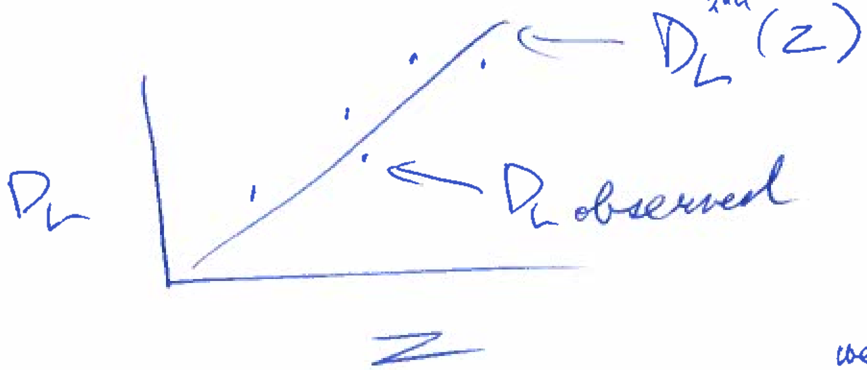
$$= a_0 \frac{zc}{a_0 H_0}\Big[1 - \tfrac{1}{2}(1+q_0)z + \ldots\Big](1+z)$$

see p. 4057

CL-19

$$D_L = \frac{zc}{H_0}\Big[1 + (\tfrac{1}{2} - q_0)z + \ldots\Big]$$

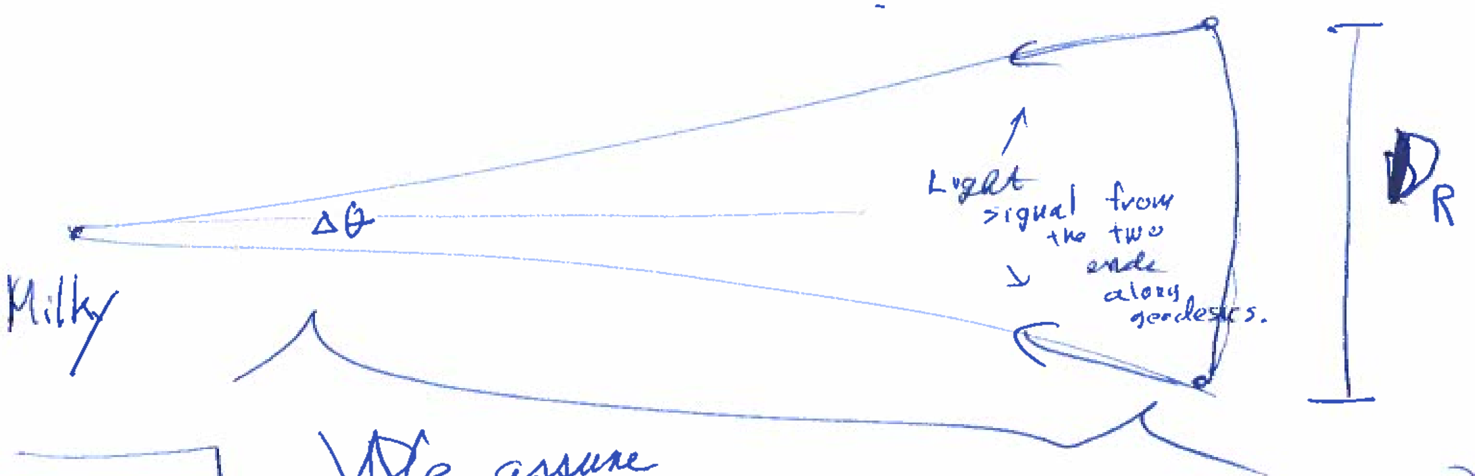
So



So finally we can solve for H_0 and q_0 for small z from a fit to observations but we have no ability to fit a_{og} and find curvature information

9) Angular Diameter Distance

– a direct observable if you have a standard ruler of proper length D_R



Curvature is general $K = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$

We assume $\Delta\theta$ is small enough (small angle approximation) that arc length and geodesic ("straight") ruler are the same to negligible error — ~~Some one~~ has looked into this, but not Weinberg – 421–423.

Observable — define this

$$D_A \equiv \frac{D_R}{\Delta\theta}$$

← we know D_R somehow e.g., size scale of BAO.

← we observe $\Delta\theta$

But we have to connect it to theory.

D_R is a length at one instant in time along a tangential curve relative to a ~~—~~ comoving distance $\underline{\underline{r}}$ from us.

4066

from RW metric

$$dD_R = a\,dr\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$$

set to zero by choice of ~~...~~ our polar axis

$$D_R = a r \Delta\theta$$

$$= a_0 r \Delta\theta \left(\frac{a}{a_0}\right)$$

$$= \frac{a_0 r \Delta\theta}{1+z} \quad \text{or} \quad \Delta\theta = \frac{D_R}{a_0 r}(1+z)$$

$$\therefore D_A = \frac{D_R}{\Delta\theta} = \begin{cases} \dfrac{a_0 r(z)}{1+z} \\[2ex] \dfrac{a_0 \chi(z)}{1+z} \quad \text{if } k=0 \\ \quad = \dfrac{c\,F(z)}{1+z} \end{cases}$$

r(z) for light signal connection

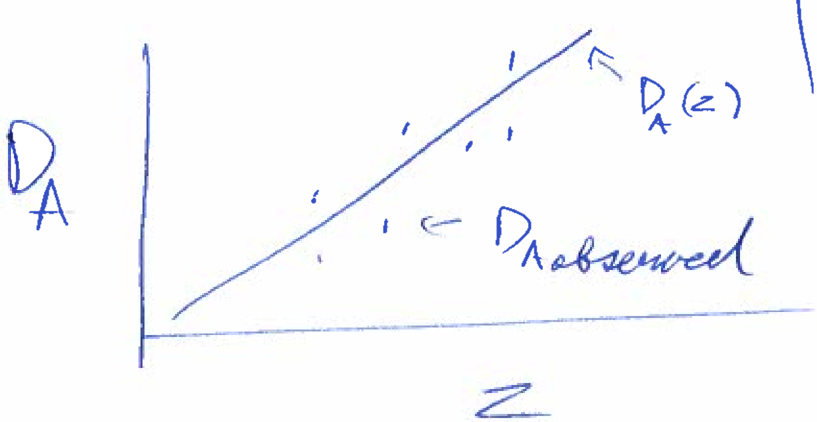
CL-19, Weinberg 422

$a r = \frac{a_0}{1+z} r$

If $k \neq 0$, then $a = a_{0g}$ (see p. 4063 - 4064) is part of fit and a model parameter

and a_0 is not part of the fit — not a model parameter

(by the same argument as on p. 4063)



Small z limit

(see p.4057)

$$D_A = \frac{a_0 r(z)}{1+z} = \frac{a_0 \frac{zc}{a_0 H_0}}{1+z}\left[1 - \tfrac{1}{2}(1+q_0)z + \dots\right]$$

$$= \frac{zc}{H_0}\left[1 - \left(\tfrac{3}{2} + q_0\right)z + \dots\right] \quad \text{to 2nd order}$$

So

$$D^{\text{2nd}}_{\text{proper}} = \frac{zc}{H_0}\left[1 - \tfrac{1}{2}(1+q_0)z + \dots\right] \quad (p.\,4059)$$

$$D^{\text{2nd}}_{L} = \frac{zc}{H_0}\left[1 + \left(\tfrac{1}{2} - q_0\right)z + \dots\right] \quad (p.\,4064)$$

$$D^{\text{2n}}_{A} = \frac{zc}{H_0}\left[1 - \left(\tfrac{3}{2} + q_0\right)z + \dots\right]$$

$$D^{1\text{st}\,z} = \frac{zc}{H_0} \quad \text{in all cases}$$

Since $v_{rec} = H_0 D_{proper}$ in general

see p.4058

$$v^{1\text{st}\,z}_{\cancel{rec}} = H_0 D^{1\text{st}\,z}$$

$$v_{red} = H_0 D^{1\text{st}\,z}$$

The asymptotic Hubble's law proven by Lemaitre 1927

and we less formally show it on p.4056

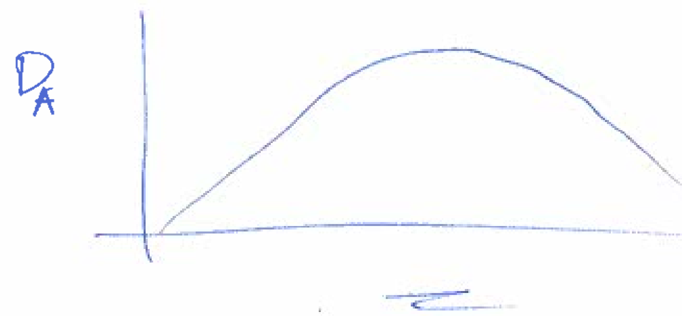
4068)

Curiosity of $D_A = \frac{a_0 r(z)}{1+z}$

is ~~that~~ it doesn't have to grow monotically with z!

$\hookrightarrow$ it can shrink!

and it does for ΛCDM



as an exact example consider ~~the flat~~ de Sitter universe

$$\chi = \frac{zc}{H_0 a_0} = r(z)$$

so $r = \chi$

$$\therefore D_A = \frac{zc}{H_0}\frac{1}{1+z}$$

Plateaus.

What does this mean?

From p. 4066

$$D_R = \frac{a_0 r(z)\Delta\theta}{1+z}$$

$$\Delta\theta = \frac{(z+1)}{r(z)}\frac{D_R}{a_0}$$

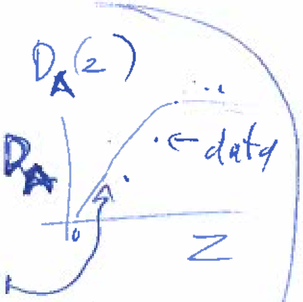
(which fixed) so $\Delta\theta(z)$ can grow.

Standard ruler can start
looking bigger farther away
it is at cosmic present.
↳ recall light
started on cosmic past.
Sort of seeing it when closer.

Two points:

a) It still gets fainter
with z and so fades
from view

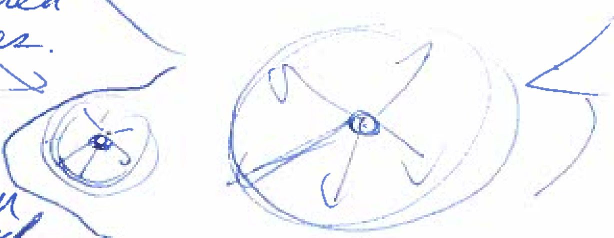
b) The small angle approximation
breaks down if ~~the~~ object is
too big on sky.



Model, so far ΛCDM fits data but much more data to come from Euclid (2023 – 2029 ~ wik) RST (2027 – 2032) (wik)

(Standard Ruler we have BAOs
baryonic acoustic oscillations

The dense regions emitted spherical waves. The acoustic waves stopped at recombination z = 1090 when photons flowed freely. The shells all overlap but impose a scale — small excess in galaxy pairs at D_R



imposed ~~[illegible]~~ ~~str~~ structure
on matter.
Since 2-point correlation
excess at ~~150 Mpc at cosmic present~~

$$D_R = D_{R_0} \frac{1}{(z+1)}$$

~150 Mpc to
~0.15 Mpc at recombination

So standard ruler at
at z
but they shrink to the past.

D_R fiducial value D_{R_0} = 150 Mpc

4070
h)

20230408

Etherington Reciprocity Theorem (Wiki)
AKA Distance Duality Relation

$D_L = a_0 r(z)(1+z)$
(p 4062)

$D_A = \frac{a_0 r(z)}{1+z}$ (p. 4066)

Etherington 1933
but Richard C. Tolman 1881–1948 gave the correct formula and suggested its use in cosmology

$$\frac{D_L}{1+z} = a_0 r(z) = D_A(1+z)$$

or $\frac{D_L}{D_A} \equiv (1+z)^2$ (Applies to general curvature $k = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$)

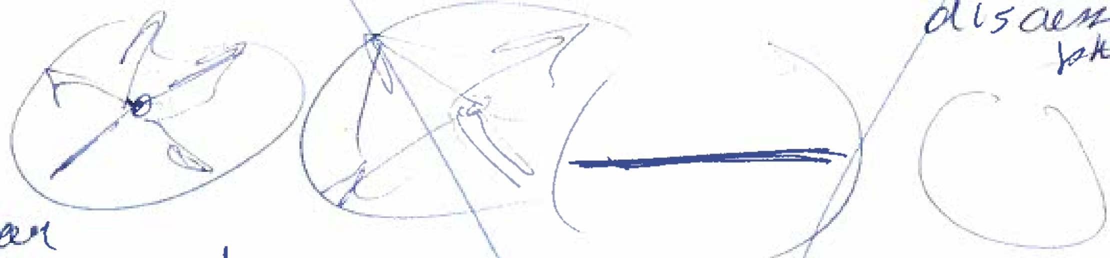
Independent of overall scale and has been verified within error.
Something bad wrong if it failed.
— exotic physics
e.g. photons not conserved
GR wrong etc.

Do we have a standard ruler ~~4044~~ [illegible]

Yes Baryonic Acoustic Oscillation (BAOs)

some early phase imprint (probably discern the physics later when I know more)

kind of cellular nature on Large-Scale structure



$z = 0$

~~$10 z$~~

to $z = z_{recomb} = 11000$?

$$L_{ruler} = 490 \frac{a}{a_0} \text{ Mly}$$

$$= 1\cancel{8}0 \frac{a}{a_0} \text{ Mpc}$$

$$= \cancel{150 \frac{1}{1+z} \text{ Mpc}}$$

scales with a/a_0

ruler then decreased relative to ruler now

Wik

~~But BAOs~~ they expand with universe

— so standard for any z

Not a solid ruler

$$\text{So } D_{BAO} = \frac{L_{ruler}}{\Delta\theta} = \frac{150 \, ruler_0 \left(\frac{1}{1+z}\right)}{\Delta\theta}$$

So have been used to help determine universe model

— so far agree with ΛCDM model

2 ~~9~~ 4076

9 Variational Calculus

In QM, and for geodesics in GR, not min, don't

But there is a Hunk question.

Just stationary Great circle

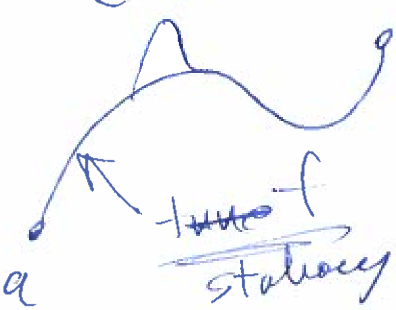
$$I = \int_a^b f(x_i, \dot{x}_i, \ldots, t)\, dt$$

set of coordinate and derivative

path parameter Not necessarily time but could be

find f ~~that~~ makes I stationary

little blip anywhere



→ max, min, inflexion

Won't get global necessarily. Do all possibilities get all: global & locals

~~global or~~ ~~just~~ ~~local~~

$$x_{i\,\text{varied}}(t) = x_i(t) + \alpha\, \eta_i(t)$$

stationary

variation parameter

general function of the coordinate

No variation at end-points

except $\eta_i(a) = \eta_i(b) = 0$

Let's consider $f(x_i, \dot{x}_i, t)$

coord. 1st derivative

This form is good for geodesics & Hamilton principle in classical mech.

40~~74~~ 2017 Sep 22

2 fixed ends
2 ⇒ BCs at two time
but same as 2 BCs at start for 2nd order DE

Uses –
1) Eqn of motion 1 endpoints
2) Geodesics, GR and others
3) Path Integral formulation of QM
4) Fermat's Principle to Reflection/refraction

Path integrals Formulation of QM → particle can be thought of as following all paths but only along stationary ones do waves add coherent – other paths lead to incoherence and cancellation

(Feynman etc.)

But is this just an emergent principle? which true if all paths really were ~~followed~~ but others paths collapse – decoherence.

– long argument
– Anyway a ~ 2000 atom molecule has been diffracted
Fein et al, 2019 Nature physics

4073 | ~~45~~ 55

$$\frac{dI}{d\alpha} = \int_a^b \left[\underbrace{\frac{\partial f}{\partial x_i}\frac{\partial x_{iv}}{\partial \alpha}}_{\eta_i} + \underbrace{\frac{\partial f}{\partial \dot{x}_i}\frac{\partial \dot{x}_{vi}}{\partial \alpha}}_{\dot{\eta}_i} \right] dt$$

use integration by parts

zero at endpoints

$$\left.\frac{\partial f}{\partial \dot{x}_i}\eta_i\right|_a^u - \int \left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)\right]\eta_i dt$$

for stationarity

$$0 = \frac{dI}{d\alpha} = \int_a^b \underbrace{\left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)\right]}_{\text{must be zero for stationary}} \eta_i\, dt$$

Notice total Not partial derivative

Not a constrained stationary path

must be zero for stationary ~~path~~ since η_i is general → any little blip except zero at end point

$$0 = \frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)$$

(a differential equation for each coordinate x_i

→ Called Euler's equation Art-928