3.12 Gauss' Law for the Linear Force

The linear force gives us the classical analogue to the GR cosmological constant.

For Inverse-Square law versions, see p. 3004 ~~and~~ 3012

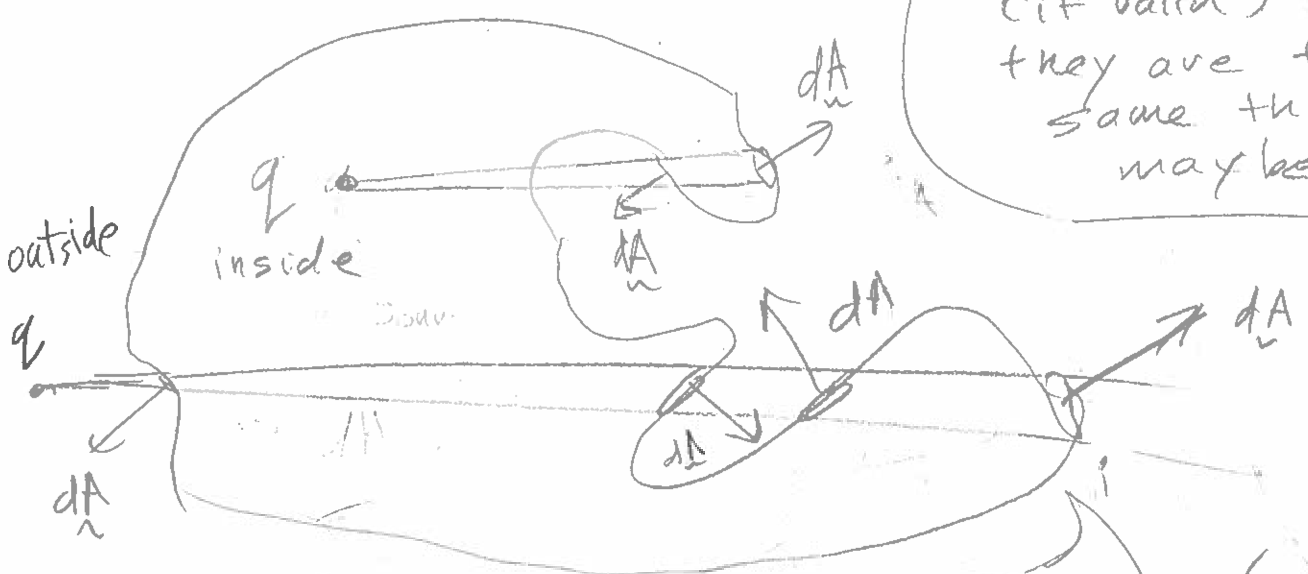
Generic linear force field for a point charge q

$$\underset{\sim}{f} = q\,\underset{\sim}{r}$$

with potential $\phi = -\frac{q}{2} r^2$: Arf-104

The source and receiver "charge" may be different, but 3rd law (if valid) implies they are the same though may be disguised.

Gaussian closed surface



General contribution

$$\hat{r}\cdot d\underset{\sim}{A}_i = |d\underset{\sim}{A}_i| \cos\theta = \pm r^2 d\Omega$$

$$\left[\underset{\sim}{f}\cdot d\underset{\sim}{A}\right]_i = \left[q r(\pm r^2 d\Omega)\right]_i$$

+ for outward going
− for inward going

In spherical coordinates $d\Omega = \sin\theta\, d\theta\, d\phi$ of course

3066)

2016 jan 24

$$\left[\underset{\sim}{f}\cdot dA\right]_i = \left[3q\ \frac{r^3}{3}(\pm d\Omega)\right]_i$$

$$= \left[3q \int_{r_{i-1}}^{r_i} r^2\, dr\, d\Omega\right]_i$$

$$= 3q\left[\pm V(r_i, r_{i-1})\right]_i$$

$\therefore$ Along any cone

$$f\cdot dA = \sum_i \left[\underset{\sim}{f}\cdot dA_i\right] = 3q\, V_{cone}$$

cone inside to the last outgoing surface

$$\therefore \oint f\cdot d\underset{\sim}{A} = 3q\, V_{\text{Gaussian surface}}$$

q anywhere inside, outside, at infinity contributing to $\underset{\sim}{f}$

Summing up all point charges

$$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = 3QV$$

where Q is all charge in the universe contributing to $\underset{\sim}{f}$

Gauss' Law for the linear force QED

The linear force is a very long range force

Another Proof

$$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = \int \nabla\cdot\underset{\sim}{f}\, dV = 3qV$$

Then sum on all point charges again

by Gauss' Thm p. 3009

divergence in spherical coordinates (Art-104)

$$\nabla\cdot f = \nabla\cdot(q\, r\, \hat{r}) = q\frac{1}{r^2}\frac{\partial(r^2)r}{\partial r} = 3q$$ for a point charge

Actually, you can use Gauss' Theorem to prove Gauss law too sort of.

$$\oint \underline{f}\cdot d\underline{A} = \oint \nabla\cdot\underline{f}\, dV = \int q \frac{1}{r^2}\frac{\partial(r^2 h)}{\partial r}\, dV$$

Point charge

q. $\quad \underline{f} = q\,\underbrace{h(r)}_{\text{general function of } r}\,\hat{r}$

$$\nabla\cdot\underline{h} = \left[\frac{\partial h}{\partial r} + \frac{2}{r}h\right]$$

If $h = r^n$

$$\frac{\partial h}{\partial r} = n\, r^{n-1} + 2\, r^{n-1}$$

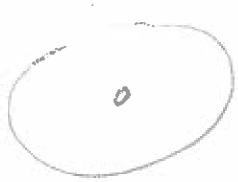
a) If $\underline{n=1}$, $\frac{\partial h}{\partial r} = 3$ as before on p. 3066

b) If $\underline{n=-2}$

$$\frac{\partial h}{\partial r} = -2r^{-3} + 2r^{-3} = 0.$$

But the is a singularity at $r=0$

Little sphere around charge and one gets a Dirac Delta function result.



$$\oint h\hat{r}\cdot d\underline{A} = r^n\, 4\pi\, r^2 = 4\pi\, r^{2+n}$$

If $n=-2$, $\quad \oint h\hat{r}\cdot d\underline{A} = 4\pi$

$$\therefore \nabla\cdot\underline{h} = 4\pi\,\delta(r)$$ as on p. 3011

In general $\oint \underline{h}\cdot d\underline{A} = \int \nabla\cdot\underline{h}\, dV = \begin{cases} 4\pi & \text{if charge inside} \\ 0 & \text{if charge outside} \end{cases}$

3068

c) Are there any other cases of n that lead to interesting divergences and results

$n=1$ leads to constant divergence

$n=-2$ leads to Dirac Delta divergence

It seems not. Only position dependent by inside/outside.

All other power-law cases lead to an r dependence, and so not a position independent Gauss' Law.

But how about $h = e^{r^n}$?

$$\nabla \cdot \underset{\sim}{h} = n r^{n-1} h + \frac{2}{r} h$$

$$= [n r^{n} + 2] \frac{h}{r}$$

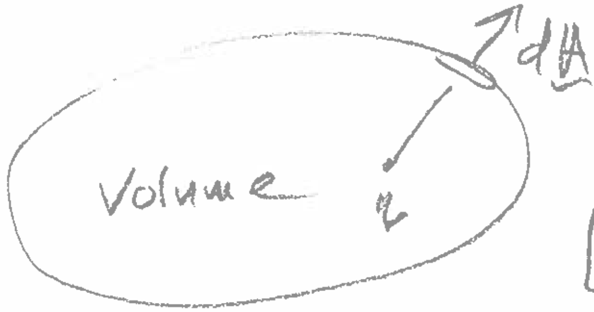
seems to lead to nothing.

So maybe there are only two Gauss laws: the Gauss' law and the linear force Gauss' law

3.13 Shell Theorem for the Linear Force Gauss' Law

$$\oint \underline{f} \cdot d\underline{A} = 3QV \text{ (see p 3066)}$$

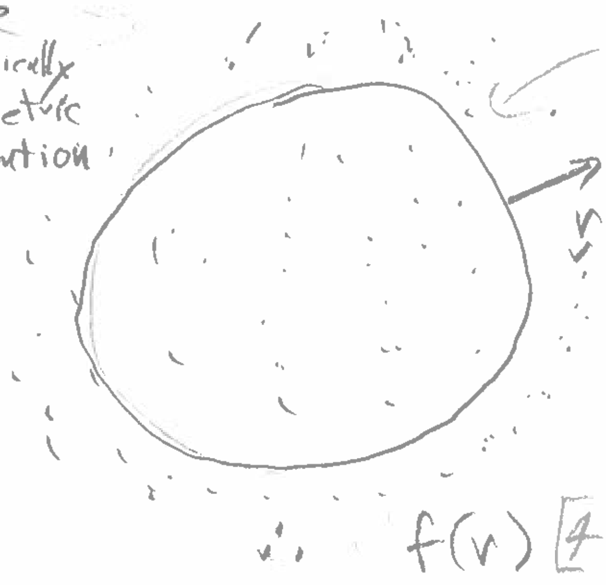
All q anywhere inside and outside contributing to $\underline{f}$ (So NOT all charge in universe)



probably only direct solutions for $\underline{f}$ in 3 cases of high symmetry! planar, cylindrical, spherical

Let's consider spherically symmetric distribution.

charges in the spherically symmetric distribution



Gaussian Surface.

By symmetry

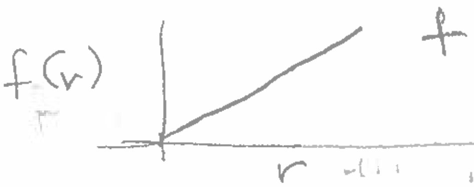
$$\underline{f}(\underline{r}) = f(r)\,\hat{r}$$

$f(r)$ can be +ve or -ve

$$\oint \underline{f} \cdot d\underline{A} = f(r)\left[4\pi r^2\right]$$

$$\therefore\ f(r)\left[4\pi r^2\right] = 3Q\left(\frac{4\pi}{3}\right) r^3$$

$$f(r) = Qr \quad \text{and} \quad \underline{f}(r) = Qr\hat{r} = Q\underline{r}$$



and so just as if all charge were point charge at the center.

3080

Let's assume only one kind of charge and for point charges

$$\underset{\sim}{F}_{ij} = (q_i \hat{\underset{\sim}{r}}_{ij}) q_j = q_i q_j \hat{r}_{ij}$$

$$= -q_i q_j \hat{r}_{ji} = -\underset{\sim}{F}_{ji}$$

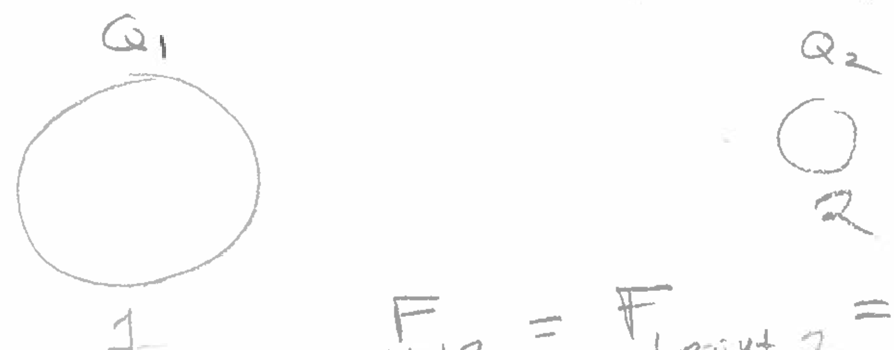
and so 3rd law holds.

What two systems of point charges?

$$\underset{\sim}{F}_{12} = \sum_{ij} q_i q_j \underset{\sim}{r}_{ij} = -\sum_{ij} q_i q_j \underset{\sim}{r}_{ji} = -\underset{\sim}{F}_{21}$$

and so 3rd law holds again

What of two spherically symmetric charge systems? Exactly Analogous to Proof on p. 3015 for gravity.



$$\underset{\sim}{F}_{12} = \underset{\sim}{F}_{1,\text{point }2} = -\underset{\sim}{F}_{2,1\text{point}}$$

$$= -F_{2\text{point},1\text{point}} = \underset{\sim}{F}_{1\text{point},2\text{point}}$$

But there is distinction from the gravity case.

In gravity case the spherically symmetric mass distributions can't overlap for a simple result since

$$\underset{\sim}{g} = -\frac{G M_{enclosed}}{r^2}\hat{r}$$

For the linear

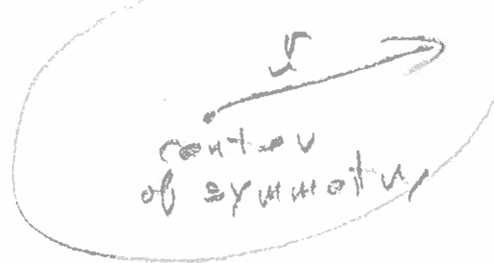
$$\underset{\sim}{F} = Q \underset{\sim}{r}$$

where Q is all charge in the spherically symmetric distribution. So the two systems can overlap and even coincide, but magically each one stays spherically symmetric

3.14 The Newtonian Cosmological Constant Force AKA the Lambda (Λ) Force

For a spherically symmetric linear force charge Q

$$\underset{\sim}{f} = Q \underset{\sim}{r}$$



But if the charge distribution is uniform out to R

$$\underset{\sim}{f} = \left[\left(\frac{4\pi}{3}R^3\right)\rho\right]\underset{\sim}{r}$$

If $R \to \infty$, $\underset{\sim}{f} \to \infty$.

We hypothesis that

$$\frac{4\pi}{3}R^3\rho \longrightarrow \frac{\Lambda}{3}$$

← Lambda

where Λ is a constant and, in fact, the Newtonian analog the cosmological constant of GR

"Charge" positive always and so a pushing force

For simplicity in discussion, I consider the Cos. Constant the simplest form of dark energy even though it is an aspect of gravity. Constant dark energy the next simplest. Both are conflated as Λ since the have the same effect in the Friedmann eqn, though they differ in other aspects

[In Einstein Field equations, (p.3034) Λ is the coefficient ~~of~~ term and is an aspect of gravity. It's long range effect is not explicitly a force, but by being a requirement for all spacetime]

The $\frac{1}{3}$ factor is for consistency with GR.

The Λ force field is

$$\underset{\sim}{f} = \frac{\Lambda}{3}\underset{\sim}{r}$$

and the origin for $\underset{\sim}{r}$ is any point in unbounded space.

Hard to believe in a force that grows to infinity. But in GR, ($\Lambda g_{\mu\nu}$ is just a term in the Einstein field eqns (p.3034) that apply ~~in~~ all spacetime. Recall $\underset{\sim}{f}_g = -\frac{GM}{r}\hat{r}$ goes to zero only at ∞ which is also a bit hard to believe, but is just an aspect of Poisson's eqn $\nabla^2\phi = 4\pi G\rho$

[For flat or hyperbolic space, unbounded means infinite space. Hyperspherical space finite but unbounded. We'll discuss this a bit later. But pure Newtonian space is always flat]

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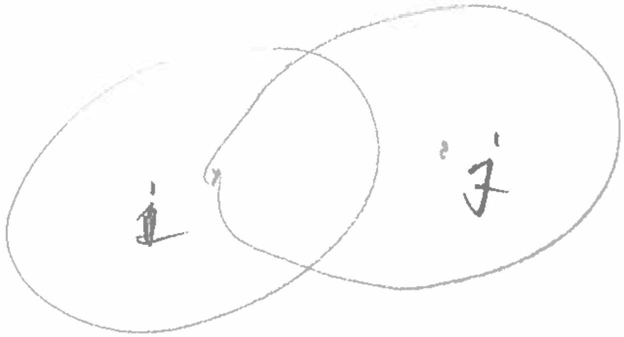
3072

Does $\underset{\sim}{f} = \frac{\Lambda}{3}\underset{\sim}{r}$ satisfy 3^{rd} law?

Maybe not and not explicitly at least



$$\underset{\sim}{F}_{ij} = \frac{\Lambda}{3} r_{ij} m_j$$

$$= -\frac{\Lambda}{3} r_{ji} m_j$$

$$\neq -\frac{\Lambda}{3} r_{ji} m_i \quad \text{unless } m_i = m_j$$

But the source of $\underset{\sim}{F} = \frac{\Lambda}{3}\underset{\sim}{r}$

is NOT a point in space

but all space or all mass in space

So $\underset{\sim}{F}_{ij}$ pushes on m_j

but m_j doesn't push on m_i

(which just happens to be at the origin of $\underset{\sim}{F}_{ij}$), but on all space

on all mass in space. But this is just

a Newtonian analog view which may have no real meaning in GR.

3074

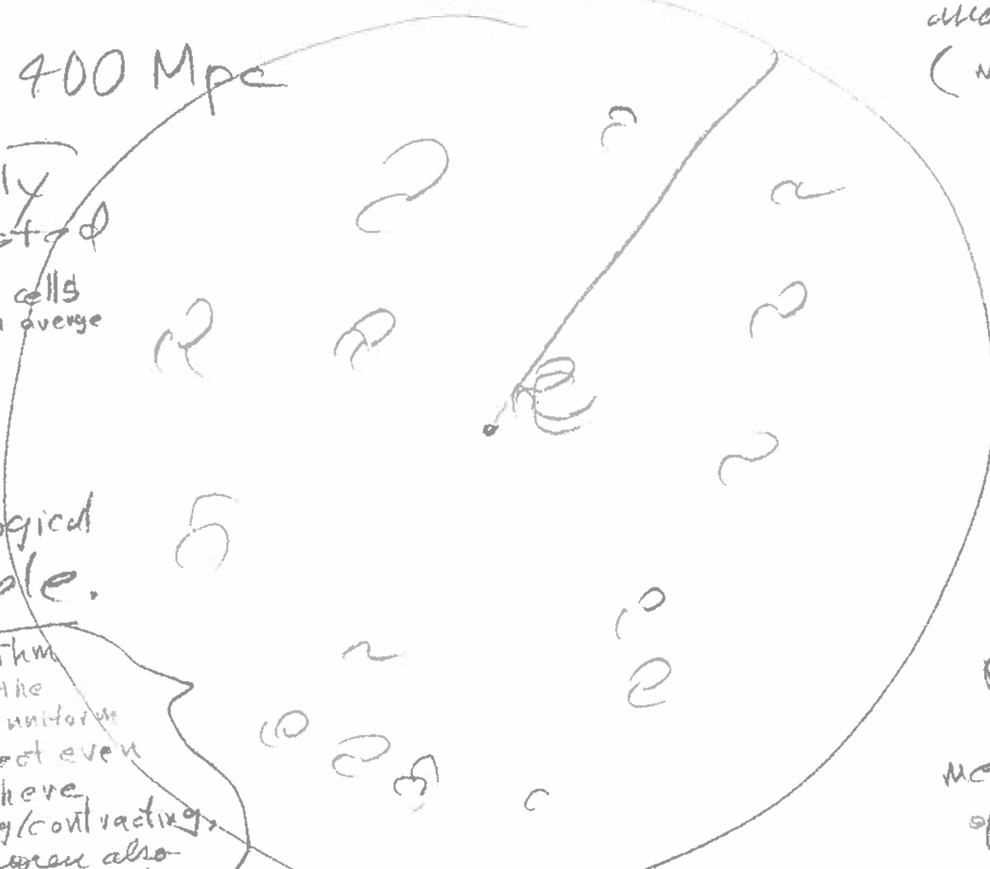
2026 jan 24

(Blank for now)

Take any point in space and draw a big sphere around it. Given large scale homogeneity and isotropy (which is called the cosmological principle) the center is the center of mass (CM) for the sphere and participants in the mean expansion of the universe.

$R \simeq 400$ Mpc

Commonly suggested scale for all cells to be same on average and so to roughly satisfy cosmological principle.



Birkhoff thm tells us the outside if uniform has no effect even if the sphere is expanding/contracting. Birkhoff's theorem also applies when there is a Λ term since it can be moved to $T_{\mu\nu}$ as a dark energy

The CM is the origin of a comoving center-of-mass inertial frame Newtonian limit, so the sphere is Not so large as to require GR corrections

See p. 3085

A particle at CM will generally have a peculiar velocity.

If expansion or contraction NOT too fast for Newtonian limit

The Local Group has peculiar velocity $v = 620(15)$ km/s to $l = 271.9(20)°$, $b = 30(3)°$

known from CMB dipole

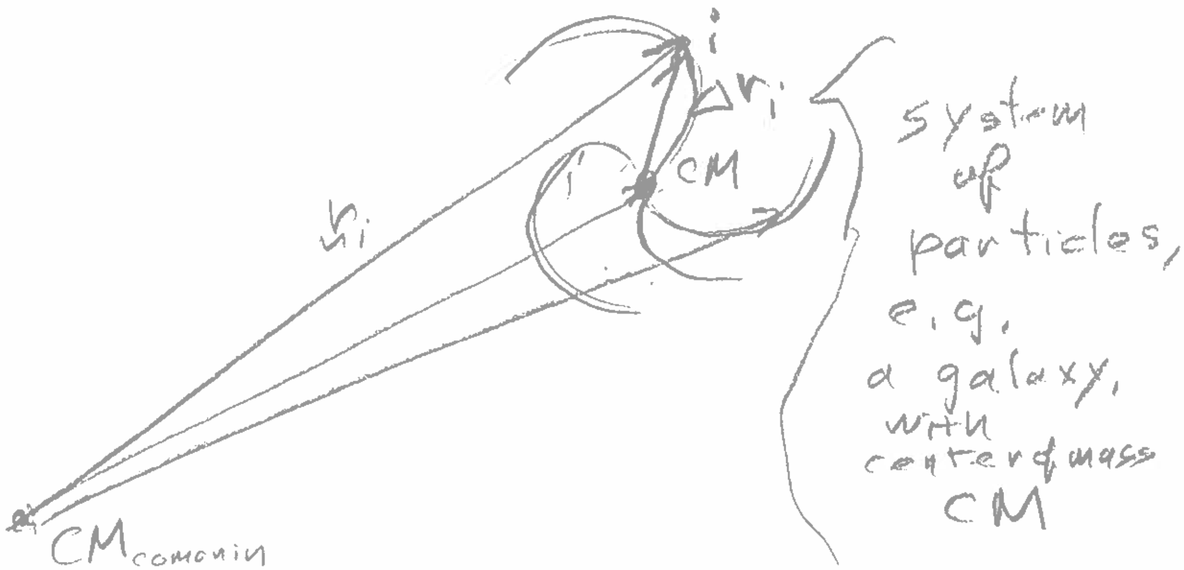
Galactic longitude

Galactic latitude

(Wiki: CMB) which is in Ursa Major (Google AI)

3076)

So



origin in a comoving CM frame for example

Cosmological constant force from $CM_{comoving}$ on particle i

$$\underset{\sim}{F}_i = \frac{\Lambda}{3} \underset{\sim}{r}_i m_i$$

on just a CM in a larger center of mass frame

Force on whole system

$$\underset{\sim}{F} = \sum_i F_i = \frac{\Lambda}{3} \sum_i \underset{\sim}{r}_i m_i$$

$$= \frac{\Lambda}{3} \sum_i (\underset{\sim}{r}_{cm} + \Delta \underset{\sim}{r}_i) m_i$$

relative to CM of the system

$$= \frac{\Lambda}{3} \underset{\sim}{r}_{cm} m$$

since $\sum_i \Delta \underset{\sim}{r}_i m_i = 0$ (by definition of CM)

<u>Upshot</u> for any system of particles the Λ-~~force~~ just pushes out from it's CM and for internal motions

the externall net Λ-force makes **no** contributions to internal motions

(The internal Λ-force $\frac{\Lambda}{3}\Delta r_i\, m_i$ is a bit like the Λ analogue to the tidal force (see p. 3039))

So what is the internal Λ-force effect on a system, e.g., a galaxy including its dark matter halo?

$$\underset{\sim}{f} = \underbrace{-\frac{GM(r)}{r^2}\hat{r}}_{\text{gravity}} + \underbrace{\frac{\Lambda}{3}\underset{\sim}{r}}_{\Lambda\text{ force}}$$

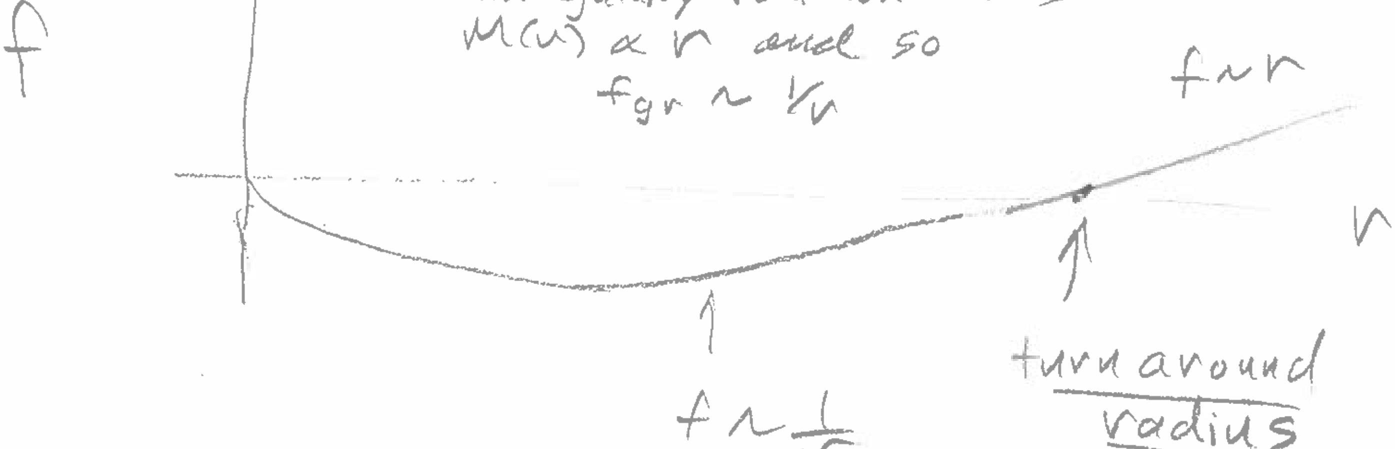
gravity for enclosed mass to r assuming spherical symmetry which is roughly for dark matter halos

Note

$$\frac{v^2}{r} = \frac{GM(r)}{r^2} \quad \text{for a circular orbit}$$

$$v = \sqrt{GM(r)/r}$$

for flat galaxy rotation curves $M(r) \propto r$ and so $f_{gr} \sim 1/r$



beyond the turnaround radius particles are unbound

3078

Now for the Λ-CDM model, Wik gives

$$\Lambda = 1.1056 \times 10^{-52}\ \mathrm{m}^{-2}\ c^2$$

$$[\Lambda] = s^{-2} \text{ units}$$

$$[f_\Lambda] = s^{-2}\,\mathrm{m} = \mathrm{m}/s^2$$
$$= \mathrm{N}/\mathrm{kg}$$

So force per unit mass

Now

$$f = -\frac{GM(r)}{r^2}\hat{r} + \frac{\Lambda}{3}\underline{r}$$

$$= \left[-1.3939\times10^{-13}\ \mathrm{m}/s^2\right]\left[\frac{M(r)}{10^{12} M_\odot}\right]\left[\frac{1\ \mathrm{Mpc}}{r}\right]^2$$

$$+ \left[1.0220\times10^{-13}\ \mathrm{m}/s^2\right]\left[\frac{r}{1\ \mathrm{Mpc}}\right]$$

fiducial numbers for a large galaxy.

Actually, when a galaxy reaches $M \sim 10^{12} M_\odot$ it is usually quenched (star formation has turned off)

$$a_{circular} = \cancel{\frac{v^2}{r}}$$

$$\cancel{a_{circular} = \left(\frac{4}{3}\right)\times10^{-13}\ \mathrm{m}/s^2\ \frac{\left(\frac{v}{200\ \mathrm{km/s}}\right)^2}{(r/1\ \mathrm{Mpc})}}$$

Fiducial acceleration for galaxies at $r = 10$ kpc

$$\cancel{\approx \frac{(200\times10^3\ \mathrm{m/s})^2}{3\times10^{20}\ \mathrm{m}}\frac{\left(\frac{v}{200\ \mathrm{km/s}}\right)^2}{(r/10\ \mathrm{kpc})} = \left(\frac{4}{3}\right)\times10^{-10}\ \mathrm{m}/s^2\ \frac{\left(\frac{v}{200\ \mathrm{km/s}}\right)^2}{(r/10\ \mathrm{kpc})}}$$

So where is the turnaround radius?

$$f = 0 = C_1 \hat{M} \hat{r}^{-2} + C_2 \hat{r}$$

in fiducial units

$\tilde{r} = \left(\frac{r}{1\,\mathrm{Mpc}}\right)$

$$\hat{r} = \left(-\frac{C_1}{C_2}\hat{M}\right)^{1/3} = \left(-\frac{C_1}{C_2}\frac{M}{10^{12} M_\odot}\right)^{1/3}$$

large Galaxy

$$\hat{r} = [1.10981\ldots\ \mathrm{Mpc}]\left(\frac{M}{10^{12} M_\odot}\right)^{1/3}$$

large galaxy cluster (wiki)

$$= [11.0981\ldots\ \mathrm{Mpc}]\left(\frac{M}{10^{15} M_\odot}\right)^{1/3}$$

Example Galaxy M87 center of Virgo Cluster

Estimate: At radius 50 kpc encloses $6 \times 10^{12} M_\odot$

— but where does the dark matter halo mass fall to density of the intracluster medium?

Google AI say virial radius $= 1.3\ \mathrm{Mpc} = R_{200}$ where halo density is $200 * \rho_{\text{critical density of universe}}$

which for some reason is take as a good radius for galaxies

enclosed mass $1.3 \times 10^{14} M_\odot$

$\hat{r}(1.3 \times 10^{14} M_\odot) \cong [1.1\ \mathrm{Mpc}](130)^{1/3} \approx 5.5\ \mathrm{Mpc}$

So well beyond the virial radius.

3774b

Note on acceleration scale for galaxies

typical rotation curve plateau velocity at 10 kpc

Recall from p. 3078

$$a_{circular} = \frac{v^2}{r} = \frac{(200 \times 10^3 \text{ m/s})^2}{3 \times 10^{20} \text{ m}} \frac{\left(\frac{v}{200 \text{ km/s}}\right)^2}{\left(\frac{r}{10 \text{ kpc}}\right)}$$

1 kpc = 3.0857 $\times 10^{19}$ m

$$= \left(1.3 \times 10^{-10} \text{ m/s}^2\right) \frac{\left(\frac{v}{200 \text{ km/s}}\right)^2}{\left(\frac{r}{10 \text{ kpc}}\right)}$$

$$= 1.3 \times 10^{-13} \text{ m/s}^2 \frac{\left(\frac{v}{200 \text{ km/s}}\right)^2}{\left(\frac{r}{1 \text{ Mpc}}\right)^2}$$

comparable to fiducial acceleration scales for large galaxies at 1 Mpc (see p. 3078)

Also $r = a r_0$ Scaling of cosmic distances with cosmic scale factor $a(t)$

$\dot{r} = \dot{a} r_0$

$\ddot{r} = \ddot{a} r$

$= (-q_0 a_0 H_0^2) r$ (Liddle - 53)

q_0 is the deceleration parameter

$$= .55 \cdot 1 \cdot \left(70 \frac{\text{km/s}}{\text{Mpc}}\right)^2 \left(\frac{-q_0}{.55} \frac{a_0}{1} \cdot \left(\frac{H_0}{70}\right)^2\right) \cdot 1 \text{ Mpc} \left(\frac{r}{1 \text{ Mpc}}\right)$$

$$= .55 \cdot 49 \times 10^8 \text{ m}^2/\text{s}^2 \left(\frac{1}{\text{Mpc}}\right) * \left(\frac{1 \text{ Mpc}}{3 \times 10^{22}}\right) \left[\left(\frac{-q_0}{.55}\right) \frac{a_0}{1} \left(\frac{H_0}{70}\right)^2\right] \left(\frac{r}{1 \text{ Mpc}}\right)$$

$$= 10^{-13} \text{ m/s}^2 \left[\left(\frac{-q_0}{.55}\right) \frac{a_0}{1} \left(\frac{H_0}{70}\right)^2\right] \left(\frac{r}{1 \text{ Mpc}}\right)$$

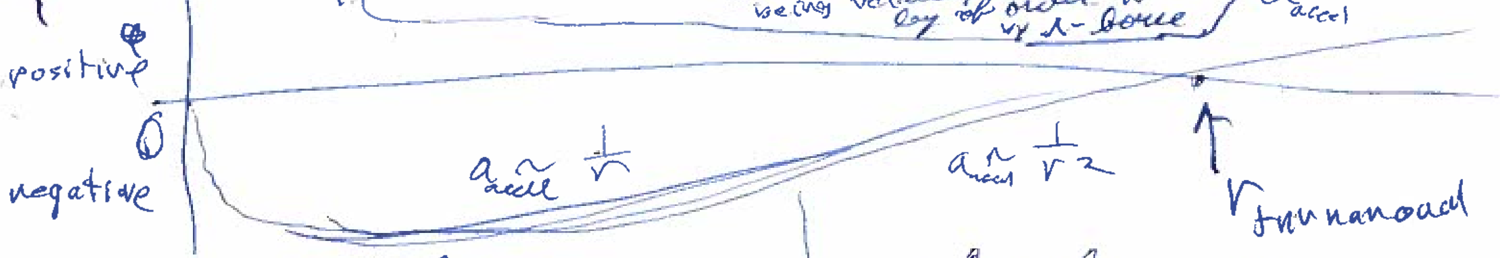
So probably not coincidently the cosmic acceleration per Mpc is also $\sim 10^{-13}$ m/s2

Supplemental Notes

Comparable 1 Mpc is fiducial nearest neighbor galaxies distance

Well the Λ dominates the evolution now and galaxies ~~beyond the~~ + separated by more than the turnaround radius are being violently accelerated by of order 10^{-11} km/s by Λ-force

[2025 jan 04] [3079c]



plateau region

Rotation curves of galaxies plateau

$v \triangleq 200$ km/s $\cong 2 \times 10^5$ m/s

outside most of the mass — But we don't know where that is or really how much)

mass joins the expansion of the universe (Kordis + Pavlidou (2024) use name turnaround radius)

Recall the circular velocity law (centripetal acceleration law)

$$mg = F = \frac{mv^2}{r}$$

$$\therefore \quad a = \frac{v^2}{r}$$

$$a \propto \frac{1}{r}$$

in this region

$h = \frac{H_0}{100}$ Cimatti-14

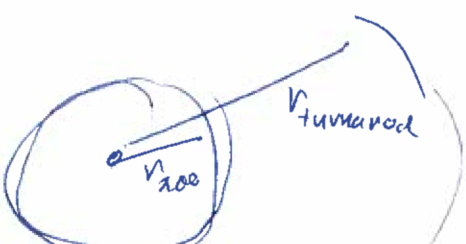
Cimatti p. 205

$\Delta_c = 200$ means $\frac{\rho}{\rho_{crit}} = 200$ Cimatti p. 203

Virial formula for order of estimate

$$r_{200} = (206.3 \text{ kpc}) \left(\frac{M_{200}}{10^{12} M_\odot} \right)^{1/3} \left(\frac{h}{0.7} \right)^{-2/3}$$

$$v_{200} = (144.4 \text{ km/s}) \left(\frac{M_{200}}{10^{12} M_\odot} \right)^{1/3} \left(\frac{h}{0.7} \right)$$



$$\frac{\rho(r)}{\rho_{critical\ of\ universe}} = \Delta_c = 200$$

fiducial radius for cut off virialization.

How much mass here — by shell theorem it doesn't affect inside much

3079d

For a given M_{200} (in units of $10^{12} M_\odot$), r_{200} is the order of virilzation radius (virialization radius) for galaxies at cosmic time present i.e. cosmological redshift $z = 0$

within ~~the~~ radius ~~the~~ mass distribution is in quasi ~~eq~~ steady state and if the outside is roughly ~~spherically~~ symmetric the shell theorem (gravity and cosmological constant), then outside of virial radius has little effect on inside other than cosmological of rest of universe

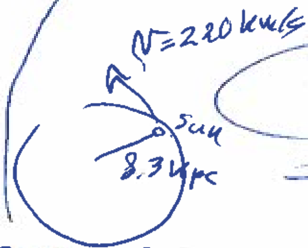
In fact, tracers of acceleration only let us measure out to ~ 200 — 300 kpc, and so not near the turnaround radius in general.

⇒ So how ~~that~~ far out mass including dark matter is observationally rather inaccessible.

2024 jan 04 — 3079e

see p. 3076 derivation

Note as $r \to r_{turnaround}$
$= 1.~~10~~ 888$ Mpc (see p. 3076)
for $M = 10^{12} M_\odot$



$a_{\text{~~sun~~}} = 1.8898.. \times 10^{-13}$ km/s
$a_{tc} = 8.7383... \times 10^{-17}$ km/s^2 / Mpc
$\lesssim 10^{-16}$ km/s / Mpc

Comparable

– Most of sun accel is NOT relative to our comoving frame

(see p. 3076)

the two field strengths or an acceleration goes ~~well~~ to $\sim 10^{-16}$ km/s^2

well below the $a_{MOND} = 1.2 \times 10^{-13}$ km/s^2

MOND which we may discuss a bit more ~~latt~~ later has no dark matter and uses a new acceleration/inertia behavior for very low acceleration $a \lesssim a_{MOND} = 1.2 \times 10^{-13}$ km/s^2

So where the turnaround radius is if MOND is true, I don't know. Someone probably has studied the question

fiducial value.

MOND ~~suppo~~ is a great counter theory to dark matter.

And ~~no one~~ it explains why can't directly detect dark matter

It supporters show that ~~some~~ version or another explains a lot of galaxy behavior.

But it doesn't work so well

There is more!!

~~3080~~ 3080

3079+

2025 jan 09

extra insert pages 3080a → 3080d

Eagle, Illustris, etc.

for galaxy clusters.

Also ~~the large~~ large scale structure simulations use dark matter and ~~get~~ evolve large scale ~~structure~~ from the primordial era to cosmic present ($z=0$, $t_0 = 13.8$ Gyr ~~after~~ Big Bang)

They must parametrize a lot of subgrid physics

pretty well. At present, it seems better treatments in such simulations may remove all significant discrepancies and leave no place for MOND.

But we still don't know what the dark matter particle is.

I sometimes wonder if we live in the worst of all possible worlds: There is dark matter exotic particles, dark matter primordial black holes and MOND.

Also a cosmological constant, constant dark energy and dynamical dark energy

Repeat page $\boxed{\text{2026 jan 24}}$ $\boxed{3079g}$

So where is the turnaround radius?

$$f = 0 = C_1 \hat{M} \hat{r}^{-2} + C_2 \hat{r}$$

in fiducial units

$$\left.\tilde{r} = \left(\frac{r}{1\,\mathrm{Mpc}}\right)\right\} \quad \hat{r} = \left(-\frac{C_1}{C_2}\hat{M}\right)^{1/3} = \left(-\frac{C_1}{C_2}\frac{M}{10^{12}M_\odot}\right)^{1/3}$$

$$\hat{r} = \left[1.10981\ldots\ \mathrm{Mpc}\right]\left(\frac{M}{10^{12}M_\odot}\right)^{1/3}$$

Example Galaxy M87 center of Virgo Cluster

__Estimate:__ At radius 50 kpc encloses $6 \times 10^{12} M_\odot$

— but where does the dark matter halo mass fall to density of the intracluster medium?

Google AI say virial radius = 1.3 Mpc = R_{200}

where halo density is 200 * critical density of universe

... which for some reason is take as a good radius for galaxies

enclosed mass $1.3 \times 10^{14} M_\odot$

$\hat{r}(1.3\times 10^{14} M_\odot) \cong [1.1\ \mathrm{Mpc}]\,(130)^{1/3} \cong 5.5\ \mathrm{Mpc}$

So well beyond the virial radius.

3080

2026 jan 24

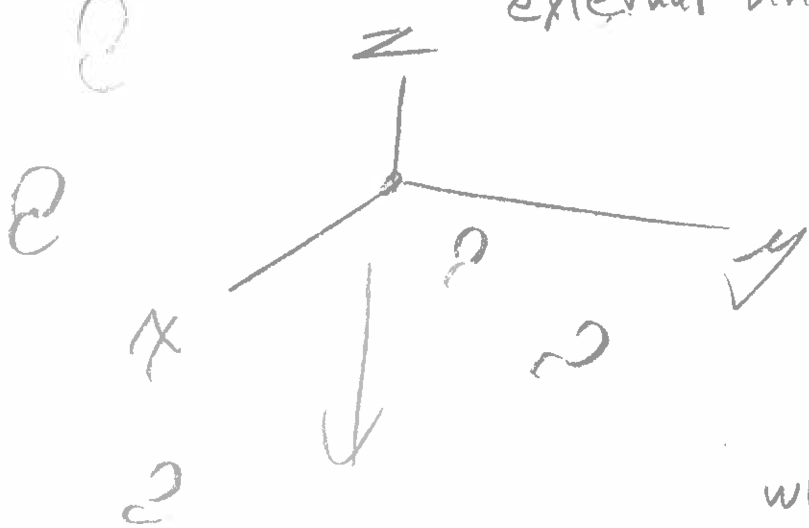
Largely a Newtonian physics discussion

3.15 Inertial Frames and Inertial Forces

General Relativity (GR) tells us what inertial frames are (correcting the pre-GR concept) and all other physical law is specified relative to inertial frames

[Probably, this last statement can be quibbled, but I suspect all quibbling amounts to using workarounds that effectively use inertial frames]

An elementary inertial frame is a free-fall frame in an external uniform gravitational field Not rotating relative to the observable universe (bulk mass of the observable universe) which defines an absolute state of Nonrotation



Notes (2026jan24) 3081

a) Maybe in intense gravity fields near black holes frames rotating relative to the observable universe are also inertial frames but this fine point is beyond my knowledge

b) The Λ-force (if it really exists) is always present in inertial frames.

c) If the external gravitational field is NOT uniform, you need to the tidall force

d) A local frame Not accelerated relative to a local inertial frame is also an inertial frame. There is an infinite family of them and for analysis you choose a convient one.

[Local meaning in same place or nearly the same place as usual]

3082

e) A local frame accelerated relative to a local inertial frame is a non-inertial frame

f) You can always analyze motions "in" a non-inertial frame ~~by~~ switching to a local inertial frame, Probably a larger frame.

Thinking of the Newtonian physics limit

g) Or you can convert the non-inertial frame to an inertial frame by invoking inertial forces

e.g.)

$$\underset{\sim}{F}_{net} = m\underset{\sim}{a}$$

frame acceleration

$$= m[\Delta \underset{\sim}{a}_i + \underset{\sim}{a}_f]$$

$$\left[\underset{\sim}{F}_{net} - \underbrace{m\underset{\sim}{a}_f}\right] = m\Delta\underset{\sim}{a}$$

inertial force $-m a_f$

$$\underset{\sim}{F}_{\substack{net \\ extended}} = m\Delta\underset{\sim}{a}$$

Inertial forces are sometimes
called fictitious forces,
but I disfavor that
name.
They are (linearly) mass-dependent forces
like gravity and in GR
gravity is an <u>inertial force</u>
So inertial forces are relationships
between mass-energy and
spacetime structure.
Ordinary forces are
relationships between
mass-energy including in
fields of force
(e.g., the electromagnetic field)
h) <u>Rotating Frames</u> (relative to the
observable
universe)
are tricky
because they are a
continuum of non-inertial frames

2026 jan 24

They are treated by the special inertial forces

Per unit mass

i) centrifugal force $-\underset{\sim}{\omega}\times(\underset{\sim}{\omega}\times\underset{\sim}{r})$

ii) coriolis force $-2\underset{\sim}{\omega}\times\underset{\sim}{v}$

iii) Euler force $-\frac{d\underset{\sim}{\omega}}{dt}\times\underset{\sim}{r}$

$\underset{\sim}{r}$ and $\underset{\sim}{v}$ measured in the rotating frame

Pressure-supported astro-bodies (stars, planets, moons, asteroids, etc.) are always rotating, and so treating there internal motions are almost always done using these inertial forces

Except on a small enough scale you can usually neglect them and just treat the very local frame (e.g., the ground) as defining an inertial frame good enough for your purposes.

Supplementary Notes | 2025 jan 05 | 30846 ~~3080~~a

(in) Inertial Frames

They have a physical nature — the structure of space

What are they actually?

I think ~~the~~ modern understanding is they are ~~the~~ reference frames for all physical laws {All refenced to inertial frames} except general relativity which tells us what they are, but we have to specify them for Newtonian calculations
(They emerge in a GR calculation and so are not specified

Some might quibble, but I think that after quibbling ~~that~~ the first statement stands.

But what are they intrinsically?

Isaac Newton 1643–1726 and everyone up to 1915 or so { Original Newtonian physics

Absolute Space was the singular fundamental inertial frame.

~~All fra~~ → the frame in which ~~the~~ stars were at rest on average

All frames not accelerated with respect to Absolute Space were also inertial frames.

General Relativity Available after 1915 { and modern version of Newtonian physics

The is NO singular fundamental inertial frame.

All frames in free fall in a uniform external gravitational field (including the Λ-field if it exists) with no other external forces acting are ideal inertial frames.

They are also not rotating with respect to the ~~bulk~~ observable universe (except maybe in strong gravitational fields like near black holes)

Other any ~~other~~ references frames could be converted to inertial frames using inertial forces (~~which are the~~ effect of accelerations) $-m_{body} * a_{accel}$

inertial force field $-a_{accel}$

And this was done for ground frames and what I call center-of-mass frames and celestial frames (see p. 3081) and the celestial mechanics people always got the right answers to ~~[illegible]~~ in the classical limit which is where they all worked before circa 1915

~~Traditional~~ Inertial ~~force~~ or inertial ~~m body experiencing~~

These ideal frames are closely approached by spacecraft in free fall and ~~plane~~ most planetary systems which are small compare to variations in galaxy fields.

Often using tidal forces and inertial forces you can vastly extend inertial frames to all cases you like excepting only when real general relativity effects are noticeable and if you restricting yourself to Newtonian physics you ~~than~~ have to have low enough relative velocities.

I think a nonrelativistic general approach to finding such (non-ideal) inertial frames ~~are~~ is to use center-of-mass frames. (CM frames)

Any non-relativistic noninertial frame can be turned into a center-of-mass inertial frame I think.

Special cases are what I call celestial frames (used in celestial mechanics), comoving frames (which are the cosmological upper limit of celestial frames), and rotating frames (used for rotating compact ~~pressure~~ ~~supported~~ bodies: e.g., planets and stars)

See p. 3081 for these

Not supported by macroscopic kinetic energy

An oddity of Newtonian physics is that

$$m_{grav} = m_{inertial}$$

charge of gravity force | resistance to acceleration

$\therefore$ If only gravity acts on a body

$$\underset{\sim}{F}_{net} = m\underset{\sim}{g} = m\underset{\sim}{a}$$

and so $\underset{\sim}{a} = \underset{\sim}{g}$

– the acceleration equals the gravitational field.

In Newtonian physics there is an unexplained coincidence.

Now inertial forces are always $-m\underset{\sim}{a}_{accel}$

$$\underset{\sim}{F}_{net} = ma$$
$$= m(\Delta g + a_{accel})$$

$$F_{net\,real} - m a_{accel} = m\Delta g$$

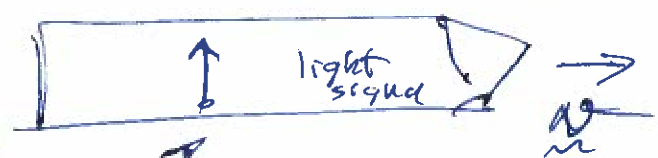
Newton's 2nd law for a noninertial frame.

Einstein concluded that in closed box you cannot distinguish gravity and inertial forces so Gravity is in a sense an inertial force

In fact, No way in principle to distinguish uniform $\underset{\sim}{g}$ and uniform $\underset{\sim}{g}$

The sameness of gravitational mass and inertial mass is an axiom (or corollary ~~thereof~~ thereof) of general relativity.

A Gedanken experiment that goes back to Einstein (I think) is imagine a rocket ship



$\underset{\sim}{v}$

the light beam sent perpendicular follows a straight line as it should for an inertial frame

unaccelerated and so defines an inertial frame

now accelerating at $\underset{\sim}{a}_{accel}$

beam should follow a curved path relative the rocket since it follows a straight line path relative to a coincident inertial frame.

But if matter (with rest mass) responds to gravitation just as to acceleration, then so should light.

~~308~~ 30

(2017 jan 09)

Light ~~should~~ beams should be bent by gravity.

Now the idea that gravity could affect light goes back to the 18th century or even Newton I think.

So there are simple Newtonian hypotheses to predict the bending of light.

But in fact they are wrong quantitatively

Gravitational lensing quantitatively requires a general relativity calculation.

The 1919 ~~Solar~~ eclipse measurement verified the bending of light beam passing near the Sun, but the key point was the GR prediction was confirmed within error and the other hypotheses weren't.

Actually that was very favorable eclipse for those measurements since there were many stars to measure — the Hyades Cluster was aligned with the Sun, and so the confirmation was not significantly ~~verified~~ improved for decades.

The 1919 Eclipse verification made Einstein famous for the general public.

3.16 Center-of-Mass Inertial Frames

CM Frames

all in the classical limit

A name I invented because it needed to exist since we use them all the time for kinetic-energy-supported astro-bodies (galaxy clusters, galaxies, star clusters, planetary systems, moon systems, etc.)

CM frames can be used in other contexts but probably not often.

There is a hierarchy of these systems, but every level has a broad distribution, and those evolve with the evolving universe.

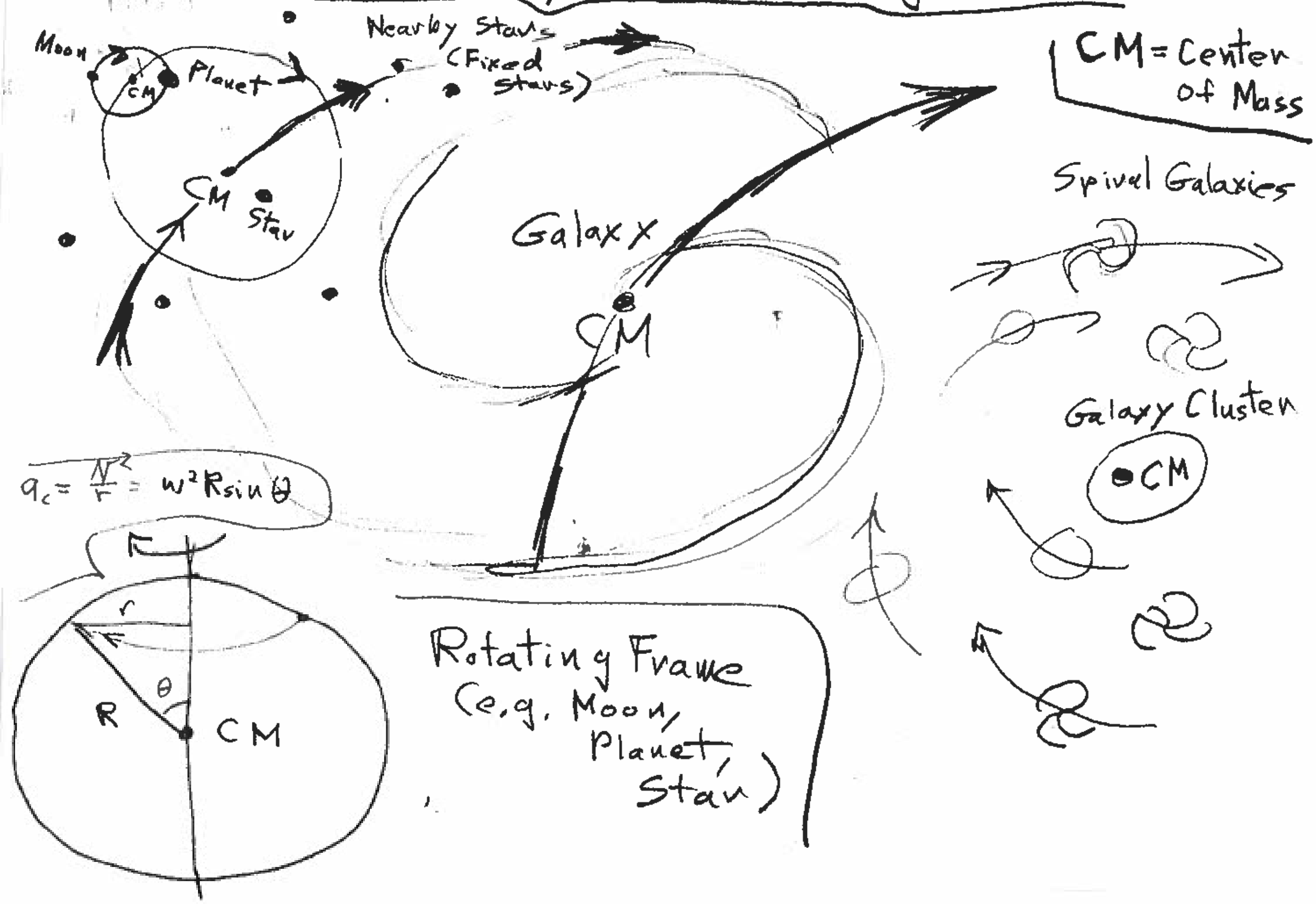
Primordial density fluctuations in dark matter mostly set cosmic voids and galaxy superclusters (which are mostly unbound) and universe expansion may limit the size of bound galaxy clusters and super massive black hole feedback and and merger history may control to a degree largest galaxies.

3086

2026jan24

Center-of-Mass ~~Free-Fall~~ Inertial Frames (CM ~~COMFFI~~ Frames) in a Hierarchy and Rotating Frames

DJ Jeffery
UNLV
2021



Large-scale structure formation is a major branch of modern cosmology that in this course we only touch on. There has been huge progress in the last 50 years, but lots more to do especially as JWST and the ongoing/upcoming giant survey keep making major data dumps.

[I predicted 30 years ago that in the future data management would become a major bore]

CM Frames

Note it is natural to use the CM frame as a synonym for the system in most contexts

General system of particles

total mass $m = \sum_i m_i$

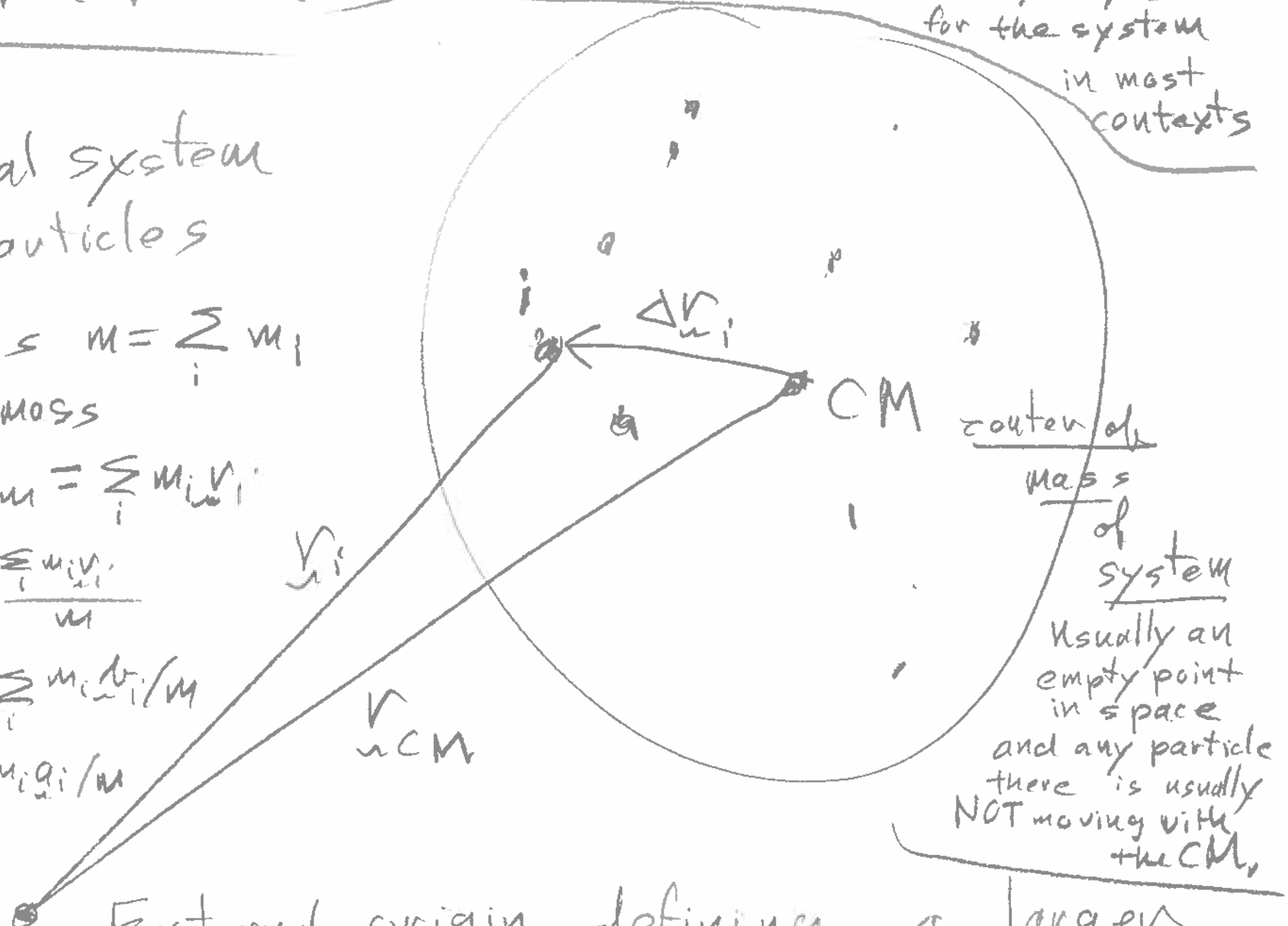
center of mass

$$m \underset{\sim}{r}_{cm} = \sum_i m_i \underset{\sim}{r}_i$$

or $$\underset{\sim}{r}_{cm} = \frac{\sum_i m_i \underset{\sim}{r}_i}{m}$$

$$\underset{\sim}{v}_{cm} = \sum_i m_i \underset{\sim}{v}_i / m$$

$$\underset{\sim}{a}_{cm} = \sum_i m_i \underset{\sim}{a}_i / m$$



External origin defining a larger inertial frame. The largest ones being the comoving CM frames discussed on p. 3075

Equation 1

Apply Newton's 2nd Law to particle i

Lambda force see p. 3071 – 3079

$$\underbrace{\underset{\sim}{F}_{ext\,i} + F_{int\,i}}_{} + m_i\Big(\underset{\sim}{g}_{ext\,i} + \sum_{j \neq i} \underset{\sim}{g}_{ji}\Big) + \underbrace{\frac{\Lambda}{3}(\underset{\sim}{r}_{cm} + \Delta \underset{\sim}{r}_i) m_i}_{\frac{\Lambda}{3} r_{cm} m + \frac{\Lambda}{3} \Delta r_i m_i}$$

$$= m_i(\Delta \underset{\sim}{a}_i + \underset{\sim}{a}_{cm})$$

Excluding gravity and Λ-force

For kinetic-energy supported astro-bodies usually negligible except sometimes the magnetic force.

3088

2026 jan 24

— separating gravity and the Λ-force is sensible since they are always long-range forces and act differently than other forces and each other.

But in a GR sense, a true cosmological constant force is part of gravity

— the electromagnetic force can be long but often isn't,

— pressure, friction, normal force, etc., are contact forces

Important for pressure-supported bodies (planets, stars, nebulae, etc.) and rigid bodies due to chemical bonds

Equation 2 Sum over all particles in the system

$$\sum_j \underset{\sim}{F}_{ext_j} + 0 + \sum_j m_j \underset{\sim}{g}_{ext_j} + 0 + \frac{\Lambda}{3} \underset{\sim}{r}_{cm} M + 0$$

assume $\sum_j F_{int_j}$ cancel by 3rd law or some more general law

internal gravity cancels by 3rd law

The internal Λ-forces cancel by CM definition and NOT by the 3rd law

$$= M \underset{\sim}{a}_{cm} + 0$$

cancels by CM definition

Recall, relative to an external inertial frame origin

Note $-M \underset{\sim}{a}_{cm}$ is an inertial force on the CM and the LHS is collectively an "inertial force" that we are about to subtract off

Equation 3

$$\frac{\text{Equation 1}}{m_i} - \frac{\text{Equation 2}}{m} = \text{Equation 3}$$

The tidal force

$$\frac{F_{ext\,i}}{m_i} - \frac{\sum_j F_{ext\,j}}{m} + \frac{F_{int\,i}}{m_i} + \left(g_{ext\,i} - \frac{\sum_j m_j g_{ext\,j}}{m} \right)$$

$$+ \sum_{\substack{j \\ j\neq i}} g_{ji} + \underbrace{\frac{\Lambda}{3} \Delta r_i}_{\text{the } \Lambda\text{-force tidal force sort of}}$$

(see p. 3077)

$$= \Delta a_i$$

Note these are part of the inertial force specified on p. 3088

But they both have a dependence on m_i though it may be weak in many cases so they are sort of partial inertial forces.

A clean separation into ordinary forces and inertial forces does not happen.

Tidal Force Special Cases

a) If g_{ext} is uniform over the whole system, then

$$F_{tidal} = g_{ext} - \frac{\sum_j m_j g_{ext}}{m}$$

$$= g_{ext} - g_{ext} = 0$$

In this case the CM frame is an elementary inertial frame in a GR sense.

b) Say the 1st order expansion is adequate

$$g_{ext\,k} = g_{ext\,k0} + \sum_\ell \Delta x_\ell \left.\frac{\partial g_{ext\,k}}{\partial x_\ell}\right|_0$$

coordinate index NOT particle

for the whole system where 0 labels the CM.

3090

2026 jan 24

$$\frac{\sum_j m_j g_{ext\,j\,k}}{M} = g_{ext\,k\,0} + \sum_\ell \frac{\partial g_{ext\,k}}{\partial x_\ell}\Big|_0 \sum_j m_j \Delta x_{j\ell}$$

j is a particle index

0 by definition of CM

$$= g_{ext\,k\,0}$$

$$\therefore \frac{\sum m_j \underset{\sim}{g}_{ext}}{M} = \underset{\sim}{g}_{ext\,CM}$$

Then the tidal force is

$$\underset{\sim}{g}_{tidal\,i} = \underset{\sim}{g}_{ext\,i} - \underset{\sim}{g}_{ext\,CM}$$

which is a result usually assumed for planets and other bodies small compared to variations in the external gravitational field

Note say $\underset{\sim}{g}_{tidal} = g_{ext\,i} - g_{ext\,CM}$ holds, non-gravity forces are zero, the mass distribution of the system is spherically symmetric, and particle i is at the CM, then $\sum_{j \neq i} \underset{\sim}{g}_{ji} = 0$ by shell theorem

$$\underset{\sim}{g}_{tidal\,i} = 0, \quad \frac{\Delta}{3}\Delta \underset{\sim}{v}_i = 0 \text{ since } \Delta \underset{\sim}{v} = 0,$$

and so $\Delta \underset{\sim}{a}_i = 0$ and so particle i is unaccelerated relative to the CM, but $\underset{\sim}{v}_i \neq 0$ in general.

3.17 Celestial CM Frames

In celestial mechanics, one can usually neglect the "other forces" (the $\underset{\sim}{F}$ forces on p. 3089 Equation 3), ~~then~~ Equation 3 for particle i reduces to

$$\underbrace{\left(g_{ext\,i} - \frac{\sum_j m_j\, g_{ext\,j}}{m} \right)}_{g_{tidal\,i}} + \sum_{\substack{j \\ j\neq i}} \underset{\sim}{g}_{ji} + \frac{\Lambda}{3}\Delta \underset{\sim}{r}_i = m_i \Delta \underset{\sim}{a}_i$$

One would need this for galaxies, galaxy clusters, and galaxy superclusters

But these may not be treatable as CM frames. You may need a large-scale structure simulation box on a comoving frame (see p. 3093)

For planetary systems, with stars, planets, etc., treated as point masses, the external gravity field is usually so uniform that the tidal force is negligible (see p. 3089) and the Λ-force completely negligible.

So the equation of motion reduces to

$$\sum_{\substack{j \\ j\neq i}} \underset{\sim}{g}_{ji} = m_i \Delta a_i$$

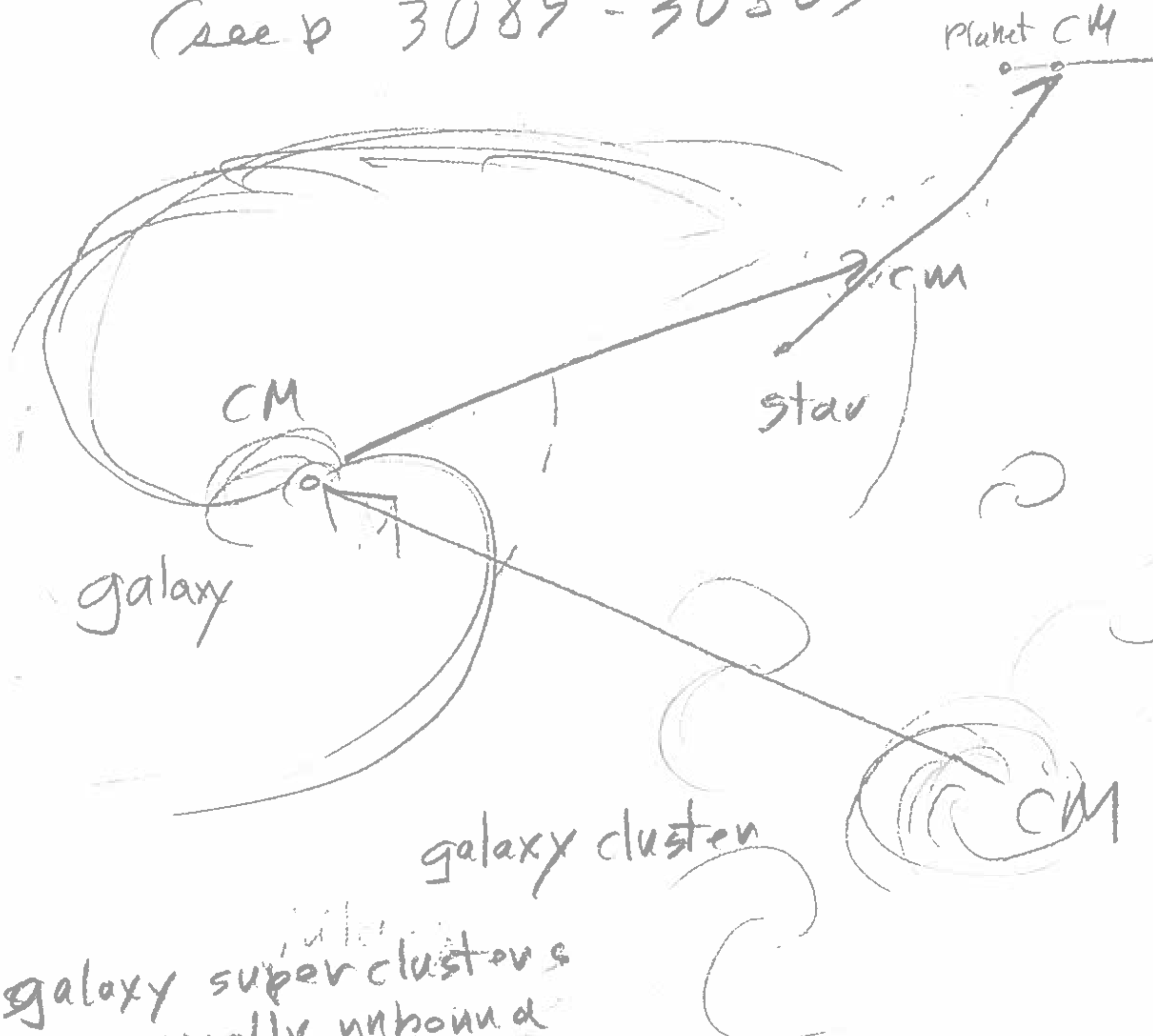
For moon systems, the external gravity of the parent star may not be negligible, and so the equation of motion would be $\underset{\sim}{g}_{tidal\,i} + \sum_{\substack{j \\ j\neq i}} \underset{\sim}{g}_{ji} = m \Delta \underset{\sim}{a}_i$

2026 jan 24

Equation 2 on p. 3088 applies to the whole system's motion in the external larger CM frame

$$\sum_j m_j \, g_{ext_j} + \frac{\Lambda}{3} \underset{\sim}{r}_{cm} M = M \underset{\sim}{a}_{cm}$$

As already discussed, one has a whole hierarchy of celestial CM frames (see p 3085-3086)



Note gravitationally bound systems do NOT expand with universe. The Expansion is a growth of space between bound systems.

galaxy superclusters are usually unbound and amorphous and ill-defined (defined in the eye of the beholder) and so (as said on p. 3091) probably can't be treated by CM frames. You need -

G R perspective on inertial frames

From a GR perspective all there are close to elementary inertial frames:
Free-fall frames in a uniform gravitational field

The hierararchy of CM frames tops out NOT a single univeral absolute inertial frame but at infinite continuum of comoving frames (see p. 3095)

No universal inertial frame

But there is an absolute rotation measured to bulk mass-energy of observable universe. At least for the observable universe away from strong gravity of Black holes

Newton's own perspective used before GR

There is Absolute Space — a singular fundamental inertial frame.

All CM frames attached to systems of astro-bodies are non-inertial frames converted to inertial frames by an inertial forces

$-m\,\underset{\sim}{a}_{cm}$

that cancels

$m\,\underset{\sim}{g}_{ext}$

which is partially what we did on p. 3088.

And this worked perfectly well and so celestial mechanics pre-1915 as it does now from our GR perspective

3099)

of course, the old Newtonian Perspective did not work for cosmology as Newton himself partially understood in unpublished work from c. 1700

→ But by this year Newton was 58 and moving beyond active science and did not pursue cosmological questions.

Instead he became Master of the English Mint and made money (which can all agree was a good thing)

The hierarchy tops out with spherical comoving ~~frames~~ , but they can have various sizes

Comoving frames are Not gravitationally bound unlike the smaller CM frames on p 3092

We are here assuming the cosmological principle — the universe on large enough scales is homogeneous and isotropic. A basic principle/axiom of most modern cosmology of the observable universe

The CM participates in the mean expansion of the universe. It comoves with the expansion.

$r \sim 400\,\mathrm{Mpc}$

CM of comoving frame Large enough to be spherical symmetric on average and so CM at center, But NOT so large that you need to go beyond Newtonian physics.

Birkhoff's theorem tells us if everything outside is uniform enough we can analyze everything inside without outside effects

Of course these are ideal limits that might be hard to get close enough too in practice. The Stiskalek et al. 2026 paper shows how hard it is for the comoving frame centered on us (i.e., the local group)

3096

2026jan24

Notes

a) Traditionally, superclusters and voids are a bit in the eye of the beholder.

~~There~~ a bit amorphous in shape.

The Laniakea Super Cluster is defined by a specific rule where you freeze the motions and follow stream lines of test particles from watersheds.

It includes the Virgo supercluster

That may be a useful definition, but seems a bit hard to apply and Stiskalek et al, 2026 indicate such definition bit arbitrary with only small physical meaning.

b) What is the homogeneity scale for the observable universe = R_H

Sawala et al, 2025 says there is no sharp value

At the 10 Mpc scale, you clearly have gravitationally bound or strongly interacting systems.

Compare to fiducial turnaround radius of galaxy cluster on p.3079?. About same probably not a coincidence

Cimatti-115 large galaxy clusters virial radius ~ 3 Mpc virial mass $10^{15} M_\odot$

At 1 Gpc scale, you have patterns determined by your selection criteria which are the same as randomly placing galaxies

And this is consitent with the Λ-CDM model and the cosmological Principle

No sharp line between the two behaviors

cMpc means comoving Mpc

R_H = Yadav scale = $\frac{370}{h_{70}}$ cMpc (Yadav 2010)

$h_{70} = \frac{H_0}{70 \frac{km/s}{Mpc}}$

Scales with universal expansion

from a fractal argument is often sited as homogeneity scale

Reasonable and well between 10 Mpc ~~and~~ 1 Gpc

From studying Peculiar velocities

Giani... Davis... etal (2026)

suggest $R_H = \frac{180 \text{ cMpc}}{h_{70}}$ as the homogeneity scale

R_H being a scale for some sphere or cube

3098)

Recently people have identified gigaparsec structures — Great Arc, Great Ring Lopez et al (2012, 2024).

in Sawala et al's (2025) view, these are patterns — real, but just found by a selection criterion. — different patterns would occur for different selection criteria.

They are not of physical importance

c) Peculiar Velocities + Recession Velocities