~~3.15~~ 3.18 | 2025 jan 5 | 3 20

# The Friedmann Equation

1) Introduction

After a long warm up in gravity, inertial frames, the Cosmological constant force, etc,

## A Standard Form — ~~the~~ most standard form

Not form ~~for~~ finding exact solutions you need scaled form

(there are others)

Li-55

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

Cosmological constant

Lambda

$G = 6.67430(15)\times10^{-11}$ MKS units

The most poorly known fundamental constant (?)

$H = \frac{\dot a}{a}$

Hubble parameter – at cosmic present

$H = H_0 = h_{70}\left(70\ \frac{km/s}{Mpc}\right)$

(The Hubble constant)

It's an inverse time quantity and is the relative rate of ~~E~~ Expansion of ~~the~~ universe

us 1 Mpc — 70 km/s

us 2 Mpc — 140 km/s

$t = \frac{Mpc}{70\ km/s} = \frac{1000}{70} \approx 14\ Gyr$

mean density of all mass-energy here in mass units

$\rho = \rho(a)$

To my knowledge no one considers $\rho$ as an explicit function of time or anything else. What else could it be a function of? A real question

k is a constant of integration called (Li-32) curvature

Can be can ±ve and also $k \to kc^2$ another form

An integral of motion

$k>0$ hypersphere

$k=0$ flat

$k<0$ hyperbolic

But only GR tells us

The solution is $a = a(t)$ cosmic scale factor

$r = r_0 a(t)$, t is cosmic time

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It's a 1st order ordinary, nonlinear differential equation

(nonlinear means solutions do NOT add to ~~a~~ solutions)

autonomous equation (Wik)

→ No explicit dependence on the independent variable cosmic time $t$

[unless you impose an explicit time dependence.]

Being 1st order it has limited abilities to describe universal ~~...~~. It's ~~...~~ about ~~...~~ which require 2nd order DE's ... our usual understanding. But, in fact, the acceleration equation doesn't give any more dynamics, so the symmetries imposed rather than order of the Friedmann equation may be the real dynamic limitation

Physical distances = can be measured with a ruler at one instant in time (Cosmic time)

Other distance measures turn up
luminosity distance
- angular distance
cosmic time itself
- cosmological redshift

$r$

All distances $r$ that participate in the mean expansion of the universe obey

$$r = r_0 \, a(t)$$

$$r(\text{cMpc}) = \frac{r\,(\text{Mpc})}{a(t)}$$

comoving Mpc

Note
$$\dot{r} = r_0 \, \dot{a}(t)$$
$$\ddot{r} = r_0 \, \ddot{a}(t)$$

$r_0$ are the comoving distances
- they are constant in time and by convention equal physical distances at cosmic present $t_0$

Actually, there is a physical way of specifying "$a$" also ~~but~~ and we will consider that when we consider cosmic curvature

Then it has units of length

$a$ has no units and by convention $a(t = t_0) = 1$

cosmic present

When one considers curved space there is form of $a$ with units

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$v = \dot{r} = \dot{a} r_0$

Perhaps universe stranger if they couldn't be added.

However, in a Newtonian physics sense they are sort of the same and can be added and sort of but when bother ~~...~~

recession velocities
— NOT ordinary velocities
— rate of growth of space in a GR sense
— or ~~speed~~ speed between separating inertial frames
(comoving frames as I call them)

Ideal comoving frames, not my CM comoving frames

There is no limit on $v = \dot{r}$

The speed of light is the fastest speed anything physical can move relative to an inertial frame
— light moves at this speed in a vaccuum and it is invariant ⟹ same for all observers ⟹ one of ~~the~~ the axioms of special relativity (SR)

past you at one place in an inertial frame

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In fact, ~~the~~ theoretical Hubble's Law is trivially shown

Friedmann (1922) must have known this ~~...~~ and de Sitter (1917) too, but Lemaître (1927) seems to be the first to explicitly publish it.

$$r = a(t)\, r_0$$

$$v = \dot{r} = \dot{a}\, r_0 = \frac{\dot{a}}{a}\, a r_0 = H r$$

where $H = \frac{\dot{r}}{r} = \frac{\dot{a}}{a}$

i.e. $v = H r$

$v$: true recession velocity

$H$: Hubble parameter at some instant in cosmic time

$r$: physical distance at one instant in cosmic time

~~The observed~~

But $v$ and $r$ are NOT directly observable except asymptotically as $r \to 0$ or cosmological redshift $z \to 0$

That is where Hubble's law is experimentally verified as we'll show ~~later~~ later.

In the asymptotic limit. ~~limit~~

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# The Friedmann equation

is not in most senses a dynamics equation (Equation of motion, Wik

— there are no forces in it.

it's an integral of ~~motion~~ equation ~~with~~ K being the integral itself

curvature and as it's name suggests it describes the curvature of space

$k > 0$ hyperspherical space

$k = 0$ flat space
Euclidean space

$k < 0$ ~~k~~ hyperbolic space

But the Second Friedmann equation (the acceleration equation) doesn't give you any more dynamics, so maybe imposed symmetries (i.e., cosmological principle) that limits its range of behavior

However, ~~its~~ curvature cannot be derived in Newtonian physics, and so we will rely on GR to tell us about curvature.

In fact, the Friedmann equation ~~start~~ is a sort of energy balance equation, but in GR interpretation it loses it's simple meaning I think.

It's a balance of something that is only energy in small scale classical limit sense,

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## ~~2)~~ 3.19 Derivation of Friedmann Eq.

a) GR derivation Alexander Friedmann 1922.

Independently, Georges Lemaître 1927

Einstein himself seems to be the only one to notice both — but in the 1920s he was not interested in cosmology.

In 1934, Milne & McCrea gave a Newtonian Physics derivation

Which requires some special hypotheses only justified ultimately in GR and doesn't give any notion of curvature.

A 19th century physicist without ~~GR~~ and SR could have derived it in essence.

But none did.

If the Friedmann eqns is true, they imply that theory which in fact is GR.

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complete
general theory of gravity and motion under gravity, ~~If~~ ~~They're true they~~ imply it exists, and it does and it's general relativity)

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But we'll try (without being too pendantic) as what that person would have done,
Let's call them Max for James Clerk Maxwell (1831–1879), who could've done it if his mind ever wandered that way.

But we'll do it ideally as he might have presented to an Audience Not with all the meanders of actual research as done.

## b) Motivation and Points for Max

— If the universe is infinite and mostly full of matter (stars ~~to~~ Max, galaxies ~~to~~ us) that on a large enough scale is homogeneous & isotropic (~~the~~ cosmological principle — name invented by e.a. Milne (1935) but in various forms the idea went back a long way, Einstein, de Sitter, Friedmann, Lemaitre even earlier in some sense) what holds it up.

can't infinite, eternal, homogeneous static by Olbers paradox

— Can it be static on average? (in stable equilibrium)

— Newton himself wondered about that and briefly

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and in unpublished work tried to find a universal balance

but he couldn't and no one else either

unstable unless making the system infinite magically stabilized it.

Maybe an extra hypothesis was needed to keep stars at rest on average in Newton's absolute space.

— But also the universe was not in thermodynamic equilibrium

so why should it be static in a dynamical sense?

clear in a thermodynamic sense, but in a more primitive sense known for a long time Olbers Paradox — first discussed by Edmond Halley

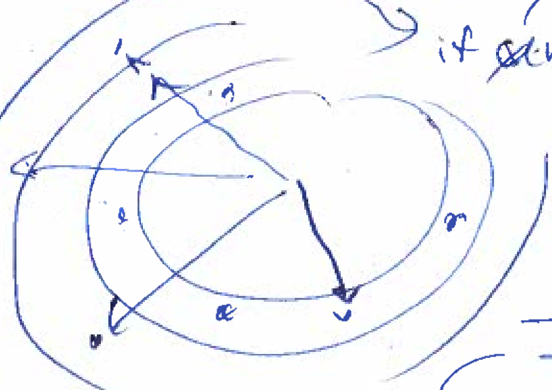
Conclusion sky should be as bright as a star surface but it's Not

if universe static in all senses infinitely old and by cosmol. principle then in every direction you should see a star.

— Area of shells $\propto r^2$

— inverse square fall of light $\propto 1/r^2$

cancelation.



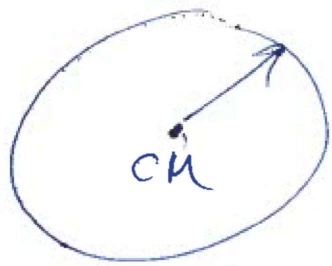
Halley said the idea was due to some one unnamed and presented in 1721 to the Royal Society with elderly Newton presiding (North - 377)

Basic Derivation (2025 Jan 05) 3209

c) So Max hypothesizes an infinite universe with all matter smoothed out into a perfect fluid of uniform density at ~~any time~~ all times.

— No viscosity

— can have pressure, but no gradients since uniform.




Say expanding or contracting everywhere and every point is in free fall under gravity in a sense

Max guesses there must all points ~~all~~ ~~de~~fine inertial frames (comoving frames in the term I use)

Then Max says the shell thm applies at every point

This goes beyond old Newtonian physics

Max guesses there must some more general theory of motion under gravity than Newtonian physics but knows Newtonian physics is a valid limit of it (except for absolute space) and so is trying to generalize from Newtonian physics to capture a portion (portion) of the more general theory e.g., general relat

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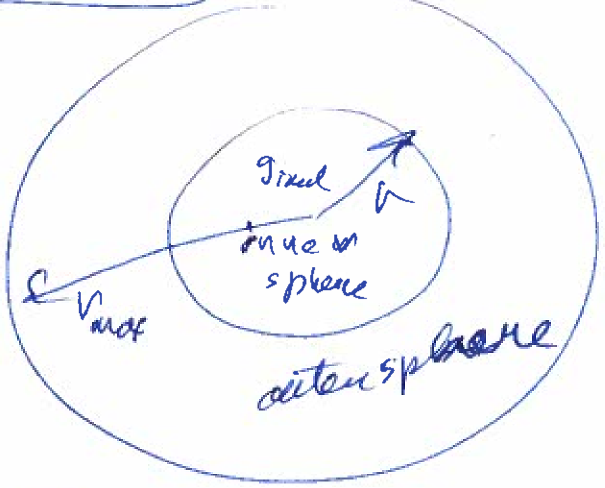
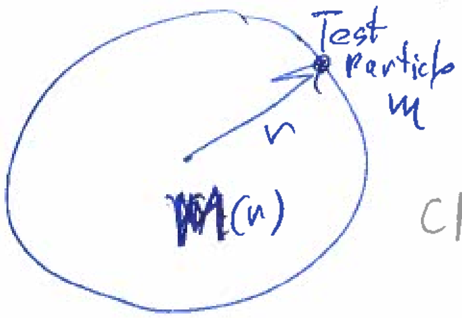



~~ginsid only~~

$g(r)$ only depends on $M(r)$ not on mass beyond — True in Newtonian physics and Birkhoff tells us $r << r_{scale}$ of universe does imply ~~the gas~~ empty spherical cavity is flat Minkowski space in which Newtonian physics can be done ~~if spherical sym~~ velocities and gravity is weak enough.

Then let $r_{max}$ ~~go~~ go to infinity and $g(r)$ stays ~~the~~ same by hypothesis.

Then every point in infinite space allows a CM comoving frame centered on it.



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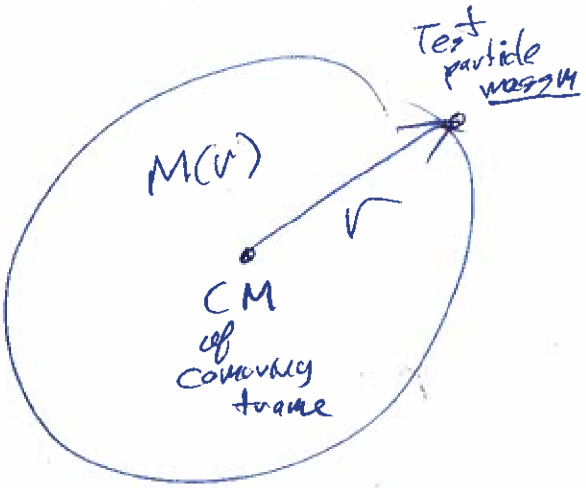
Max could try to apply $F = ma$ to test particle and probably does a lot hunting around, but arrives at a conservation of mechanical energy approach to get an integral of motion.

In fact, we are imagining Max after all kinds of false starts and loops in his thinking




$$\underset{\sim}{g}(r) = -\frac{GM(r)}{r^2}\hat{r}$$

$$V_G = -\frac{GM(r)}{r} = -\frac{4\pi}{3}G\rho r^2$$

assume $M(r)$ initially is conserved as $r$ changes with expansion/contraction of sphere

$$\boxed{M(r) = \frac{4\pi}{3}\rho r^3}$$

given mass conserved we get $\rho \propto \frac{1}{r^3}$

After some playing around Max also includes a $\Lambda$ force field

Max knows the linear force has a symmetry with gravity and it's a single force and so may be universal.

$$\underset{\sim}{f} = \frac{\Lambda}{3} r\hat{r}$$

$$V_\Lambda = -\frac{1}{2}\frac{\Lambda}{3}r^2$$

and also a general PE $U$.

Apply conservation of mechanical energy to a test particle of mass m

$$E = \frac{1}{2}mv^2 - \frac{GM(r)}{r}m - \frac{1}{2}\frac{\Lambda}{3}mr^2 + U$$

$$= \frac{1}{2}mv^2 - \frac{4\pi}{3}G\rho m r^2 - \frac{1}{2}\frac{\Lambda}{3}mr^2 + U$$

$V \to -\infty$ at $r = \infty$ but $v \to \infty$ to conserve E mech

3212 | 2025 jan 04

Now the scaling behavior of the universe cannot depend on the peculiarities of a test particle

$m, r, v, E$

somehow all must vanish.

is called curvature, it is a constant of integration, somehow set as initial condition or boundless (quasi-boundless) universe it is constrained maintain the cosmological principle. Et before g Bang, how?

Max defines $r = a(t)\, r_0$

$\dot{r} = \dot{a}(t)\, r_0$ where $a(t)$ is ~~the universal~~ cosmic scale factor.

Divides three by $\frac{1}{2} m r_0^2$

$$\frac{2E}{m r_0^2} = \dot{a}^2 - \frac{8\pi G}{3}\rho a^2 - \frac{\Lambda}{3} a^2 + \frac{U}{\frac{1}{2} m r_0^2}$$

We ~~demand~~ demand this ~~were a~~ to be a universal constant for all particles or else the equation can't be universal

Max defines

$$k = -\frac{2E}{m r_0}$$

(Lr—24)

The minus is because Max is clairvoyant about GR.

If then this would a universal expression

Expect this spoils things unless it has peculiar properties

~~Say U = V(a)~~

Say $U = V(a)\, m [r^{(n)}]^2$ where $r^{(n)}$ is the nth order derivative of $r$.

We've ~~already~~ already got gravity and the linear force. ~~~~ U would have to be something odd with No obvious motivation from known forces

So Max ansatzes it must be zero.

In fact a reason for the linear force is that it does satisfy the requirements for a universal eqn., it must exist!!!?

Rewriting, we get ~~Friedmann's equation (FE)~~ Friedmann's equation (FE) in standard form

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

K is called the curvature (Liddle-33) But really $-k/a^2$ is how curvy space is. See lecture.

~~If~~ If mass is conserved $\rho \propto \frac{1}{a^3}$ (see p. 3211)

~~Mass Conservation~~ Mass Not Conserved

1) But Max wonders what if mass is not conserved.

What if it mysteriously disappears/appears and what can ~~that~~ depend on in any obvious way? $\longrightarrow$ depends on "a",

Curvy in the sense of how flat at an instant in time measured by ... fixed ... fixed ... volume ... grows further

~~Why~~ Why does Max wonder about mass nonconservation? His annual faculty evaluation at Cambridge Univ. is due and he is desperately procrastinating

Note, nonconservation depending on scale "a", NOT on explicit time or anything else

Max ansatz $\rho = \rho_0 \left(\frac{r_0}{r}\right)^n = \rho_0 a^{-n}$ where $n$ is ~~a~~ a general power.

But what should be the potential energy (or potential)

Max tries $\underset{\sim}{g} = -\frac{4\pi}{3} \frac{G\rho_0 r^3 \left(\frac{r_0}{r}\right)^n}{r^2} \propto r^{1-n}$

$$\therefore V = \frac{4\pi}{3} G\rho_0 r_0^n \frac{r^{2-n}}{2-n}$$

$$= \frac{4\pi}{3} G\rho_0 \left(\frac{r_0}{r}\right)^n \frac{r^2}{2-n}$$

$$= -\frac{4\pi}{3} G\rho r^2 \frac{1}{n-2}$$

This is just like on p. 3211 Except $\frac{1}{n-2}$

and we recover that case if $n=3$.

Not worrying about $n=2$ case which leads to $V$ logarithmic and we are going to dispense with all $n$ except $n=3$ anyway

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But Max says this potential is different for every kind of mass nonconservation (we've considered)

and so leads to a different Friedmann Eqn

[and at this point Max wonders who is Friedmann?]

But mass ~~and it's effect~~ ~~can only~~ must be unique or we have no guidance.

fact, is ansatz leads to sort antiguavity.

So Max ansatzes

$$V = -\frac{4\pi}{3} G \rho r^2$$

for all cases of mass variation (nonconservation)

and $n=3$ which has mass conservation so, See 3211

It works for mass conservation. It must be same in all cases.

"must" in physics means "maybe".

At this point, the Friedmann eqn must be somehow be more general ~~than~~ conservation of mechanical energy in Newtonian physics but whatever ~~more~~ general physics applies, it is still an integral of motion.

Note ... says constant even though mass-energy NOT conserved.

Conservation Equations or Integrals of Motion

Note the Friedmann equ. is a sort of a 'k' conservation equation analoguous to the mechanical energy conservation equation of classical mechanics

The equation is an integral of motion

$$E_{mech} = KE + PE \text{ constant}.$$

Which is not an accident given our derivation (see p. 3206 – 3212).

Consider thrown ball.

Note as usual $g = |\vec{g}|$, $\vec{g} = -g\hat{y}$

$$E = \tfrac{1}{2} m v_y^2 + mgY$$

$$v_y^2 = \frac{2E}{m} - 2gY$$

Note: $2 v_y \cdot a_y = 0 - 2g v_y$

$a_y = -g$ which is a dynamical equation of motion.

And the Friedmann acceleration equation is the analog as we show on p. 3220.

a) Note, for simple 1-d motion, the conservation eqn does give us all the solution just itself.

$$\frac{dy}{dt} = \pm\sqrt{\frac{2E}{m} - 2g y}$$

$$\frac{dy}{\sqrt{\frac{2E}{m} - 2gy}} = \pm dt$$

$$\frac{2\sqrt{\frac{2E}{m} - 2gy}}{-2g} = \pm \Delta t$$

$$\frac{2E}{m} - 2gy = (g\Delta t)^2$$

$$y = \frac{E}{mg} - \tfrac{1}{2} g \Delta t^2$$

3216

Say $y(t=0)=0$ and $y_{max} = \frac{E}{mg} = \frac{\frac{1}{2}mv_0^2}{mg}$

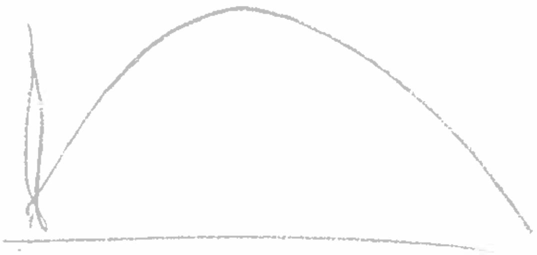
$$0 = \frac{1}{2}\frac{v_0^2}{g} - \frac{1}{2}g(0-t_0)^2 \qquad = \frac{1}{2}\frac{v_0^2}{g}$$

$$t_0 = \frac{v_0}{g}$$

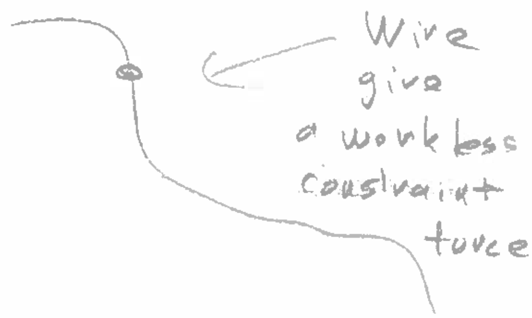
$\therefore \; y = \frac{1}{2}\frac{v_0^2}{g} - \frac{1}{2}g\left(t - \frac{v_0}{g}\right)^2$ is the whole solution given initial condition.

b) But what if not simple 1-d motion?

Trajectory 2-d motion



Bead on a frictionless wire




In both cases, there a broader range of solutions and you need more information to solve

$E = KE + PE$ is NOT sufficient.

c) The Friedmann equation is it seems like case, It's strictly 1-d and seems no place for workless constraint forces

The expanding sphere derivation seems to be like case (a).

So the Friedmann eqn. gives all solutions itself and the Friedmann acceleration eqn. just gives acceleration $\ddot{a}$ for given "a" without knowing solutions. (see p. 3220)

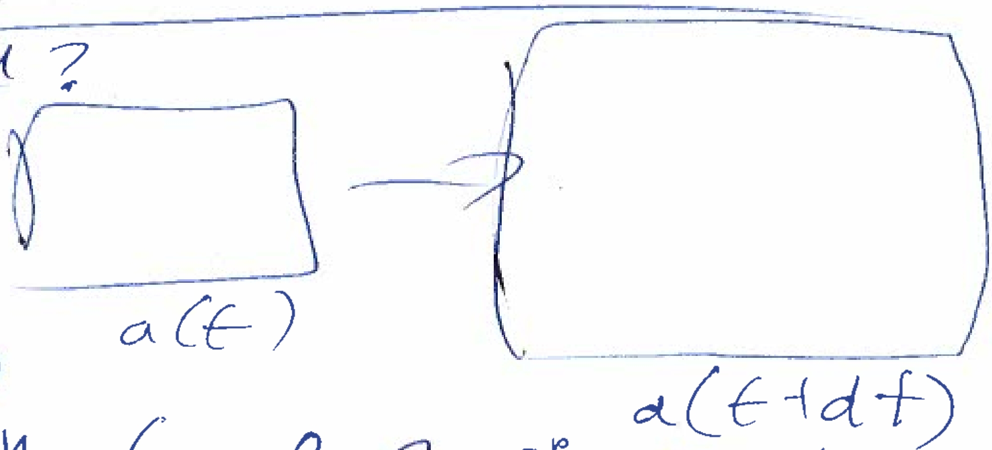
fact we are ... tions the acceleration equation that are NOT solutions of Friedmann eqn. These ... keep ...

But say $v_2^2 = \frac{2E}{m(y)} - 2gy$; i.e., you let mass be a function of y. Then the solutions are vichon and this the case of the Friedmann eqn.

(... 3212) and are cosmic scale factor solution

3.2.1 Mass-Energy Nonconservation | 2025 jan 05 | 3.2.1

But where does mass nonconservation arise from?




Well ~~the~~ form assumed is $\rho = \rho_0 a^{-n}$ (or $\Omega = \Omega_{0p} x^{-p}$ in terms of J 2026 (My review paper)) says changes ~~with~~ volume,

What changes with volume

Being one of ~~the~~ developers of 19th century statistical ~~classical~~ mechanics Max knows ~~the~~ 2nd law of Thermodynamics (Wik: 2nd law / 1st law Energy available)

$$dE = TdS - PdV + \mu dN$$

thermal (internal) energy; Temperature; entropy; pressure; Volume; chemical potential; particles

Actually Probably some clairvoyance fog this is needed by Max ... ↓ but we know about entropy somewhat (Wik)

Differential form where denominator is anything, but usually time $dt$

So energy can change with volume (an inside ~~thing~~ source)

And Max doesn't want heat energy or particles from outside because there's no outside in an infinite universe

3218 (2025jan05)

But says Max, this is the 19th century, and so mass and energy are different things.

<u>But</u> Max also discovered Maxwell's equations of electromagnetism and derived $c \triangleq 3 \times 10^8$ m/s as the unique velocity of electromagnetic ~~waves~~ waves

and $mc^2$ has dimensions of energy like $\frac{1}{2}mv^2$,

and so Max say's let's assume mass is a form of energy; ~~[illegible]~~ $E = mc^2$ at least for perfect fluids

...el ...ere ...s ...me ...nd ...f Perfect fluid stuff with pressure of some kind

and so with $dS=0$ (adiabatic) and no chemical potential ~~or~~ particles

$$c^2 dM = dE = -P\,dV$$

$$d(\rho V) = -\frac{P}{c^2}\,dV \quad \text{and} \quad V \propto a^3$$

$$V\,d\rho + \rho\,dV = -\frac{P}{c^2}\,dV$$

$$d\rho = -\frac{dV}{V}\left(\rho + \frac{P}{c^2}\right)$$

$$d\cancel{V} \;\; \frac{da}{a}$$

$$dV \approx 3a^2\,da$$

$$\frac{dV}{V} \approx 3\frac{da}{a}$$

...mc² ...the ...n ...fluid ...ing ...ae.

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$$d\rho = -3\frac{da}{a}\left(\rho + \frac{p}{c^2}\right)$$ < $\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$

Now what says Max.
Simplest idea → simplest equation of state (EOS)

$$p = w c^2 \rho$$

where $w$ is a constant

(just the $w$ parameter of the cosmological perfect fluid in modern astro jargon — No other name it seems)

Note, this pressure can be formal. It doesn't have to push/pull on anything including itself and the form of the energy it destroys or creates could be anything e.g., particle energy, particles

$$\therefore \quad d\rho \propto -3\frac{da}{a}(\rho + w\rho)$$

$$\frac{d\rho}{\rho} = -3\frac{da}{a}(1+w)$$

$$\ln\rho = -3(1+w)\ln a$$

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$$

of Fulvio Melio

The assessment specified possibilities,

$\rho_c\left(\frac{a_0}{a}\right)^3$

$\rho\left(\frac{a_0}{a}\right)^4$

$\rho_0$

$\rho_0\left(\frac{a_0}{a}\right)^2$

NR matter

$w = 0$ "matter"

ER matter

$w = \frac{1}{3}$ "radiation" (as well groove lattice)

$w = -1$ constant dark energy or cos. constant

$w = -\frac{1}{3}$ $R_h = ct$ universe

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3.22

# The acceleration Equation or 2nd Friedmann Eq.

2nd order DE, but it does NOT give any behavior NOT in the Friedmann equation itself. So dynamics is limited. See p. 3216 conservation Equations

From p. 3213

$$\dot{a}^2 = \frac{8\pi G \rho a^2}{3} - \cancel{k} + \frac{\Lambda}{3} a^2$$

Differentiate

curvature term vanishes. It is

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left[\dot{\rho} a^2 + 2\rho a \dot{a}\right] - 0 + 2\frac{\Lambda}{3} a\dot{a}$$

$-3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right)$

Without the pressure and $E = mc^2$, Max, EOS would not know what $\dot{\rho}$ is except for $n = 3$.

It was a constant ~~of~~ of integration ~~fudge~~ ~~integral~~ set ~~equation~~. before big era, before inflation, but now?

divide thru by $2\dot{a}$

$$\ddot{a} = \frac{4\pi G}{3}\left[-3a\left(\rho + \frac{P}{c^2}\right) + 2\rho a\right] + \frac{\Lambda}{3} a$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho + \frac{3P}{c^2}\right] + \frac{\Lambda}{3} \quad \text{(Li-27)}$$

If $P = w\rho c^2$, $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho[1 + 3w] + \Lambda/3$

or with $r = r_0 a$ ~~and $r_0$ cancelled~~ substituted and $r_0$ cancelled out.

$$\ddot{r} = -\frac{4\pi G}{3}\left[\frac{\rho r^3}{r^2} + \frac{3P}{c^2} r\right] + \frac{\Lambda}{3} r$$

$$\ddot{r} = -\frac{GM(r)}{r^2} - \frac{4\pi G}{3}\left(3\frac{P}{c^2}\right) r + \frac{\Lambda}{3} r$$

shell theorem force for spherical symmetric mass distribution. | what? | $\Lambda$-force

far as know, the acceleration useful to with exists and is way know without knowing $a(t)$, p. 3216 conservation equations 3215 or acceleration ... or ... tions, ... on 3221 allows to analyze which terms ... acceleration, coasting, deceleration.

Max wonders what the heck is the middle term

You do not get that from Newtonian physics

But recall he introduced non conservation of mass (p. 3213)

introduced a special potential energy (p. 3214),

ansatzed $E = mc^2$ (p. 3218),

did thermodynamics (p. 3218),

and introduced a special equation of state (p 3219)

$$P = wc^2\rho$$

which led to $\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$ (p. 3219)

Now

$$\dot{a}^2 = \frac{8\pi G a^2}{3}\sum_i \rho_{0i}\left(\frac{a_0}{a}\right)^{3(1+w_i)} - k + \frac{\Lambda}{3}a^2 \quad \text{(p. 3220)}$$

and we can find $\ddot{a}$ without explicit pressure

$$a^2\left(\frac{a_0}{a}\right)^{3(1+w_i)} = a_0^{3(1+w_i)}a^{-(1+3w_i)}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\sum_i \rho_{0i}\left(\frac{a_0}{a}\right)^{3(1+w_i)} a\left[-(1+3w_i)\right]\dot{a} + 0 + \frac{2}{3}\Lambda a\dot{a}$$

$$\ddot{a} = -\frac{4\pi G}{3}\sum_i \rho_{0i}\left(\frac{a_0}{a}\right)^{3(1+w_i)} a\left[(1+3w_i)\right] + \frac{\Lambda}{3}a$$

A real dynamics equation of motion.

Define $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ which is constant and $w_\Lambda = -1$

$$\ddot{a} = -\frac{4\pi G}{3}\left[\sum_i \rho_{0i}\left(\frac{a_0}{a}\right)^{3(1+w_i)}\left[(1+3w_i)\right]\right]a$$

$$\frac{4\pi G}{3}\left[\frac{\Lambda}{4\pi G}\right] = \frac{4\pi}{3}\left[2\rho_\Lambda\right] = \frac{4\pi}{3}\rho_\Lambda\left[-(1+3w_\Lambda)\right]$$

with $w_\Lambda = -1$

So for the terms:

$1 + 3w_i > 0$ gives deceleration

$1 + 3w_i = 0$ or $w_i = -1/3$ gives coasting

$1 + 3w_i < 0$ gives acceleration

and recall from p. 3219

$p = 3(1+w)$ where $p$ is power NOT pressure

$w = P/3 - 1$

| $w$ | $p$ | |
|---|---|---|
| $2/3$ | $p=5$ | Nonrelativistic KE density & free streaming particles |
| $1/3$ | $p=4$ | radiation (extreme relativistic particles) |
| $0$ | $p=3$ | matter (rest mass energy) |
| $-1/3$ | $p=2$ | $R_a = ct$ universe or curvature or cosmic strings |
| $-2/3$ | $p=1$ | quintessence of some kind |
| $-1$ | $p=0$ | cosmological constant or constant dark energy perfect fluid |

Note as Google AI tells us and as we know from examples (see p. ), the acceleration eqn can have solutions that do NOT satisfy the Friedmann eqn itself, and so can't have a constant $k$, the requirement for a universal solution (see p. 3212). Such solutions are NOT therefore are NOT cosmic scale factor solutions. But for a given "a" value, the acceleration eqn gives the correct $\dot{a}$ and $\ddot{a}/a$.

a(t) satisfying the Friedmann eqn. is necessary & sufficient for being a cosmic scale factor, satisfying the acceleration eqn. is necessary and sufficient.

Note $1 + 3w < 0$ or $p < -\frac{1}{3}$ gives an acceleration which sort of an antigravity. In the classical interpretation of the Friedmann eqn, these terms here "Potential energy" decreasing as "a" → ∞ and so to keep the balance we have "$\dot{a}$" (~ kinetic energy) increasing to ∞.

But what does pressure mean in

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$$ (see p. 3217-3218)

Perhaps, No more than in a adiabatic perfect fluid ($dS=0$) with chemical potential $= 0$ or $dN = 0$ or both

$$dE = -PdV$$

Ordinarily, we ~~think of~~ pressure doing $PdV$ work, but here it can be just some kind of coefficient ~~that~~ controls changes in mass-energy with volume.

Consider photons: ~~as~~ we show in Lecture 4, $E_{ph} \propto \frac{1}{a}$

and $E_{ph} \propto \frac{1}{a^4}$
energy density

but the free streaming photons are not pushing on anything (below the level of pair creation),
There energy seemingly vanishes.

## 3.23 The Cosmological Constant and Dark Energy

"What of dark energy?

It's taken as uniform in boundless space

$$P = w\rho_c = -\rho_c$$

$w = -1$ (see p. 3222)

and so that suggests a "pulling in force", but what does this "force" act on.

— Itself?
— It's uniform, so no pressure gradients
— maybe it doesn't pull on anything.
— maybe just a formal negative pressure

In GR, the Einstein field Equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein tensor

Cosmological constant

metric tensor

stress-energy Tensor

DE true at all points in space above quantum gravity level

In this form, $\Lambda g_{\mu\nu}$ is an aspect of gravity.

Schrödinger's idea had no immediate impact (Einstein didn't think much of it) but Georges Lemaitre used the idea in 1934 (WiK: Vacuum energy)

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But it was noticed by none other than Erwin Schrödinger in 1918 ~~that for~~ a perfect fluid (no viscosity) (O'Raifeartaigh et al 2017, p. 32)

Carroll -172

$$T_{\mu\nu} = (\rho + P/c^2) U_\mu U_\nu + P g_{\mu\nu}$$

(perfect fluid)

4-velocities

with $P_{(PF)} = -\rho_{(PF)} c^2$ would give

$$\underset{(PF)}{T_{\mu\nu}} = 0 - \rho g_{\mu\nu}$$

Carroll-172 says $\rho_{PF}$ like this must be constant throughout space-time to be Lorentz invariant

and so

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

$$= \frac{8\pi G}{c^4}\Big(T_{\mu\nu} - \underbrace{\frac{\Lambda c^4}{8\pi G}}_{\rho_{PF}} g_{\mu\nu}\Big)$$

$$= \frac{8\pi G}{c^4}\Big(T_{\mu\nu} + \underset{PF}{T_{\mu\nu}}\Big)$$

Quantum Field Theory likes a perfect fluid like this that is vacuum energy, but it's simple predictions give $\Lambda_{vac} \cong 10^{120}\, \Lambda_{observed}$ which is a significant overestimate

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Of course, you can tune $\Lambda_{vacuum}$ to you what is needed and just say it's Anthropic i.e., we would NOT have a chance being here if $\Lambda_{vac} \neq \Lambda_{obs}$, but nobody likes Anthropic arguments much — they always seem to dissipate into endless philosophical musing — of course, they could be true for $\Lambda$, but we have no guidance on the distribution of $\Lambda$ in a multiverse.

But AI threatens to solve quantum gravity but perhaps with a theory beyond human comprehension

Of course, a true Cosmological constant and a true constant dark energy (a true vacuum energy?) must be different somehow potentially observably though maybe indirectly. (since $\rho = \rho_0 x^{-p}$ is now allowed for all $p \geq 0$ not just $p = 3$ and $p = 0$)

Actually, I like a true cosmological constant since only it and gravity seem to be allowed as universal forces by the Newtonian derivation of the Friedmann equ. (see p. 3211) allowing mass non-conservation makes this feature not so clear. Of course, there could be constant or varying dark energy and a cosmological constant. After all, who needs Occam's razor.

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~~3225~~ 3227

There must be some observable differences between a true cosmological constant and a dark energy perfect fluid.

But I haven't ~~noticed~~ looked for any discussion.

Note gravity and cosmological constant force have ~~that~~ odd symmetry (see p. 3053) we've discussed and seem to be the only non-exotic forces allowing the Friedmann equation (see p. 3212)

So $\Lambda$ or ~~dark~~ energy perfect fluid.

The question is moot.

Both are just called $\Lambda$ conflating the two concepts.

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## 3.24 Scaled Form of the Friedman Equation

a) $$H^2 = \left(\frac{\frac{da}{dt}}{a}\right)^2 = \frac{8\pi G}{3}\underbrace{\rho_{ME}}_{\text{mass-energy}} - \underbrace{\frac{k}{a^2}}_{\text{curvature term}} + \frac{\Lambda}{3}$$

I prefer $a$ to a. "a" looks like a constant to me.

Choose a fiducial time labeled by $t_0$ which is usually cosmic present, but it could be any time.

At $t_0$, we have $H = H_0$, $a = a_0$

Usually set to 1, but there are other useful choices.

Note, $\frac{H^2}{H_0^2}$ is dimensionless

$$\therefore H^2 = H_0^2 \underbrace{\frac{8\pi G}{3H_0^2}}_{\therefore\ \text{dimensionless}}\rho_{ME} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

We define $\rho_c = \rho_{critical} = \frac{3H_0^2}{8\pi G}$

$$[\rho_c] = \frac{T^{-2}}{EL/M^2} = \frac{T^{-2}}{\frac{ML^2}{T^2}L/M^2} = \frac{M}{L^3} \text{ correct dimensions}$$

$$H^2 = H_0^2\left[\frac{\rho_{ME}}{\rho_c} - \frac{k}{H_0^2 a^2} + \frac{\Lambda}{3H_0^2}\right]$$

We define density parameters (representations of density components)

$$\Omega_{ME} = \frac{\rho_{ME}}{\rho_c}$$

minus sign to make all $\Omega$'s explicitly positive

$$\Omega_k = -\frac{k}{H_0^2 a^2} \text{ and so } \rho_k = \rho_c\Omega_k = -\frac{\rho_c k}{H_0^2 a^2}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2} \text{ and so } \rho_\Lambda = \rho_c\Omega_\Lambda = \frac{\Lambda}{8\pi G}$$ as on p. 3221

total $\Omega = \Omega_{ME} + \Omega_k + \Omega_\Lambda$

$= \Omega_{ME\Lambda} + \Omega_k$ with $\Omega_{ME\Lambda} = \Omega_{ME} + \Omega_\Lambda$

$$\therefore H^2 = H_0^2[\Omega] = H_0^2[\Omega_{ME\Lambda} + \Omega_k]$$

at $x_0$ where $H = H_0$, we have $1 = \Omega_{ME\Lambda 0} + \Omega_{k0}$

$$\therefore \Omega_{k_0} = 1 - \Omega_{ME\Lambda 0}$$

If $\Omega_{ME\Lambda} = 1$, $\Omega_k = 0$

If $\Omega_{ME\Lambda_0} \neq 1$, $\Omega_{k_0} \neq 0$

But only general relativity tells us this

If $\Omega_{k_0} > 0$, $k < 0$ and there is -ve curvature (hyperbolic space)

If $\Omega_{k_0} < 0$, $k > 0$ and there is +ve curvature (hyperspherical space, finite boundless space

The reversal of signs is annoying. Just accept it. Lesser of two evils

so $\Omega_{ME\Lambda} = 1$ is *flat* space or $\rho_{ME\Lambda} = \rho_c$

b) $\rho_{critical} = \dfrac{3H_0^2}{8\pi G} = \begin{cases} (0.920374\ldots \times 10^{-26})\,\text{kg/m}^3 \left(\frac{H_0}{70}\right)^2 \\ (0.135\,7834\ldots\, 10^{12})\,\frac{M_\odot}{\text{Mpc}^3} \left(\frac{H_0}{70}\right)^2 \end{cases}$

$\Longrightarrow \sim\frac{1}{10}$ mass of a large galaxy per $\text{Mpc}^3$

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As we know, evidence (from some analysis) points to $\Omega_{ME\Lambda 0} \simeq 1$

Planck 2018 give $|\Omega_{k}| < 0.005$ (Google AI),

but some people have estimated higher and lower values.

Standard Inflation predicts $\Omega_{ME\Lambda}$ very close to 1,

but not exactly 1, $|\Omega_{k0}| \lesssim 10^{-5}$ (Google AI)

Recall in Friedmann equation $H^2 = \ldots - \frac{k}{x^2} \ldots$

and the super expansion of inflation flattens the inflation region.

Driven by something that acts like a super $\Lambda$.

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lecture 4.
as $\frac{k}{x^2}$
s smaller, the 'curviness'
: measure with a fixed
llen by ~~E+M~~ forces

But how is k set?

can it be changed?

Apparently, Not for a homogeneous, isotropic boundless universe in GR

c) Eliminating $H_0$ from the Formal solution

$$H^2 = H_0^2 \sum_i \Omega_i(x)$$

{ Sum on all different density components including curvature and $\Lambda$

Let $d\tau = H_0\,dt$ and use scaled time $\tau$ where $\tau_0 = H_0 t_0$.

$$h^2 = \left(\frac{\frac{dx}{d\tau}}{x}\right)^2 = \sum_i \Omega_i(x)$$

{ Note $H(\tau_0) = H_0$ and so $h(\tau_0) = 1$

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_i \Omega_i(x) \quad \text{and} \quad 1 = \sum_i \Omega_i(x = x_0)$$

where I usually mean $\frac{dx}{d\tau} = \dot{x}$

Say there are I density parameter constants (where we assume each density parameter is controlled by one density parameter constant e.g. $\Omega_i = \Omega_{i0}\left(\frac{x_0}{x}\right)^{p_i}$.

There can be other density parameter formulae that are NOT (inverse) power laws, but I'm not going to consider them.

e.g., CPL density parameter from $w(x) = w_0 + w_a(1-x)$ where $x = \frac{1}{1+z}$

$\frac{\rho_{DE}}{\rho_{DE_0}} = x^{-3(1+w_0+w_a)} e^{-3w_a(1-x)}$

(Abdul-Karim 2005 p. 5)

Power-law and exponential dependence on $x$

There is some physical motivation

Note, we have effectively replaced one of the density parameter constants as a free parameter, so I-1 ~~free density~~ parameter constants plus $H_0$, make I controlling parameters.

Note a density parameter constant is a density-parameter parameter, but that sounds obscure.

But we make the $H_0$ disappear by scaling time: e.g. $H_0 dt = d\tau$.

So solving $h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_i \Omega_i(x)$ gives a scaled solution which may be all you want for studying solution behavior.

$h$ is scaled Hubble parameter where $h = h_0 = 1$ at $\tau_0$

But if you want real time behavior, then you have to know or set $H_0$.

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d) Actually, you can set $\rho_c$ as a controlling parameter,

then $\rho_c = \frac{3H_0^2}{8\pi G}$

Note, the Hubble time is Not $t_0$ in general

given $\sqrt{\frac{8\pi G \rho_c}{3}} = H_0$ or $t_{\mathrm{H}_0} = \sqrt{\frac{3}{8\pi G \rho_c}}$

In this case, $\rho_c$ sets the time scaling $H_0 dt = d\tau$.

But in any case, $t_0$ ~~or~~ $\tau_0$ is usually an output: i.e., a part of the solution.

So $h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_i \Omega_i(x)$

is solved for $\tau(x)$ and/or $x(\tau)$,

and $\tau' = \tau(x_0)$ or $x_0 = x(\tau_0)$

is part of the solution.

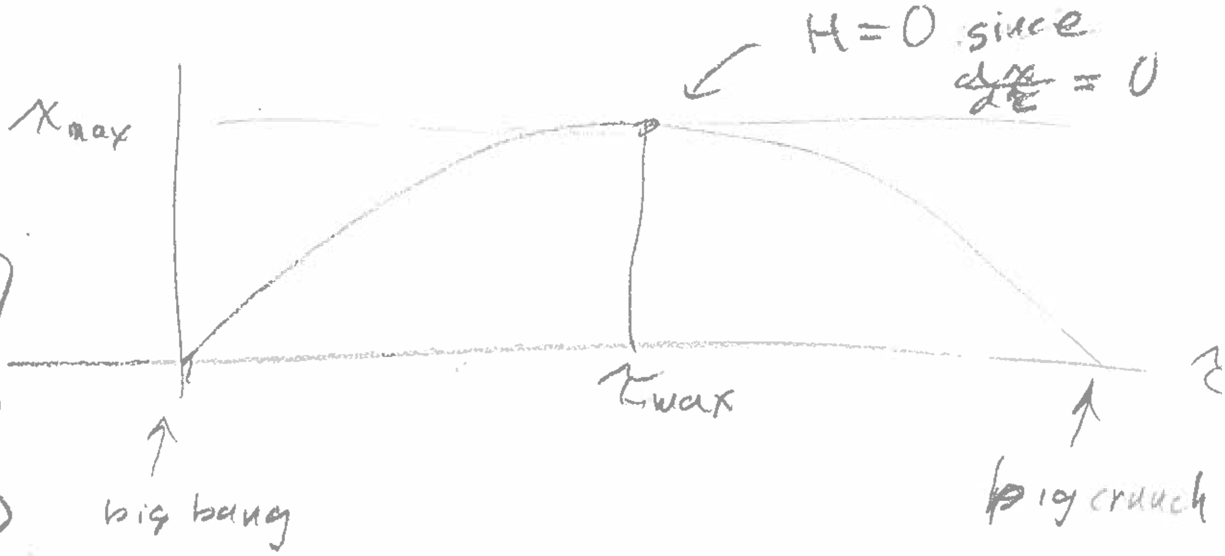
e) Usually $x_0$ is set by $x_0 = 1$

$\therefore \left(\frac{\dot{x}}{x}\right)^2 = 1 = \sum_i \Omega_i(x_0 = 1)$

But this is NOT the only useful choice.

For example, if $\Omega_{K0} < 0$, you can solutions with a maximum

We consider these solutions in lecture 5 on advanced Friedmann equation solutions




In this case, you may prefer to choose $x_{max}(\tau_{max}) = 1$ where $\sum_i \Omega_i (x = x_{max} = 1) = 0$

Which means $x_0 \neq 1$

$x_0$ is still where $H = H_0$

But $\sum_i \Omega_i (x_{max} = 1) = 0$ means $\Omega_{ME\Lambda} > 0$ and $\Omega_K < 0$ at $\tau_{max}$

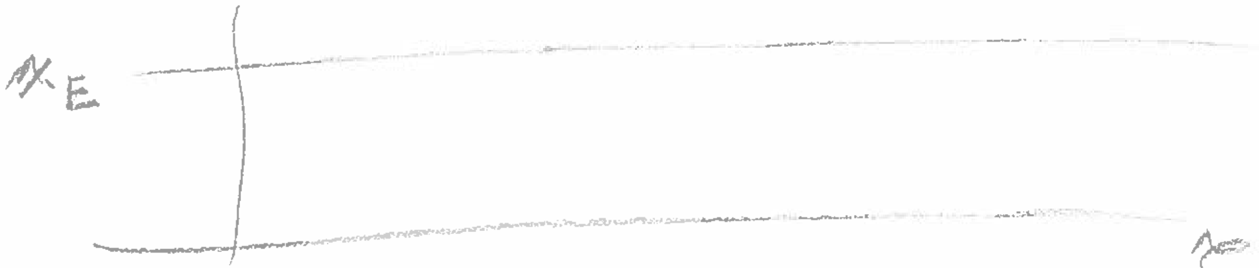
Then you could set $\rho_{critical} = \underbrace{\rho_{ME\Lambda\,min}}$

then $H_0 = \sqrt{\dfrac{3}{8\pi G \rho_c}} = \sqrt{\dfrac{3}{8\pi G \rho_{ME\Lambda\,min}}}$ Which occurs at maximum expansion.

In this case, $h^2 = \left(\dfrac{\dot{x}}{x}\right)^2 = 1 = \sum_i \Omega_i (x = x_0 \neq 1)$

Note $h = 1$ and $\tau = \tau_0$ may never occur in the solution. { See p. 3230 definition of $\tau_0$: i.e., $h(\tau_0) = 1$.

Also there is the Einstein (static) universe



In this case $h^2 = \left(\frac{\dot{x}}{x}\right)^2 = 0 = \sum_i \Omega_i \,(x = x_E)$

where $x_E$ can be set $x_E = 1$.

~~In this case~~, $h = 0$ always and never 1, and thus there is no $\tau_0$ (see p. 3230 where $h(\tau_0) = 1$)

## 3.2.5 The Deceleration Parameter $q$

$q$ is a sort of scaled form of the acceleration equation (p. 3220) (but probably not as useful as that even) with an annoying historical minus sign.

$$q = -\frac{\frac{d^2x}{dt^2}}{x}\,\frac{1}{H^2} = -\frac{\frac{d^2x}{dt^2}\,x}{\left(\frac{dx}{dt}\right)^2} = -\frac{\ddot{x}\,x}{(\dot{x})^2}$$

Hubble parameter

$$= -\frac{\ddot{x}}{x}\,\frac{1}{H^2/H_0^2} = -\frac{\ddot{x}}{x}\,\frac{1}{h^2}$$

Where $\ddot{x} = \frac{d^2x}{d\tau^2}$

and $\dot{x} = \frac{dx}{d\tau}$

and recall

$H_0\,dt = d\tau$

Hubble constant for some. See p. 3230

Why the minus sign? For the 1930s - 1990s most assumed that the universe was decelerating and to get a positive quantity they added a minus sign. Bad mistake.

In 1970, Allan Sandage wrote an article
"Cosmology & the search for Two numbers"
They hypothesized in those days that
$H_0$ and $q_0$ might be enough
to determine $x(t) = x(\tau)$
and $\tau_0$ and $t_0$ for the actual universe.
We'd still like to know them
to high accuracy of course,
but nowadays $q(z)$ is probably
most seen as diagnostic for
overall solutions (e.g., Lodha 2025
"Extended Dark energy
analysis using DESI DR2
BAO measurements)

Let's hypothesize

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_i \Omega_{i0} x^{-p_i}, \text{ where } x_0 = 1$$

$$\therefore \quad \dot{x}^2 = \sum_i \Omega_{i0} x^{-p_i+2}$$

$$2\dot{x}\ddot{x} = \left[\sum_i \Omega_{i0} x^{-p_i+1}(-p_i+2)\right]\dot{x}$$

See p. 3234

$$\ddot{x} = \sum_i \Omega_{i0} x^{-p_i+1}(-p_i/2+1)$$

$$q = -\frac{\ddot{x}x}{(\dot{x})^2} = \frac{\sum \Omega_{i0} x^{-p_i+2}(p_i/2-1)}{\sum_i \Omega_{i0} x^{-p_i+2}} = \frac{\sum_i \Omega_{i0} x^{-p_i}(p_i/2-1)}{\sum_i \Omega_{i0} x^{-p_i}}$$

$$q_0 = \sum_i \Omega_{i0}(p_i/2 - 1) \quad \text{with } x = x_0 = 1$$

$$0.3(3/2-1) + 0.7(-1)$$

$$= 0.15 - 0.7 = -0.55$$

a well known result

for $\Lambda$-CDM model
with $\Omega_{30} = \Omega_{M0} = 0.3$
and $\Omega_0 = \Omega_\Lambda = 0.7$

Now again (from p. 3235)

$$q = \frac{\sum_i \Omega_{i0} x^{-p_i}\left(\frac{p}{2} - 1\right)}{\sum_i \Omega_{i0}\, x^{-p_i}}$$

Note $p/2 < 1$, or $p < 2$,
gives a negative term and thus
a positive accelerating effect.
A sort of antigravity

$p = 2$ gives a coasting effect

$p > 2$ gives a positive term
and so a deceleration

# 3.26 Elementary Solutions of the Friedmann Equation

## Single density component solutions

$$h = \frac{\dot{x}}{x} = \pm\sqrt{\sum_p \Omega_{p_0} x^{-p}}$$

Recall $\dot{x} = \frac{dx}{d\tau}$ and $H_0 dt = d\tau$ and here we choose $x_0 = 1$. (Note we do not have to make this choice. See p 3232)

where $p \geq 0$ but not necessarily an integer, but physically only integer $p$ seem well motivated.

Recall, $\Omega_p = \Omega_{p_0} x^{-p}$ is a density parameter. $\Omega_{p_0}$ is a density parameter constant.

Note at $\tau_0$ where $x_0 = 1$, $h = h_0 = 1$ and $\sum_p \Omega_{p_0} = 1$,

If only one density component then $\Omega_{p_0} = 1$.
But for our derivation, we will leave $\Omega_{p_0}$ unevaluated so that we can see where it is.

Note, ~~the~~ $\pm$ means there are always expanding/contracting branches,
But unlike most autonomous 1st order ordinary DEs, the Friedmann equation solutions

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... can have stationary points at finite time. See p. and HomeWk 3 for a discussion of stationary points of autonomous 1st order ordinary DEs..

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However, physically $x \le 0$ is not allowed. No meaning. However, we do formally include $x=0$ as a limit in discussions and formalism.

But with very implausible density components, you can get an a pure Friedmann Eqn. cyclic universe:

Awkward to exclude it.

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p. 82) just
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new cycle.
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avior into Friedmann eqn
use ~~ex~~ generalized Friedmann Eqn..

Never goes to zero


Since we live in an expanding universe, the negative branch solutions have less interest and we won't consider them here.

o solve
write
the
DE
as

$$d\tau = \frac{dx}{x\sqrt{\sum_p \Omega_p x^{-p}}}$$

$$d\tau = \frac{dx}{\sqrt{\sum_p \Omega_p x^{-p+2}}}$$

In fact, always numerically and (I think) always analytically you solve for $\tau(x)$ first and invert to get $x(\tau)$. NOT all exact $\tau(x)$ can be inverted to exact $x(\tau)$.

(2024 jan 24)



a) For a single density component $p \neq 0$

$$d\tau = \frac{dx}{\sqrt{\Omega_{p0}}} \frac{1}{x^{-p/2+1}} = \frac{x^{p/2-1}\,dx}{\sqrt{\Omega_{p0}}}$$

solution $\tau = \frac{1}{\sqrt{\Omega_{p0}}} \frac{x^{p/2}}{p/2}$ choosing $\tau(x=0)=0$ for a Big Bang singularity origin or a point origin in older jargon, A point in time origin is meant.

age of universe $\tau_0 = \frac{2}{p}$ with $x=1$, $\Omega_{p0}=1$

If you measure $H_0$ and $q_0$ (which gives $p = 2(q_0+1)$), then you know $t_0$, the age of the universe

$$t_0 = \frac{2}{p}\frac{1}{H_0}$$

$$t_0 = \frac{2}{p} t_{Hubble} \text{ and so } t_{Hubble} = \frac{p}{2} t_0$$

The Hubble time. Note, only for $p=2$, do we have $t_0 = t_{Hubble}$

Inverse solution

$$x(\tau) = \left[\sqrt{\Omega_{p0}}\left(\frac{p}{2}\right)\tau\right]^{2/p}$$

$$h = \frac{\dot{x}}{x} = \frac{2}{p}\frac{1}{\tau}$$

$$h_0 = \frac{2}{p}\frac{1}{\tau_0} = \frac{2}{p}\frac{1}{2/p} = 1$$

(see p. 3237 for the same result in general)

$$q = \frac{\sum_p \Omega_{p0}\, x^{-p}\,(p/2-1)}{\sum_p \Omega_{p0}\, x^{-p}} \text{ in general (see p. 3235)}$$

$q = p/2 - 1$ and so is a constant for a single density component

$\therefore q_0 = q = p/2 - 1$ and $p = 2(q_0+1)$

$$\begin{cases} <0 \text{ for } p<2, \text{ acceleration} > 0 \\ 0 \text{ for } p=2, \text{ coasting (acceleration} = 0) \\ >0 \text{ for } p>2, \text{ acceleration} < 0, \text{ deceleration case.} \end{cases}$$

b) For $p=0$ things are a bit different.

In this case as we see in just a moment, there is no single time $t_0$ where $H=H_0$ or $h=1$ and so we do NOT get $x_0=1$

$$d\tau = \frac{dx}{\sqrt{\Omega_{p0}}\, x^{+1}}$$

Solution

$$\tau = \frac{1}{\sqrt{\Omega_{p0}}} \ln x + C$$

Inverse solution $x_0 e^{\sqrt{\Omega_{p0}}\,\tau}$ and there is no point origin

This is the exponential de Sitter universe.

7 ... second ... FR cosmological model

It's pure cosmological constant universe

Recall $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$ (see p. 3228) $= \Omega_{p0} = 1$, then $H_0 = \sqrt{\Lambda/3}$

Now $x = x_0 e^{\tau} = x_0 e^{H_0 t} = x_0^{\sqrt{\Lambda/3}\, t}$

$\therefore \frac{\dot{x}}{x} = 1$ or $\frac{\left(\frac{dx}{dt}\right)}{x} = H_0$ and the Hubble parameter is a constant equal to the Hubble constant at all time.

$q = q_0 = P/2 - 1$ (see p. 3239)

$= -1$ for $p=0$

Actually, the P=0 gives an example of a solution of the acceleration eqn, that is NOT a solution of ~~the~~ Friedmann eqn, and therefore cannot keep k constant which we require for a universal solution (see p. 3212 and 3222),

Satisfying the acceleration eqn. is necessary for being a cosmic scale factor solution, but NOT sufficient. Satisfying the Friedmann eqn. is necessary and sufficient

If there is just $\Omega_\Lambda$, then the scaled Friedmann eqn. is

$$\frac{\dot{x}}{x} = \pm\sqrt{\Omega_\Lambda}$$ (see p. 3230)

by what we mean by cosmic scale factor

$$\dot{x} = \pm\sqrt{\Omega_\Lambda}\, x$$

$$x = x_0 e^{\pm\sqrt{\Omega_\Lambda}\tau}$$

and the scaled acceleration eqn. is

$$\ddot{x} = \Omega_\Lambda x$$ (see p. 3235)

which is satisfied by $x = x_0 e^{\pm\sqrt{\Omega_\Lambda}\tau}$,

but also by

$$x = x_0 \sinh(\sqrt{\Omega_\Lambda}\,\tau)$$

$$\dot{x} = \sqrt{\Omega_\Lambda}\, x_0 \cosh(\sqrt{\Omega_\Lambda}\,\tau)$$

which gives $\sqrt{\Omega_\Lambda}\, x_0 \cosh(\sqrt{\Omega_\Lambda}\,\tau) \neq \pm\sqrt{\Omega_\Lambda}\, x$

but does give $\ddot{x} = \Omega_\Lambda x_0 \sinh(\sqrt{\Omega_\Lambda}\,\tau)$

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Another case is the Einstein (static) universe.

Note,

$x^{(2)}$, $x^{(3)}$, etc are derivatives with respect to $\tau$ NOT $x$.

$$\dot{x} = \sqrt{\sum_p \Omega_{p0} x^{-p+2}}$$ (see p. 3235)

$$\ddot{x} = \sum_p \Omega_{p0} x^{-p+1} (-P/2+1)$$ (p. 3235)

So $\ddot{x} = f(x)$ and say $\ddot{x} = f(x_E) = 0$

Is $x_E$ the Einstein universe solution? NOT necessarily. $\dot{x}(\tau) = 0$ for all $\tau$ to be that.

$\ddot{x} = x^{(2)} = f(x)$

and $\ddot{x} = f(x_E) = 0$

$x^{(3)} = \frac{df}{dx}\dot{x}$, let $\frac{df}{dx} = g(x)$ and $x^{(3)}(x_E) = g(x_E)\dot{x}(x_E) = 0$?

$$x^{(n)} = \sum_{l=0}^{n-3} \binom{n-3}{l} g^{(n-3-l)} \dot{x}^{(l)}$$

Leibniz rule generalization of product rule (Arf-667)

$$= \ldots + g^{(0)} \dot{x}^{(n-3)}$$

$$= \ldots + g\, x^{(n-2)}$$

These terms must all have at least one factor of $\dot{x}$

So if $\dot{x}_E = 0$, all terms are zero

and if $x_E^{(n-2)} = 0$ their term is zero too

Implied proof by induction

So if $\ddot{x}_E = 0$ all $x_E^{(n\ \text{even})} = 0$.

And since $\dot{x}_E = 0$, all $x^{(n\ \text{odd})} = 0$.

The upshot is if $\ddot{x}_E = 0$ the acceleration equation is satisfied,

but $\dot{x}_E = 0$ for all $x_E^{(n)} = 0$ for the Einstein universe to be realized.

Thus both $\dot{x}_E = 0$ and $\ddot{x}_E = 0$ are needed for the Einstein universe.

However, $x(\tau)$ that satisfies only one $\dot{x}_E = 0$ or $\ddot{x}_E = 0$ will be a solution if it satisfies the Friedmann Eqn. Satisfying the accel. eqn is necessary Not sufficient.

c) Special features of the elementary solutions for $p>0$

$$\Omega_p = \Omega_{p0}\, a^{-p}$$

$$= \Omega_{p0}\left(\left[\sqrt{\Omega_{p0}}\left(\frac{p}{2}\right)\tau\right]^{2/p}\right)^{-p}$$

See p. 3239

$$= \Omega_{p0}\,\Omega_{p0}^{-1}\left(\frac{p}{2}\right)^{-2}\tau^{-2}$$

$$= (p/2)^{-2}\,\tau^{-2}$$

$\Omega_p \propto \tau^{-2}$ in all cases of $p>0$

Note this is result does not hold for multiple density component solutions for $\sum_{p>0} \Omega_{p_0} a^{-p}$ cases

of course $\Omega_\Lambda = \Omega_{p=0} = \Omega_{0,p=0}$

i.e., the density parameter of $\Lambda$ is a constant.

d) Note Friedmann Equ. is Nonlinear.

So a sum of solutions is NOT a solution in general.

There are some special cases where a sum of solutions is a solution, but only for certain coefficients, but not general coefficients.

But the elementary solutions don't sum to solutions ever I think.

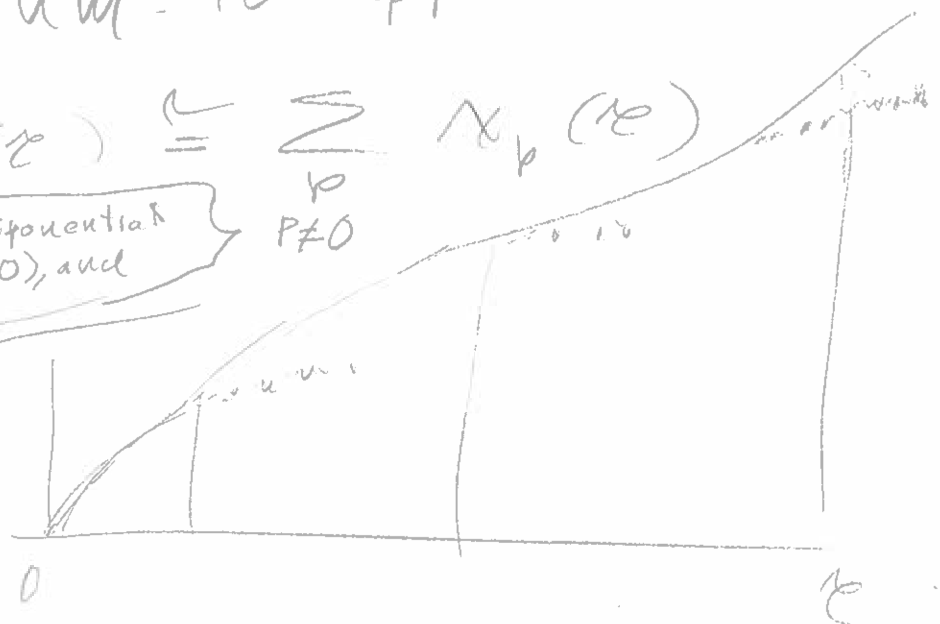
The only cases I know exact solutions are sums of exact solutions

32.44.

But in some cases the elementary solutions can sum to approximate solutions

$$x(t) \cong \sum_{p \neq 0} x_p(t)$$

p=0 gives an exponential solution (see p 32.40), and so you need way to verify this statement.




$$\dot{x} = x\sqrt{\sum_p \Omega_{p0} x^{-p}}$$

The various ~~terms~~ become dominant at different times, and so a crude solution of phases can exist.

In fact, I've wasted far too much time finding high accuracy approximations for certain parameter choices. Perfectly worthless I think now.

But I think such solutions are all very qualitative and for quantitative accuracy, you need exact solutions or high accuracy numerical solutions.

Actually, I believe I have found (I think) an educationally useful qualitative approximate solution formalism. But it is just to show how different behaviors emerge, not for any accuracy. I probably won't bore the class with it.

2025 jan 04

~~3 2 2 5~~ 3 2 4 5

$$h^2 = \frac{\rho}{\rho_{critical}} = \underbrace{\Omega}_{\text{Omega parameter}}$$

{ Omega = density parameter

~~the density~~

The scaled Friedmann equation.

$$= \underbrace{\Omega_{ME}}_{\text{mass energy}} + \Omega_k + \Omega_\Lambda$$

At cosmic present where $x=1$

Recall $h = \frac{H}{H_0}$

$$h^2 = 1 = \Omega_{ME_0} + \Omega_{k0} + \underbrace{\Omega_\Lambda}_{\text{always const}}$$

$$\Omega_{k0} = 1 - \Omega_{ME0} - \Omega_\Lambda$$

iff these sum to -1

then $\Omega_{k0} = 0$

$$\Omega_{k0} = -\frac{3k}{8\pi G a_0^2} \frac{1}{x^2}$$

$x=1$ at cosmic present

So $\Omega_{k0} = 0$ implies the universe is flat.

We'll look more a curvature of the universe later.

Note $k>0$ is +ve curvature but $\Omega_k < 0$

$k<0$ is -ve curvature but $\Omega_k > 0$

A very annoying change of sign.

~~3226~~
3246

2025 jan 05

density parameter are considered?

e) What kind of ~~Omega~~ are ~~considered~~.

Typically, inverse-power law ones

$$h^2 = \Omega = \sum_{p=5}^{0} \Omega_{p0}\, x^{-p}$$

(But see p. 3231 for the CPL density parameter.)

Note 6 parameters needed. If you knew $H_0$, then you could ~~eliminate~~ fix one of them.

$$= \Omega_{50} x^{-5} + \Omega_{40} x^{-4} + \Omega_{30} x^{-3} + \Omega_{20} x^{-2} + \Omega_{10} x^{-1} + \Omega_{00} x^{-0}$$

e.g., -2, 0

p = 0 for $\Lambda$ cosmological constant or dark energy (Je-II)

p = 1 for some quintessence theories

3239

p = 2 for curvature, cosmic strings (some theories), $R_h = ct$ universe of Fulvio Melia.

p = 3 "matter" in the cosmological sense of matter at rest or nearly in comoving frames
all baryonic and dark matter
— really means non-relativistic matter

p = 4 "radiation" for radiation, but also any stuff moving at extreme relativistic speeds ~ c

p = 5 for nonrelativistic KE density
as for free streaming nonrelativistic cosmic neutrinos (p. 3102)
Probably always negligible to cosmic scale factor $x$ evolution

(2029 jan 05) ~~3229~~ 3247

~~Density decline for elementary solution~~

$$\Omega_p = \Omega_{p0} x^{-p}$$

$$= \Omega_{p0}\left[\sqrt{\Omega_{p0}}\,\frac{p}{2}\,\tau\right]^{-2} = \frac{(2/p)^2}{\tau^2}$$

$$\propto \tau^{-2}$$

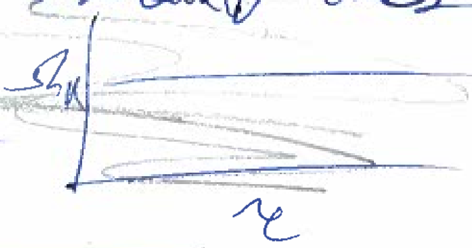
So log $\Omega_p$ on log $\tau$

for all $p \neq 0$ there is an inverse square decline

But just for elementary solutions, not multi-component ones

$$\Omega_{p=0} = \Omega_\Lambda = 1 \text{ for single component}$$

a constant of course



f) Special cases of interest

$p=4$ applies to early universe before $\lesssim 50$ kyr when universe was radiation dominated

$$x(\tau) = \left[\Omega_{40}(2)\,\tau\right]^{\frac{1}{2}} \sim \tau^{\frac{1}{2}}$$

See p. 3239 for the general formula for $p>0$.

~~[illegible]~~

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see p. 3237 for general formula

$p = 3$

$$x = \left[\sqrt{\Omega_{30}}\ \frac{3}{2}\ \tau\right]^{2/3} \sim \tau^{2/3}$$

This is for matter and is called the Einstein-de-Sitter universe (1932)

EdS universe

...ot the Einstein ...tatic ...niverse 1917) or ... Sitter ~~universe~~ 1917)

They proposed it as ~~the~~ simplest of all ~~universes~~ expanding universe models

$\Omega_K$ (i.e. $p=2$) $= 0$   So flat

$\Omega_\Lambda = 0$   (So no cosmological constant)

Just one density component matter

and so $\Omega_{30} = \dfrac{\rho}{\rho_{crit}} = 1$

In fact identifying

$$\rho_{crit} = \frac{3 H_0^2}{8\pi G}$$

was one of the main points of the paper of EdS.

Einstein & de Sitter were deeply impressed by specifying the critical density

~~They~~ They never wrote the $x(\tau)$ formula in the paper (O'Raifeartaigh p. 6-7)

A contemporary cosmologist ~~Heckmann~~ Otto Heckmann described ~~[illegible]~~ the paper as ~~not~~ very profound

2024 jan 05

~~3233~~ 3249

# Age of universe of EdS

Einstein in 1933 review article thought simple models such as the EdS model were too simple to extrapolate to $t=0$, and so was unconcerned about the age problem (O'Raif-23) McCrea (1984) say generally cosmologists call 10 of them, thought the's

$$\tau(a_c = 1) = \frac{1}{\sqrt{\Omega_{3,0}}}\,\frac{1}{3/2}$$

$= 1$ since only matter

$$= \frac{2}{3}$$

$$t = \frac{2}{3} H_0^{-1}$$

see p 3094d Natural unit

Precisely because it was the simplest model and data allowed it in 1960–1990s, it was then the standard cosmology

$$\frac{2}{3}\,\frac{1}{500}\,\frac{\text{Mpc}}{(\text{km/s})} = \frac{2}{3}\cdot 2\ \text{Gyr}$$

1932 value

This was too short for Earth $t_{age} \gtrsim$ 2–3 Gyr in those days

$$= \frac{4}{3}\ \text{Gyr}$$

Only Lemaître with his Lemaître tried to deal with this well, Eddington too with the Lemaître-Eddi. universe.

$$\frac{2}{3}\,\frac{1}{70}\,\frac{\text{Mpc}}{\text{km/s}} = \frac{2}{3}\cdot 14\ \text{Gyr}$$

2025 value

$$\simeq 10\ \text{Gyr}$$

But much too short

Globular cluster age $\gtrsim$ 13 Gyr

~~Recal p. 3094d~~

Natural time unit

$$\frac{1\ \text{kpc}}{1\ \text{km/s}} \simeq 0.98\ \text{Gyr} \simeq 1\ \text{Gyr}$$

$$\therefore \frac{1000}{70} \simeq 14\ \text{Gyr}$$

$$\frac{1\ \text{kpc}}{1\ \text{km/s}} = \frac{3\times10^{16}\ \text{km}}{1\ \text{km/s}}$$

$$= 3\times10^{16}\ \text{s}\left(\frac{1\ \text{year}}{\pi\times10^{7}\ \text{s}}\right)$$

$$\simeq 10^{9}\ \text{y} \simeq 1\ \text{Gyr}$$

$$\frac{2}{3}\,\frac{1}{50} = \frac{2}{3}\cdot 20 = 13\ \text{Gyr}$$

OK

50 ⟷ 100

1960–1990s

$$\frac{2}{3}\cdot 10 = 7\ \text{Gyr}$$

Too short

~~3232~~

3250

But even before the discovery of the acceleration of Universe in 1998, the EdS universe ~~was~~ falling out of favor

$\Omega_{M_0} \simeq 0.3$ ~~This~~ This was being estimated for baryonic and dark matter

But ~~the~~ inflation paradigm said $\Omega_0 = 1$ to very high accuracy (Though Not exactly 1)

∴ ~~And~~ And

$H_0 \sim 60-80$

So the range was narrowing & getting close to ruling out the EdS age. even if the universe was flat

So EdS fell out ~~and~~ of favor and ~~with~~ with the acceleration of universe proof ~~and~~ for good.

we now ~~have~~ Λ-CDM univer

But from 50 kyr to 10 Gyr in Λ-CDM there is a matter dominated universe EdS phase.

So there is an EdS phase. But now we are in the de Sitter phase if Λ-CDM is true.

$P = 0$ (mostly recapitulates p. 3240)

$$x = x_0 e^{\sqrt{\Omega_\Lambda}\,\tau}$$

de Sitter found this solution from GR in 1917 but not finding the Friedmann equation — It was a tricky effort

$$\sqrt{\Omega_\Lambda}\,\tau = \cancel{\sqrt{\Lambda 8\pi G}}\, H_0 t$$

see p. 3224

$$= \sqrt{\frac{\Lambda/(8\pi G)}{\frac{3 H_0^2}{8\pi G}}}\, H_0 t = \sqrt{\Lambda/3}\; t$$

$$\therefore\; x = x_0 e^{\sqrt{\Lambda/3}\; t}$$

$$H_{dS} \quad \frac{\dot{x}}{x} = \sqrt{\Lambda/3}$$

$\therefore$ The invariant Hubble parameter of ~~the~~ de Sitter universes is

$$H_{deSitter} = \sqrt{\Lambda/3}$$

Λ-CDM value $\Omega_\Lambda \doteq .7$

$$= \sqrt{\Omega_\Lambda}\, H_0$$
$$= \sqrt{0.7} \cdot 70 = 0.85 \cdot 70 \doteq 60 \ \frac{\text{km/s}}{\text{Mpc}}$$

~~3234~~ 325.2

The de Sitter universe had no matter
→ it was a pure ~~cosmological constant~~ universe

But it did predict ~~the~~ expansion of universe and the cosmological redshift and ~~Edwin~~ Hubble was aware of ~~that~~ in the 1920s.

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Of course, it hasn't gone away
Since t = 10 Gyr we've been in a Λ dominated universe and if the Λ-CDM model is extrapolated to the future, we asymptotically approach a cold, low density de Sitter universe.

Also inflation era is predicted to be a de Sitter phase.

## 3.27 Non-Elementary Solutions: A First Look

a) Just a first look since we cover non-elementary exact solutions in Lecture 5.

First recall scaled Friedmann equation

$$\frac{\dot{x}}{x} = \pm\sqrt{\sum_p \Omega_{p0} x^{-p}} \quad (p.\,3228)$$

for (inverse) power law density components $p \geq 0$

where at $\tau_0$, $\frac{\dot{x}}{x} = 1$

and we have chosen $x = 1$ at $\tau_0$ (which is usual, but not required, see p. 3232)

b) Numerically one always solves for $\tau(x)$

$$d\tau = \pm\frac{dx}{x\sqrt{\sum_p \Omega_{p0} x^{-p}}}$$

using the midpoint method or Rung-kutta with small enough steps for the accuracy required.

c) To my knowledge (not exhaustive), there are no exact solutions for 4 or more components.

For 1 component, there is always an exact solution (see p. 3239 -3240).

For 2 components, there is an infinite set of exact solutions, but only for certain sets of powers

see lectures

Note, if $P_2 > 0$ and $P_0 = 0$
then there is always $r(x)$ and $x(r)$ exact solutions

But there other cases too

For 3 components, there is also an infinite set of exact solutions, but only for certain sets of powers.

See Lecture 5

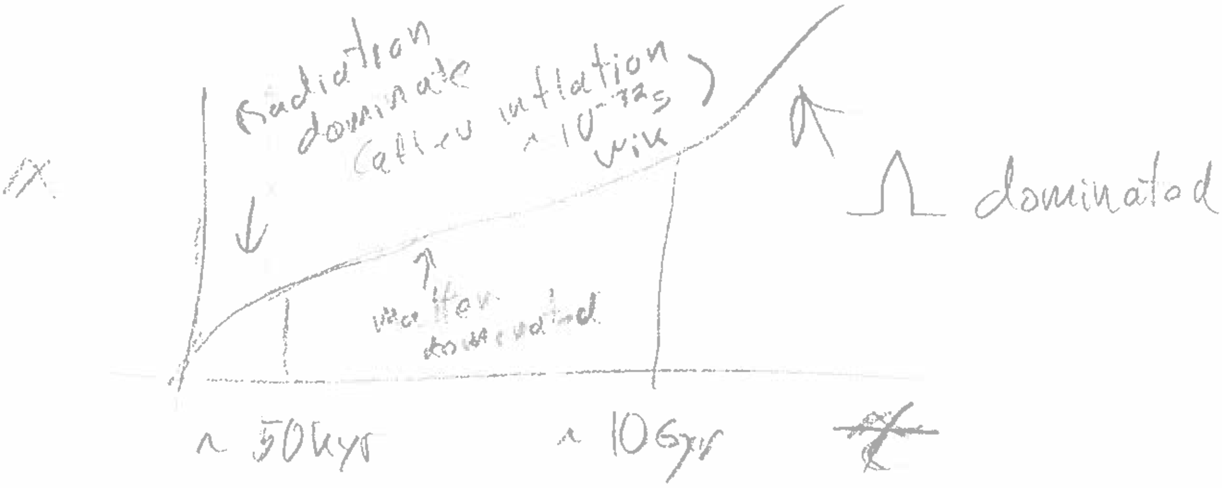
Note, if $P_2 > 0$, $P_1 = P_2/2$, $P_0 = 0$
then there are exact solutions $r(x)$ and $x(r)$.

But there are other exact solutions too.

Of course, only a few of these exact solutions are of physical interest.

d) Radiation - Matter - Λ universe

This is the Λ-CDM universe




This model is still definitely in contention (c. 2026) — despite many analyses for dynamical dark energy since many others say those analyses are NOT decisive.

There is NO exact solution that spans all phases. ~~A~~ complex solution of a cubic

**But** There is $t(x)$ and $x(t)$ exact for radiation-matter universe

and $t(x)$ and $x(t)$ exact for matter-Λ universe

3256] You can just join the two exact solutions in mid matter phase where radiation and $\Lambda$ are both negligible.

The radiation phase is so brief that ~~the~~ matter-$\Lambda$ exact solution can be regarded as the $\Lambda$-CDM model solution.

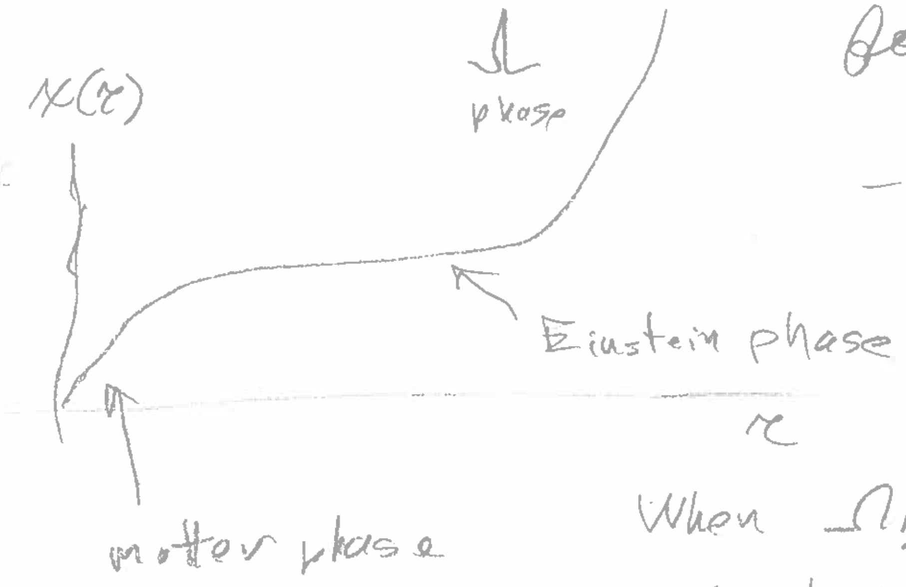
e) <u>The Historically Interesting Lemaitre universe</u>

circa early 1930s

$$\frac{\dot{x}}{x} = \sqrt{\Omega_{30}x^{-3} + \Omega_{20}x^{-2} + \Omega_0}$$

$\Omega_{20} < 0$

for $k > 0$ and so positive curvature — a hyperspherical, finite, unbounded universe




When $\Omega_{30}x^{-3} + \Omega_{20}x^{-2} + \Omega_0 \cong 0$, you have the Einstein nearly-static phase.

You can adjust $\Omega_{30}$, $\Omega_{20}$, $\Omega_0$ to make ~~the~~ Einstein phase as long as you like.

This allowed Lemaitre to avoid the age of universe problem discussed on p. 3247 for the EdS universe and similar cosmic scale factor solutions.

Also Lemaitre also argued that the Einstein phase was when collapse of density fluctuation in the original gas of particles fragmented from his primeval atom picture would lead to the formation of galaxies.

He did explore the GR collapse solution quantitatively with what is called the Lemaitre-Tolman metric.

Of course, this work in the 1930s was all pre-computer and so structure formation computer simulations were beyond is scope

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2026 jan 24

The Lemaitre universe has no exact solution, and so Lemaitre probably did approximation or tedious by hand Midpoint method numerical calculations to produce his solutions,

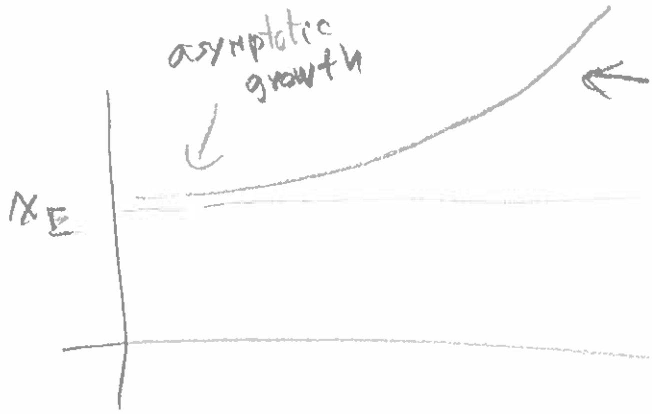
But the analogous radiation-curvature-$\Lambda$ universe

satisfying $$\frac{\dot{x}}{x} = \sqrt{\Omega_{40}x^{-4} + \Omega_{20}x^{-2} + \Omega_0}$$

does have an exact solution and we will cover that in Lecture 5.

## f) Eddington-Lemaître Universe

also $$\frac{\dot{x}}{x} = \sqrt{\Omega_{03}x^{-3} + \underbrace{\Omega_{02}x^{-2}}_{\Omega_{02}<0} + \Omega_0}$$




← $\Lambda$ phase

← Einstein Static solution for $\Omega_{03}x_E^{-3} + \Omega_{02}x_E^{-2} + \Omega_0 = 0$

Lemaitre found the solution
that at $t = -\infty$ is the Einstein universe
but the Einstein universe
is unstable to global perturbation

and one guesses local perturbations
lead to overall expansion
with local collapses
to form galaxies

Eddington favor this model
to avoid the point origin and
the age problem altogether and put
(see p. 3247)
whatever created the Einstein phase
at some undetermined
long age universe

Again there is no exact solution
but the analogous
radiation - curvature - $\Lambda$ universe
does have an exact solution
that we will discuss
in lecture 5

3260

# 9) Approximate Solutions

I wastod a lot of time trying to find good approximate solution. For special parameter choices it can be done.

But one would like a general approximate solution even if only a good qualitative one, but only educational interest.

Actually I've found a ...

If you want high accuracy, you find an exact solution if possible and if not us high accuracy numerical solution by the midpoint method, Runge-Kutta, or other means.

Very qualitatively for solutions that expand from a point origin

semi-general qualitatively good approximation. It may be covered in lecture 5

$$x = \sum_i^{I-2} \Omega_{i0} x^{-p_i} + x(p_{I-1}, p_I)$$

$p_i$ decrease with index $i$

Exact if $I = 1$,

Good asymptotically for $p_1$ and $p_I$

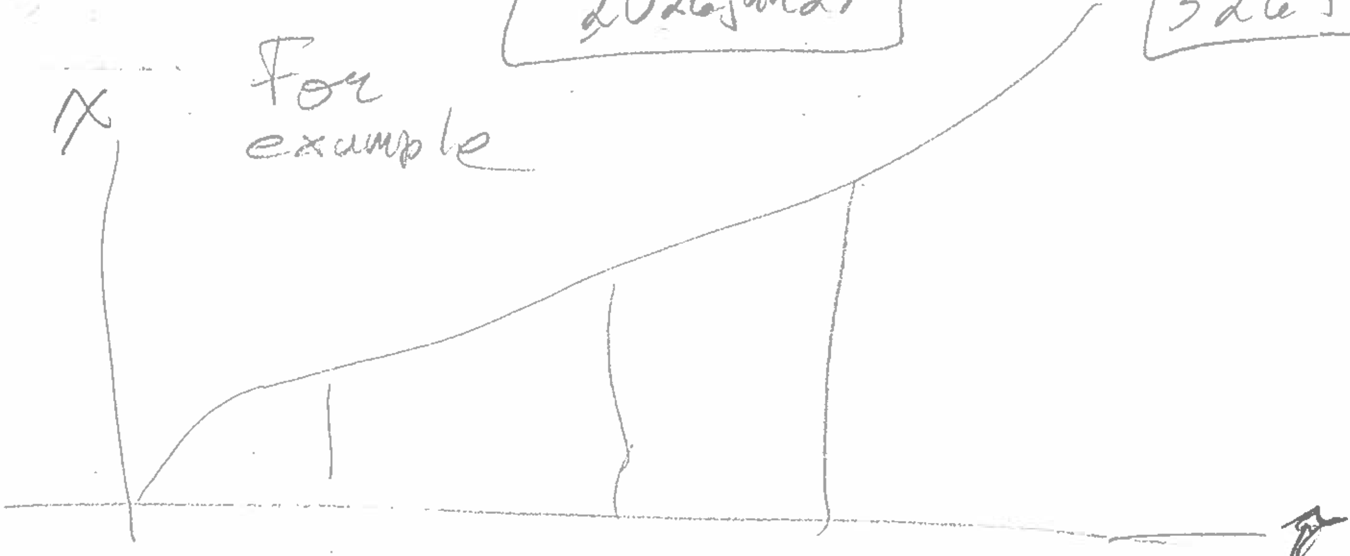
Goodish in phases where one term in $\dot{x} = x\sqrt{\sum_i \Omega_{i0} x^{-p_i}}$ dominates

Use if $p_I = 0$.

There is always an exact solution for inverse-power law density parameter of two density components if one is $p = 0$

Recall, the elementary $p = 0$ solution never goes to zero

For example




$P=4$ $\quad$ $P=3$ $\quad$ $P=2$ $\quad$ $P=1$ and $0$

Radiation $\quad$ matter $\quad$ curvature $\quad$ quintessence and $\Lambda$

The larger the power $P$, the earlier it's phase of dominance.

The phase transitions are specified by equality points. This is true of exact solution (analytic or numerical) too, of course.

$\tau_1 < \tau_2 < \tau_3$ ... $\tau_4$

$$\Omega_{i0}\, a^{-P_i} = \Omega_{i+1,0}\, a^{-P_{i+1}}$$

where $P_i > P_{i+1}$

$$\Omega_{i,0} = \Omega_{i+1,0}\, a^{P_i - P_{i+1}}$$

$$a_{\text{equality } i} = \left[\frac{\Omega_{i,0}}{\Omega_{i+1,0}}\right]^{\frac{1}{P_i - P_{i+1}}}$$

The end of the ith phase

3262 | 2026 jan 24 | 3xx

An educationally interesting ~~case~~
is

$$\frac{\dot{x}}{x} = \sqrt{\sum_{p=4}^{0} \Omega_{p_0} x^{-p}}$$

at least to consider for a moment

There is an exact solution

$x(\eta)$ (Steiner-9) an auxillary variable

$d\eta = \frac{d\tau}{x}$ is conformal time
(Steiner-6
aside from constants)

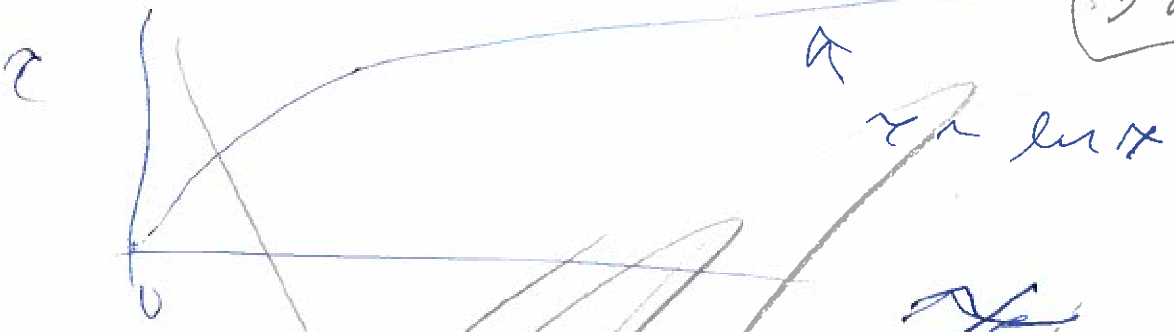
But $x(\eta)$ is exact in terms
of the Weierstrass elliptical $\wp$-function
(Steiner-7, 9)

A special function that
is only analytic by
definition,
but that is also true of $e^x$, $\sin x$, $\cos x$
$\sinh x$, etc.

But for a complete solution,
you need $\tau(\eta) = \int_0^\eta x(\eta')\,d\eta'$
which can only be done numerically in general
(Steiner-11)

(2025 Jan 09)

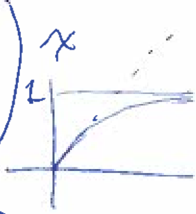
~~3241~~ 3263




But time is the coordinator of all physics, so we'd like even very crudely to ~~have~~ some sort of analytic approximation $x(\tau)$

At the moment (but maybe not forever) I suggest

linear growth from 0 and saturation at ∞




arctangent funct

$$x(\tau) = \prod_{p=4}^{1} \left[ \underbrace{(1-\delta_{p,4})}_{0 \text{ for } p=4} + x_p(\tau_p)\,\mathrm{atan}\left(\frac{x_p(\tau)}{x_p(\tau_p)}\right)\right] e^{\sqrt{\Omega_4}\,\tau}$$

for $\tau \ll \tau_p$: $x_p(\tau)$

for $\tau \gg \tau_p$: $x_p(\tau_p)$ saturation

At best this could be sort of qualitative and maybe not good at all.

I was tempted to test it, but a lied down till ~~the~~ feeling went away.

~~3242~~ 3264 | 2025 jan 05

3, 28

(Ordinary)

# ~~70~~ 1st Order Autonomous Differential Equations

Have No stationary Points except at Infinity and constant solutions (a continuum of stationary Points)

There can have stationary points NOT at infinity. { except for special cases which aren't that rare and include the Friedmann Eq.

a) Proof Except for the special cases

Let $x' = f(x)$

where $x' = \frac{dx}{dt}$ and $t$ is the independent variable

and $f(x_i) = 0$ and so $x_i$ is a zero of $f(x)$. There can be a set of zeros in general $\{x_i\}$.

We assume $f(x)$ is infinitely differentiable with respect to $x$ (not $t$) at $x_i$

∴ We can Taylor expand around $x_i$

$$\Delta x' = f(x) = \underset{\substack{\| \\ 0 \\ \text{since } f(x_i)=0}}{f_0} + \Delta x f_1 + \Delta x^2 f_2 + \ldots$$

$$f_i = \frac{1}{i!}\left.\frac{d^i f}{dx^i}\right|_{x_i} \text{ etc.}$$

$x' = \Delta x'$

since $\frac{dx}{dt} = \frac{d\Delta x}{dt}$

We truncate to lowest non-zero order. $\boxed{2025 \text{ jun } 05}$ ~~3A4~~ 3265

$\therefore \Delta x' = \Delta x^{\ell} f_{\ell}$, where all $f_{k<\ell} = 0$

different by t

$$\Delta x'' = \ell \Delta x^{\ell-1} \frac{d\Delta x}{dt} f_{\ell}$$ ~~where we drop $f_\ell$ constant for simplicity~~

$$\propto \ell \Delta x^{\ell-1} \Delta x^{\ell}$$

$$\propto \Delta x^{2\ell-1}$$ ~~where we drop coefficients for simplicity~~

$$\Delta x''' \propto (2\ell-1)\Delta x^{2\ell-2}\Delta x'$$

$$\propto (2\ell-1)\Delta x^{2\ell-2}\Delta x^{\ell}$$

$$\propto \Delta x^{3\ell-2}$$

The pattern is clear: every time differentiated add $\ell$ to the exponent and subtract 1

$$\Delta x^{(n)} = \cancel{\Delta x^{n\ell-(n-1)}}$$

$$\propto \Delta x^{n\ell-(n-1)}$$

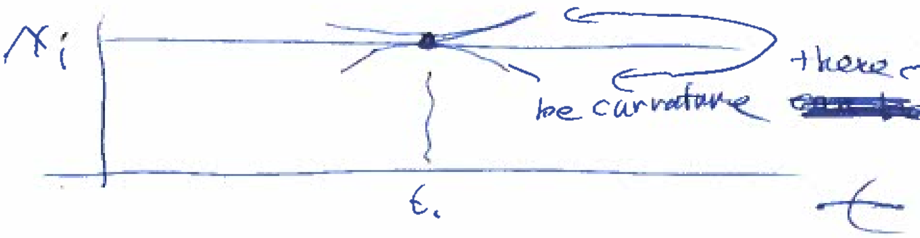
$$\propto \Delta x^{(n-1)\ell+1}$$

where $n \geq 1$, $\ell \geq 1$

$\therefore$ the exponent $p = (n-1)\ell + 1 \geq 1$

$\therefore \Delta x^{(n)}(\Delta x = 0) = 0$

$\therefore x^{(n)}(x_i) = 0$ to lowest order in $\Delta x$ and therefore all orders $t$.

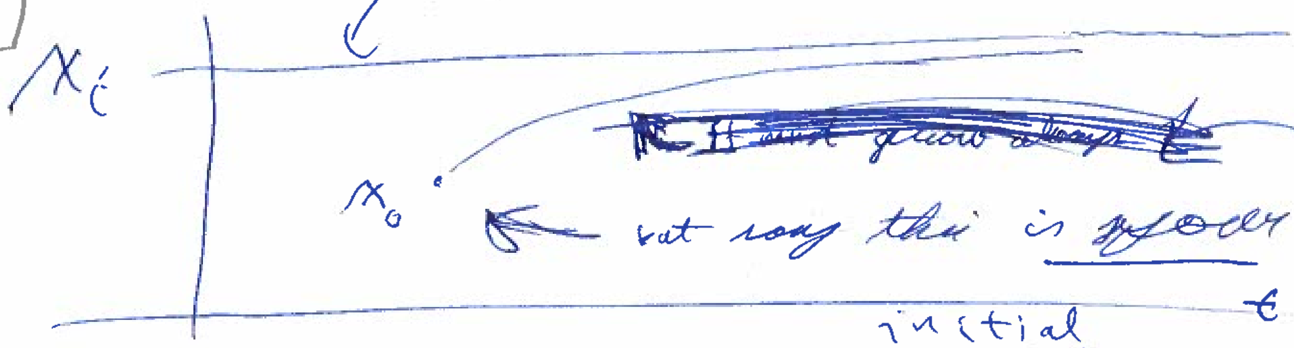



there cannot be curvature about $t_i$ point. So all time points are $t$ ~ a constant solution.

3~~244~~

3266

$x(t) = x_i$ constant solution



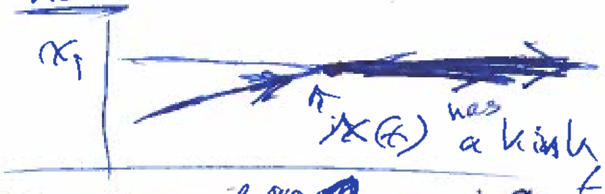

but say this is your initial condition

and $f(x < x_i) > 0$ then $x(t)$ must always grow

— intuitively (?)

it is ~~clear~~ that $x(t)$ (with initial condition $x_0 < x_i$) can only reach $x_i$ at $t \to \infty$ since all orders of $x^{(n)}$ must be zero when $x(t)$ reaches $x_i$

Note




if $x(t)$ reaches $x_i$ at finite time.

But can we prove definitely?

Yes. Proof

Let $y = \pm(x - x_i) = \pm\Delta x$ where +ve case for $x > x_i$, -ve case for $x < x_i$, and so $y > 0$ always

$$\Delta x' = \Delta x^{\ell} f_{\ell}$$

$$y' = (\pm 1)^{\ell-1} y^{\ell} f_{\ell}$$

Case 1 $\quad \ell = 1$ meaning $f_1 \neq 0$

$$y' = y f_1$$

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(~~3245~~)

(3267)

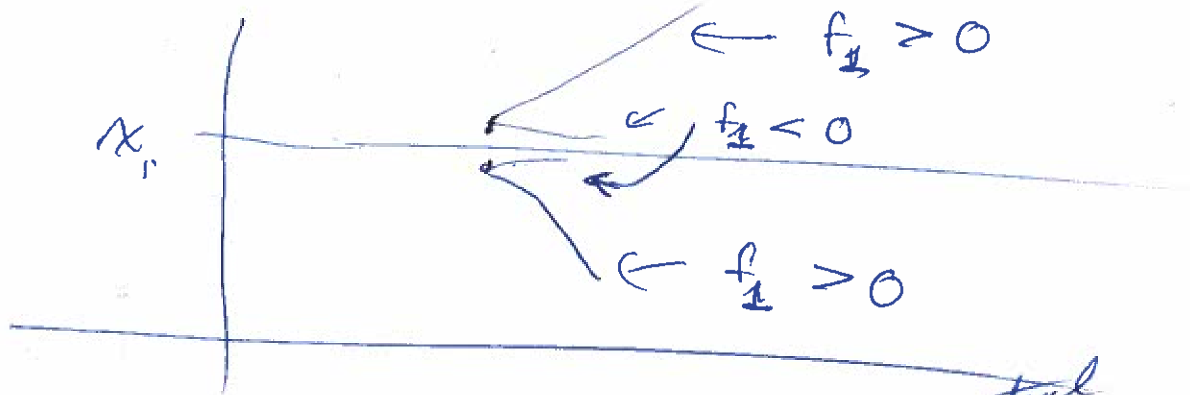
$$\frac{dy}{y} = f_1\,dt$$

$$\ln(y/y_0) = f_1(t - t_0)$$

$$y = y_0 e^{f_1(t-t_0)}$$

($\pm 1$) cancels on both sides

$$\Delta x = \Delta x_0 e^{f_1(t-t_0)}$$




So

$f_1 < 0$ leads to exponential convergence but the stationary point is at infinity

A stable case of the constant solution since perturbations damp out

$f_1 > 0$ leads to divergence which is only exponential ~~to~~ lowest order.

Case $l \geq 2$

$$\frac{dy}{y^l} = (\pm 1)^{l-1} f_l$$

$$\frac{y^{-l+1} - y_0^{-l+1}}{-l+1} = (\pm 1)^{l-1} f_l (t - t_0)$$

An unstable ~~as~~ the constant solution since perturbations lead to divergence

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32681

$$\Delta x = y = \frac{1}{\left[(-l+1)(\pm 1)^{l-1} f_l (t-t_0) + y_0^{-l+1}\right]^{\frac{1}{l-1}}}$$

$-l+1<0$

since $l \geq 2$

$y_0 > 0$ by definition

$(\pm 1)^{l-1} f_l < 0$ for convergence

and $>0$ divergence to infinity

but only to lowest order in $l$

~~if~~ if $l$ is odd

$l-1$ is even

$(\pm 1)^{l-1} = 1$

then $f_l < 0$, converges

~~gives convergy~~ (stable constant solution)

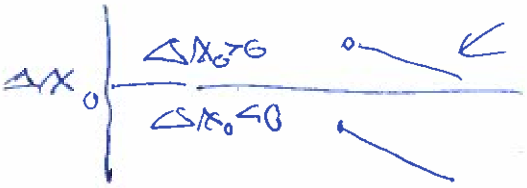
$f_l > 0$

diverges (unstable constant solution)

if $l$ is even, $l-1$ is odd

$(\pm 1)^{l-1} = \pm 1 \begin{cases} +ve & \Delta x_0 > 0 \\ -ve & \Delta x_0 < 0 \end{cases}$

$f_l < 0$ for converge




$f_l < 0$ divergen

divergence

$f_l > 0$


$f_l > 0$

convergent

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Note if ~~the~~ ~~constant~~ solution is only stable for perturbations of one sign, then it is ~~usually~~ unstable ~~since~~ if perturbations of any sign can happen.

It's ~~stable~~ if ~~the~~ ~~the~~ divergent perturbation are somehow constrained Not to happen.

b) Special Cases where there are stationary points NOT at infinity

But only one kind of special case I know of that is relevant to the Friedmann Eq.

Say $x' = [f(x)]^{\frac{1}{k}}$ where $k>0$ to avoid pointless generality

and $x'(x=x_i) = [f(x_i)]^{\frac{1}{k}} = 0$

$x' = x^{(n-1)}$ so $n \geq 1$ and integer

expand $f(x)$ which is infinitely differential about $x_i$ where $f(x_i) = 0$

So $\ell \geq 1$ and integer

$$\Delta x' = [\Delta x^{\ell} f_{\ell} + \Delta x^{\ell+1} f_{\ell+1} + \ldots]^{\frac{1}{k}}$$

where $\ell$ is the lowest nonzero term and $\ell \geq 1$ and integer

$$= [\Delta x^{\ell} f_{\ell}]^{\frac{1}{k}} [1 + O(\Delta x)]^{\frac{1}{k}}$$

This term goes to zero as $\Delta x \to 0$ and so can be dropped

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$$\therefore x' = [x^{\ell} f_{\ell}]^{\frac{1}{k}}$$ to lowest order in $\ell$

$$x' \propto x^{\frac{\ell}{k}}$$

$$x'' \propto \ell x^{\frac{\ell}{k}-1} x' = x^{2\frac{\ell}{k}-1}$$

$$x''' \propto x^{2\frac{\ell}{k}-2} x^{\frac{\ell}{k}} = x^{3\frac{\ell}{k}-2}$$

$$\vdots$$

$$x^{(n)} \propto x^{\frac{\ell}{k}n-(n-1)} = x^{(\frac{\ell}{k}-1)n+1}$$

$\frac{\ell}{k} = \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots$

1 is just on p3244.

~~usually~~ ~~the other case~~

$= 2,\ k=2$

$\Rightarrow$ most likely Einstein universe case in fact

If $\frac{\ell}{k} = 1$, then $x' \propto x^1$ and we are back to p. 3244 with only a stationary point at $\infty$ as ~~and~~ solution and exponential convergence/divergence

If $\frac{\ell}{k} > 1$, then exponent $p = (\frac{\ell}{k}-1)n+1$ increases for ever with $n$ and all $x^{(n)}(x_i) = 0$ and again we have only stationary points at $\infty$ as on p. 3243

$= x^{\frac{\ell}{k}}$

and $\frac{\ell}{k}$

need not be an integon

it if it is

~~sadly~~

on

p. 3244

3245

If $\frac{\ell}{k} < 1$, then eventually we get $\bigcirc x^{-2}$ and as $\Delta x \to 0$ we have a singularity which is not a stationary point and may be unphysical

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Unless ~~for~~ $\frac{l}{k} < 1$ there is a stopping $n$ ~~such that~~ ~~p~~ $0 = (\frac{l}{k} - 1)n + 1$

~~An all the~~ stopping $n$,

$$x^{(n)} = C$$

Integrating n times

$$x = Ct^n + \dots$$

~~but the constants~~ of integration have to be such ~~that~~ at $t_i$, $x = x_i$ and $x^{(n-1)} = 0$ and $x^{(n)} = C$

The stopping $n$ formula is

$$\frac{l}{k} = 1 - \frac{1}{n}$$

$$= 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1$$

$n=1$ $n=2$ $n \to \infty$

This means $x'(x_i) = C \neq 0$ and ~~that~~ is not allowed by the hypothesis ~~that~~ $x'(x_i) = 0$.

The physically ~~most~~ relevant ~~for~~ $l = 1$ and $k = 2$ and this turns up in Friedmann Equation cases.

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3272

( ~~325~~ 2025 Jan 05 )

~~F~~Friedmann eqn

$$\frac{\dot{x}}{x} = \pm\sqrt{\sum_{p=9}^{0} \Omega_{p,0} x^{-p}}$$

Most relevant cases

$\frac{k}{l} = \frac{2}{2}$ and only ~~stationary~~ points at $\infty$ & constant solution

Einstein universe — constant solution

$\frac{k}{l} = \frac{1}{2}$ and a finite stationary point.

Lemaitre universe — no stationary point at all, so no constant solution

except in the parameters go from one to the other,

I know of no others, but they probably exist, but maybe only for unlikely density behaviors

Einstein universe a limiting case of Lemaitre universe

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~~3271~~ 3273

## c) A super-interesting exception

A $k=2$
from ~~p. 3xxx~~ p. 3270

$$x' = f(x) \quad \text{and} \quad f(x_i) = 0$$

$$x'' = \frac{df}{dx}\cancel{x}' \quad \text{but} \neq 0 \text{ at } x = x_i$$

$$x'' = \frac{df}{dx} f$$

Let $$\frac{df}{dx} f = g(x)$$

$$\frac{1}{2} f^2 = \int g\,dx + C$$

$$f = \pm\sqrt{2\left(\int g\,dx + C\right)}$$

where $g(x)$ and $C$ are chosen . . .
so ~~that~~ $f(x_i) = 0$
But $f(x)$ must always be re...

There are many possible solutions for $g(x)$
but the most obvious one (after a little playing around) is . . . . .

OK, a lot of playing around

~~$g(x) = 2Dx$ which is not zero at $x = x_i$~~

~~$\int g\,dx = Dx^2$ and $D = -C$~~

~~$\therefore f = \pm\sqrt{2C(1 - x^2)}$~~

~~$\frac{df}{dx} = \pm \dots$~~

~~3272~~

3274

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$$x'' = \frac{df}{dx} f = -Cx = -Gw^2 x$$

parameterize $C$ as $Gw^2$

omega

$$\left(\frac{\Delta x}{x_i}\right)^2 \left(\frac{x_i + \Delta x}{x_i}\right)^2 \doteq 1 + 2\frac{\Delta x}{x_i}$$

~~then $x'' = -w^2 x$~~

~~and the general solution is an old friend~~

~~$x = A \sin wt + B \cos wt$~~

~~$x' = w(A \cos wt - B \sin wt)$~~

~~but this is not the original 1st order DE and has two integration constant.~~

~~So let's say $B = 0$ and then~~

~~$x = wA(\pm\sqrt{1 - \sin^2 wt})$~~

....is $f = \pm w x_i \sqrt{1 - \left(\frac{x}{x_i}\right)^2}$

which is 0 at $x = x_i$

Note this is $\frac{\ell}{L} = \frac{1}{2} = 1/2$ case on p. 3250

$$\frac{df}{dx} = \pm w x_i \frac{-\frac{x}{x_i^2}}{\sqrt{\phantom{x}}}$$

$$x'' = f\frac{df}{dx} = (w x_i)^2 \left(-\frac{x}{x_i^2}\right)$$

which is not 0 at $x = x_i$

$x'' = -w^2 x$ which is an old friend.

Solution matching the condition

$x = x_i \cos(wt)$ which is a maximum at $t_i = 0$

Note

$$x' = -w x_i \sin(wt) = -w x_i [\pm\sqrt{1 - \cos^2 wt}]$$

$$= -w x_i (\pm\sqrt{1 - \left(\tfrac{x}{x_i}\right)^2}) = f(x)$$