3.12 Gauss' Law for the Linear Force

The linear force gives us the classical analogue to the GR cosmological constant.

For Inverse-Square law versions, see p. 3004 ~~and~~ 3012

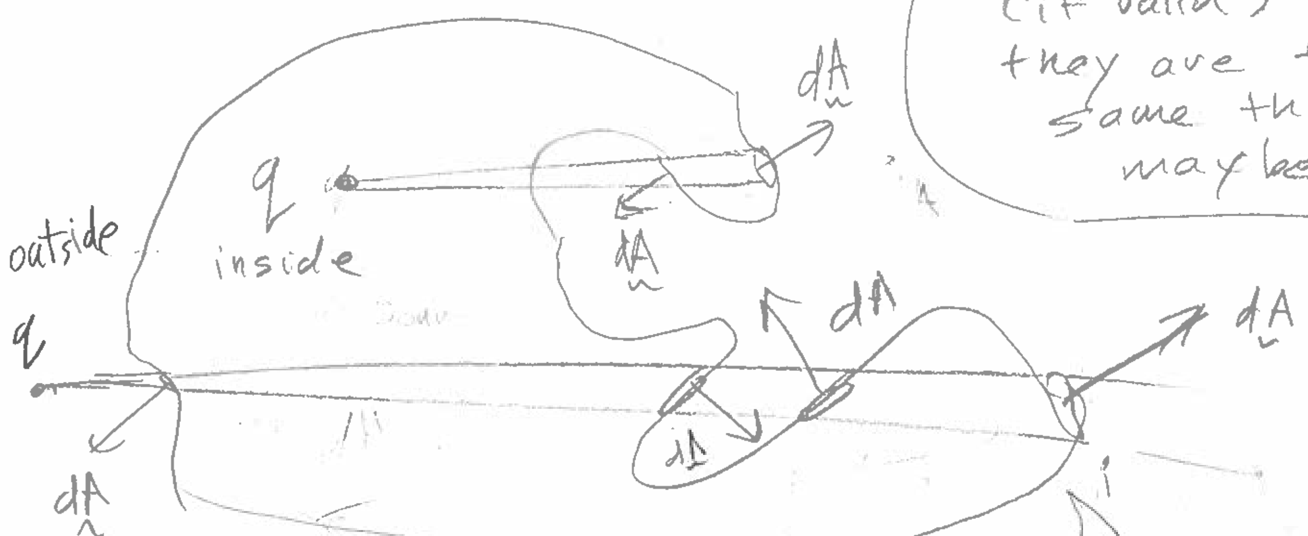
Generic linear force field for a point charge q

$$\underset{\sim}{f} = q \underset{\sim}{r}$$

with potential $\phi = -\frac{q}{2} r^2$ Arf-104

The source and receiver "charge" may be different, but 3rd law (if valid) implies they are the same though may be disguised.

Gaussian closed surface



General contribution

$\hat{r} \cdot d\underset{\sim}{A}_i = |d\underset{\sim}{A}_i| \cos\theta = \pm r^2 d\Omega$

$$\left[\underset{\sim}{f} \cdot d\underset{\sim}{A}\right]_i = \left[q r (\pm r^2 d\Omega)\right]_i$$

+ for outward going
− for inward going

In spherical coordinates $d\Omega = \sin\theta \, d\theta \, d\phi$ of course

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$$\left[\underset{\sim}{f}\cdot dA\right]_i = \left[3q\ \frac{r^3}{3}(\pm d\Omega)\right]_i$$

$$= \left[3q \int_{r_{i-1}}^{r_i} r^2\,dr\,d\Omega\right]_i$$

$$= 3q\left[\pm V(r_i - r_{i-1})\right]_i$$

$\therefore$ Along any cone

$$f\cdot dA = \sum_i\left[\underset{\sim}{f}\cdot dA_i\right] = 3q\,V_{\text{cone}}$$

inside to the last outgoing surface

$$\therefore\ \oint f\cdot d\underset{\sim}{A} = 3q\,V_{\text{Gaussian surface}}$$

q anywhere inside, outside, at infinity contributing to $\underset{\sim}{f}$

Summing up all point charges

$$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = 3Q\,V$$

where Q is all charge in the universe contributing to $\underset{\sim}{f}$

Gauss' Law for the linear force QED

The linear force is a very long range force

Another Proof

$$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = \int \nabla\cdot\underset{\sim}{f}\,dV = 3qV$$

Then sum on all point charges again

by Gauss' Thm p. 3009

divergence in spherical coordinates (Arf-104)

$$\nabla\cdot f = \nabla\cdot(q\,r\,\hat{r}) = q\,\frac{1}{r^2}\,\frac{\partial(r^2)r}{\partial r} = 3q \text{ for a point charge}$$

Actually, you can use Gauss' Theorem to prove Gauss law too sort of.

$$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = \oint \nabla\cdot \underset{\sim}{f}\, dV = \int q \frac{1}{r^2}\frac{\partial (r^2 h)}{\partial r}\, dV$$

Point charge

q $\quad \underset{\sim}{f} = q\, h(r)\, \hat{r}$

general function of r

$$\nabla\cdot \underset{\sim}{h} = \left[\frac{\partial h}{\partial r} + \frac{2}{r} h\right]$$

If $h = r^n$

$$\frac{\partial h}{\partial r} = n r^{n-1} + 2 r^{n-1}$$

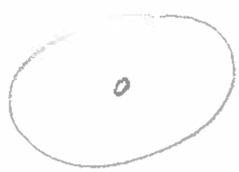
a) If $\underline{n=1}$, $\frac{\partial h}{\partial r} = 3$ as before on p. 3066

b) If $\underline{n=-2}$

$$\frac{\partial h}{\partial r} = -2r^{-3} + 2r^{-3} = 0.$$

But the is a singularity at $r=0$

Little sphere around charge and one gets a Dirac Delta function result.



$$\oint h\hat{r}\cdot d\underset{\sim}{A} = r^n\, 4\pi r^2 = 4\pi r^{2+n}$$

If $n=-2$, $\quad \oint h\hat{r}\cdot d\underset{\sim}{A} = 4\pi$

$$\therefore\ \nabla\cdot \underset{\sim}{h} = 4\pi\,\delta(r)$$

as on p. 3011

In general $\oint \underset{\sim}{h}\cdot dA = \int \nabla\cdot \underset{\sim}{h}\, dV = \begin{cases} 4\pi & \text{if charge inside} \\ 0 & \text{if charge outside}\end{cases}$

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c) Are there any ~~other~~ cases of n that lead to interesting divergences and results

$n=1$ leads to constant divergence

$n=-2$ leads to Dirac Delta divergence

It seems not. Only position dependent by inside/outside.

All other power-law cases lead to an r dependence, and so not a position independent Gauss' Law.

But how about $h = e^{r^n}$?

$$\nabla \cdot \underset{\sim}{h} = n r^{n-1} h + \frac{2}{r} h$$

$$= [n r^n + 2] \frac{h}{r}$$

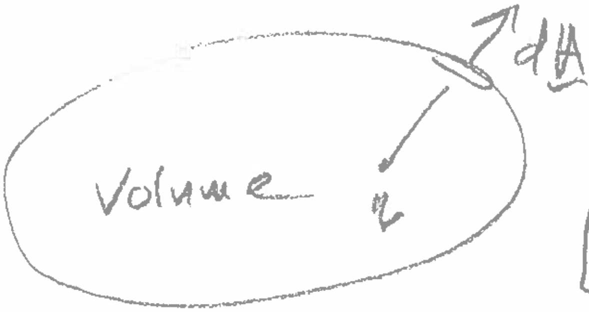
seems to lead to nothing.

So maybe there are only two Gauss laws { the Gauss' law and the linear force Gauss' law

3.13 Shell Theorem for the Linear Force Gauss' Law

$$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = 3\underbrace{Q}_{\text{All } q \text{ anywhere inside and outside contributing to } \underset{\sim}{f}}V \quad \text{(see p 3066)}$$

(So NOT all charge in universe)

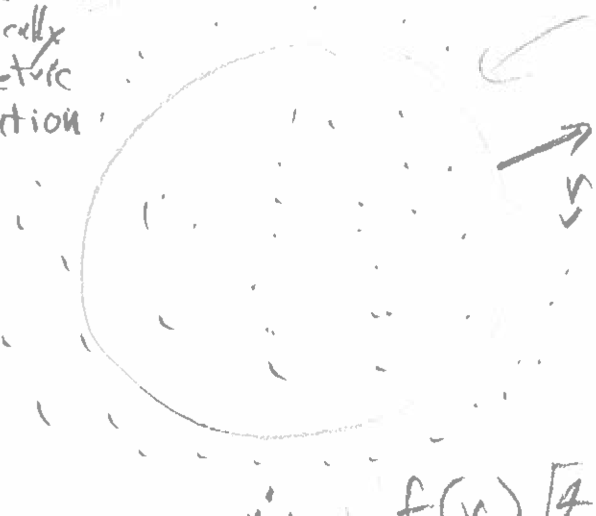


Probably only direct solutions for $\underset{\sim}{f}$ in 3 cases of high symmetry: planar, cylindrical, spherical

Let's consider spherically symmetric distribution.

charges in the spherically symmetric distribution

Gaussian Surface.



By symmetry

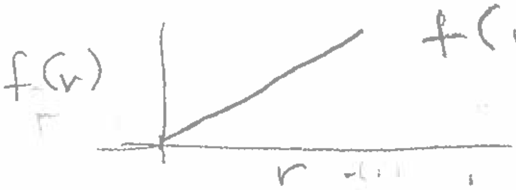
$$\underset{\sim}{f}(\hat{r}) = f(r)\,\hat{r}$$

$f(r)$ can be +ve or -ve

$$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = f(r)[4\pi r^2]$$

$$\therefore\ f(r)[4\pi r^2] = 3Q\left(\tfrac{4\pi}{3}\right)r^3$$

$$f(r) = Qr \quad \text{and} \quad \underset{\sim}{f}(r) = Qr\hat{r} = Q\underset{\sim}{r}$$



and so just as if all charge were point charge at the center.

Let's assume only one kind of charge and for point charges

$$\mathbf{F}_{ij} = (q_i \hat{\mathbf{r}}_{ij}) q_j = q_i q_j \hat{r}_{ij}$$

$$= -q_i q_j \hat{r}_{ji} = -\mathbf{F}_{ji}$$

and so 3rd law holds.

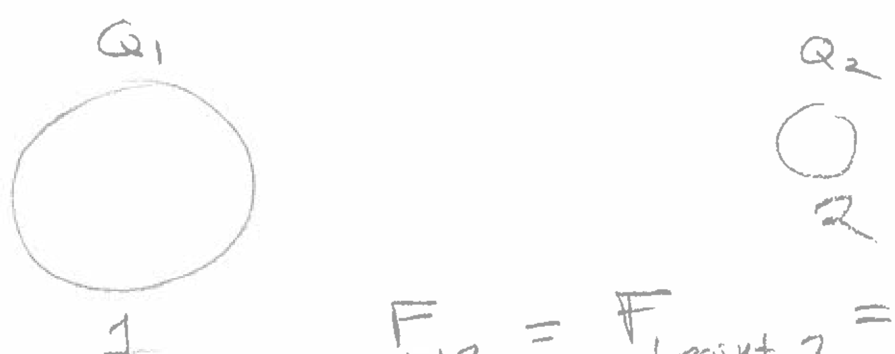
What two systems of point charges?

$$\mathbf{F}_{12} = \sum_{ij} q_i q_j \mathbf{r}_{ij} = -\sum_{ij} q_i q_j \mathbf{r}_{ji} = -\mathbf{F}_{21}$$

and so 3rd law holds again

What of two spherically symmetric charge systems?

Exactly Analogous to Proof on p. 3015 for gravity.



$$\mathbf{F}_{12} = \mathbf{F}_{1,\text{point } 2} = -\mathbf{F}_{2,1\text{point}}$$

$$= -\mathbf{F}_{2\text{point},1\text{point}} = \mathbf{F}_{1\text{point},2\text{point}}$$

But there is distinction from the gravity case.

In gravity case the spherically symmetric mass distributions can't overlap for a simple result since

$$\underset{\sim}{g} = -\frac{G M_{enclosed}}{r^2}\hat{r}$$

For the linear

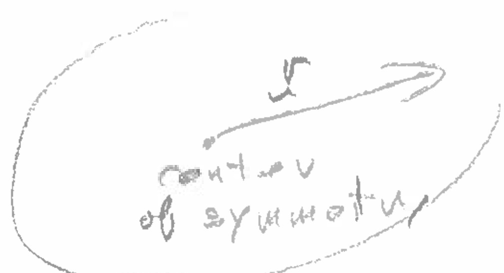
$$\underset{\sim}{F} = Q\underset{\sim}{r}$$

where Q is all charge in the spherically symmetric distribution. So the two systems can overlap and even coincide, but magically each one stays spherically symmetric

3.14 The Newtonian Cosmological Constant Force AKA the Lambda (Λ) Force

For a spherically symmetric linear force charge Q

$$\underset{\sim}{f} = Q\underset{\sim}{r}$$



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But if the charge distribution is uniform out to R

$$\underset{\sim}{f} = \left[\left(\frac{4\pi}{3}R^3\right)\rho\right]\underset{\sim}{r}$$

If $R \to \infty$, $\underset{\sim}{f} \to \infty$.

We hypothesis that

$$\frac{4\pi}{3}R^3\rho \longrightarrow \frac{\Lambda}{3}$$

← Lambda

where Λ is a constant and, in fact, the Newtonian analog the cosmological constant of GR

"charge" positive always and so a pushing force

For simplicity in discussion, I consider the Cos. Constant the simplest form of dark energy even though it is an aspect of gravity. Constant dark energy the next simplest. Both are conflated as Λ since the have the same effect in the Friedmann eqn. though they differ in other aspects

[In Einstein Field equations (p.3034), Λ is the coefficient ~~of~~ term and is an aspect of gravity. It's long range effect is not explicitly a force, but by being a requirement for all spacetime]

The $\frac{1}{3}$ factor is for consistency with GR.

The Λ force field is

$$\underset{\sim}{f} = \frac{\Lambda}{3}\underset{\sim}{r}$$

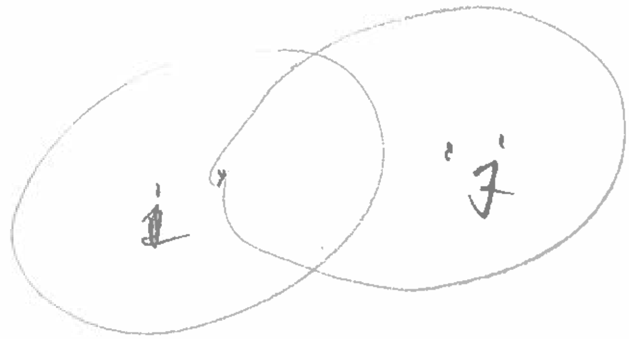
Hard to believe in a force that grows to infinity. But in GR, Λ ~~gun~~ is just a term in the Einstein field eqns (p.3034) that apply ~~in~~ all spacetime. Recall $\underset{\sim}{f}_g = -\frac{GM}{r}\hat{r}$ goes to zero only at ∞ which is also a bit hard to believe, but is just an aspect of Poisson's eqn $\nabla^2\phi = 4\pi G\rho$

and the origin for $\underset{\sim}{r}$ is any point in unbounded space.

[For flat or hyperbolic space, unbounded means infinite space. Hyperspherical space finite but unbounded. We'll discuss this a bit later. But pure Newtonian space is always flat]

Does $\underset{\sim}{f} = \frac{\Lambda}{3} \underset{\sim}{r}$ ~~s~~ satisfy 3rd law?

Maybe not and not explicitly at ~~least~~



$$\underset{\sim}{F}_{ij} = \frac{\Lambda}{3} \underset{\sim}{r}_{ij} m_j$$

$$= -\frac{\Lambda}{3} r_{ji} m_j$$

$$\neq -\frac{\Lambda}{3} r_{ji} m_i \quad \text{unless } m_i = m_j$$

But the source of $\underset{\sim}{F} = \frac{\Lambda}{3} \underset{\sim}{r}$

is NOT a point in space

but all space or all mass in space

So $\underset{\sim}{F}_{ij}$ pushes on m_j

but m_j doesn't push on m_i (which just happens to be at the origin of $\underset{\sim}{F}_{ij}$), but on all space on all mass in space. But this is just a Newtonian analog view which may have no real meaning in GR.

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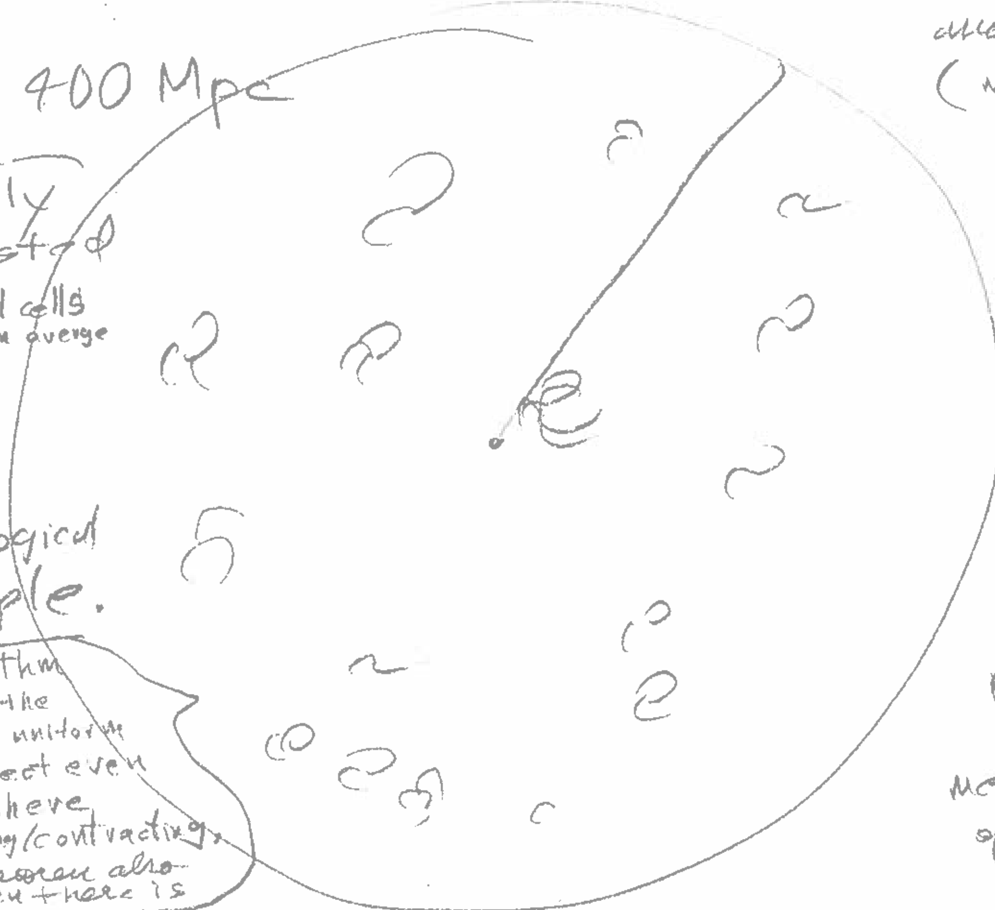
Take any point in space and draw a big sphere around it.

Given large scale homogeneity and isotropy (which is called the cosmological principle)

$R \cong 400$ Mpc

Commonly suggested scale for all cells to be same on average and so to roughly satisfy cosmological principle.



the center is the center of Mass (CM) for the sphere and participants in the mean expansion of the universe.

Birkhoff thm tells us the outside if uniform has no effect even if the sphere is expanding/contracting. Birkhoff's theorem also applies when there is a Λ term since it can be moved to $T_{\mu\nu}$ as a dark energy

The CM is the origin of a comoving inertial frame in the Newtonian limit, so the sphere is Not so large as to require GR corrections

A particle at CM will generally have a peculiar velocity.

The Local Group has peculiar velocity $v = 620(15)$ km/s to $l = 271.9(20)^\circ$, $b = 30(3)^\circ$

Known from CMB dipole

Galactic longitude

Galactic latitude

(Wiki: CMB) which is in Ursa Major (Google AI)

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so

origin in a comoving CMF frame

Cosmological constant force from $CM_{comoving}$ on particle i

$$\underset{\sim}{F}_i = \frac{\Lambda}{3}\underset{\sim}{r}_i m_i$$

Force on whole system

$$\underset{\sim}{F} = \sum_i F_i = \frac{\Lambda}{3}\sum_i \underset{\sim}{r}_i m_i$$

$$= \frac{\Lambda}{3}\sum_i (\underset{\sim}{r}_{cm} + \Delta \underset{\sim}{r}_i) m_i$$

relative to CM of the system

$$= \frac{\Lambda}{3}\underset{\sim}{r}_{cm} m$$

since $\sum_i \Delta \underset{\sim}{r}_i m_i = 0$

<u>Upshot</u> for any system of particles the Λ-~~force~~ just pushes out from it's CM and for internal motions

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the externall net Λ-force makes no contributions

So what is the internal Λ-force effect on a system, e.g., a galaxy including its dark matter halo?

$$\underset{\sim}{f} = \underbrace{-\frac{GM(r)}{r^2}\hat{r}}_{\text{gravity}} + \underbrace{\frac{\Lambda}{3}\underset{\sim}{r}}_{\Lambda\text{ force}}$$

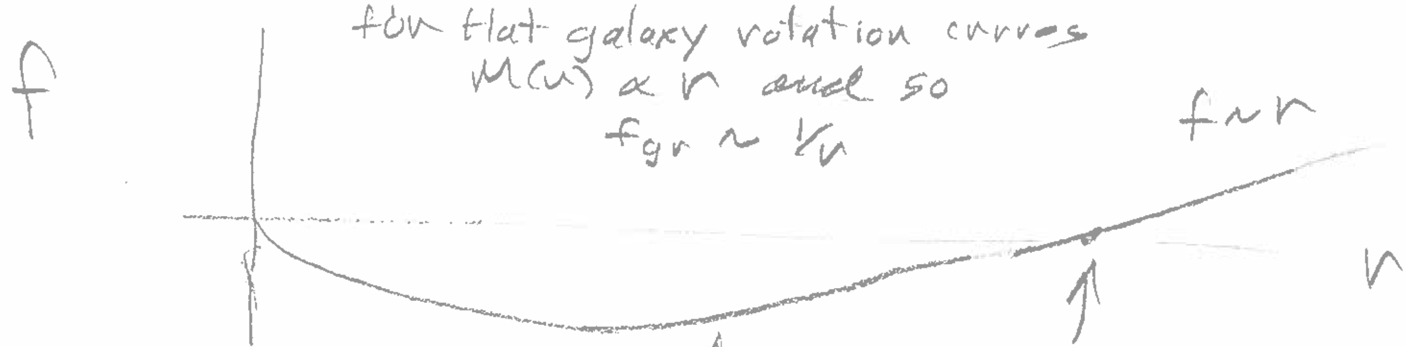
gravity for enclosed mass to r assuming spherical symmetry which is roughly for dark matter halos

Note

$$\frac{v^2}{r} = \frac{GM(r)}{r^2} \quad \text{for a circular orbit}$$

$$v = \sqrt{GM(r)/r}$$

for flat galaxy rotation curves $M(r) \propto r$ and so $f_{gr} \sim 1/r$



$f \sim \frac{1}{r}$

turnaround radius

beyond the turnaround radius particles are unbound

Now for the Λ-CDM model, Wik gives

$$\Lambda = 1.1056 \times 10^{-52}\ m^{-2}\ c^2$$

$$[\Lambda] = s^{-2} \text{ units}$$

$$[f_\Lambda] = s^{-2}\, m = m/s^2$$
$$= N/kg$$

So force per unit mass

Now

$$f = -\frac{G M(r)}{r^2}\hat{r} + \frac{\Lambda}{3}\underset{\sim}{r}$$

$$= \left[-1.3939 \times 10^{-13}\ m/s^2\right]\left[\frac{M(r)}{10^{12} M_\odot}\right]\left[\frac{1\ \mathrm{Mpc}}{r}\right]^2$$

$$+ \left[1.0220 \times 10^{-13}\ m/s^2\right]\left[\frac{r}{1\ \mathrm{Mpc}}\right]$$

fiducial numbers for a large galaxy.

Actually, when a galaxy reaches $M \sim 10^{12} M_\odot$ it is usually quenched (star formation has turned off.)

$$a_{circular} = \frac{v^2}{r}$$

Fiducial acceleration for galaxies at $r = 10$ kpc

$$\cong \frac{(200 \times 10^3\ m/s)^2 \left(\frac{v}{200\ km/s}\right)^2}{3 \times 10^{20}\ m\ (r/10\ \mathrm{kpc})} = \left(\frac{4}{3} \times 10^{-10}\ m/s^2\right)\frac{\left(\frac{v}{200\ km/s}\right)^2}{(r/10\ \mathrm{kpc})}$$

So where is the turnaround radius?

$$f = 0 = C_1 \hat{M} \hat{r}^{-2} + C_2 \hat{r}$$

in fiducial units

$$\hat{r} = \left(-\frac{C_1}{C_2} \hat{M} \right)^{1/3} = \left(-\frac{C_1}{C_2} \frac{M}{10^{12} M_\odot} \right)^{1/3}$$

$$\hat{r} = \left[1.10981\ldots \text{ Mpc} \right] \left(\frac{M}{10^{12} M_\odot} \right)^{1/3}$$

Example Galaxy M87 center of Virgo Cluster

Estimate: At radius 50 kpc encloses $6 \times 10^{12} M_\odot$

— but where does the dark matter halo mass fall to density of the intracluster medium?

Google AI say virial radius $= 1.3$ Mpc $= R_{200}$ where halo density is $200 *$ $\rho_{\text{critical density of universe}}$

... which for some reason is take as good radius for galaxies enclosed mass $1.3 \times 10^{14} M_\odot$

$$\hat{r}(1.3 \times 10^{14} M_\odot) \stackrel{\sim}{=} \left[1.1 \text{ Mpc} \right] (130)^{1/3} \doteq 5.5 \text{ Mpc}$$

So well beyond the virial radius.