~~3.17~~ 3.18 | 2025 jan 01 | 3 20

The Friedmann Equation

1) Introduction

After a long warm up in gravity, inertial frames, the Cosmological constant force, etc,

A Standard Form — ~~the~~ most standard form

Not form ~~for~~ finding exact solutions — you need scaled forms

(there are others)

Li-55

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

Cosmological constant Λ

Lambda

$G = 6.67430(15) \times 10^{-11}$ MKS units — The most poorly known fundamental constant (?)

$H = \frac{\dot{a}}{a}$

Hubble parameter — at cosmic present

$H = H_0 = h_{70}\left(70 \frac{\text{km/s}}{\text{Mpc}}\right)$

(The Hubble constant)

It's an inverse time quantity and is the relative rate of Expansion of ~~the~~ universe

mean density of all mass-energy here in mass units

$\rho = \rho(a)$

To my knowledge no one considers ρ as an explicit function of time or anything else. What else could it be a function of? A real question

k is a constant of integration called (Li-33) curvature

Can be can $\pm$ve and also

$k \to kc^2$

another form

$k>0$ hypersphere

$k=0$ flat

$k<0$ hyperbolical

But only GR tells us this

An integral of motion

us — 1 Mpc → 70 km/s

us — 2 Mpc → 140 km/s

$t = \frac{\text{Mpc}}{70\ \text{km/s}} = \frac{1000}{70} \simeq 14\ \text{Gyr}$

The solution is

$a = a(t)$ cosmic scale factor

$r = r_0\, a(t)$, t is cosmic time

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It's a ordinary, nonlinear 1st order differential equation

(nonlinear means solutions do NOT add to ~~of~~ solutions)

autonomous equation (Wik)

→ No explicit dependence on ~~the~~ independent variable cosmic time t

[unless you impose ~~an~~ an explicit time dependence.]

Being 1st order it has limited abilities to describe universal ~~order~~. It's Not about forces which require 2nd order DE's in our usual understanding.

But, in fact, the acceleration equation doesn't give any more dynamics. So the symmetries imposed rather than ~~order~~ of the Friedmann equation may be the real dynamic limitation

Physical distances = can be measured with a ruler at one instant in time (Cosmic time)

Other distance measures turn up
- luminosity distance
- angular distance
- cosmic time itself
- cosmological redshift

r

All distances r that participate in the mean expansion of the universe

~~obey~~

$$r = r_0 \, a(t)$$

r_0(cMpc) = $\frac{r\,(\mathrm{Mpc})}{a(t)}$

comoving Mpc

Note

$$\dot{r} = r_0 \, \dot{a}(t)$$

$$\ddot{r} = r_0 \, \ddot{a}(t)$$

r_0 are the comoving distances
- they are constant in time and by convention equal physical distances at cosmic present t.

Actually, there is a physical way of specifying "a" also ~~but~~ and we will consider that when we consider cosmic curvature

Then it has units of length

a has no units and by convention $a(t=t_0) = 1$

cosmic present

When one considers curved space there is form of a with units

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$v = \dot{r} = \dot{a} r_0$

recession velocities
— NOT ordinary velocities
— rate of growth of space in a GR sense
— or ~~speed~~ between separating inertial frames (comoving frames as I call them)

Perhaps universe stranger if they couldn't be added.

However, in a Newtonian physics sense, they are sort of the same and can be added and subtracted but when both are $\ll c$

Ideal comoving frames, not my CM comoving frames

There is no limit on $v = \dot{r}$

The speed of light is the fastest speed any thing physical can move relative to an inertial frame

past you at one place in an inertial frame

— light moves at this speed in a vaccuum and it is invariant $\Longrightarrow$ same for all observers $\Longrightarrow$ one of the axioms of special relativity (SR)

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In fact, ~~the~~ theoretical Hubble's Law is trivially shown

$$r = a(t)\, r_0$$

$$v = \dot{r} = \dot{a}\, r_0 = \frac{\dot{a}}{a}\, a r_0 = H r$$

where $H = \frac{\dot{r}}{r} = \frac{\dot{a}}{a}$

Friedmann (1922) must have known this ~~but Lemaitre~~ and de Sitter (1917) too, but Lemaitre (1927) seems to be the first to explicitly publish it.

$$\therefore \quad \underbrace{v}_{\text{true recession velocity}} = \underbrace{H}_{\text{Hubble parameter at some instant in cosmic time}}\ \underbrace{r}_{\text{physical distance at one instant in cosmic time}}$$

~~The observed~~

But v and r are NOT directly observable except asymptotically as $r \to 0$

or cosmological redshift $z \to 0$

That is where Hubble's law is experimentally verified as we'll show ~~later~~ later.

In the asymptotic limit.

(2025 jan 05)

L3205

The Friedmann equation is not in most senses a dynamics equation (Equation of motion, Wiki) — there are no forces in it.

it's an integral of motion equation ~~with~~ K being integral itself

curvature and as it's name suggests it describes the curvature of space

But the Second Friedmann equation (the acceleration equation) doesn't give you any more dynamics, so maybe imposed symmetries (i.e., cosmological principle) that limits its range of behavior

However, curvature ~~its~~ cannot be derived in Newtonian physics, and so we will rely on GR maybe to tell us about curvature.

$k > 0$ hyperspherical space

$k = 0$ flat space
Euclidean space

$k < 0$ hyperbolic space

In fact, the Friedmann equation ~~stae~~ is a sort of energy balance equation, but in GR interpretation it loses it's simple meaning I think.

It's a balance of something that is only energy in small scale classical limit sense, 3206 ~~2024~~ Jan 05

~~2)~~ 3.19 Derivation of Friedmann Eq.

a) GR derivation Alexander Friedmann 1922.

Independently, Georges Lemaître 1927

Einstein himself seems to be the only one to notice both — ~~but in the~~ 1920s he was not interested in cosmology.

In 1934, Milne & McCrea gave a Newtonian Physics derivation

Which requires some special hypotheses only justified ultimately in GR and doesn't give any Notion of curvature.

A 19th century physicist without ~~GR~~ and SR could have derived it in essence.

But none did.

Einstein & de Sitter in 1927 both missed discovering FE and had to find their cosmological models by klutzien means. They'd have saved a lot time if they'd discovered it.

The exting hypotheses do NOT give a complete general theory of gravity and of motion under gravity, ~~If~~ ~~They~~ ~~true, they~~ imply it exists, and ~~Lesser~~ and it's general ~~relativity~~)

If the Friedmann eqn is true, they imply that theory which in fact is GR.

But we'll try (without being too pendantic) as ~~what~~ that person would have done,
Let's call them Max for James Clerk Maxwell (1831–1879), who could've done it if his mind ever wandered that way.

But we'll do it ~~I~~deally as he might have presented to an Audience Not with all the meandering of actual research as done.

b) Motivation and Points for Max

- If the universe is infinite and mostly full of matter (stars ~~to~~ Max, galaxies ~~to~~ us) that on a large enough scale is homogeneous & isotropic (~~the~~ cosmological principle – name invented by E.A. Milne (1935) but in various forms the idea went back a long way, Einstein, de Sitter, Friedmann, Lemaitre, Even earlier in some sense), what holds it up.
- Can it be static on average? (in stable equilibrium)
- Newton himself wondered about that and briefly ~~in~~

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and in unpublished work tried to find a universal balance

but he couldn't and no one else either

(unstable unless making the system infinite magically stabilized it.)

Maybe an extra hypothesis was needed to keep stars at rest on average in Newton's absolute space.

— But also the universe was not in thermodynamic equilibrium

clear in a ~~thermodynamic sense~~, but in a more primitive sense

so why should it be static in a dynamical sense?

known for a long time

Olbers Paradox — first discussed by Edmond Halley

conclusion sky should be as bright as a star surface but it's Not

if universe static in all senses infinitely old and obey cosmol. principle then in every direction you ~~should see~~ a star.

Halley said the idea was due to someone unnamed and presented in 1721 to the Royal Society with elderly Newton presided (North - 377)

— Area of shells $\propto r^2$
— inverse square fall of light $\propto 1/r^2$

cancelation.

Basic Derivation

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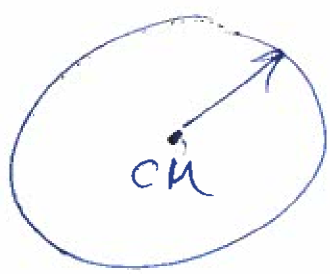
c) So Max hypothesizes an infinite universe with all matter smoothed out into a perfect fluid of uniform density at ~~any time~~ all times.

— No viscosity

— can have pressure, but no gradients since uniform.



Say expanding or contracting everywhere and every point is in free fall under gravity in a sense

Max guesses these must all define inertial frames (comoving frames in the term I use)

Then Max says the shell thm applies at every point

This goes beyond old Newtonian physics

Max guesses there must some more general theory of motion under gravity than Newtonian physics but knows Newtonian physics is a valid limit of it (except for absolute space) and so is trying to generalize from Newtonian physics to capture a portion of the more general theory. L.

32[illegible]0

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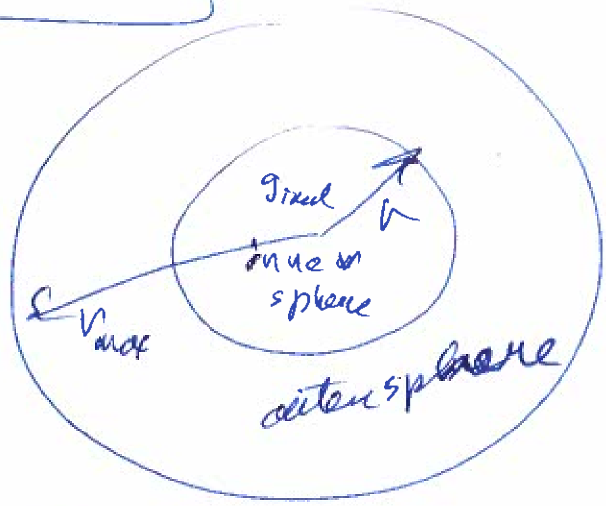
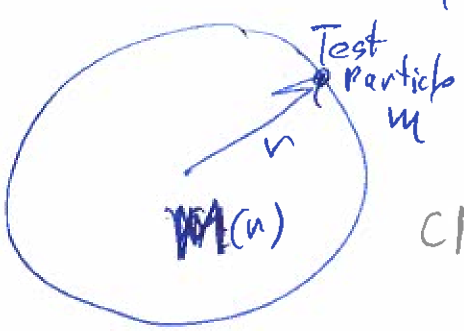


Then let r_{max} ~~go~~ go to infinity and $g(r)$ stays ~~the~~ same by hypothesis.

Then every point in infinite space allows a CM comoving frame centered on ~~it~~ it.



~~g inside only~~

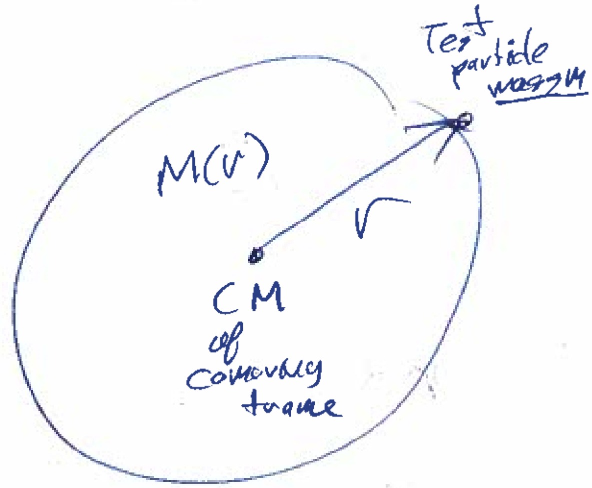
$g(r)$ only depends on $M(r)$ not on mass beyond

— True in Newtonian physics

and Birkhoff tells us $r << r_{scale}$ of universe does imply ~~all spa~~ empty spherical cavity is flat Minkowski space in which Newtonian physics can be done ~~if spherical sym~~ velocities and gravity is weak enough.

Max could try to apply $F = ma$ to test particle and probably does a lot hunting around, but arrives at a conservation of mechanical energy approach to get an integral of motion.

In fact, we are imagining Max after all kinds of false starts and loops in his thinking



$$\underset{\sim}{g}(r) = -\frac{GM(r)}{r^2}\hat{r}$$

$$V_G = -\frac{GM(r)}{r} = -\frac{4\pi}{3}G\rho r^2$$

assume $M(r)$ is conserved as r changes with expansion/contraction of sphere

$$\boxed{M(r) = \frac{4\pi}{3}\rho r^3}$$

given mass conserved we get $\rho \propto \frac{1}{r^3}$

After some playing around Max also includes a Λ-force field

$$\underset{\sim}{f} = \frac{\Lambda}{3} r \hat{r}$$

$$V_\Lambda = -\frac{1}{2}\frac{\Lambda}{3} r^2$$

Max knows the linear force has a symmetry with gravity and it's a simple force and so may be universal.

and also a general PE U.

Apply conservation of mechanical energy to a test particle of mass m

$$E = \frac{1}{2}mv^2 - \frac{GM(r)}{r}m - \frac{1}{2}\frac{\Lambda}{3}mr^2 + U$$

$$= \frac{1}{2}mv^2 - \frac{4\pi}{3}G\rho m r^2 - \frac{1}{2}\frac{\Lambda}{3}mr^2 + U$$

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Now ~~the~~ scaling behavior of the universe cannot depend on the peculiarities of a test particle m, r, v, E

somehow all must vanish.

Max defines $r = a(t) r_0$

$\dot{r} = \dot{a}(t) r_0$ where $a(t)$ is ~~the universal~~ cosmic scale factor.

Divides thru by $\frac{1}{2} m r_0^2$

$$\frac{2E}{m r_0^2} = \dot{a}^2 - \frac{8\pi G}{3} \rho a^2 - \frac{\Lambda}{3} a^2 + \underbrace{\frac{U}{\frac{1}{2} m r_0^2}}$$

Or else we can't have all particle distances scale with 'a'?

We demand this ~~...~~ to be a universal constant for all particles

If then this would a universal expression

Expect this spoils things unless it has peculiar properties

Say ~~$U = V(a)$~~

Say $U = V(a)\, m [r^{(n)}]^2$ where $r^{(n)}$ is the nth order derivative of r.

Max defines $k = -\frac{2E}{m r_0}$

(Lr-24)

The minus is because Max is clairvoyant about GR.

We've ~~already~~ already got gravity and the linear force. U would have to be something odd with no obvious motivation from known forces

So Max ansatzes it must be zero.

In fact a reason for the linear force is that if does satisfy the requirements for a universal eqn. it must exist!!!?

Rewriting, we get Friedmann's equation (FE) in standard form

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

~~If~~ mass is conserved $\rho \propto \frac{1}{a^3}$ (see p. 3211)

~~Mass Conservation~~ Mass Not Conserved

1) But Max wonders what if mass is not conserved.
What if it mysteriously disappears/appears and what can ~~that~~ depend on in any obvious way? ⟶ depends on "a".

[~~Why~~ does Max wonder about mass nonconservation? His annual faculty evaluation at Cambridge Univ. is due and he is desperately procrastinating]

Max ansatz $\rho = \rho_0 \left(\frac{r_0}{r}\right)^n$, $\rho_0 a^{-n}$ where n is a general power

But what should be the potential energy? (or potential)

Max tries $g = -\frac{4\pi}{3}\frac{G\rho_0 r^3 \left(\frac{r_0}{r}\right)^n}{r^2} \propto r^{1-n}$

$$\therefore V = \frac{4\pi}{3} G\rho_0 r_0^n \frac{r^{2-n}}{2-n}$$

$$= \frac{4\pi}{3} G\rho_0 \left(\frac{r_0}{r}\right)^n \frac{r^2}{2-n}$$

$$= -\frac{4\pi}{3} G\rho r^2 \frac{1}{n-2}$$

This is just like on p. 3211 Except $\frac{1}{n-2}$

and we recover that case if $n=3$.

Not worrying about $n=2$ case which leads to V logarithmic and we are going to dispense with all n except n=3 anyway.

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But ~~M~~ Max say ~~this~~ potential is different for every kind of mass nonconservation (we've considered)

and ~~so~~ to ~~leads to~~ a different Friedmann Eqn

[and at this point Max wonders who is Friedmann?]

But mass and it's ~~effect~~ ~~coul only~~ must be unique or we have no guidance.

So Max ansatzes

$$V = -\frac{4\pi}{3} G \rho r^2$$

for all cases of mass variation.

And $n=3$ which has mass conservation too, Seen 3211

It works for mass conservation. It must be same in all cases. "must" in physics means "maybe"

At this point, the Friedmann eqn must be somehow be more general ~~than~~ conservation of mechanical energy in Newtonian physics — but whatever ~~more~~ general physics applies, it is still an integral of motion.

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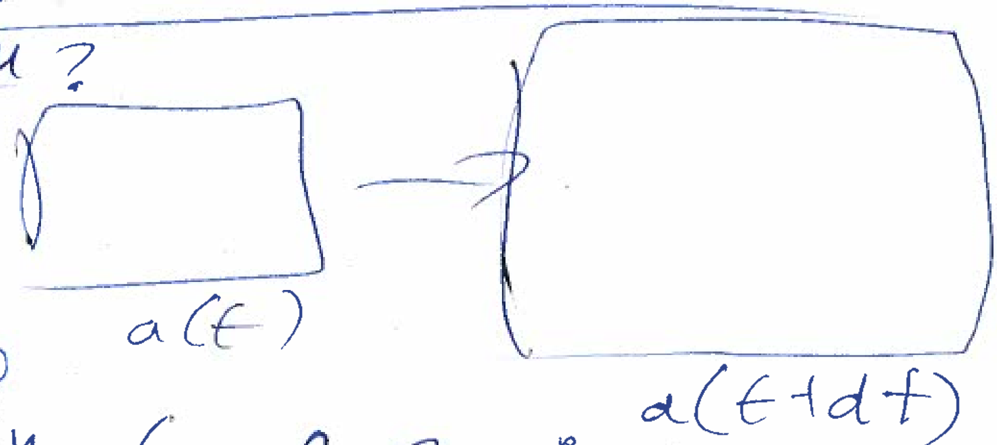
But where does mass nonconservation arise from?



Well ~~the~~ form assumed is $\rho = \rho_0 a^{-n}$ (or $\Omega = \Omega_{0p} x^{-p}$ in ~~terms~~ of J2026) (My review paper)

says changes ~~with~~ volume,

What changes with volume.

Being one of ~~the~~ developers of 19th century statistical ~~classical~~ mechanics Max knows ~~the~~ 2nd law of Thermodynamics (1st law) (Wik: 2nd law Energy available)

Actually probably some clairvoyance for this is needed by Max [illegible] ↓ But we know about entropy somewhat (Wik)

$$dE = TdS - PdV + \mu dN$$

thermal (internal) energy; Temperature; entropy; pressure; Volume; chemical potential; particles

Differential forms where denominator is anything, but usually time

So energy can change with volume — an inside ~~thing~~ source

And Max doesn't want heat energy or particles from outside because there are no outside ... outside universe, one hopes.

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But says Max, this is ~~the~~ 19th century, and so mass and energy are different things.

But Max also discovered Maxwell's equations of electromagnetism and derived $c \doteq 3 \times 10^8\ m/s$ as the unique velocity of electromagnetic ~~waves~~ waves

and mc^2 has dimensions of energy like $\frac{1}{2}mv^2$,

and so Max say's let's assume mass is a form of energy; ~~that~~ $E = mc^2$ at least for perfect fluids

And there is some kind of perfect fluid stuff with pressure of some kind

and so

and $E = mc^2$ works for the perfluid if nothing else.

$$c^2 dM = dE = -P\,dV$$

$$d(\rho V) = -\frac{P}{c^2}\,dV \quad \text{and} \quad V \propto a^3$$

$$V\,d\rho + \rho\,dV = -\frac{P}{c^2}\,dV$$

$$d\rho = -\frac{dV}{V}\left(\rho + \frac{P}{c^2}\right)$$

$d\cancel{V}$

$dV \propto 3a^2\,da$

$\frac{dV}{V} = 3\frac{da}{a}$

(2025 jan 05)

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$$dP = -3\frac{da}{a}\left(\rho + \frac{p}{c^2}\right)$$ $\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$

Now what says Max.
Simplest idea → simplest equation of state (EOS)

$$p = w c^2 \rho$$

where w is a constant (just the w parameter of ~~the~~ the cosmological perfect fluid in modern astro jargon — No other name it seems)

Note, this pressure can be formal. It doesn't have to push/pull on anything including itself and the form of the energy it ~~does~~ destroys or creates could be anything ~~e.g.~~, particle energy, particles

$$\therefore\ d\rho \propto -3\frac{da}{a}(\rho + w\rho)$$

$$\frac{d\rho}{\rho} = -3\frac{da}{a}(1+w)$$

$$\ln\rho = -3(1+w)\ln a$$

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$$

of Fulvio Melio

The commonest specified possibilities,

$\rho_0\left(\frac{a_0}{a}\right)^3$ — $w = 0$ "matter" NR matter

$\rho\left(\frac{a_0}{a}\right)^4$ — $w = \frac{1}{3}$ "radiation" ER matter (as well move later)

ρ_0 — $w = -1$ constant dark energy or cos. constant

$\rho_0\left(\frac{a_0}{a}\right)^2$ — $w = -\frac{1}{3}$ $R_h = ct$ universe

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3.20 The acceleration Equation or 2nd Friedmann Eq.

2nd order DE, but it doesn't give any behavior NOT in the Friedmann equation itself. So dynamics is limited

From p. 3213

$$\dot{a}^2 = \frac{8\pi G \rho a^2}{3} - k + \frac{\Lambda}{3} a^2$$

Differentiate

curvature term vanishes

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left[\dot{\rho} a^2 + 2\rho a \dot{a}\right] - 0 + 2\frac{\Lambda}{3} a \dot{a}$$

$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$

Without the pressure and $E = mc^2$, Max, EOS would not know what $\dot{\rho}$ is except for $n = 3$.

It was an integral of motion.

divide thru by $2\dot{a}$

$$\ddot{a} = \frac{4\pi G}{3}\left[-3a\left(\rho + \frac{p}{c^2}\right) + 2\rho a\right] + \frac{\Lambda}{3} a$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho + \frac{3p}{c^2}\right] + \frac{\Lambda}{3} \qquad \text{(Li-27)}$$

or with $r = r_0 a$ and r_0 ~~cancels out~~ canceled out

$$\ddot{r} = -\frac{4\pi G}{3}\left[\frac{\rho r^3}{r^2} + \frac{3p}{c^2} r\right] + \frac{\Lambda}{3} r$$

$$\ddot{r} = -\frac{GM(r)}{r^2} - \frac{4\pi G}{3}\left(3\frac{p}{c^2}\right) r + \frac{\Lambda}{3} r$$

shell theorem force for spherical symmetric mass distribution.

what?

Λ-force

Max wonders what the heck is the middle term. You don't get that from Newtonian physics. 13219

But recall he introduced

- mass non-conservation, (p. 3213)
- a special form of ~~potential~~ (p. 3214)
- $E = mc^2$ for his perfect fluid, (p. 3216) but maybe nothing else.

So maybe the pressure term is OK. It arises in the more general theory — i.e., general relativity.

I wonder about it too, but general relativity gives it and the Einstein field equations from which Friedmann equation is derived without special hypotheses has it

$$\underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} + \underbrace{\Lambda}_{\text{cosmological constant}} \underbrace{g_{\mu\nu}}_{\text{metric tensor}} = \frac{8\pi G}{c^4} \underbrace{T_{\mu\nu}}_{\text{stress energy tensor}}$$

(momentum energy tensor For perfect fluid (MIK)

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)u^\alpha u^\beta + p g^{\alpha\beta}$$

So pressure comes from GR.

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~~3.21~~ 3.21

~~7)~~ Cosmological Constant and dark energy perfect fluid

(i.e., constant dark energy)

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{gen} - \frac{k}{a^2} + \frac{\Lambda}{3}$$ {See p. 3213

$H^2 = \frac{8\pi G}{3}\left[\rho + \frac{\Lambda}{8\pi G}\right] - \frac{k}{a^2}$

~~def $\rho_{gen} = \rho + \rho_{w=0}$ other a constant density no matter how volume changes see p. 3217~~

~~$= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3}$~~

~~Both constant terms.~~

Thus, we identify } $\therefore$ $\rho_\Lambda = \frac{\Lambda}{8\pi G}$

or $\Lambda = 8\pi G\rho_\Lambda$ (Li-56)

← see p. 3217

But our simple equation of state

$p = w\rho c^2$ with $w = -1$ needed for constant density (see p. 3217)

$\therefore$ $p = -\rho_\Lambda c^2$

$\rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$

So if you interpret the cosmological constant as a ~~form of matter~~ perfect fluid, it has a negative pressure.

But what does that mean? | But where does it come from?

~~It doesn't~~ It pulls in and if you expand it, energy ~~comes~~ goes into the perfect fluid not out.

But what does it pull in on. Nothing we know of Maybe itself

In what form is the mass-energy? of ~~the~~ perfect fluid stuff? some kind of particle ~~that~~ doesn't clump under gravity. Dark photon (weak) ~~Probably not the thought of meaning~~

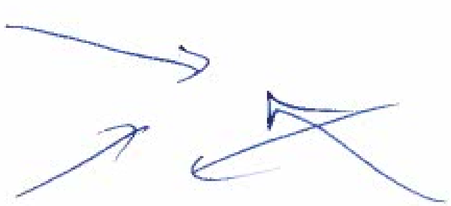
only, but in the cosmological principle sense, the perfect fluid is ~~constant~~ density and there is no pressure gradient.

Maybe it doesn't pull on itself.

For example ~~photons~~ Thermal equilibrium photons have a pressure (+ve pressure)

$$P = \frac{1}{3} a T^4$$

radiation constant



photon gas, but without pair creation they just pass each other without interaction.

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So ~~the~~ negative pressure of
of dark energy perfect fluid
may be purely formal.

Quantum field theorists (QFT)
~~think~~ vacuum field energy
with -ve pressure
is a nifty idea
but natural QFT theories
of it predict

$$\Lambda_{QFT} \cong 10^{120} \Lambda_{observed}$$

one of largest theoretical discrepancies in physics.

Historically, E. Schrödinger in 1918
(before he was famous)
suggested the cosmological constant
could be a perfect fluid
with -ve pressure

and Georges LeMaitre
also suggested a -ve pressure
perfect fluid as a replacement
of the cosmological constant.
(Wik: Vacuum Energy)

There must be some observable differences between a true cosmological constant and a dark energy perfect fluid.

But I haven't ~~noticed~~ looked for any discussion.

Note gravity and ~~cosmological constant~~ force have ~~that~~ odd symmetry (see p. 3053) we've discussed and seem to be the only non-exotic forces allowing the Friedmann equation (see p. 3212)

So Λ or ~~dark~~ energy perfect fluid.

The question is moot.

Both are just called Λ conflating the two concepts

~~3.22~~

3) The Other Form of the Friedmann Equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_{\text{mass-energy}}}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$= \frac{8\pi}{3}G\Big[\rho_{ME} \underbrace{- \frac{3k}{8\pi G a^2}}_{\equiv \rho_k} + \underbrace{\frac{\Lambda}{8\pi G}}_{\equiv \rho_\Lambda}\Big]$$

$$= \frac{8\pi G}{3} G\rho$$

p. 3201
3213

Define $\frac{a}{a_0} = x$

I can't stand working with "a" as variable (it's a constant like the radiation constant)

Let

$\tau = \frac{t}{t_{H_0}} = H_0 t$, a scaled time

where H_0 is the Hubble constant, but it could be a general one Not just our $H_0 \cong 70 \frac{km/s}{Mpc}$

Let $\dot{x} \equiv \frac{dx}{d\tau}$

~~Let $h = H/t_H$~~

$h = H/H_0$

$$h^2 = \left(\frac{\dot{x}}{x}\right) = \frac{8\pi G}{3H_0^2}\rho$$

$$\rho_{\text{critical}} = \frac{3H_0^2}{8\pi G} = \begin{cases} (0.9203874.. \times 10^{-26})\,kg/m^3 \left(\frac{H_0}{70}\right)^2 \\ 0.1359834.. \times 10^{12} \frac{M_\odot}{Mpc^3}\left(\frac{H_0}{70}\right)^2 \\ \sim\frac{1}{10} \text{ of a large galaxy in a } Mpc^3 \end{cases}$$

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3.22 Scaled Form of the Friedman Equation

$$H^2 = \left(\frac{\frac{dx}{dt}}{x}\right)^2 = \frac{8\pi G}{3}\rho_{\substack{mass\\-energy}} - \underbrace{\frac{k}{x^2}}_{\text{curvature term}} + \frac{\Lambda}{3}$$

I prefer x to a ... "a" looks like a constant to me

Choose a fiducial time labeled by t_0 which is usually cosmic present, but it could be any time

At t_0, we have $H = H_0$, $x = x_0$

Can be set 1 but doesn't have to be

Note, $\frac{H^2}{H_0^2}$ is dimensionless

$$\therefore\ H^2 = H_0^2 \underbrace{\frac{8\pi G}{3H_0^2}}_{\therefore \text{ dimensionless}} \rho_{M0} - \frac{k}{x^2} + \frac{\Lambda}{3}$$

We define $\rho_c = \rho_{critical} = \frac{3H_0^2}{8\pi G}$

$$[\rho_c] = \frac{T^{-2}}{E\,L/M^2} = \frac{T^{-2}}{\frac{ML^2}{T^2}L/M^2} = \frac{M}{L^3} \text{ correct dimensions}$$

$$H^2 = H_0^2\left[\frac{\rho_{ME}}{\rho_c} - \frac{k}{H_0^2 x} + \frac{\Lambda}{3H_0^2}\right]$$

We define density parameter (a representation of density components)

$$\Omega_{ME} = \frac{\rho_{ME}}{\rho_c}$$

minus sign to make all Ω_i's explicitly positive

$$\Omega_K = -\frac{k}{H_0^2 x^2} \quad \text{and so} \quad \rho_K = \rho_c\Omega_K = -\frac{\rho_c k}{H_0^2 x^2}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2} \quad \text{and so} \quad \rho_\Lambda = \rho_c\Omega_\Lambda = \frac{\Lambda}{8\pi G}$$

as on p. 3220

$$\text{total } \Omega = \Omega_{ME} + \Omega_K + \Omega_\Lambda$$

$$= \Omega_{ME\Lambda} + \Omega_K \quad \text{with } \Omega_{ME\Lambda} = \Omega_{ME} + \Omega_\Lambda$$

$\therefore H^2 = H_0^2[\Omega] = H_0^2[\Omega_{ME\Lambda} + \Omega_k]$

at $x = x_0$, $1 = \Omega_{ME\Lambda 0} + \Omega_{k0}$

$\therefore \Omega_{k_0} = 1 - \Omega_{ME\Lambda 0}$

If $\Omega_{ME\Lambda} = 1$, $\Omega_k = 0$.

If $\Omega_{ME\Lambda} \neq 1$, $\Omega_k \neq 0$

If $\Omega_k > 0$, $k < 0$ and there is -ve curvature (hyperbolic space)

If $\Omega_k < 0$, $k > 0$ and there is +ve curvature (hyperspherical space)

The reversal of signs is annoying.
Just accept it. Lesser of two evils

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$$h^2 = \frac{\rho}{\rho_{critical}} = \underbrace{\Omega}_{\text{Omega } \cancel{\text{the density}} \text{ Parameter}}$$
{ Omega = density Parameter

The scaled Friedmann equation.

$$= \underbrace{\Omega_{ME}}_{\text{mass energy}} + \Omega_k + \Omega_\Lambda$$

at cosmic present where $x=1$

Recall $h = \frac{H}{H_0}$

$$h^2 = 1 = \Omega_{ME_0} + \Omega_{k_0} + \underbrace{\Omega_\Lambda}_{\text{always constant}}$$

$$\Omega_{k_0} = 1 - \underbrace{\Omega_{ME_0} - \Omega_\Lambda}_{\text{if these sum to } -1}$$

~~then~~ $\Omega_{k_0} = 0$

$$\Omega_{k_0} = - \frac{3k}{8\pi G a_0^2} \frac{1}{x^2}$$
$\hookrightarrow x = 1$ at cosmic present

So $\Omega_{k_0} = 0$ implies the universe is flat.
We'll look more a curvature of the universe later.

A very Annoying change of sign.

Note $k>0$ is +ve curvature but $\Omega_k < 0$
$k<0$ is -ve curvature then but $\Omega_k > 0$

3226 | 2025 jan 05

What kind of Omega are ~~[illegible]~~ considered.

Typically inverse-power law ones

$h = \cancel{\Omega}$

$$h^2 = \Omega = \sum_{P=4}^{0} \Omega_{P0} x^{-P}$$

Note 5 parameters needed.

If you knew H_0, then you could ~~eliminate~~ fix one of them,

e.g., $\Omega_{2,0} = k$ as on p. 3225

$$= \Omega_{40} x^{-4} + \Omega_{30} x^{-3} + \Omega_{20} x^{-2} + \Omega_{10} x^{-1} + \Omega_{00} x^{-0}$$

$P = 0$ for Λ cosmological constant or dark energy, (Je-11)

$P = 1$ for some quintessence theories

$P = 2$ for curvature, cosmic strings (some theories),
$R_h = ct$ universe of Fulvio Melia.

$P = 3$ "matter" in the cosmological sense
of matter at rest or nearly in comoving frames
all baryonic and dark matter
— really means non-relativistic matter

$P = 4$ "radiation" for radiation, but
also any ~~[illegible]~~ stuff moving at extreme relativistic speeds $\sim c$

3.23 The Deceleration Parameter q

Really a form of the Acceleration equation (p. 3218)
with a historical minus sign.

The scaled Friedmann equation (p. 3224)

$$\left(\frac{\dot{x}}{x}\right)^2 = \sum_p \underbrace{\Omega_{p0}}_{\text{density parameter constant where } p \text{ are general powers}} x^{-p} = \text{RHS}$$

$$\dot{x} = \frac{dx}{d\tau} \quad \text{where} \quad \underbrace{d\tau = H\,dt}_{\text{scaled time}}$$

$$\dot{x}^2 = \sum_p \Omega_{p0} x^{-p+2}$$

$$2\dot{x}\ddot{x} = \sum_p \Omega_{p0}(-p+2)x^{-p+1}\dot{x}$$

$$\frac{\ddot{x}}{x} = \frac{1}{2}\sum_p \Omega_{p0} x^{-p}(-p+2) = \sum_p \Omega_{p0} x^{-p}\left(1-\frac{p}{2}\right)$$

$$q = -\frac{\ddot{x}}{x} = \sum_p \Omega_{p0} x^{-p}(p/2 - 1) \quad \text{for power-law density components}$$

$$q \equiv \underbrace{-\frac{\ddot{x}}{x}}_{} = \underbrace{-\frac{\left(\frac{d^2x}{dt^2}\right)}{x H^2} = -\frac{\frac{d^2x}{dt^2}\,x}{\left(\frac{dx}{dt}\right)^2}}_{}$$

(Liddle-53)

The minus is because for most of the 20th century, people assumed $\ddot{x} < 0$ and then the minus gave the positive quantity they wanted

Actually, the general definition independent of the form of the RHS

3228

(2026 jan 24)

In 1970 Allan Sandage wrote an article "Cosmology: the Search for Two numbers" those being H_0 and q_0. Then they thought that might be enough to determine the parameters of the Einstein-de Sitter (EdS) universe (the favored Friedmann equ. solution then. See p 3232–3234) which is NOT the Static Einstein Universe NOR the exponential de Sitter universe

Nowadays $q(z)$ is diagnostic for testing possible $a(t)$ solutions fit to data

For fiducial Λ-CDM parameters

$$\Omega_{m_0} = 0.3 \ , \ \Omega_\Lambda = 0.7$$
$$\scriptstyle p=3 \qquad\qquad p=0$$

$$\therefore \ q_0 = -\frac{1}{2}\sum_p \Omega_{p0}(-p+2)$$

$$= -\frac{1}{2}\left(0.3 + 0.7\cdot(2)\right)$$

$$q_0 = -\frac{1}{2}(1.1) = -0.55$$

The minus sign shows its a positive acceleration

You see this number sometimes and wonder where it comes from.

Elementary Solutions of the Friedmann Equation

Single density component solutions

$$\frac{\dot{x}}{x} = \pm\sqrt{\sum_p \Omega_p x^{-p}}$$

where $d\tau = H_0 dt$
and at t_0, $x_0 = 1$

$$h = \frac{\dot{x}}{x} = 1 \quad \text{or} \quad \frac{\frac{dx}{dt}}{x} = H_0$$

$\therefore$

$$\sum_p \Omega_{p_0} = 1$$

If only one density component then $\Omega_{p_0} = 1$.
But for our derivation we will leave Ω_{p_0} unevaluated so that we can see where it is.

Note the $\pm$ means there are always expanding/contracting branches,
But unlike most 1st order autonomous DEs the Friedmann equation

3230

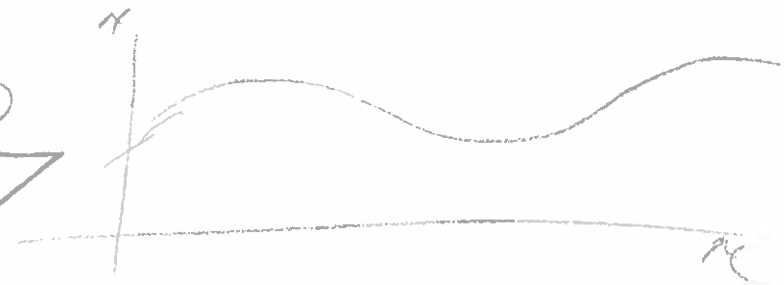
does x can have stationary points ~~at~~ finite time, See p. ______ for a discussion of stationary points of 1st order autonomous DEs

However, physically $x < 0$ is not allowed, No meaning.

Often cyclic universes? Bondi 1961 (p. 82) just assumes Big Crunches start a new cycle. Other modern cyclic universes must somehow introduce exotic behavior into Friedmann eqn or use a generalized Friedmann Eqn.

But with very implausible density components you can get an a true Freidmann Eqn cyclic universe:

Never goes to zero



Since we live in an expanding universe, the negative branch solutions have less interest and we won't consider them here

$$d\tau = \frac{dx}{x\sqrt{\sum_p \Omega_p x^{-p}}}$$

$$d\tau = \frac{dx}{\sqrt{\sum_p \Omega_p x^{-p+2}}}$$

In fact, numerically and (I think) always analytically you solve for $\tau(x)$ first.

(2025 jan 24)

3231

a) For a single density component $p \neq 0$

$$d\tau = \frac{dx}{\sqrt{\Omega_{p_0}}\, x^{-p/2+1}} = \frac{x^{p/2-1}\,dx}{\sqrt{\Omega_{p_0}}}$$

$$\tau = \frac{1}{\sqrt{\Omega_{p_0}}}\,\frac{x^{p/2}}{p/2}$$

choosing $\tau(x=0)=0$ for a Big Bang singularity origin or a Point origin in olden jargon

age of universe $\tau_0 = \frac{2}{p}$ with $x=1$, $\Omega_{p_0}=1$

$$t_0 = \frac{2}{p}\,\frac{1}{H_0}$$

$$t_0 = \frac{2}{p}\,t_{\text{Hubble}},\ t_{\text{Hub}} = \frac{p}{2}t_0$$

The Hubble time

$$x(\tau) = \left[\sqrt{\Omega_{p_0}}\left(\frac{p}{2}\right)\tau\right]^{2/p}$$

$$h = \frac{\dot{x}}{x} = \frac{2}{p}\,\frac{1}{\tau}$$

$$h_0 = \frac{2}{p}\,\frac{1}{\tau_0} = 1$$ (see p. 3229 for the same result in general)

$$q = -\frac{\ddot{x}}{x} = \sum_p \Omega_{p_0} x^{-p}\left(\frac{p}{2}-1\right)$$ (see p. 3227

$\therefore$ Here $q = \Omega_{p_0} x^{-p}(p/2-1)$

$$q_0 = \frac{p}{2} - 1$$

3232

b) For $p = 0$, things are a bit different.

$$d\tau = \frac{dx}{\sqrt{\Omega_{p0}}\, x^{+1}}$$

$$\tau = \frac{1}{\sqrt{\Omega_{p0}}} \ln x + C$$

Thus $x = x_0 e^{\sqrt{\Omega_{p0}}\,\tau}$ and there is no point origin

This is the exponential de Sitter universe.

It's pure cosmologgical constant universe

If we set $\Omega_{p0} = 1$,

then $x = x_0 e^{\tau} = x_0 e^{H_0 t}$

$$\frac{\dot{x}}{x} = H_0$$ and the Hubble parameter is a constant equal to the Hubble constant here defined to be at $\tau = 0$ since the universe is eternal

$$q = -\frac{\ddot{x}}{x} = \Omega_{p0}(p/2 - 1)$$
$$= -1 = q_0$$

So the deceleration parameter is also a constant for all time.

Recall $\rho_{critical} = \frac{3H_0^2}{8\pi G}$ (p 3224)

and $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ (p. 3220)

$$\Omega_\Lambda = \Omega_{p=0} = \frac{\rho_\Lambda}{\rho_{critical}} = \frac{\Lambda}{3H_0^2}$$

$$\therefore \quad x = x_0 e^{\sqrt{\Omega_{p0}}\,\tau} = x_0 e^{\sqrt{\frac{\Lambda}{3H_0^2}}\,\tau}$$

$$x = x_0 e^{\sqrt{\Lambda/3}\,t}$$

where recall $d\tau = H_0 dt$

2026jan27

Special feature of the elementary solutions for $p \neq 0$

$$\Omega_p = \Omega_{p0}\, a^{-p}$$

$$= \Omega_{p0}\left(\left[\sqrt{\Omega_{p0}}\left(\frac{p}{2}\right)\tau\right]^{2/p}\right)^{-p}$$

see p. 3231

$$= \Omega_{p0}\,\Omega_{p0}^{-p/2}\left(\frac{p}{2}\right)^{-p}\tau^{-2}$$

Note this is result does not hold for multiple density component solutions for $\sum_{p>0} \Omega_{p_0} a^{-p}$ cases

$\Omega_p \propto \tau^{-2}$ in all cases.

of course $\Omega_\Lambda = \Omega_{p=0} = \Omega_{0,p=0}$

i.e., the density parameter of Λ is a constant.

Note Friedmann Equ. is Nonlinear.

So the sum of solutions is NOT a solution in general.

There are some special cases where a sum of solutions is a solution, but only for certain coefficients not general coefficients,

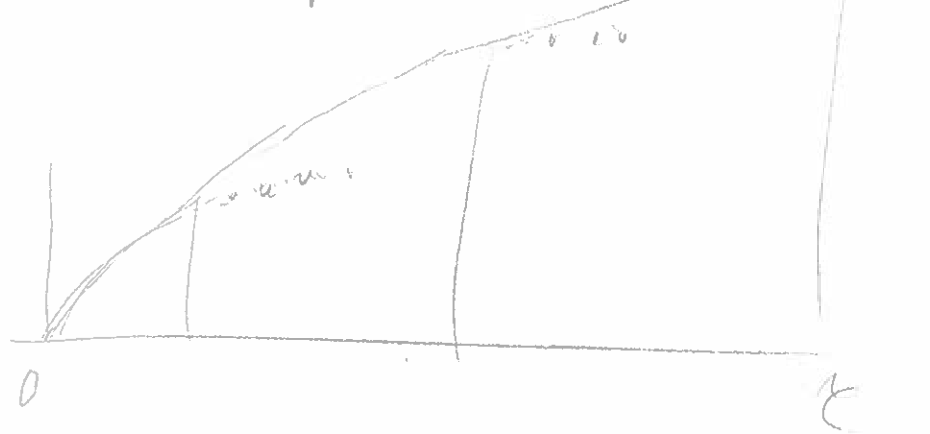
But the elementary solutions don't sum to solutions ever I think.

The only cases I know of.

3234)

But in some cases the elementary solutions can sum to approximate solutions

$$x(t) \simeq \sum_p x_p(t)$$



$$\dot{x} = x\sqrt{\sum_p \Omega_{p0} x^{-p}}$$

The various ~~terms~~ become dominate at different times and so a crude solution of phases can exist.

In fact, I've wasted far too much time finding high accuracy approximations for certain parameter choices. Perfectly worthless I think now.

But I think such solutions are all very qualitative and for quantitative accuracy, you need exact solutions or high accuracy numerical solutions.

Actually, I believe I have found (I think) an educationally useful qualitative solution formalism. But it is just to show how different behaviors emerge, not for any accuracy. I probably won't bore the class with it.