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3b The Friedmann Equation

1) Introduction

After a long warm up in gravity, inertia frames, the cosmological constant force, etc.

Not form for finding exact solutions (need scale for)

A Standard Form — ~~the~~ most standard form (there are others)

Li-55

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

Λ cosmological constant

Lambda

$G = 6.67430(15)\times10^{-11}$ MKS units — The most poorly known fundamental constant (?)

$H = \frac{\dot a}{a}$ Hubble parameter

– at cosmic present $H = H_0 = h_{70}\left(70\ \frac{km/s}{Mpc}\right)$

It's an inverse time quantity and is the relative rate of Expansion of ~~the~~ universe

us → 1 Mpc 70 km/s

us → 2 Mpc 140 km/s

$= \frac{Mpc}{70\ km/s} = \frac{1000}{70} \simeq 14\ Gyr$

mean density of all mass-energy here in mass units

$\rho = \rho(a)$

To my knowledge no one considers ρ as an explicit function of time or anything else. What else could it be a function of? A real question

k is a constant of integration called (Li-33) curvature

Can be can ±ve and also

$k \to kc^2$ another form

$k>0$ hyperspherical, $k=0$ flat, $k<0$ hyperbolical. But only GR tells us this

An integral of motion

The solution is $a = a(t)$ cosmic scale factor

$r = r_0\, a(t)$, t is cosmic time

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It's a 1st order ordinary, nonlinear differential equation

(nonlinear means solution do NOT add to a solution)

autonomous equation (Wik)

↳ No explicit dependence on ~~the~~ independent variable cosmic time t

[unless you impose an explicit time dependence.]

Being 1st order it has limited abilities to describe universal order. It's NOT about forces which require 2nd order DE's in our usual understanding

Physical distances = can be measured with a ruler at one instant in time (cosmic time)

Other distance measures turn up
- luminosity distance
- angular distance
- cosmic time itself
- cosmological redshift

r

All distances r that participate in the mean expansion of the universe obey

$$r = r_0\, a(t)$$

$$r(\text{cMpc}) = \frac{r\,(\text{Mpc})}{a(t)}$$

comoving Mpc

Note

$$\dot{r} = r_0\, \dot{a}(t)$$

$$\ddot{r} = r_0\, \ddot{a}(t)$$

r_0 are the comoving distances
- they are constant in time and by convention equal physical distances at cosmic present

Actually, there is a physical way of specifying "a" also ~~but~~ and we will consider that when we consider cosmic curvature

a has no units and by convention $a(t=t_0)=1$

cosmic present

When one considers curved space there is [illegible] of a with units

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$v = \dot{r} = \dot{a} r_0$

recession velocities
— NOT ordinary velocities
— rate of growth of space in a GR sense
— or ~~speed~~ speed between separating inertial frames (comoving frames as I call them)

There is no limit on $v = \dot{r}$

Perhaps universe stranger if they couldn't be added.

However, in a Newtonian physics sense, they are sort of the same and can be added and subtracted but when both are $\ll c$

The speed of light is the fastest speed anything physical can move relative to an inertial frame
— light moves at this speed in a vaccuum and it is invariant ⟹ same for all observers ⟹ one of ~~the~~ the axioms of special relativity (SR)

past you at one place in an inertial frame

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In fact, ~~the~~ ~~theoretical~~ Hubble's Law is trivially shown

Friedmann (1922) must have known this ~~but Lemaitre~~ and de Sitter (1917) too, but Lemaitre (1927) seems to be the first ~~first~~ to explicitly publish it.

$$r = a(t)\, r_0$$

$$v = \dot{r} = \dot{a}\, r_0 = \frac{\dot{a}}{a}\, a r_0 = H r$$

where $H = \frac{\dot{r}}{r} = \frac{\dot{a}}{a}$

i.e. $v = H r$

true recession velocity

Hubble parameter at some instant in cosmic time

physical distance at one instant in cosmic time

~~The observed~~

But v and r are NOT directly observable except asymptotically as $r \to 0$ or cosmological redshift $z \to 0$

That is where Hubble's law is experimentally verified as we'll show ~~later~~ later.

In the asymptotic limit.

The Friedmann equation is not in most senses a dynamics equation (Equation of motion, Wik) — there are no forces in it.

— it's an integral of ~~motion~~ equation ~~with~~ K being the integral ~~itself~~

curvature and as it's name suggests it describes the curvature of space

However, curvature ~~K~~ cannot be derived in Newtonian physics and so we will rely on GR to tell us

$k > 0$ hyperspherical space

$k = 0$ flat space
Euclidean space

$k < 0$ ~~k~~ hyperbolic space

In fact, the Friedmann equation ~~stat~~ is a sort of energy balance equation, but in GR interpretation it loses it's simple meaning ~~I~~ think.

It's a balance of something that is only energy in small scale classical limit sense, 3206 [2024]

2) Derivation of Friedmann Eq.

a) GR derivation Alexander Friedmann 1922

Independently, Georges Lemaître 1927

Einstein himself seems to be the only one to notice both — ~~but~~ in the 1920s he was not interested in cosmology.

In 1934, Milne & McCrea gave a Newtonian Physics derivation

Which requires some special hypotheses only justified ultimately in GR and doesn't give any notion of curvature.

A 19th century physicist without ~~GR~~ and SR could have derived it in essence. But none did.

Einstein & de Sitter in 1917 both missed discovering FE and had to find their cosmological models by klutzion means. They'd have saved a lot of time if they'd discovered it.

The extra hypotheses do NOT give a complete general theory of gravity and motion under gravity. If true, they ~~They~~ imply it exists, and ... and it's general relativity)

But we'll try (without being too pendantic) as what that person would have done,
Let's call them Max for
James Clerk Maxwell (1831–1879)
who could've done it if his mind ever wandered that way

But we'll do it ideally as he might have presented to an Audience Not with all the meanders of actual research as done.

b) Motivation and Points for Max

— If the universe is infinite and mostly full of matter (stars ~~to~~ Max, galaxies ~~to~~ us) that on a large enough is homogeneous & isotropic
(~~the~~ cosmological principle
— name invented by e.a. Milne but in various forms the idea went back a long way, Einstein, de Sitter Friedmann, Lemaitre even earlier in some sense,)
What holds it up.

— Can it be static on average? (in stable equilibrium)

— Newton himself wondered about that and briefly

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and in unpublished work tried to find a universal balance

but he couldn't and no one else either.

unstable unless making the system infinite magically stabilized it

<u>Maybe</u> an extra hypothesis was needed to keep stars at rest on average in Newton's absolute space.

— But also the universe <u>was</u> <u>not</u> in thermodynamic equilibrium

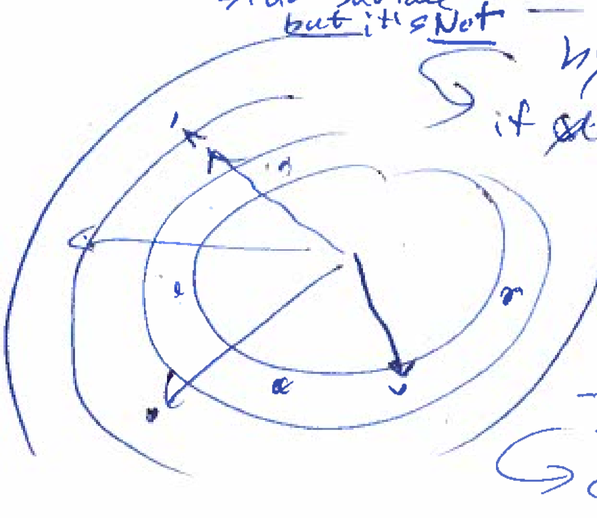
so why should it be static in a dynamical sense

clear in a thermodynamic sense but in a more primitive sense known for a long time

Olbers Paradox — first discussed by Edmond Halley

Conclusion sky should be as bright as a star surface but it's <u>Not</u>

if universe static in all senses infinitely old and obey cosmol. principle then in every direction you should see a star.



— Area of shells $\propto r^2$

— inverse square fall of light $\propto 1/r^2$

cancelation

Basic Derivation

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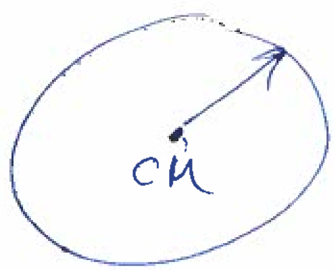
c) So Max hypothesizes an infinite universe with all matter smoothed out into a perfect fluid of uniform density at any time.

— No viscosity

— can have pressure, but no gradients since uniform



Say expanding or contracting everywhere and every point is in free fall under gravity in a sense

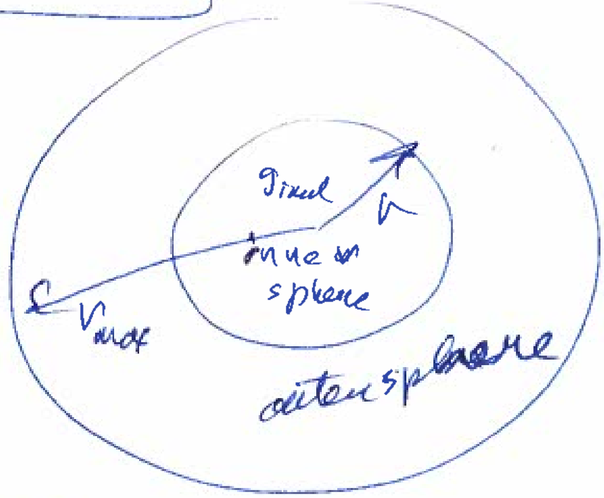
Max guesses these must all points ~~all~~ define inertial frames (comoving frames in the term I use)

Then Max says the shell thm applies at every point

This goes beyond old Newtonian physics

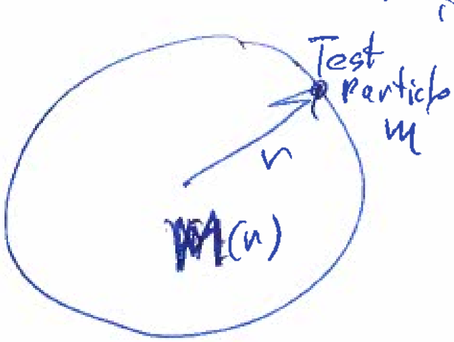
Max guesses there must some more general theory of motion under gravity than Newtonian physics but knows Newtonian physics is a valid limit of it (except for absolute space) and so is trying to generalize from Newtonian physics to capture a portion of the more general theory.

32[illegible]0



Then let r_{max} ~~go~~ go to infinity and $g(r)$ stays ~~the~~ same by hypothesis.

Then every point in infinite space allows a comoving frame centered on it



~~g inside only~~

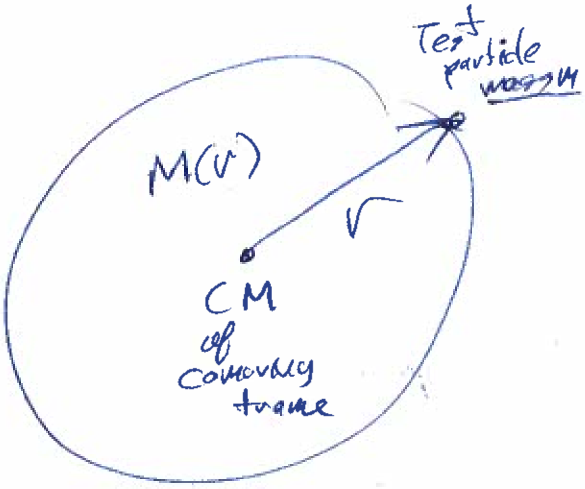
$g(r)$ only depends on $M(r)$ not on mass beyond

— True in Newtonian physics

and Birkhoff tells us $r << r_{scale\ of\ universe}$ does imply ~~this spa~~ empty spherical cavity is flat Minkowski space in which Newtonian physics can be done ~~if spherical sym~~

Max could try to apply $F = ma$ to test particle and probably does a lot hunting around, but arrives at a conservation of mechanical energy approach to get an integral of motion

In fact, we are imagining Max after all kinds of false starts and loops in his thinking



$$\underset{\sim}{g}(r) = -\frac{GM(r)}{r^2}\hat{r}$$

$$V_G = -\frac{GM(r)}{r} = -\frac{4\pi}{3}G\rho r^2$$

assume $M(r)$ is conserved as r changes with expansion/contraction of sphere

$$\boxed{M(r) = \frac{4\pi}{3}\rho r^3}$$

given mass conserved we get $\rho \propto \frac{1}{r^3}$

After some playing around Max also includes a Λ-force field

$$\underset{\sim}{f} = \frac{\Lambda}{3} r\,\hat{r}$$

$$V_\Lambda = -\frac{1}{2}\frac{\Lambda}{3}r^2$$

Max knows the linear force has a symmetry with gravity and it's a single force

and also a general PE U:

Apply conservation of mechanical energy to test particle of mass m

$$E = \frac{1}{2}mv^2 - \frac{GM(r)}{r}m - \frac{1}{2}\frac{\Lambda}{3}mr^2 + U$$

$$= \frac{1}{2}mv^2 - \frac{4\pi}{3}G\rho m r^2 - \frac{1}{2}\frac{\Lambda}{3}mr^2 + U$$

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Now the scaling behavior of the universe cannot depend on the pecularities of a test particle m, r, v, E

somehow all must vanish.

Max defines $r = a(t)\, r_0$

$\dot{r} = \dot{a}(t)\, r_0$ where $a(t)$ is a universal scale factor.

Divides thru by $\frac{1}{2} m r_0^2$

$$\frac{2E}{m r_0^2} = \dot{a}^2 - \frac{8\pi G}{3}\rho a^2 - \frac{\Lambda}{3} a^2 + \frac{U}{\frac{1}{2} m r_0^2}$$

Or else we can't have all particle distances scale with "a"?

~~We~~ ~~demand~~ this ~~were a~~ to be a universal constant for all particles

Max defines

$$k = -\frac{2E}{m r_0}$$

(hr-24)

The minus is because Max is clairvoyant about GR.

Then this would a universal expression

Except this spoils things unless it has peculiar properties

We've ~~already~~ already got gravity and the linear force. U would have to be something odd with NO obvious motivation from known forces

So Max ansatz it must be zero.

~~Say $U = V(a)$~~

Say $U = V(a)\, m [r^{(n)}]^2$

where $r^{(n)}$ is the nth order derivative of r

In fact a reason for the linear force ~~is~~ that it does satisfies this requirement for a universal one, it must exist!!!?

Rewriting, we get Friedmann's equation (FE) in standard form

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

If mass is conserved $\rho \propto \frac{1}{a^3}$ (see p. 3211)

Mass Conservation

2) But Max wonders what if mass is not conserved.
What if it mysteriously disappears/appears and what can ~~that~~ depend on in any obvious way? ⟶ depends on "a".

Why does Max wonder about mass nonconservation? His annual faculty evaluation at Cambridge Univ. is due and he is desperately procrastinating

Max ansatz $\rho = \rho_0 \left(\frac{r_0}{r}\right)^n = \rho_0 a^{-n}$ where n is general

But what should be the potential energy? (or potential)

$$g = -\frac{4\pi}{3}\frac{G\rho_0 r^3 \left(\frac{r_0}{r}\right)^n}{r^2} \propto r^{1-n}$$

$$\therefore V = \frac{4\pi}{3} G\rho_0 r_0^n \frac{r^{2-n}}{2-n}$$

$$= \frac{4\pi}{3} G\rho_0 \left(\frac{r_0}{r}\right)^n \frac{r^2}{2-n}$$

$$= -\frac{4\pi}{3} G\rho r^2 \frac{1}{n-2}$$

This is just like on p. 3211 Except $\frac{1}{n-2}$

and we recover that case if $n = 3$

Not worrying about $n=2$ case which leads to V logarithmic and we are going to dispense with all n except $n=3$ anyway.

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But ~~Max~~ Max say ~~this~~ potential is different for every kind of mass nonconservation (we've considered)

and ~~so~~ so leads ~~to~~ to a different Friedmann Equ

[and at this point Max wonders who is Friedmann?]

But mass and it's ~~effect~~ ~~con only~~ must be unique or we have no guidance.

It works for mass conservation. It must be same in all cases.
"must" in physics means "maybe".

So Max ansatzes

$$V = -\frac{4\pi}{3} G \rho r^2$$

for all cases of mass variation.

And n=3 which has mass conservation too, see 3211

At this point, the Friedmann eqn must be somehow be more general ~~than~~ conservation of mechanical energy in Newtonian physics — but whatever ~~more~~ more general physics applies, it is still an integral of motion.

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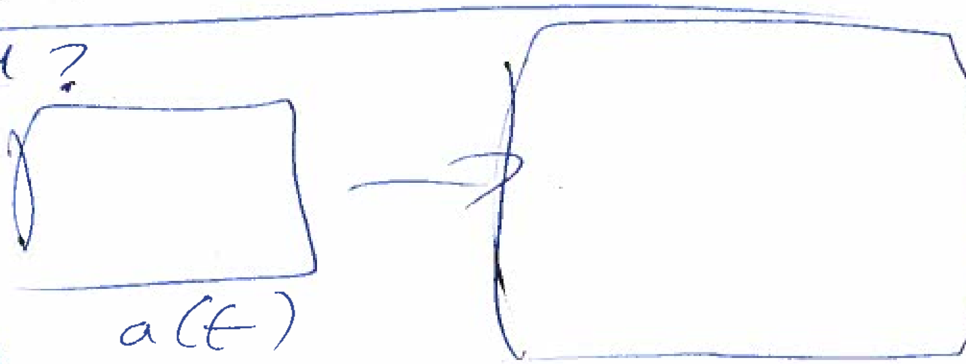
e) But where does mass non conservation arise from?



Well ~~the~~ form assumed $\rho = \rho_0 a^{-n}$ (or $\Omega = \Omega_{0p} x^{-p}$ in ~~terms~~ of J2026)

says changes ~~with~~ volume,

What changes with volume.

Being one of ~~the~~ developers of 19th century statistical ~~classical~~ mechanics Max knows ~~the~~ 2nd law of Thermodynamics (Wik: 2nd law Energy available) 1st law

Actually Probably some clairvoyance for this is needed by Max [illegible]

↓ But we know about entropy somewhat (Wik)

$$\underbrace{dE}_{\text{thermal (internal) energy}} = \underbrace{T}_{\text{Temperature}}\,\underbrace{dS}_{\text{entropy}} - \underbrace{P}_{\text{pressure}}\,\underbrace{dV}_{\text{Volume}} + \underbrace{\mu}_{\text{chemical potential}}\,\underbrace{dN}_{\text{particles}}$$

Differential forms where denominator is anything, but usually time

So energy can change with volume (an inside ~~thing~~ source)

And Max doesn't want heat energy or particles from outside (outside universe) we hope

3216 (2025 jan 05)

But says Max, this is the 19th century, and so mass and energy are different things.

But Max also discovered Maxwell's equations of electromagnetism and derived $c \triangleq 3 \times 10^8$ m/s as the unique velocity of electromagnetic ~~waves~~ waves

and mc^2 has dimensions of energy like $\frac{1}{2}mv^2$

And there is some kind of perfect fluid stuff with pressure of some kind

and so Max say's let's assume mass is a form of energy; ~~that~~ $E = mc^2$ at least for perfect fluids

and so

and $E=mc^2$ works for the perfluid if nothing else

$$c^2 dM = dE = -P\,dV$$

$$d(\rho V) = -\frac{P}{c^2}\,dV \quad \text{and} \quad V \propto a^3$$

$$V\,d\rho + \rho\,dV = -\frac{P}{c^2}\,dV$$

$$d\rho = -\frac{dV}{V}\left(\rho + \frac{P}{c^2}\right)$$

$$dV \propto 3a^2\,da$$

$$\frac{dV}{V} \approx 3\frac{da}{a}$$

(2025 Jan 05)

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$$d\rho = -3\frac{da}{a}\left(\rho + \frac{p}{c^2}\right)$$ < $\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$

Now what says Max.
Simplest idea → simplest equation of state (EOS)

$$p = w c^2 \rho$$

Note, this pressure can be formal. It doesn't have to push/pull on anything including itself and the form of the energy it ~~[illegible]~~ destroys or creates could be anything e.g., particle energy, particles

where w is a constant
(just ~~the~~ w parameter of the cosmological perfect fluid in modern astro jargon — No other name it seems)

$$\therefore\ d\rho \propto -3\frac{da}{a}(\rho + w\rho)$$

$$\frac{d\rho}{\rho} = -3\frac{da}{a}(1+w)$$

$$\ln\rho = -3(1+w)\ln a$$

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$$

of Fulvio Melia

The ~~commonest~~ specified possibilities,

$\rho_0\left(\frac{a_0}{a}\right)^3$ NR matter
$w = 0$ "matter"

$\rho\left(\frac{a_0}{a}\right)^4$ ER matter
$w = \frac{1}{3}$ "radiation" (as well move later)

ρ_0
$w = -1$ constant dark energy or cos-constant

$\rho_0\left(\frac{a_0}{a}\right)^2$
$w = -\frac{1}{3}$ R=ct universe

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The acceleration Equation or 2nd Friedmann Eq.

From p. 3213

$$\dot{a}^2 = \frac{8\pi G \rho a^2}{3} - k + \frac{\Lambda}{3} a^2$$

Differentiate

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left[\dot{\rho} a^2 + 2\rho a \dot{a}\right] - 0 + 2\frac{\Lambda}{3} a\dot{a}$$

curvature term vanishes

$-3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$

Without the pressure EOS and $E=mc^2$, Max, would not know what $\dot{\rho}$ is except for $n=3$.

It was an integral of motion.

divide thru by $2\dot{a}$

$$\ddot{a} = \frac{4\pi G}{3}\left[-3a\left(\rho + \frac{p}{c^2}\right) + 2\rho a\right] + \frac{\Lambda}{3} a$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho + \frac{3p}{c^2}\right] + \frac{\Lambda}{3} \qquad \text{(Li-27)}$$

or with $r = r_0 a$ and r_0 cancels all

$$\ddot{r} = -\frac{4\pi G}{3}\left[\frac{\rho r^3}{r^2} + \frac{3p}{c^2} r\right] + \frac{\Lambda}{3} r$$

$$\ddot{r} = -\frac{GM(r)}{r^2} - \frac{4\pi G}{3}\left(3\frac{p}{c^2}\right) r + \frac{\Lambda}{3} r$$

shell theorem force for spherical symmetric mass distribution.   what? Λ-force

32(9

Max wonders what the heck is the middle term. You don't get that from Newtonian physics.

But recall he introduced

- mass non-conservation, (p. 32(3))
- a special form of ~~potential~~ (p. 32(4))
- $E = mc^2$ for his perfect fluid, (p. 32(6)) but maybe nothing else.

So maybe the pressure term is OK. It arises in the more general theory — i.e., general relativity.

I wonder about it too, but general relativity gives it

and ~~the~~ field equation from which Friedmann equation is derived without special hypotheses has it

$$\underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} + \underbrace{\Lambda}_{\text{cosmological constant}} \underbrace{g_{\mu\nu}}_{\text{metric tensor}} = \frac{8\pi G}{c^4} \underbrace{T_{\mu\nu}}_{\text{stress energy tensor (momentum energy tensor)}}$$

For perfect fluid (wiki)

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) u^\alpha u^\beta + p g^{\alpha\beta}$$

So pressure comes from GR.

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9) Cosmological Constant and dark energy perfect fluid (i.e., constant dark energy)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{gen} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$H^2 = \frac{8\pi G}{3}\left[\rho + \frac{\Lambda}{8\pi G}\right]$

$-\frac{k}{a^2}$

~~def $\rho_{gen} = \rho + \rho_{w=0}$ other a constant density no matter how volume changes~~

~~$= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3}$ see p. 32(7)~~

~~Both constant terms.~~

Thus, we identify $\therefore$ $\rho_\Lambda = \frac{\Lambda}{8\pi G}$

or $\Lambda = 8\pi G \rho_\Lambda$ (Li-46)

But ~~that to that~~ our simple equation of state

$P = w\rho c^2$ with $w = -1$ for constant density (see p. 32(7))

$\therefore P = -\rho_\Lambda c^2$

$\rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$

(2025 jan 04 notes)

So if you interpret the cosmological constant as a ~~form of matter~~ perfect fluid, it has a negative pressure.

But what does that mean?

~~It doesn't~~ It pulls in and if you expand it, energy goes into the perfect fluid not out.

But where does it come from?

But what does it pull in on.

Nothing we know of

Maybe itself only, but in the cosmological principle sense, the perfect fluid is ~~constant~~ density and there is no pressure gradient.

Maybe it doesn't pull on itself.

In ~~what form~~ is the mass-energy? ~~the~~ perfect fluid stuff? some kind of particle ~~that~~ doesn't clump under gravity. Dark photon (wik)

~~Probably~~ not thought of meaning

Thermal equilibrium photons have a pressure (+ve pressure)

$$P = \frac{1}{3} a T^4$$

radiation constant

photon gas, but without pair creation they just pass each other without interaction.

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So ~~the~~ negative pressure of of dark energy perfect fluid may be purely formal.

~~A~~ Quantum field theorists (QFT) ~~think~~ vacuum field energy with -ve pressure is a nifty idea but natural QFT theories — of it predict

$$\Lambda_{QFT} \cong 10^{120} \Lambda_{observed}$$

one of largest theoretical discrepancies in physics.

Historically, E. Schrödinger in 1918 (before he was famous) suggested the cosmological constant could be a perfect fluid with -ve pressure

and Georges LeMaitre also suggested a -ve pressure perfect fluid as a replacement ~~of~~ of the cosmological constant. (WiK)

(Vacuum Energy)

~~There~~ must be some
observable differences
between a true cosmological constant
and a dark energy perfect fluid

But I haven't ~~noticed~~ looked
for any discussion.

Note gravity and cosmological
~~constant force~~
none ~~that~~ odd symmetry
we've discussed
and seem to be the only
non-exotic forces
allowing the Friedmann equation
(see p. 3212)

So Λ or ~~dark~~ energy perfect fluid.

The question is moot.

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3) The Other Form of ~~the~~ Friedmann Equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_{\text{mass-energy}}}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$= \frac{8\pi}{3} G \Big[\rho_{\text{ME}} \underbrace{- \frac{3k}{8\pi G a^2}}_{\equiv \rho_k} + \underbrace{\frac{\Lambda}{8\pi G}}_{\equiv \rho_\Lambda} \Big]$$

p. 3201, 3213

$$= \frac{8\pi}{3} G\rho$$

Define $\frac{a}{a_0} = x$

I can stand working with "a" or variable (it's a constant like the radiation constant)

Let

$$\tau = \frac{t}{t_{H_0}} = H_0 t \text{, a scaled time}$$

Let $\dot{x} \equiv \frac{dx}{d\tau}$

where H_0 is ~~the~~ Hubble constant, but it could be a general one. Not just our $H_0 \cong 70\ \frac{\text{km/s}}{\text{Mpc}}$

~~Let $h = H/t_H$~~

$h = H/H_0$

$$h^2 = \left(\frac{\dot{x}}{x}\right) = \frac{8\pi G}{3H_0^2}\rho$$

$$\rho_{\text{critical}} \simeq \frac{3H_0^2}{8\pi G} = \begin{cases} (0.9203874.. \times 10^{-26})\ \text{kg/m}^3 \left(\frac{H_0}{70}\right)^2 \\ 0.1359834.. \times 10^{12}\ \frac{M_\odot}{\text{Mpc}^3} \left(\frac{H_0}{70}\right) \end{cases}$$

~1/10 of a large galaxy in a Mpc^3

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$$h^2 = \frac{\rho}{\rho_{critical}} = \Omega$$

The scaled Friedmann equation.

Omega ~~the~~ density parameter

$$= \underbrace{\Omega_{ME}}_{\text{mass energy}} + \Omega_k + \Omega_\Lambda$$

At cosmic present

Recall $h = \frac{H}{H_0}$

$$h^2 = 1 = \Omega_{ME_0} + \Omega_{k_0} + \underbrace{\Omega_\Lambda}_{\text{always const}}$$

$$\Omega_{k_0} = 1 - \underbrace{\Omega_{ME_0} - \Omega_\Lambda}_{\text{if these sum to } -1}$$

~~then~~ $\Omega_{k_0} = 0$

$$\Omega_{k_0} = - \frac{3k}{8\pi G a_0^2} \frac{1}{x^2}$$

$\hookrightarrow x = 1$ at cosmic present

So $\Omega_{k_0} = 0$ implies the universe is flat. We'll look more a curvature of the universe later.

Note $k > 0$ is +ve curvature but $\Omega_k < 0$ then $k < 0$ is −ve curvature but $\Omega_k > 0$

Annoying change of sign.

3226 | 2025 Jan 05

What kind of Omega are considered.
Typically inverse-power law ones

~~h =~~

$$h^2 = \Omega = \sum_{p=4}^{0} \Omega_{p0} x^{-p}$$

Note 5 parameters needed.
If you knew H_0, then you could ~~eliminate~~ fix one of them, e.g., $\Omega_{2,0} = k$ as on p.3225

$$= \Omega_{40} x^{-4} + \Omega_{30} x^{-3} + \Omega_{20} x^{-2} + \Omega_{10} x^{-1} + \Omega_{00} x^{-0}$$

$p = 0$ for Λ cosmological constant or dark energy, (Je-II)

$p = 1$ for some quintessence theories

$p = 2$ for curvature, cosmic strings (some theories), $R_h = ct$ universe of Fulvio Melia.

$p = 3$ "matter" in the cosmological sense of matter at rest or nearly in comoving frames
all baryonic and dark matter
— really means non-relativistic matter

$p = 4$ "radiation" for radiation, but also any stuff moving at extreme relativistic speeds $\sim c$

(2025jan05) (3227

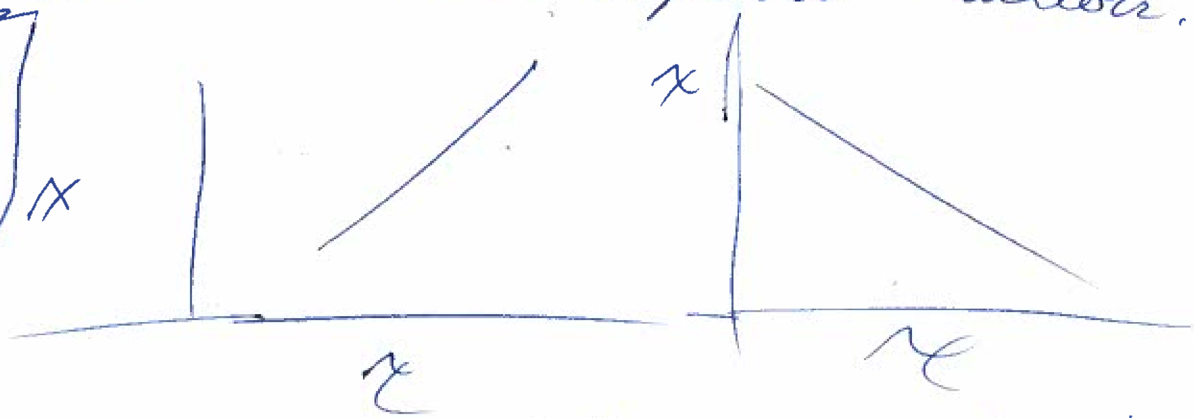
4) Elementary Solutions (single density components) of the Friedmann equation (FE)

$$\frac{\dot{x}}{x} = \pm\sqrt{\sum_{p=4}^{0} \Omega_{p0}\, x^{-p}}$$

Scale ϕ form I remember. Best for analytic solutions

always +ve and negative solution.

There are some more complex solutions that can transition between these but only once I think, No pure FE cyclic universes



The contracting solutions are of little interest ~~little~~ since we live in an expanding universe.

The FE is nonlinear, and so added solutions are NOT solutions in general (and almost always NOT solution)

Single Component FE

$$h = \frac{\dot{x}}{x} = \sqrt{\Omega_{p0}\, x^{-p}}$$

No cases I know of

$\Omega_{p0} = 1$ for single component of course

Except $\Omega_{k,0} < 0$ for +ve curvature has no real solution

which is easy to solve.

$$d\tau = \frac{dx}{x\sqrt{\Omega_{p0}\,x^{-p}}}$$

$$= \frac{dx}{\sqrt{\Omega_{p0}}\;x^{-p/2+1}}$$

$\tau_{age} = \frac{2}{p}$

$\tau_{age} = \frac{2}{p}\frac{1}{H_0}$ in all cases

$$\tau = \frac{1}{\sqrt{\Omega_{p0}}}\frac{x^{p/2}}{p/2}$$

where we set $\tau(x=0)=0$ or $x(\tau=0)=0$

$\tau(x)$ solution

$\tau(x=1)$ = age of universe if so-called

$$x(\tau) = \left[\sqrt{\Omega_{p0}}\,\frac{p}{2}\,\tau\right]^{2/p}$$

~~Since~~ these have a "Big Bang Singularity" or in the older jargon a point origin (in time)

$q = -\frac{\ddot{x}}{x}\frac{1}{h^2}$

$q_{p=0} = -1$

general except for $p = 0$

$$\tau = \frac{1}{\sqrt{\Omega_{p0}}}\ln x + C$$

$$x = x_0\, e^{\sqrt{\Omega_{p0}}\,\tau} = x_0\, e^{\sqrt{\Omega_\Lambda}\,\tau}$$

q not so important anymore, We want the whole $a(t)$

This solution has no point origin — The exponential solution for the cosmological constant.

$h_{p=0} = \sqrt{\Omega_{p0}}$

$h = \frac{\dot{x}}{x} = \frac{2}{p}\tau^{-1}$

Hubble parameter $H = \frac{2}{p}\frac{1}{t}$

deceleration parameter

$$q = -\frac{\ddot{x}\,x}{\dot{x}^2} = -\frac{\frac{2}{p}\left(\frac{2}{p}-1\right)\tau^{\frac{2}{p}-2}\,\tau^{\frac{2}{p}}}{\left(\frac{2}{p}\right)^2\left(\tau^{\frac{2}{p}-1}\right)^2} = -\frac{\left(\frac{2}{p}-1\right)}{(2/p)} = -(1-p/2) = \frac{p}{2}-1$$

-ve sign because people wanted a [illegible]

(2029 jan 05) (3229

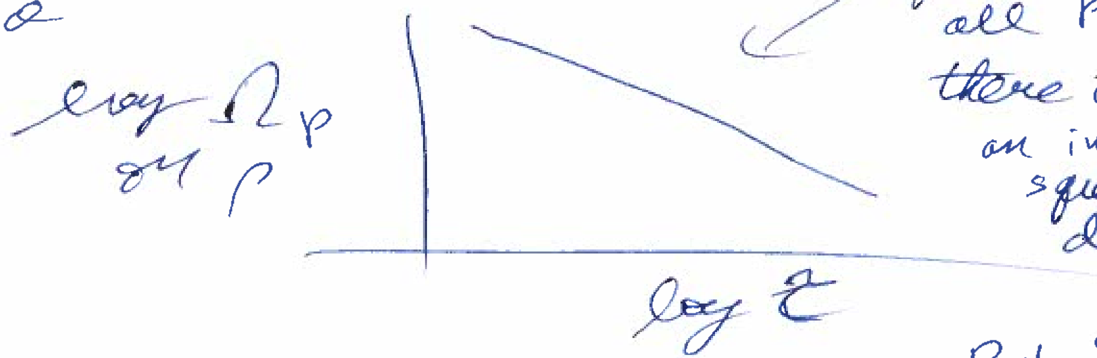
Density decline for elementary solution

$$\Omega_p = \Omega_{p0} x^{-p}$$

$$= \Omega_{p0}\left[\sqrt{\Omega_{p0}}\,\frac{p}{2}\,\tau\right]^{-2} = \frac{(2/p)^2}{\tau^2}$$

$$\propto \tau^{-2}$$

So a

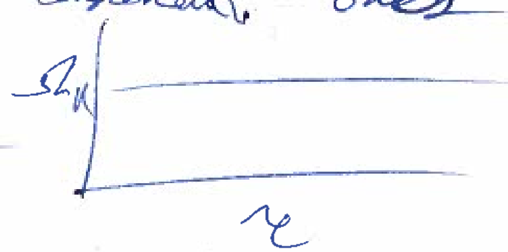


for all $p \neq 0$ there is an inverse square decline

But just for elementary solutions, Not multi-component ones

$$\Omega_{p=0} = \Omega_\Lambda = 1$$ for single component

a constant of course



Special cases of interest

$p = 4$ applies to early universe before ≈ 50 kyr when universe was radiation dominated

$$x(\tau) = \left[\Omega_{40}(2)\,\tau\right]^{\frac{1}{2}} \sim \tau^{\frac{1}{2}}$$

3230 | 2025 Jan 04

$p = 3$

$$x = \left[\sqrt{\Omega_{3,0}}\ \frac{3}{2}\ \tau\right]^{2/3} \sim \tau^{2/3}$$

This is for matter and is called the Einstein-de-Sitter universe (1932) < EdS universe

They proposed it as ~~the~~ simplest of all universes

$\Omega_{k\ (\text{ie. } p=2)} = 0$ < so flat

$\Omega_{\Lambda} = 0$

Just one density component

and so $\Omega_{30} = \dfrac{\rho}{\rho_{crit}} = 1$

In fact identifying

$$\rho_{crit} = \frac{3H_0^2}{8\pi G}$$

was one of the main points of the paper of EdS.

~~They~~ never wrote the $x(\tau)$ formula in the paper (O'Raifeartaigh p. 6-7)

A contemporary cosmologist Heckmann described the paper as not very profound)

Age of universe of EdS

$$\tau(a_c=1) = \frac{1}{\sqrt{\Omega_{3,0}}}\,\frac{1}{3/2}$$

$\Omega_{3,0} = 1$ since only matter

$$= \frac{2}{3}$$

$$\tau = \frac{2}{3} H_0^{-1}$$

see p 3094d Natural unit

$$\frac{2}{3}\,\frac{1}{500}\,\frac{\text{Mpc}}{(\text{km/s})} = \frac{2}{3}\cdot 2\ \text{Gyr} = \frac{4}{3}\ \text{Gyr}$$

1932 value

This was too short for Earth age $\gtrsim$ 2-3 Gyr in those days

$$\frac{2}{3}\,\frac{1}{70}\,\frac{\text{Mpc}}{\text{km/s}} = \frac{2}{3}\cdot 14\ \text{Gyr} \simeq 10\ \text{Gyr}$$

2025 value

But much too short

Globular cluster ages $\gtrsim$ 13 Gyr

$$\frac{2}{3}\,\frac{1}{50} = \frac{2}{3}\cdot 20 = 13\ \text{Gyr}$$

OK

$50 \leftrightarrow 100$

1960 – 1990s

$$\frac{2}{3}\cdot 10 = 7\ \text{Gyr}$$

Too short

Precisely because it was the simplest model and data allowed it in 1960–1990s, it was then the standard cosmology

Recall p. 3094d

Natural time unit

$$\frac{1\ \text{kpc}}{1\ \text{km/s}} \simeq 0.98\ \text{Gyr} \simeq 1\ \text{Gyr}$$

$$\therefore \frac{1000}{70} \simeq 14\ \text{Gyr}$$

~~32~~ 32

But even before
the discovery
of the acceleration
of Universe in 1998,
the EdS universe ~~was~~ was
falling out of favor

$$\Omega_{M_0} \simeq 0.3$$

This was being estimated for baryonic and dark matter

But the
inflation paradigm
said $\Omega_0 = 1$ to very high accuracy

(Though Not exactly 1)

∴ ~~A~~ And
$H_0 \sim 60-80$
So the range was narrowing
getting close to ruling
out the EdS age. even if the universe was flat

So EdS fell out and with acceleration
of universe proof and
we now ~~have~~ Λ-CDM universe

But from 50 kyr to 10 Gyr in Λ-CDM
there is a matter dominated universe
EdS phase.

$P = 0$

$$x = x_0 e^{\sqrt{\Omega_\Lambda}\,\tau}$$

de Sitter found this solution from GR in 1917
but not finding ~~the~~ the Friedmann equation
— It was a tricky ~~effort~~

$$\sqrt{\Omega_\Lambda}\,\tau = \cancel{\sqrt{\Lambda 8\pi G}}\, H_0 t$$

see p-3224

$$= \sqrt{\frac{\Lambda/(8\pi G)}{\frac{3H_0^2}{8\pi G}}}\, H_0 t = \sqrt{\Lambda/3}\; t$$

$$\therefore\; a = a_0 e^{\sqrt{\Lambda/3}\; t}$$

$$H_{de} \quad \frac{\dot a}{a} = \sqrt{\Lambda/3}$$

$\therefore$ The invariant Hubble parameter of ~~the~~ de Sitter universe is

$$H_{deSitter} = \sqrt{\Lambda/3}$$

Λ-CDM value $\Omega_\Lambda \doteq .7$

$$= \sqrt{\Omega_\Lambda}\, H_0$$

$$= \sqrt{0.7} \cdot 70 = 0.85 \cdot 70 \doteq 60 \;\frac{km/s}{Mpc}$$

3234)

~~The~~ de Sitter universe had no matter
→ it was a pure cosmological ~~constant~~ universe

But it did predict the expansion of universe and the cosmological redshift and ~~Edwin~~ Hubble was aware of that in the 1920s.

And implicitly it obeys Hubble's law but no one before Lemaître 1927 published explicitly for all FE models

Of course, it hasn't gone away since $t = 10$ Gyr we've been in a Λ dominated universe and if the Λ-CDM model is extrapolated to the future, we asymptotically approach a cold, low density de Sitter universe.

Also inflation era is predicted to be a de Sitter phase

5) ~~General~~ ~~Exact~~ Non-Elementary Solutions

After a foray into cosmic geometry cosmological Redshift and cosmic distance measurement, we'll look at these again

a) Just a ~~general~~ ~~1st~~ first look.

First note from p. 3227

$$\frac{\dot{x}}{x} = \sqrt{\sum_{p=4}^{0} \Omega_{p_0} x^{-p}}$$

~~that~~ and so

$$d\tau = \frac{dx}{x\sqrt{\sum_{p=4}^{0} \Omega_{p_0} x^{-p}}}$$

and so numerical solutions are always done $\tau(x)$ or $t(a)$

and not $x(\tau)$ and $a(t)$

Our one-component solution (p 3228) were also $\tau(x)$ and then inverted.

Most ~~Some~~ ~~All~~ other exact solutions are easier to solve $\tau(x)$.

~~Maybe all are. Not sure~~

I think there is ~~maybe~~ ~~one at least~~ where $x(\tau)$ is obtained first. However there are some where exact $t(x)$, but no exact $x(t)$

b) Is ~~there~~ a 5-component exact solution?

Mostly no. You can solve for $x = x(\eta)$

3236)

$a = a(\eta)$

where η is conformal time

$d\eta = \frac{c\,dt}{a}$ (Steiner 6)

but $a(\eta)$ is in (Steiner p. 9)

terms of the

Weierstrass elliptic $\wp$-function

(Steiner p. 7)

which is analytic only by definition since it's an infinite sum

(but so is the sine function but the sine function is familiar — our friend the sine function)

If you want ~~$a(t)$~~ $t(a)$

you need to numerically integrate for ~~$t(a)$~~ $dt = \frac{a}{c}\,d\eta$ and then numerically get $a(t)$

since to get $t(a)$ and $a(t)$ you need numerical solutions, I don't consider $a(\eta)$ a full analytic solution

c) Radiation-Matter Universe

However there are several important exact solutions

$a = a(t)$ for $p=4$ and $p=3$

and $t = t(a)$ radiation-matter universe

which is the universe up to $t \sim 50$ kyr

Actually $t = t(a)$ is straightforward to solve for and not bad looking

Maybe we will solve $t = t(a)$ for this case. Easiest of the non-elementary solutions. But $a = a(t)$ is NOT.

$a = a(t)$ is requires solving a cubic and looks pretty opaque.

We'll find these ~~later~~ later

But good that it exists

d) Matter-Lambda universe

$r = 3$ and $P = 0$

~~Important because the~~ is the ~~most~~ Λ-CDM solution after $t \hat{=} 50$ kyr

$a(t)$ and $t(a)$ available and ~~that~~ ~~these~~ are pretty solutions

~~We look at them later~~

In fact all $p > 0$ and $P = 0$ can be obtained

p doesn't need to be an integer

Ja-24

e) Solution for 3-component case

$P = 0$, $P = 4$ and $R = 2q$

Solar this could be

Radiation Analogue to Lemaître universe for $\Omega_{2,0} < 0$

$$\Omega_{4,0} x^{-4}, \ \Omega_{2,0} x^{-2}, \ \Omega_{\Lambda}$$

or

Analogue to Lambda-CDM with ~~radiation~~ replaced by non-physical $\Omega_{6,0} x^{-6}$

$$\Omega_{6,0} x^{-6}, \ \Omega_{3,0} x^{-3}, \ \Omega_{\Lambda}$$

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Interesting solutions mathematically and the first is analogue ~~to the~~

Lemaître universe

static Einstein phase

F ~~Radiation~~ Matter phase

Replaced by radiation phase in analogue case.

exponential phase

$$\Omega_{3,0}\,a^{-3},\ \Omega_{2,0}\,a^{-2},\ \Omega_\Lambda$$

which we will also consider

$\Omega_2 < 0$ and so a positive curvature univers

No analytic solution

which has ~~and~~ No analytic solution

And Λ-CDM ~~[illegible]~~

$$\Omega_{4,0}\,a^{-4},\ \Omega_{3,0}\,a^{-3},\ \Omega_\Lambda$$

~~So~~ no full analytic solution for the Λ-CDM universe

Which have ~~to~~ exact solutions for early and late and we have to join them together in the mid matter phase.

6) Approximate Solutions

— but not generally.

$$\frac{dx}{\wedge x} = \sqrt{\sum_{p=4}^{0} \Omega_{p,0}\, x^{-p}}$$

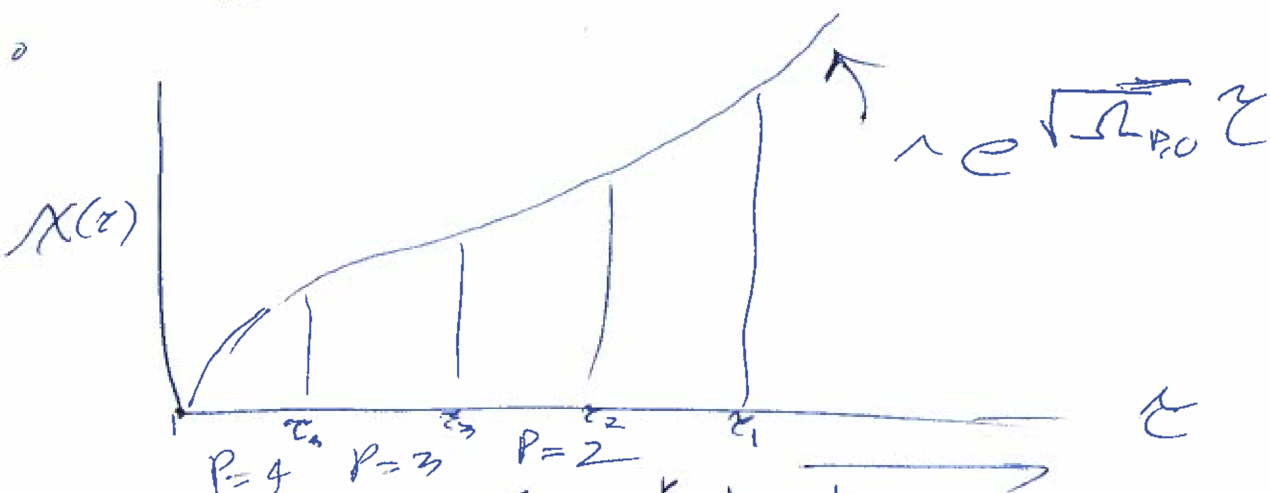
We assume $\Omega_{2,0} > 0$. Dealing with +ve curvature cases where ($\Omega_{2,0} < 0$) is another story

So the terms start with $p=4$ dominant and then have dominance phases $p=4$, $p=3$, $p=2$, $p=1$ and $p=0$

Of course a phase could be missed if $\Omega_{p,0}$ was too small or zero

Each dominance controls the slope of the growth and "starts" at the height the last one "ended" on.

So



we assume $\Omega \;\; > 0$ so −ve curvature

 . (2025 jan 06)

It would be nice if we could calculate τ_p's directly from the $\Omega_{p,0}$,
But we need to do it from the solutions (see p. 3228)
elementary solutions are all piecewise approximations.

$$\tau = \frac{1}{\sqrt{\Omega_{p,0}}}\left(\frac{2}{p}\right)\left[x^{p/2} - x_{p+1}^{p/2}\right] + \tau_{p+1}$$

and $\tau = \frac{1}{\sqrt{\Omega_{4,0}}} \ln(x) + \tau_1$

<u>But</u> what are x_p's?

$$\Omega_{p+1,0}\, x^{-(p+1)} = \Omega_{p,0}\, x^{-p}$$

$$\frac{\Omega_{p+1,0}}{\Omega_{p,0}} = x_{p+1}$$

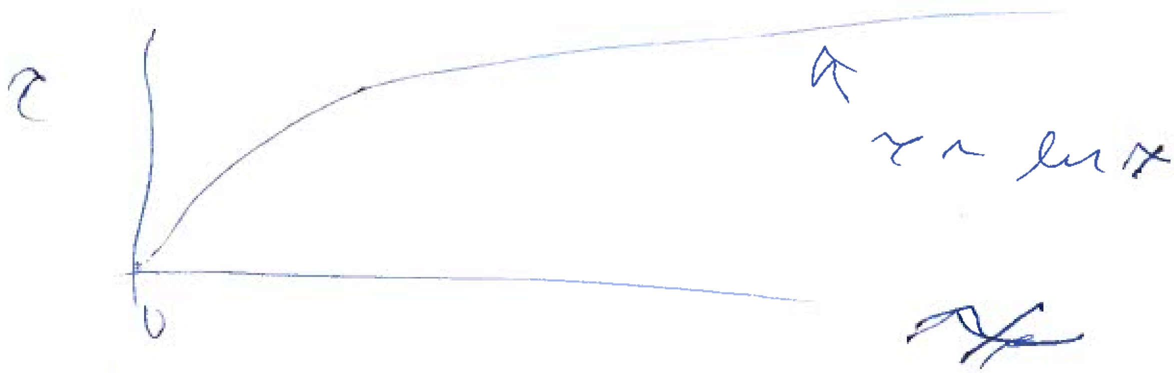
The equality x value. End of the $p+1$ phase, Start of the p phase.

Again assuming every ~~phase~~ $\Omega_{p,0}$ has a dominance phase

$$x_4 < x_3 < x_2 < x_1 < x_0 = \infty$$

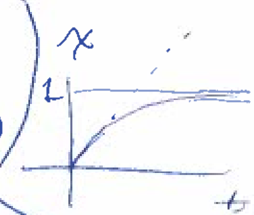
So we could calculate in piecewise fashion $\tau(x)$

[2025jan09] [3241]



But time is the coordinator of all physics, so we'd like even very crudely to have some sort of analytic approximation $x(\tau)$

linear growth from 0 and saturates at $x=1$



At the moment (but maybe not forever) I suggest

arctangent funct

$$x(\tau) = \prod_{p=4}^{1} \left[\underbrace{(1-\delta_{p,4})}_{0 \text{ for } p=4} + x_p(\tau_p)\, \mathrm{atan}\left(\frac{x_p(\tau)}{x_p(\tau_p)}\right) \right] e^{\sqrt{\Omega_4}\,\tau}$$

for $\tau \ll \tau_p$ $\quad x_p(\tau)$

for $\tau \gg \tau_p$ $\quad x_p(\tau_p)$ saturation

At best this could be sort of qualitative and maybe not good at all.

I was tempted to test it, but I lied down till ~~the~~ feeling went away.

3242 | 2025jan05

7) 1^{st} Order Autonomous Differential Equations

Have No Stationary Points
except at Infinity and
constant solution (a continuum
of stationary Points)

There can have stationary points NOT at infinity. { except for special cases which aren't that rare and include the Friedmann Eq.

a) Proof Except for the special cases

Let $x' = f(x)$

where $x' = \frac{dx}{dt}$ and t is ~~the~~ independent variable

and $f(x_i) = 0$ and so x_i is a zero of $f(x)$. There can be a set of zeros in general $\{x_i\}$.

We assume $f(x)$ is infinitely differentiable with respect to x (not t) at x_i

$\therefore$ We can Taylor expand around x_i

$$\underbrace{\Delta x'}_{x' = \Delta x'} = f(x) = \underbrace{f_0}_{\substack{= \\ 0 \\ \text{since } f(x_i)=0}} + \Delta x f_1 + \Delta x^2 f_2 + \ldots$$

$x' = \Delta x'$ since $\frac{dx}{dt} = \frac{d\Delta x}{dt}$

$f_i = \frac{1}{i!} \left.\frac{d^i f}{dx^i}\right|_{x_i}$ etc.

We truncate to lowest non-zero order.

$\therefore \Delta x' = \Delta x^{\ell} f_{\ell}$, where all $f_{k<\ell} = 0$

different by t

$$\Delta x'' = \ell \Delta x^{\ell-1} \frac{d\Delta x}{dt} f_{\ell}$$ ~~where we drop f_ℓ constant for simplicity~~

$$\propto \ell \Delta x^{\ell-1} \Delta x^{\ell}$$

$$\propto \Delta x^{2\ell-1}$$ ~~where we drop coefficients for simplicity~~

$$\Delta x''' \propto (2\ell-1)\Delta x^{2\ell-2}\Delta x'$$

$$\propto (2\ell-1)\Delta x^{2\ell-2}\Delta x^{\ell}$$

$$\propto \Delta x^{3\ell-2}$$

The pattern is clear: every time differentiate add ℓ to the exponent and subtract 1

$$\Delta x^{(n)} \propto \Delta x^{n\ell-(n-1)}$$

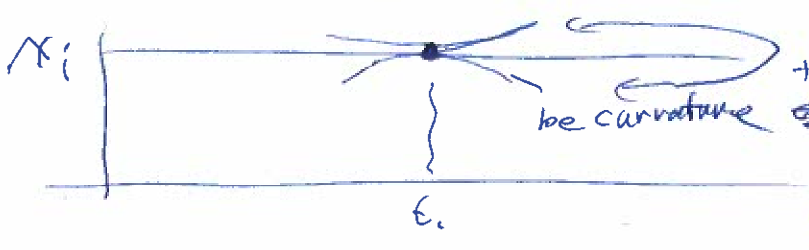
$$\propto \Delta x^{(n-1)\ell+1}$$

where $n \geq 1$, $\ell \geq 1$

$\therefore$ the exponent $p = (n-1)\ell + 1 \geq 1$

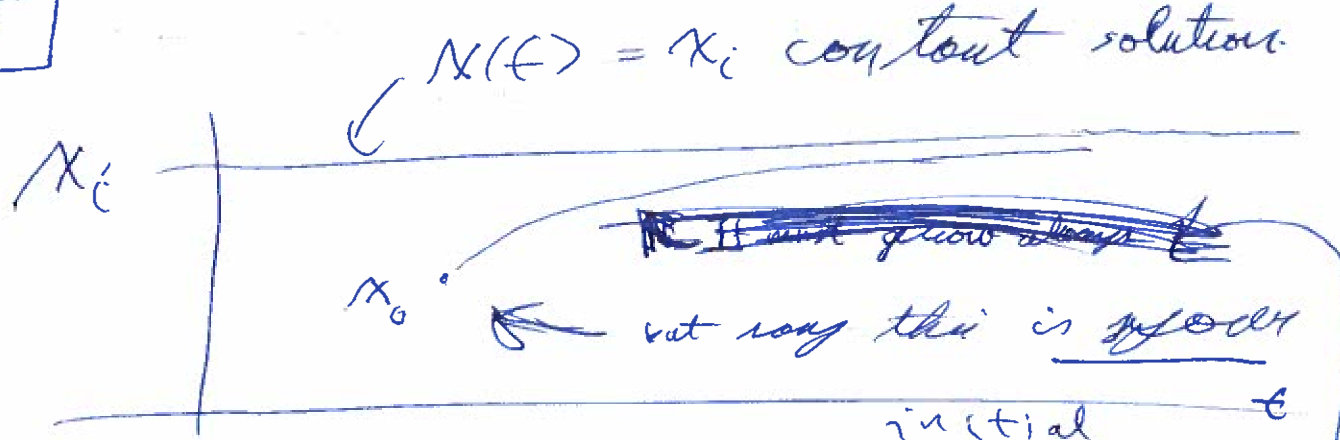
$\therefore \Delta x^{(n)}(\Delta x = 0) = 0$

$\therefore x^{(n)}(x_i) = 0$ to lowest order in Δx and therefore all orders to



there cannot be curvature about t_i point. So all ~~point~~ time points are t_i

344



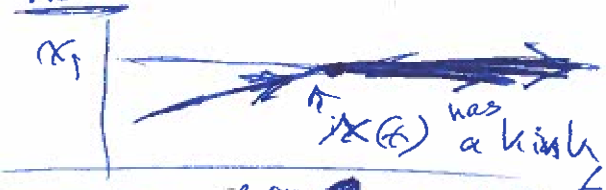
and $f(x<x_i)>0$
then $x(t)$ must always grow

– intuitively (?)
it is ~~clear~~ that $x(t)$
(with initial condition $x_0 < x_i$)
can only reach x_i at $t \to \infty$
since all orders of $x^{(n)}$ must
be zero when $x(t)$ reaches x_i

But can we
prove definitely?

Yes. Proof

Note



if $x(t)$ reaches x_i at finite time.

Let $y = \pm(x - x_i) = \pm\Delta x$ where
+ve case for $x > x_i$
-ve case for $x < x_i$
and so $y > 0$ always

$\Delta x' = \Delta x^{\ell} f_{\ell}$

$y' = (\pm 1)^{\ell-1} y^{\ell} f_{\ell}$

Case 1 $\ell = 1$ meaning $f_1 \neq 0$

$y' = y f_1$

(2025 Jan 05) 3245

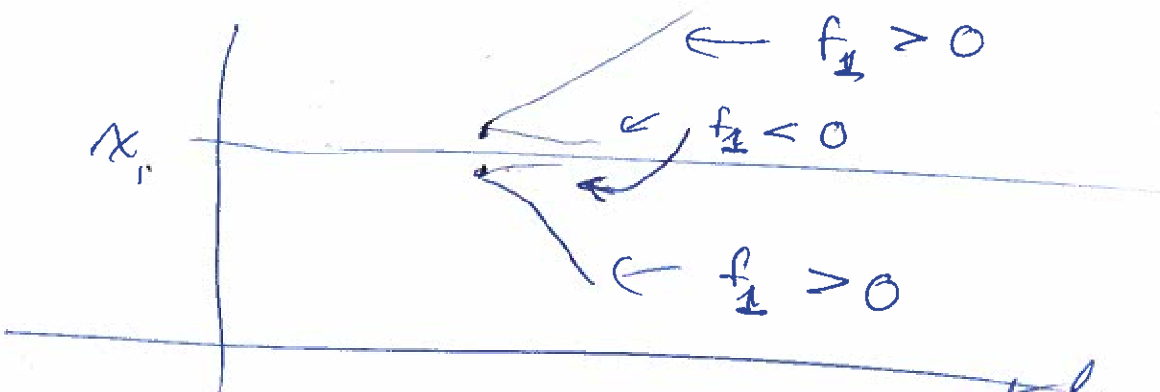
$$\frac{dy}{y} = f_1\,dt$$

$$\ln(y/y_0) = f_1(t - t_0)$$

$$y = y_0 e^{f_1(t-t_0)}$$

(±1) cancels on both sides

$$\Delta x = \Delta x_0 e^{f_1(t-t_0)}$$



So $f_1 < 0$ leads to exponential convergence but the stationary point is at infinity

A stable case of the constant solution since perturbations damp out

$f_1 > 0$ leads to divergence which is only exponential ~~to~~ to lowest order.

Case $\ell \geq 2$

An unstable case of the constant solution since perturbation lead to divergence

$$\frac{dy}{y^\ell} = (\pm 1)^{\ell-1} f_\ell$$

$$\frac{y^{-\ell+1} - y_0^{-\ell+1}}{-\ell+1} = (\pm 1)^{\ell-1} f_\ell (t - t_0)$$

3246

$$\pm\Delta x = y = \frac{1}{\left[\underbrace{(-l+1)}\,\underbrace{(\pm1)^{l-1} f_l (t-t_0)} + y_0^{-l+1}\right]^{\frac{1}{l-1}}}$$

$-l+1<0$ since $l \geq 2$

$y_0 > 0$ by definition

$\cancel{((\pm 1)^{l-1} f_l (t-t_0))}$

$(\pm1)^{l-1} f_l < 0$ for convergence and >0 divergence to infinity but only to the lowest order in l

~~if~~ if l is odd

$l-1$ is even

$(\pm1)^{l-1} = 1$

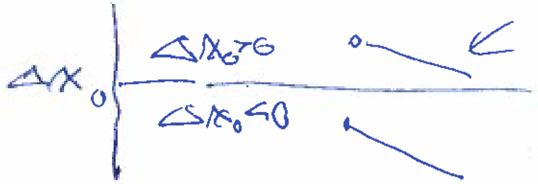
then $f_l < 0$, converges ~~gives convergy~~ (stable constant solution)

$f_l > 0$ diverges (unstable constant solution)

if l is even, $l-1$ is odd

$(\pm1)^{l-1} = \pm 1 \begin{cases} +ve & \Delta x_0 > 0 \\ -ve & \Delta x_0 < 0 \end{cases}$

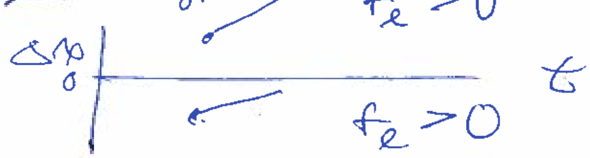
$f_l < 0$ for converges



$f_l < 0$ divergen

divergence

$f_l > 0$



$f_l > 0$

convergence

Note if ~~the~~ ~~constant~~ solution is only stable for perturbations of one sign, then it is ~~usually~~ unstable ~~since~~ if perturbations of any sign can happen.

It's ~~stable~~ if the ~~the~~ divergent perturbation are somehow constrained NOT to happen.

b) Special Cases where there are stationary points NOT at infinity

But only one kind of special case I know of that is relevant to the Friedmann Eq.

Say $x' = [f(x)]^{\frac{1}{k}}$ where $k>0$ to avoid pointless generality

and $x'(x=x_i) = [f(x_i)]^{\frac{1}{k}} = 0$

$x' = ax^{(n-1)}$ so $n \geq 1$ and integer

expand $f(x)$ which is infinitely differentiable about x_i where $f(x_i) = 0$

So $\ell \geq 1$ and integer

$$\Delta x' = [\Delta x^{\ell} f_{\ell} + \Delta x^{\ell+1} f_{\ell+1} + \ldots]^{\frac{1}{k}}$$

where ℓ is the lowest nonzero term and $\ell \geq 1$ and integer

$$= [\Delta x^{\ell} f_{\ell}]^{\frac{1}{k}} [1 + O(\Delta x)]^{\frac{1}{k}}$$

This term goes to zero as $\Delta x \to 0$ and so can be dropped

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2025 jan 05

$\therefore \quad x' = [x^\ell f_\ell]^{\frac{1}{k}}$ to lowest order in ℓ

$x' \propto x^{\frac{\ell}{k}}$

$x'' \propto x^{\frac{\ell}{k}-1} x' = x^{2\frac{\ell}{k}-1}$

$x''' \propto x^{2\frac{\ell}{k}-2} x^{\frac{\ell}{k}} = x^{3\frac{\ell}{k}-2}$

$\vdots$

$x^{(n)} \propto x^{\frac{\ell}{k}n-(n-1)}$

$= x^{(\frac{\ell}{k}-1)n+1}$

$\frac{\ell}{k} = \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots$

$1/1$ is just as on p3244. Physically of the other cases $\ell=2$, $k=2$ is most likely the Einstein universe case in fact

See p.

If $\frac{\ell}{k} = 1$, then $x' \propto x^1$ and we are back to p. 3244 with only a stationary point at ∞ ~~and~~ solution and exponential convergence/divergence

If $\frac{\ell}{k} > 1$, then exponent $p = (\frac{\ell}{k}-1)n+1$ increases for ever with n and all $x^{(n)}(x_i) = 0$ and again we have only stationary points at ∞ as on p. 3243

$x' = x^{\frac{\ell}{k}}$ and $\frac{\ell}{k}$ need not be an integer. But if it is exactly as on p.3244, 3245

If $\frac{\ell}{k} < 1$, then eventually we get $\bigcirc x^{-2}$ and as $\Delta x \to 0$ we have a singularity which is not a stationary point and may be unphysical

Unless ~~for~~ $\frac{l}{k} < 1$ there is a stopping n

such ~~that~~ $p = 0 = (\frac{l}{k} - 1)n + 1$

~~An ll~~ the stopping n,

$$x^{(n)} = C$$

Integrating n times

$$x = Ct^n + \ldots$$

~~but the constants~~ of integration have to be such ~~that~~ at t_i, $x = x_i$ and $x^{(n-1)} = 0$ and $x^{(n)} = C$

The stopping n formula is

$$\frac{l}{k} = 1 - \frac{1}{n}$$

$$= 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1$$

$n=1$ $n=2$ $n \to \infty$

This means $x'(x_i) = C \neq 0$ and ~~that~~ is <u>not</u> allowed by the hypothesis ~~then~~ $x'(x_i) = 0$.

The physically ~~most~~ relevant ~~for~~ $l = 1$ and $k = 2$ and this turns ~~up~~ in Friedmann Equation cases.

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(~~325~~ 2025 Jan 04)

~~FN~~Friedmann eqn

$$\frac{\dot{X}}{X} = \pm\sqrt{\sum_{p=q}^{0} \Omega_{p,0} X^{-p}}$$

Most relevant cases

$\frac{k}{l} = \frac{2}{2}$ and only ~~stationary~~ points at ∞ & ~~constant~~ solution

Einstein universe — constant solution

$\frac{k}{l} = \frac{1}{2}$ and a finite stationary point.

Lemaitre universe — no stationary point at all, so no constant solution

except as the parameters morph one to the other,

Einstein universe a limiting case of Lemaitre universe

I know of no others, but they probably exist, but maybe only for unlikely density behaviors

c) A super-interesting exception

A $k=2$ from p. 3248

$$x' = f(x) \quad \text{and} \quad f(x_i) = 0$$

$$x'' = \frac{df}{dx}\cancel{x}' \quad \text{but} \neq 0 \text{ at } x = x_i$$

$$x'' = \frac{df}{dx} f$$

Let

$$\frac{df}{dx} f = g(x)$$

$$\frac{1}{2} f^2 = \int g\,dx + C$$

$$f = \pm\sqrt{2\left(\int g\,dx + C\right)}$$

where $g(x)$ and C are chosen so ~~that~~ $f(x_i) = 0$

But $f(x)$ must always be real.

There are many possible solutions for $g(x)$ but the most obvious one (after a little playing around) is ......

OK, a lot of playing around

~~$g(x) = 2Dx$ which is not zero at $x = x_i$~~

~~$\int g\,dx = Dx^2$ and $D = -C < 0$~~

~~$f = \pm\sqrt{2(1 + x^2)}$~~

~~$\frac{df}{dx} = \pm$~~ ......

3252

2025 jan 05

$$x'' = \frac{df}{dx} f = -Cx = -G w^2 x$$

parameterize C as Gw^2

omega

$$\left(\frac{x_i}{x_i}\right)^2 \left(\frac{x_i + \Delta x}{x_i}\right)^2 \approx 1 + 2\frac{\Delta x}{x_i}$$

~~then~~ $\cancel{x'' = -w^2 x}$

~~and the general solution is an old friend~~

$$\cancel{x = A \sin wt + B \cos wt}$$

$$\cancel{x' = w(A \cos wt - B \sin wt)}$$

~~but this is not the original 1st order DE and has two integration constants.~~

~~So let's say B = 0 and then~~

$$\cancel{x = wA(\pm\sqrt{1 - \sin^2 wt})}$$

....is $f = \pm w x_i \sqrt{1 - \left(\frac{x}{x_i}\right)^2}$ which is 0 at $x = x_i$

Note this is $\frac{2}{4} = \frac{1}{2} = 1/2$ case on p. 3250

$$\frac{df}{dx} = \pm w x_i \frac{-\frac{x}{x_i^2}}{\sqrt{}}$$

$$x'' = f\frac{df}{dx} = (w x_i)^2 \left(-\frac{x}{x_i^2}\right)$$ which is not 0 at $x = x_i$

$x'' = -w^2 x$ which is an old friend.

Solution matching the condition

$x = x_i \cos(wt)$ which is a maximum x_i at $t_i = 0$

Note $x' = -w x_i \sin(wt) = -w x_i(\pm\sqrt{1 - \cos^2 wt})$

$= -w x_i (\pm\sqrt{1 - (\frac{x}{x_i})^2}) = f(x)$