Updated 2019jul18 | 3001b

3 Newtonian Gravity & Physics & The Friedmann Equations

Updated 2021sep

Somewhere real hand written 3001

3.1 Historical Intro ⇒ followed by intro-review — curious quirk

Notes drawn from Liddle (2015), Wik, Bondi (1960) for historical tidbits, Carroll (2004), Coles & Lucchin (2002), Ryden (2003), Weinberg (1972), Lunden (2002)

The Friedmann equations for $a(t)$ scale factor (i.e., the ~~Friedmann equation~~ plus the Friedmann acceleration equation) governs the overall (dynamics) kinematics of the observable universe — provided General Relativity (GR) is a correct theory

Eqs of motion of universal expansion 1st order ODE nonlinear (can't add solutions)

well a correct emergent theory since we believe it is ~~the~~ macroscopic limit of quantum gravity — for which there is no established theory.

Though fairly simple

Because nonlinear in general solutions of Friedmann eqn will not add to make a solution

$a_0 = 1$ convention

$v = v_0 a$

$H_0 = \frac{\dot{a}}{a}\Big|_0$ relative rate of expansion of universe, inverse time in km/s/Mpc

In fact, the Friedmann equations can be derived from Newtonian physics with plausible/natural ad hoc assumptions { If only ordinary matter is considered — Radiation & cosmological constant / Dark energy

this is for variations but all are complete, none have any weird useful for helping to develop their power

However, one needs the GR perspective to understand that they include the cosmological constant or other ...

All kinds of mass-energy with enough natural assumptions

All universal level ... Λ and ... constant energy, some ... ...

constant or other same as Λ effectively

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~~300R~~

effects of the curvature of space → and radiation & ~~Albert~~ constant & dark energy

(bit. re derived ... in Russian)

Historically, the Friedmann equations as ~~were~~ first derived by Alexander Friedmann in 1922 in Russian (Wiki; Friedmann eqs) (1888–1925)

Georges Lemaître derived ~~that~~ it independently a bit later in the 1920s (Einstein himself initially ... & Friedmann's work.

The Newtonian derivation came later remarkably by Milne & McCrea in 1934. (Bondi-75)

It's Remarkable that a lot of progress in cosmology could have been made before GR (1915), but that didn't happen.

~~Two~~ Three main hold-ups

1) People seemed to belive the universe was static on average
— Newton thought this
— odd since the universe is not in thermodynamical equlibrium
as manifested by Olbers' Paradox
— a dark night sky is inconsistent with an eternal unchanging static universe — homogeneous & isotropic too
if true in every direction you should see a star (Homewk problem ___)

many people it's hard to know, no opinion polls from the past ~1750

2) People didn't yet know there were other galaxies until 19~~24~~ when Hubble established ~~that~~ ~~the~~ Andromeda nebula was the Andromeda galaxy (Wiki Hubble;
— so then all spiral nebulae were spiral galaxies and all elliptical nebulae too since they came clustered with spirals

people had suspected this for a long time — since the 18th century (Wiki galaxy)

Even the 17th Chr. Wren in ~1660 suggested before eminent astronomers for "rubbish"!

In any case, it was hard to ~~get~~ get an expanding universe idea without knowing galaxies and observing their redshifts which came along with Slipher ~~after~~ starting in 1912 (but very slowly)

3) Need GR idea of free-fall frames being the true elementary inertial frames

Of course a ~~rigorous~~ rigorous derivation of the Friedmann equations must be from GR — but we leave that to another course — one on GR

A rigorous demonstration as to why the GR Friedmann eqn ~~GR~~ and Newtonian analogs ~~are the same~~ should be the same is given by Wells (2014)

To become an expert in cosmology you should study GR
(not me, just a specialist in teaching intro to students!)

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~~3.06~~ 3.1

But before we jump to the derivation, we do a review, here of a lot useful results — I felt I had to do these to know what I meant.

Newtonian Gravity, Gauss' Law, ~~& The Shell Theorem~~

Newton's law of universal gravitation

1 $\longrightarrow$ 2

$\underset{\sim}{r}_{12}$

under squiggle means vector

$$\hat{r}_{12} = \frac{\underset{\sim}{r}_{12}}{r_{12}}$$

$$\underset{\sim}{F}_{12} = -\frac{G m_1 m_2}{r_{12}} \hat{r}_{12}$$

an ideal limit law

— formally it's for point masses

Newton never wrote it down like this. He used an obscure formalism in the Principia (1687) and a few year later Pierre Varignon translated Newton's results into Leibniz calculus formalism, but vector notation didn't come until the later 19th century (Wiki: Euclidean Vector)

Note recognizable to us

We can derive Gauss' law but now — nonrigorously

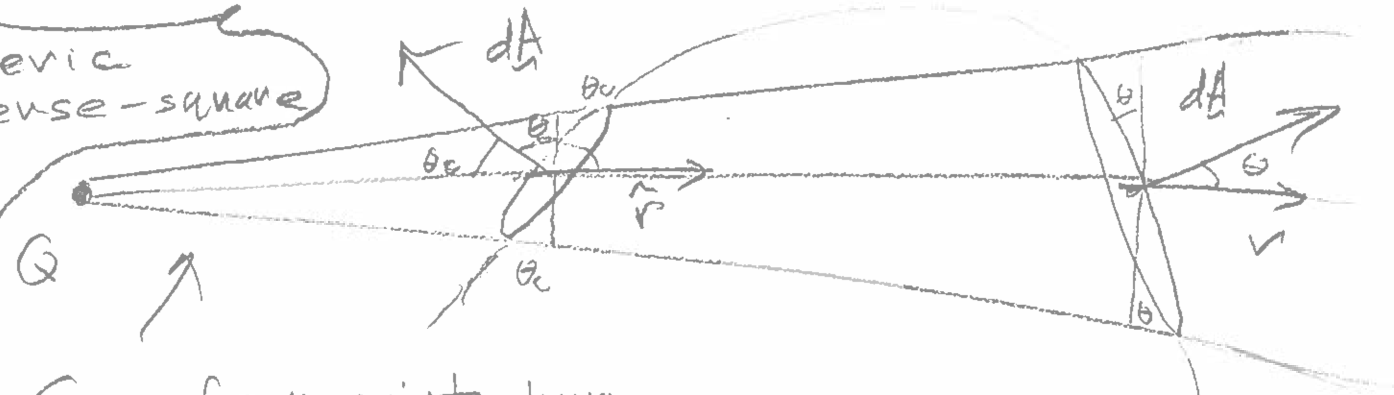
I don't think Newton ever explicitly thought of fields, but he must have thought of force per unit mass sometimes

Gauss' Law in Integral Form

For a force field (force per unit generic charge)

$$\underset{\sim}{f} = \frac{Q}{r^2}\hat{r} \quad \text{for point charge } Q$$

Generic inverse-square law force



Cone from point charge

Differential surface area vector $d\underset{\sim}{A}$ always points outward.

Close surface for which there is a definite inside and outside

Euclidean geometry tells us

$$d\Omega = \frac{dA_\perp}{r^2} \quad \text{for any radius } r$$

along the cone from the point charge

where $dA_\perp$ is the perpendicular area.

$\hat{r}$ points inward if $\theta > 90°$, $\left\{ \begin{array}{l} dA_\perp = dA\cos\theta_c \\ = dA\cos(\pi-\theta) \\ = -dA\cos\theta > 0 \end{array} \right.$

$\hat{r}$ points tangentially if $\theta = 90°$, $dA_\perp = 0$

$\hat{r}$ points outward if $\theta < 90°$, $dA_\perp = dA\cos\theta$

$$\therefore\ d\Omega = \frac{dA_\perp}{r^2} = \pm\frac{dA\cos\theta}{r^2} = \pm\frac{d\underset{\sim}{A}\cdot\hat{r}}{r^2} \quad \left\{ \begin{array}{l} + \text{ for } \hat{r} \text{ pointing outward} \\ - \text{ for } \hat{r} \text{ pointing inward} \end{array} \right.$$

$$\therefore\ (d\underset{\sim}{A}\cdot\underset{\sim}{r}/r^2) = \pm\, d\Omega$$

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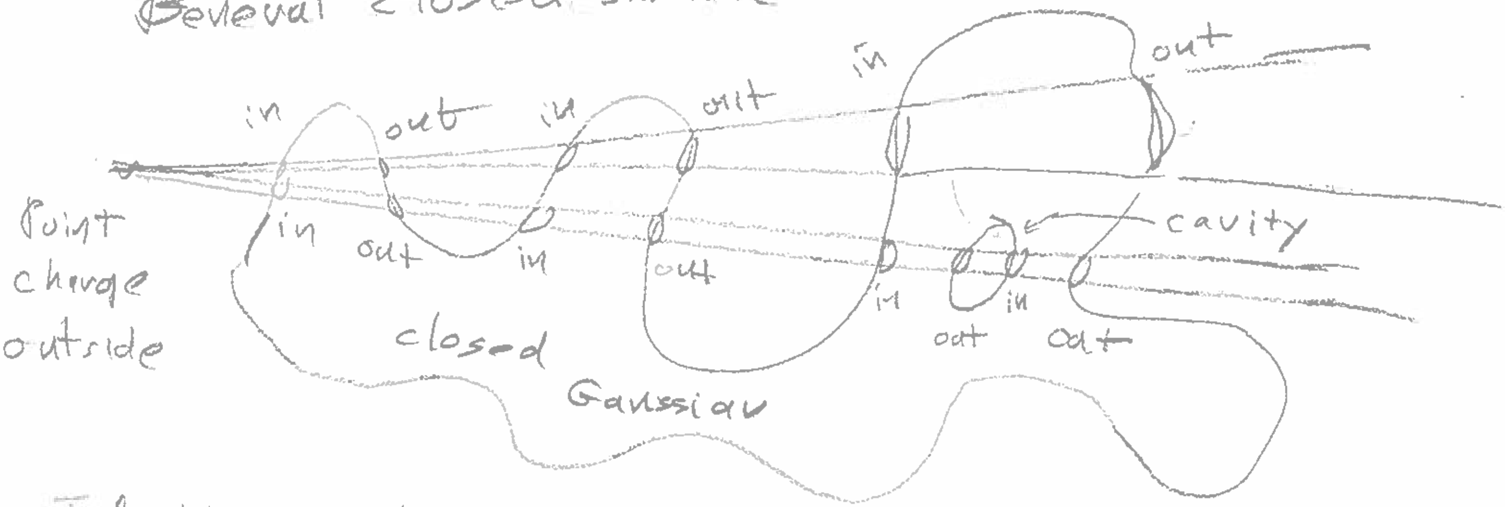
Now $\underset{\sim}{f} \cdot d\underset{\sim}{A} = \frac{Q}{r^2}\hat{r} \cdot dA = Q(\pm d\Omega)$

General closed surface



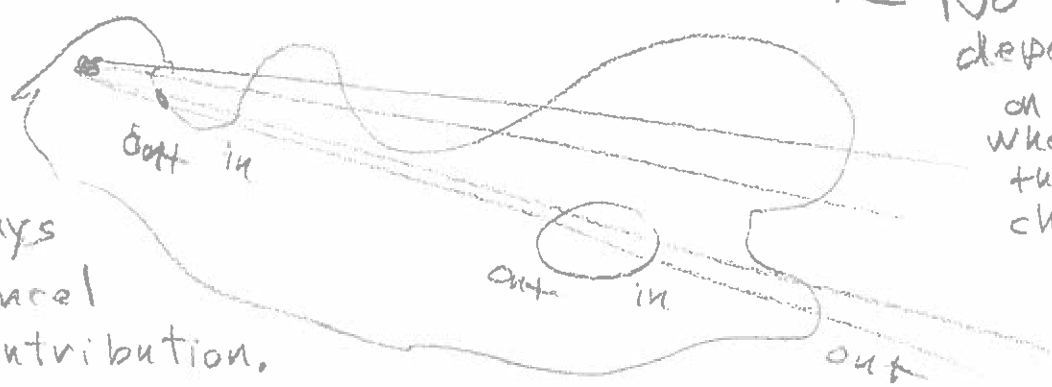
If the point charge is outside, all contributions to adding differential $\underset{\sim}{f} \cdot d\underset{\sim}{A}$ bits along any cone cancel. $f \cdot d\underset{\sim}{A} = 0$ along any cone

$\therefore \oint \underset{\sim}{f} \cdot d\underset{\sim}{A} = 0$ for charge outside

No dependence on where the charge is

But

point charge outside, there is always one non-cancel outward contribution.



$\therefore$ Along any cone $f \cdot d\underset{\sim}{A} = Q\, d\Omega$

$\therefore$ Integrating over all solid angle $\oint \underset{\sim}{f} \cdot d\underset{\sim}{A} = 4\pi Q$

Immediately Generalizing

$\oint \underset{\sim}{f} \cdot d\underset{\sim}{A} = 4\pi Q_{enclosed}$ Gauss' Law integral form

whether ever any amount of charge may be

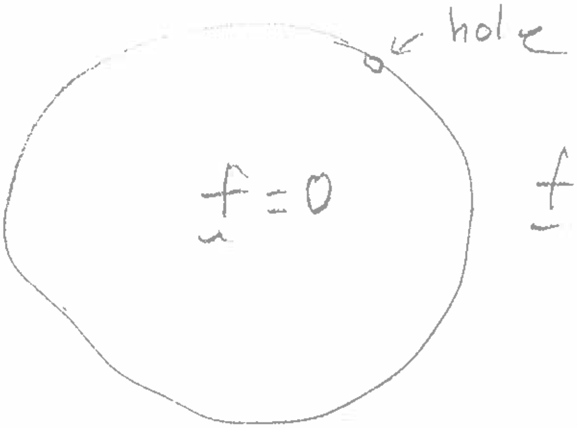
Gauss' law depends totally on the inverse-square law nature of the force.
Reality would look very different without inverse-square law forces.

The integral form of Gauss' law can be used directly to solve the force field exactly in only 3 cases of high symmetry charge distribution: planar, cylindrical, and spherical.

All good for various capacitors in E&M.

It can also be used for approximate solution to approximate symmetry and there are some special tricky systems that can be solved with it that are good for intro E&M homeworks.

example Thin spherical shell with a differential hole.



$\underline{f} = \frac{Q_{shell}}{r^2}\hat{r}$

What is the field in the hole?
Question for students

(See answer p. 3018b)

3008 | In Parallel special cases

3007

Gauss' law (Wik) integral form

Gauss' generic form (law): $\oint \vec{f}\cdot d\vec{A} = 4\pi Q_{\text{encl}}$ — only field of [illegible]

Gravity

$$\vec{g}_i = -\frac{GM_i}{r_i^2}\hat{r}_i$$

gravitational field

$Q = -GM_{in}$ — gravity always a -ve charge, so make minus explicit

M_i point source i

$\oint \vec{g}\cdot d\vec{A}$ — No Einstein summation

$$= \sum_i \oint \vec{g}_i \cdot d\vec{A} = -G\sum_i M_i\, 4\pi$$

inside

$$= -4\pi G M_{\text{inside}}$$

only -ve

minus because gravity always attractive

And mass is always +ve unlike electric charge — in total can have +ve contributions

Coulomb force

k = coulomb constant $\simeq 8.99\times10^9 \simeq 10^{10}$ MKS (Wik)

$$\vec{E}_i = \frac{kq_i}{r_i^2}\hat{r}_i$$

electric field

q_i

$\oint \vec{E}\cdot d\vec{A}$ — No Einstein summation

$$= \sum_i \oint \vec{E}_i\cdot d\vec{A} = k\sum_i q_i\, 4\pi$$

$Q = kq$

$$= 4\pi k q_{\text{inside}} = \frac{q_{\text{inside}}}{\varepsilon_0}$$

Vacuum permitivity

$\frac{1}{r^2} r^2 d\Omega$

Very special results due to special nature of inverse-square law forces

For high symmetry cases, one can use the integral Gauss' law to obtain solutions

1) spherical sym,
2) cylindrical sym
3) planar sym

and that's all I think

We'll look at the cases in a moment

Convergent evolution of physics

That both grav & EM have inverse-square forces main reason Einstein saw this unified field theory approach but was true. But no. One needs the quantum mechanics + ~~field theory~~ QFT idea to get Theory of Everything (TOE), whatever it is.

3.3 Gauss' Theorem (AKA Divergence theorem)

$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{A} = \int \nabla \cdot \underset{\sim}{F}\, dV$$

$\underset{\sim}{F}$ is a vector field. $\underbrace{\nabla \cdot \underset{\sim}{F}}_{\text{divergence}} = \frac{\partial F_i}{\partial x_i}$ } Einstein summation notation.

Sum on repeated indices. Great simplification in notation, but sometimes you have to turn it off.

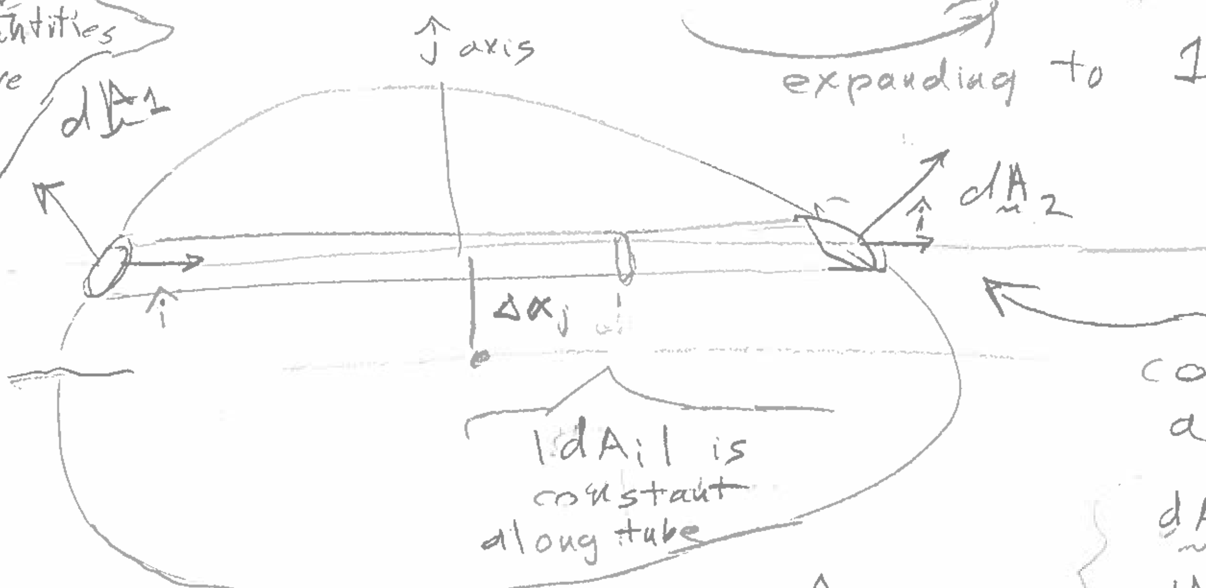
"Assume flows" in 3 orthogonal directions are independent, what vectors quantities require to be.

Consider differentially small volume V

These both have signs.

$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{A} = \int F_i\, dA_i = \int \left(F_{i0} + \frac{\partial F_i}{\partial x_j} \Delta x_j\right) dA_i$$

expanding to 1st order



$d\underset{\sim}{A}_2 \cdot \hat{j}$
$d\underset{\sim}{A}_2 \cdot \hat{k}$
} are areas in orthogonal directions

$d\underset{\sim}{A}_2 \cdot \hat{i} = dA_{2i}$

$d\underset{\sim}{A}_1 \cdot \hat{i} = dA_{1i}$

$= -dA_{2i}$

3010

For the tube with Einstein summation on i turned off

$$\int_{\text{tube}} \underset{\sim}{F}\cdot d\underset{\sim}{A} = F_{i0}\underbrace{(dA_{i1}+dA_{2i})}_{=\emptyset} + \frac{\partial F_i}{\partial x_j}\left(\Delta x_{j1}\,dA_{i1} + \Delta x_{j2}\,dA_{i2}\right)$$

No Einstein summation

$= F_{i2}\,dA_{i2} + F_{i1}\,dA_{i1}$

$= (F_{i2} - F_{i1})\,dA_{i2}$

outflow − inflow

$=$ $\frac{\partial F_i}{\partial x_j}\cdot \emptyset$ if $i \neq j$ in which case $\Delta x_{j1} = \Delta x_{j2}$

$\frac{\partial F_i}{\partial x_i}\,dV_{tube}$ if $i = j$

No Einstein summation

$(\Delta x_{i1}\,dA_{i1} + \Delta x_{i2}\,dA_{i2})$

No Einstein summation

$\underbrace{dA_{i2}(\Delta x_{i2} - \Delta x_{i1})}_{dV_{tube}}$

Rate of change of F_i along direction $\hat{i}$ times length

$\frac{\partial F_i}{\partial x_i}(\Delta x_{i2} - \Delta x_{i1})$

net change in F_i ("flow") in direction $\hat{i}$ is the difference between "inflow" and "outflow".

$\therefore$ Now integrate over all tubes in the $\hat{i}, \hat{j}, \hat{k}$ directions with total Volume V

Sum of changes of "flow" across Volume V in 3 orthogonal directions

$$\frac{\partial F_i}{\partial x_i}V + \frac{\partial F_j}{\partial x_j}V + \frac{\partial F_k}{\partial x_k}V \quad \text{with No Einstein summation}$$

$$= \frac{\partial F_i}{\partial x_i}V = \nabla\cdot\underset{\sim}{F}\,V \quad \text{with Einstein summation}$$

(sort of obvious after the proof)

Net outflow of volume surface area equal integral over volume of divergence

$$\oint \underset{\sim}{F}\cdot d\underset{\sim}{A} = \int \nabla\cdot\underset{\sim}{F}\,dV \quad \text{for differential small volume } V$$

But if you go to general volume, all internal surface $\underset{\sim}{F}\cdot d\underset{\sim}{A}$ contributions cancel pairwise.

$$\oint \underset{\sim}{F}\cdot d\underset{\sim}{A} = \int \nabla\cdot\underset{\sim}{F}\,dV \quad \text{holds for general volumes}$$

QED

must be independent of our arbitrary orientation of axes. ← is most general for all axis choices

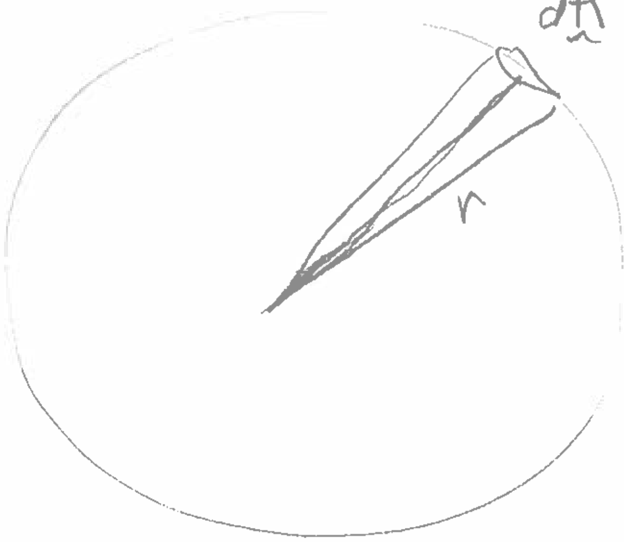
3012 | 3.4

Shell Theorem for Gravity + Birkoff's Theorem

(for electrostatics see an E+M course)

Given a spherically symmetric mass distribution Consider a Gaussian surface that is a sphere (or thin spherical shell) center on center of symmetry

$$d\underset{\sim}{A} = r^2 d\Omega\, \hat{r}$$



Now

$$\oint \underset{\sim}{g} \cdot d\underset{\sim}{A} = -4\pi G M_{encl}$$

By symmetry $\underset{\sim}{g}$ must be radial and depend only on r for magnitude and it must point inward to accommodate that negative sign.

$$\therefore \quad \underset{\sim}{g} = g(r)(-\hat{r})$$

$$\oint \underset{\sim}{g} \cdot d\underset{\sim}{A} = g(r)(-\hat{r})\, r^2 4\pi\, \hat{r} = -g\, 4\pi r^2$$

$$\therefore \quad g(r) = -\frac{4\pi G M_{encl}}{-4\pi r^2} = \frac{4\pi G M_{encl}}{r^2}$$

$$\underset{\sim}{g} = -\frac{G M_{encl}}{r^2}\hat{r}$$

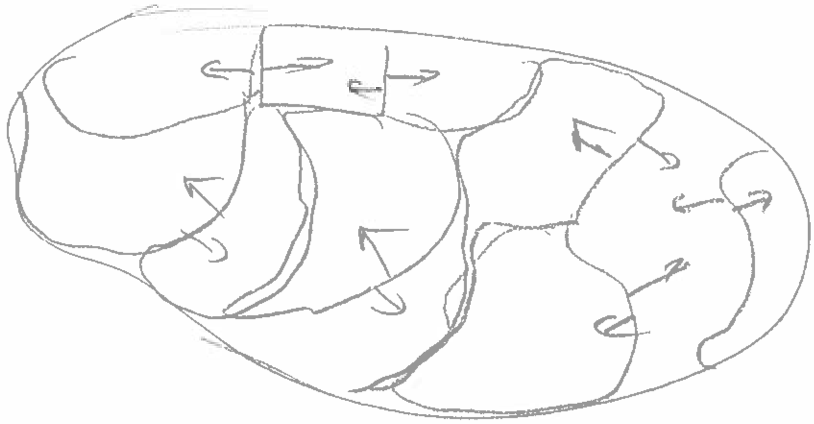
Shell Thm QED.

Corollaries a) M_{encl} is a point mass M

But Newton never wrote it like No vector notation and I think he never wrote formulas like a modern

$$\underset{\sim}{g} = -\frac{GM}{r^2}\hat{r}$$

Newton's law of Universal gravitation for a point mass. for grav. field.



Pairwise cancelation of internal $\underset{\sim}{F} \cdot d\underset{\sim}{A}$ contributions illustrated

Gauss' Law in Differential Form

$$\oint \underset{\sim}{f} \cdot d\underset{\sim}{A} = 4\pi Q_{encl}$$

Gauss' law integral form (see p. 3005) for $\underset{\sim}{f}$ an inverse-law force field

$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{A} = \int \nabla \cdot \underset{\sim}{F} \, dV$$

Gauss' theorem for a general vector field which includes force field $\underset{\sim}{f}$

Proof

$$\int \nabla \cdot \underset{\sim}{f} \, dV = \int \underset{\sim}{f} \cdot d\underset{\sim}{A} = 4\pi \, Q_{encl}$$

$$= 4\pi \int \rho \, dV$$

general volume

$\therefore$ QED $\nabla \cdot \underset{\sim}{f} = 4\pi\rho$ the differential form of Gauss' law

Gravity

$\nabla \cdot \underset{\sim}{g} = -4\pi G\rho$ (see p 3008)

Coulomb force

$\nabla \cdot \underset{\sim}{E} = 4\pi k\rho = \dfrac{\rho}{\varepsilon_0}$ < charge density (see p. 3008)

b) Finite sphere of Mass M of radius R

with $r \geq R$

$$\underset{\sim}{g} = -\frac{GM}{r^2}\hat{r} \quad \text{with } r \geq R$$

A remarkable result rooted in the inverse square law

This allows you to apply Newton's gravity law to planets as if they were point masses as long as they don't touch each other.

Newton with his klutzy math techniques had to work really hard to find this, but we find it easily from the shell theorem.

c) Inside a finite shell

Finite Shell

$$\underset{\sim}{g} = -\frac{GM_{encl}}{r^2}\hat{r}$$

but $M_{encl} = 0$

$$\underset{\sim}{g} = 0$$

What if shell extents to infinity? Pure Newtonian physics does NOT give a definite Answer. See p.3017

A remarkable result rooted in the inverse-square law. It gets a lot of use in electrostatics where spheres with zero net charge inside are easy to arrange

3014

2026 jan 24

Force of gravity which we neglected so far

$$\underset{\sim}{F}_i = m_i \underset{\sim}{g}(\underset{\sim}{r}) \quad \text{force on point mass } i$$

An axiom of the theory of Newtonian gravity

system of point masses

$$\underset{\sim}{F} = \sum_i m_i \underset{\sim}{g}(\underset{\sim}{r}_i) = \int \rho(\underset{\sim}{r})\, \underset{\sim}{g}(\underset{\sim}{r})\, dV$$

in continuum limit.

1 2

Force of system 1 on system 2

$$\underset{\sim}{F}_{12} = \sum_j m_j \underset{\sim}{g}_1(\underset{\sim}{r}_{ij}) = \sum_j m_j \sum_i \frac{G m_i}{r_{ij}^2}(-\hat{r}_{ij})$$

$$= -\sum_{ij} \frac{G m_i m_j}{r_{ij}^2} \hat{r}_{ij} = \sum_{ij} \frac{G m_i m_j}{r_{ij}^2} \hat{r}_{ji}$$

$$= -\underset{\sim}{F}_{21}$$

Two systems so No $\frac{1}{2}$ for double counting

and so Newton's 3rd law holds explicitly for gravity.

Force of Gravity Between Two spherically symmetric masses when non-overlapping

(which is really what is needed for celestial mechanics)

3rd law

$$\underset{\sim}{F}_{12} = \underset{\sim}{F}_{1\text{point},\,2} = -\underset{\sim}{F}_{2,\,1\text{point}}$$

shell thm

shell thm

$$= -\underset{\sim}{F}_{2\text{point},\,1\text{point}}$$

$$= \underset{\sim}{F}_{1\text{point},\,2\text{point}}$$

$$\therefore \quad \underset{\sim}{F}_{12} = \underset{\sim}{F}_{1\text{point},\,2\text{point}} \quad \text{QED}$$

The force of gravity between two non-overlapping sphically symmetric masses is just like between two point masses.

3016 | 2026 jun 24 | Pressure supported astro-bodies

Of course real planets, stars, etc. are only approximately spherically symmetric and at some level you need to account for that (e.g., tidal bulges, equatorial bulges)

due to centrifugal force

But what is the response to forces

$\underset{\sim}{F}_{net} = m\underset{\sim}{a}$ for a point mass

 < system of particles

$$\underset{\sim}{F}_{net} = \sum_i F_{i,ext} + \sum_{j\neq i}\sum_i \underset{\sim}{F}_{ji} = \sum m_i a_i = m\frac{\sum m_i a_i}{m}$$

$$= \sum_{\substack{ij \\ i\neq j}} \underset{\sim}{F}_{ji}$$

$$= -\sum_{\substack{ij \\ i\neq j}} \underset{\sim}{F}_{ij} \quad \text{by 3rd law}$$

$$= -\sum_{ij} \underset{\sim}{F}_{ji} \quad \text{by relabeling indexes.}$$

$A = -A$ implies $A = 0$

∴ $\underset{\sim}{F}_{net\,ext} = m\underset{\sim}{a}_{cm}$

∴ Planets, stars, etc gravitate and interact like point masses as long as they don't overlap.

2026 jan 24

3 D17

Birkoff theorem is ~~the~~ general relativity analogue to the Newtonian shell theorem

Weinberg 1972 p 337–338 & p.474

outside a spherically symmetric mass distribution, it can be treated the same for any mass concentration. Including like a black hole. But if $r >> R_{sch}$, then gravity goes to the Newtonian limit.

The mass distribution does NOT need to be static. Spherical expansion/contraction does NOT radiate gravitational waves

surroundings spherically symmetric even to infinity (which has no clear Newtonian answer in the shell theorem)

The cavity can be treated as flat if empty and it's own world of interacting masses if NOT

This is the general relativity justification for the Newtonian derivation of the Friedmann equation which we'll get too sometime soon.

3018

Why is the Newtonian shell theorem indefinite

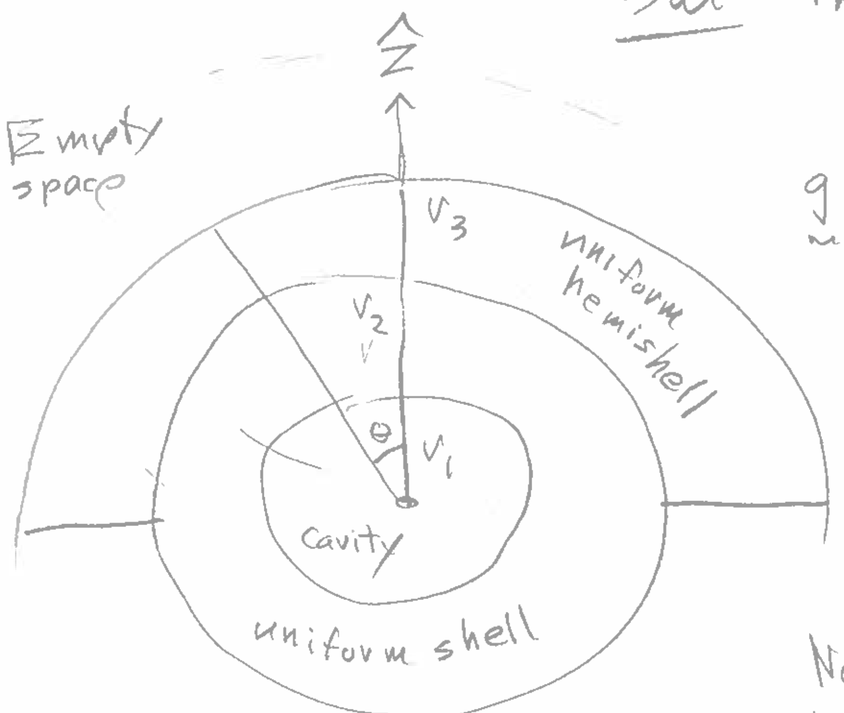
finite shell

$\underset{\sim}{g}=0$ in cavity

Most natural that

$$\lim_{r_2\to\infty} \underset{\sim}{g} = 0 .$$

But there are other limits



$$\underset{\sim}{g} = -G\int \frac{\rho}{r^2}\,d\Omega\,dr$$

$$g_z(r=0) = -G\rho\int_{r_1}^{r_2} 2\pi\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta\,dr$$

$$= -2\pi G\rho \frac{\sin^2\theta}{2}\Big|_0^{\pi/2}(r_3-r_2)$$

$$= -\pi G\rho(r_3-r_2)$$

$$[\rho(r_3-r_2)] = \frac{M}{L^2}$$

dimensionally correct

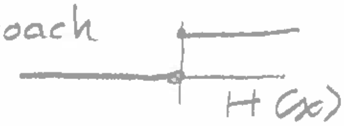
Note hemishell case Never approximates the shell case for any finite A.

Now say $r_3 = A r_{30}$
$r_2 = A r_{20}$

$$g_z(r=0) = -\pi G\rho(r_{30}-r_{20})A$$

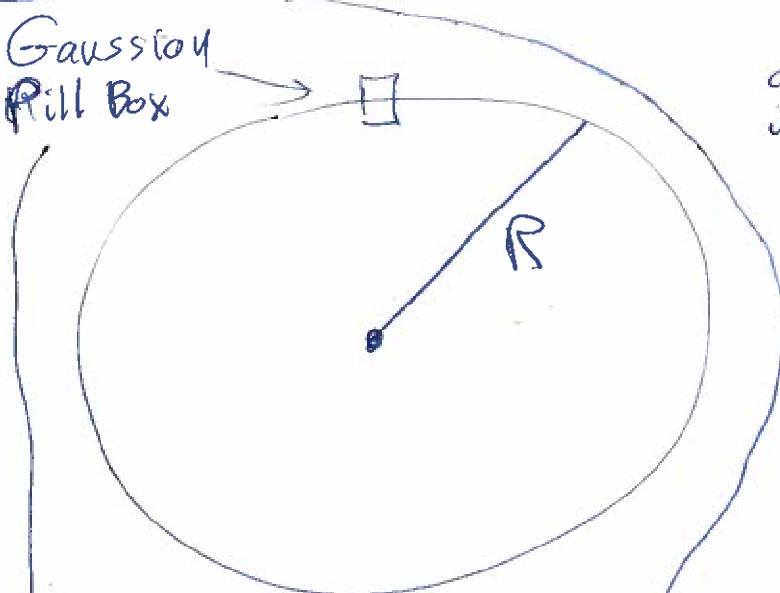
$$\lim_{A\to\infty} g_z(r=0) = \infty$$ even all space is full of uniform mass except cavity

Limit often depend on the direction of approach e.g. Heaviside step function



3.5 What is $\underset{\sim}{g}$ "inside" an infinitely thin shell? (an exercise for students) (answer to a 3007 question)

One Approach



$$\underset{\sim}{g} = \begin{cases} -\frac{GM}{r^2}\hat{r} & r > R \\ 0 & r < R \\ ? & r = R \end{cases}$$

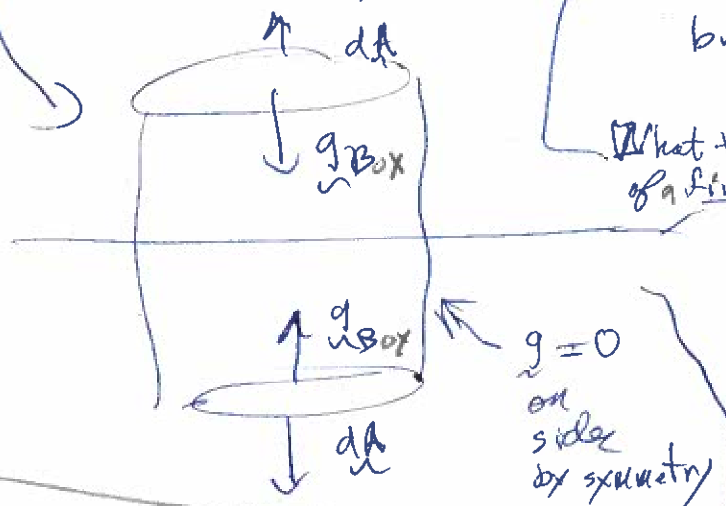
Remarkable Fact. Infinitesimal bit of shell just outside shell contributes $\frac{1}{2}$ the field there. All the rest of the shell the other half.

Is it indeterminate in all senses?

Plausibly $-\frac{1}{2}\frac{GM}{r^2}\hat{r}$, but is there a better justification?

What the physically reasonable limit of a finite shell?

It's minute on an infinite plane



Apply Gauss' law to the Pill box

$$-g_{Box}(2A) = -4\pi G M_{encl}$$

$$= -4\pi G \frac{M}{4\pi R^2} A$$

$$g_{Box} = \frac{1}{2}\frac{GM}{R^2}$$

$$\underset{\sim}{g}_{Box} = -\frac{1}{2}\frac{GM}{R^2}\begin{cases} \hat{r} & r > R \\ -\hat{r} & r < R \\ ? & r = R \end{cases}$$

Again $r = R$ case is rather indeterminate but $\underset{\sim}{g}_{Box}(r=R) = 0$ seems reasonable.

But

$$\underset{\sim}{g}_{\text{rest of shell}} = \underset{\sim}{g} - \underset{\sim}{g}_{Box}$$

$$= \begin{cases} -\frac{GM}{R^2}\left(1 - 2\frac{\delta r}{R} - \frac{1}{2}\right)\hat{r} & r > R \text{ with 1st order expansion} \\ -\frac{GM}{R^2}\frac{1}{2}\hat{r} & r < R \text{ using the shell theorem so } \underset{\sim}{g} = 0 \text{ inside shell} \\ -\frac{GM}{R^2}\frac{1}{2} & r = R \text{ by continuity.} \end{cases}$$

$\frac{1}{2} - 2\frac{\delta r}{R}$ ~~$\frac{GM}{R^2}$ for box~~

~~3014e~~
3018c

$\vec{g}_{rest}$ is all the field not due to the pill box, but due to the rest of the shell.

Although $\vec{g}$ and $\vec{g}_{PBox}$ are discontinuous at $r = R$, $\vec{g}_{rest\ of\ shell}$ just has a kink and so ~~plausibly~~ sensible to just say/define

$$\vec{g}_{rest}(r=R) = -\frac{1}{2}\frac{GM}{R^2}\hat{r}$$

so in a ~~tiny~~ tiny hole in large shell

$$\vec{g}_{hole} = \vec{g}_{rest\ of\ shell}(r=R) = -\frac{1}{2}\frac{GM}{R^2}\hat{r}$$

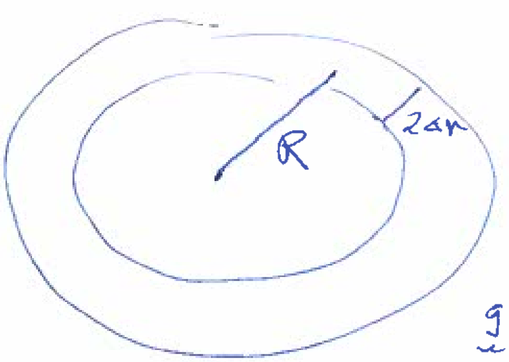
So, in fact, the tiny pill box and and the rest of the shell contribute equally just at the shell.

This procedure is ~~most~~ most convincing as the physical limit but the PillBox approach gives us confidence from another direction and gives contribution of the pill box and the rest of the shell.

Making hole ~~twice~~ smaller suggest ~~that~~ best limit for field in shell is $-\frac{1}{2}\frac{GM}{R^2}\hat{r}$

But still uneasy? What is best limit value for $\vec{g}$ in ~~shell~~ thin shell.

$$\vec{g}(r) = -\frac{G}{r^2}\int_{R-\Delta r}^{R+\Delta r}\rho(4\pi r^2)\,dr\,\hat{r}$$



$$\rho \Rightarrow \rho_{ave} = \frac{M}{\left(\frac{4\pi}{3}\right)\left[(R+\Delta r)^3-(R-\Delta r)^3\right]}$$

$$\doteq \frac{M}{\frac{4\pi}{3}R^3(6\Delta r/R)}$$

$$\vec{g}(r=R) \doteq -\frac{G}{R^2}\frac{M}{\frac{4\pi}{3}R^3(6\Delta r/R)}4\pi R^2\Delta r\,\hat{r}$$

$$= -\frac{1}{2}\frac{GM}{R^2}\hat{r} \quad \text{independent of } \Delta r$$

And so as $\Delta r \to 0$ $\vec{g}(r=R) = -\frac{1}{2}\frac{GM}{R^2}\hat{r}$ is the best limit in any practical sense.

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3.6 Potential Energy Theory

Review and a build up to understanding Newtonian Cosmology — the derivation of the Friedmann equation from Newtonian physics plus extra reasonable hypotheses.

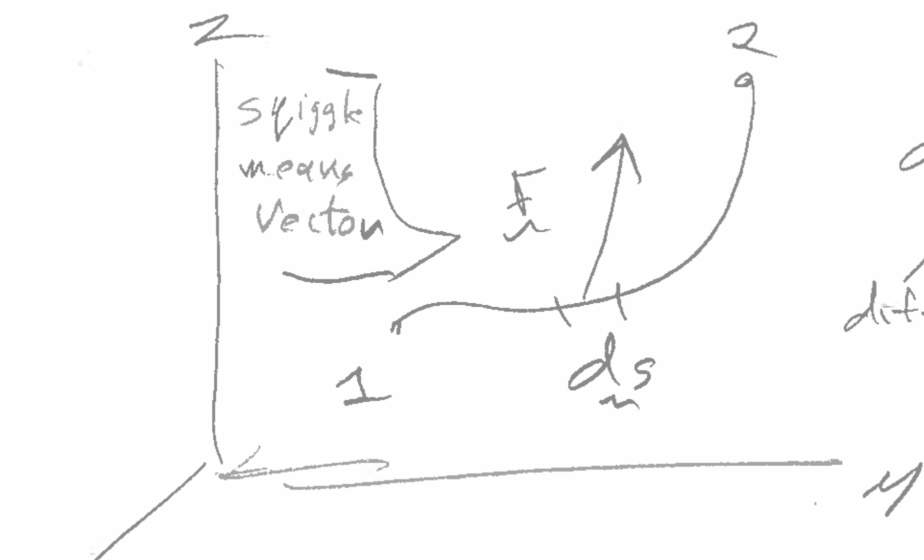
Posit a field of force $\underset{\sim}{F}$

$$\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{F}(x_i)$$

a general relativity notation where an element of a vector represents the vector

We idealize the field as time-independent

or alternatively say we can calculate the work it does in an instant of time



x

Newtonian Work definition

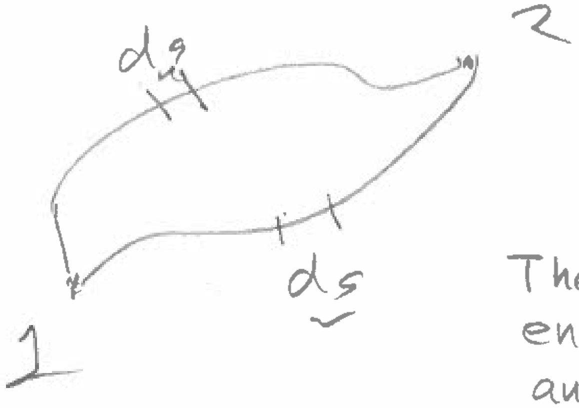
$$dW = \underset{\sim}{F} \cdot d\underset{\sim}{s}$$

differential work done by $\underset{\sim}{F}$ through differential displacement vector $d\underset{\sim}{s}$

Work is an energy transfer process.

The work transfer is by the force to somewhere else: from somewhere e.g., to kinetic energy of a body

Say the field is such that



$$W_{12} = \oint \underset{\sim}{F} \cdot d\underset{\sim}{s}$$

is path independent.

Then we can say there is an energy stored in the field and the amount stored just depends on position.
<u>But</u> we do NOT say where the energy is stored nor in what form nor is a zero point defined in pure general potential energy theory, left unspecified

and call this energy potential energy and work done by the field changes it,

Note $\underset{\sim}{F}$ is just the force of the field NOT all forces,

$$dU \equiv -dW = -\underset{\sim}{F} \cdot d\underset{\sim}{s}$$

negative sign because when ~~the~~ field of force does work, energy comes of field's store and goes somewhere else (e.g., kinetic energy) of a body

If $dW<0$, then energy goes into the field

Then $U_{12} = -\int_1^2 \underset{\sim}{F} \cdot d\underset{\sim}{s}$ or $\Delta U = -W$

and

$$dU = -\underset{\sim}{F} \cdot d\underset{\sim}{s} = -F_i \, dx_i$$

Using Einstein summation

$dU = \frac{\partial U}{\partial x_i} dx_i$, and so $\frac{\partial U}{\partial x_i} = -F_i$

$$F_i = -\frac{\partial U}{\partial x_i}$$

$$\underset{\sim}{F} = -\nabla U$$

A conservative force can be derived from a Potential energy

We call a force with a potential energy a conservative force

$\Delta U = -W$ in simplified notation

Recall your Newton Physics

$$W_{12} = \int_1^2 \underbrace{\underset{\sim}{F}_{net}}_{\text{The net external force on a body}} \cdot d\underset{\sim}{s} = \int_1^2 m\underset{\sim}{a}_{cm} \cdot ds$$

Total Work done on body

We use $F_{net} = m a_{cm}$ Which holds relative to an inertial frame, GR tells us elementary inertial frames are free-fall frames in a uniform gravitational field,

$$= \int_1^2 m \frac{d\underset{\sim}{v}}{dt} \cdot \underset{\sim}{v}\, dt = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$KE = \frac{1}{2} m v^2$$

Work-kinetic-Energy theorem

$$\Delta KE = \underbrace{W}_{\text{total Work done on body}}$$ (simplified notation)

$$W = \underbrace{W_{non}}_{\text{non-conservative work done}} + \underbrace{W_{con}}_{\text{Work done by conservative force}}$$

$$\Delta KE = W_{non} - \Delta U$$

$$W_{non} = \Delta KE + \Delta U = \underbrace{\Delta E}_{\text{change in mechanical Energy}}$$

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We define $E_{mechanical} = KE + U$

and when $W_{non} = 0$

$$\Delta E_{mech} = \Delta KE + \Delta U = 0$$

$\boxed{\text{conservation of mechanical energy holds}}$

Historical tidbit

Aristotle invented the term energy with a vague meaning. | It would have a big future

Chr. Huyghens (1629-1695) noted the quantity mv^2 was conserved in some contexts in 1652

First quantified energy in history.

G. Leibniz picked up and called vis viva a bit later.

(F. Cohen 523, 526)

The $\frac{1}{2}$ factor was discovered in the 19th century,

Galileo had some notion of work.

Newton himself never used energy, work or kinetic energy

These are implicit in Newtonian physics, but Newton did NOT notice.

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Returning to forces with potential energy (conservative forces)

Say a body subject to a conservative force has a generic charge Q for the force.

Then $\underset{\sim}{F} = Q\underset{\sim}{f}$

$\underset{\sim}{f}$ is the field itself, It's structure,

$$\underset{\sim}{f} = \frac{\underset{\sim}{F}}{Q} = -\frac{\nabla U}{Q} = -\nabla\phi$$

$$f_i = -\frac{\partial\phi}{\partial x_i}$$

I sometimes use V for potential

ϕ is called the potential

But in Quantum mechanics potential energy is called the Potential

In fact, I think people often use potential energy and potential as synonyms letting context decide the meaning

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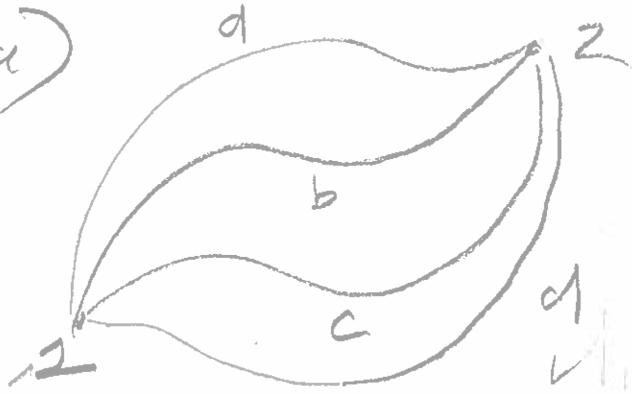
Corollaries to Path-independence for Potential energy

a)



U_{12} same for all paths a, b, c, ...

Then $U_{11} = 0$

$$U_{11} = U_{12a} + U_{21b}$$

$$= \int_{1a}^{2} \underset{\sim}{F} \cdot d\underset{\sim}{s} + \int_{2b}^{1} \underset{\sim}{F} \cdot d\underset{\sim}{s}$$

Let $ds_{new} = -d\underset{\sim}{s}$

$d\underset{\sim}{s} = -d\underset{\sim}{s}_{new}$

$$= \int_{1a}^{2} \underset{\sim}{F} \cdot d\underset{\sim}{s} + \int_{2b}^{1} \underset{\sim}{F} \cdot (-d\underset{\sim}{s}_{new})$$

$$\int_{1b}^{2} \underset{\sim}{F} \cdot (-d\underset{\sim}{s}_{new})$$

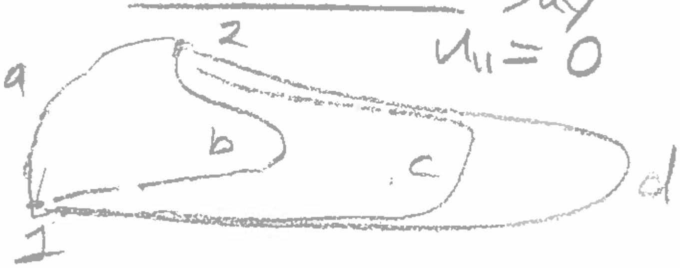
Just reversing order of adding the differential bits

$$-\int_{1b}^{2} \underset{\sim}{F} \cdot d\underset{\sim}{s}$$ dropping the subscript

$$= \int_{1a}^{2} \underset{\sim}{F} \cdot d\underset{\sim}{s} - \int_{1b}^{2} \underset{\sim}{F} \cdot d\underset{\sim}{s}$$

$= 0$ by rule U_{12} same for all paths

b) Converse Say $U_{11} = 0$



$U_{11} = 0$

$\therefore U_{12a} = -U_{21b} = -U_{21c} = \ldots$

And so U_{12} is path independent $\Rightarrow U_{12a} = U_{12b} = U_{12c} = \ldots$

b) Corollaries

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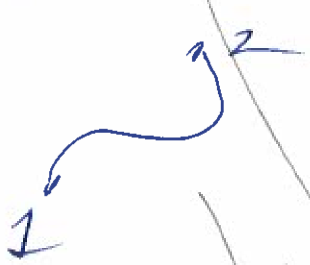
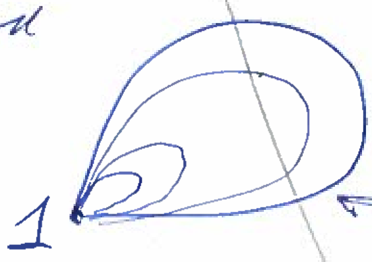


If W_{12} is path independent, then

W_{11} = Constant

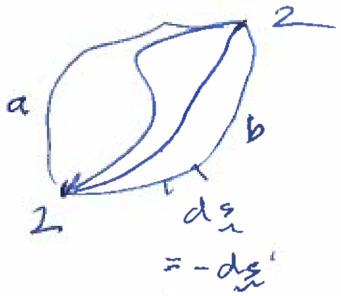
or U_{11} = Constant



Must have Constant = 0

sort of visually obvious by taking the limit as path goes to zero where the object doesn't move.

But more formally



$U_{12a} = U_{12b}$ by path independence

$$= \left[-\int_1^2 F \cdot ds\right]_b = \left[\int_1^2 F \cdot ds'\right] = \left[\int_2^1 F \cdot ds'\right]$$

$$= -\left[-\int_2^1 F \cdot ds\right]_b = -U_{21b}$$

adding bit in reverse order

$= \cancel{[-U_{21}]_b}$

∴ $U_{12a} + U_{21b} = 0$

∴ $U_{11} = 0$ and $W_{11} = 0$

Converse

If $U_{11} = 0$, then $U_{12a} = -U_{21b} = -U_{21c} = \dots$

∴ U_{12} is path independent.

3020 ③ 3.7 Fields of Force with PE?

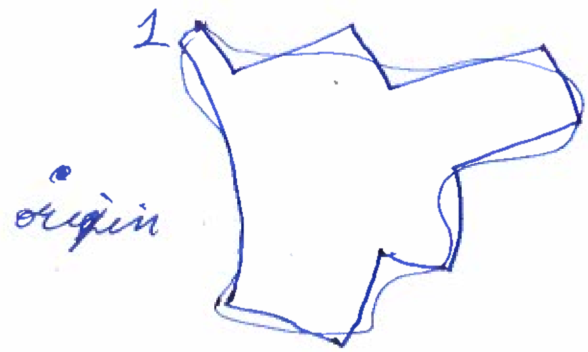
Example a central force

$$\underset{\sim}{f} = f(r)\,\hat{r}$$

Sum of fields of central force fields also conservative.

Just add up the vectors

Visually



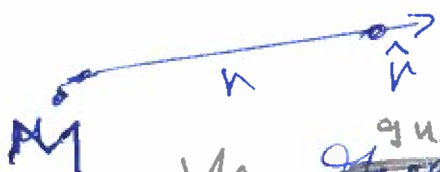
The tangential paths make zero contribution and all radial paths sum to zero on a point 1 to 1 path.

$\therefore$ All central forces are conservative.

Key example gravity

$$\underset{\sim}{g} = -\frac{GM}{r^2}\,\hat{r}$$

gravitational field of a point source



We ~~guess~~ guess $V = -\frac{GM}{r}$ is the potential

Gradient in spherical coordinates (Arf-104)

$$\nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\underbrace{\hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}}_{0}$$

$$\underset{\sim}{g} = -\nabla V = -\frac{GM}{r^2}\,\hat{r}$$

and so we guessed right the right potential.

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d) Zero-Point of PE?

Pure potential ~~theory~~ doesn't define an absolute zeropoint.

Remember the field is general, ~~[illegible]~~ V where the energy ~~is~~ unspecified

Only changes in PE occur.

There is no general reason to define ~~one~~, a zero point.

E.g. for gravity

$$V_2 = V_1 + V_0$$

a constant

$$= -\frac{GM}{r} + V_0$$

No change to predicted motions if you add a constant

However for localized sources of force

Note We assert that there is a unique total PE no matter how an system of charges are assembled. We ~~often~~ calculate ~~this~~ PE by bringing up one charge (or differential charge) at a time. This may be implicit:

$$u = \frac{1}{2}\sum_{i \neq j} u_{ij}$$

Why is PE unique, The field structure in which the energy is stored is the same no matter assembled so the PE is the same. Is there a better argument?

Example gravity of sources in any closed surface

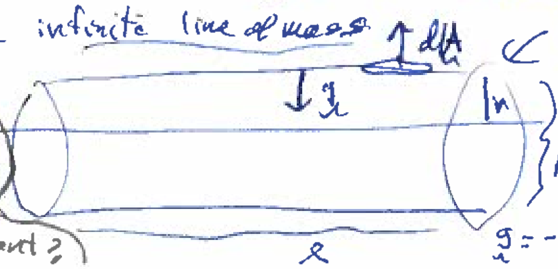
$$V(\infty) = 0$$

set infinity to be zero.

For non-localizable sources?

cylindrical symmetry

Example infinite line of mass



Gaussian surface

$$\oint g \cdot dA = -4\pi G M_{enclosed}$$

$$(-1)\, g\, \ell (2\pi r) = -4\pi G M_{enclosed}$$

$$g = -\frac{2G\lambda}{r}\hat{r}$$

$g =$ [illegible]

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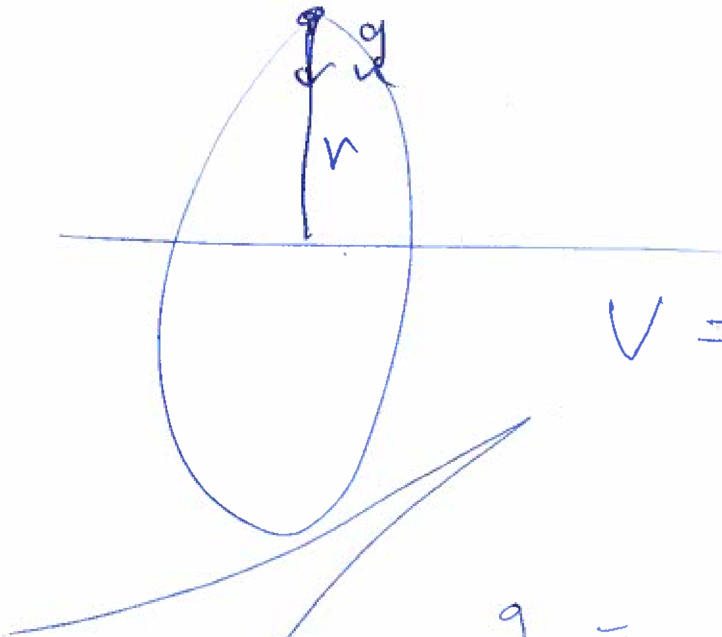
where $\lambda = \dfrac{M_{enclosed}}{l}$ is the linear mass density.

So we have $\underset{\sim}{g} = -\dfrac{2G\lambda}{r}\hat{r}$

cylindrical coordinate r



$V = 2G\lambda \ln(r/r_0)$

a reference radius

$\underset{\sim}{g} = -\hat{r}\dfrac{\partial V}{\partial r}$ (Arf-97) for cylindrical coordinate

$= -\dfrac{2G\lambda}{r}$

~~No place where potential goes to~~ zero

$V(r=0) = -\infty$

But this is cylindrical radius r

$V(r \to \infty) = \infty$ or zero

You can set $V(r=r_0)=0$, but r_0 cannot be at infinity, so in general $V(r=\infty)$ cannot be set to zero, since any finite location would be $V=\infty$ sort of.

3.8 What is Potential Energy Actually and Where is it Actually?

A) In pure potential energy theory, we don't specify. The theory is generic and abstract.

Real fundamental forces have potential energy depending on their nature AND on the particular field structure

Strong & weak nuclear force? I know nothing and we won't discuss them.

Gravity a very special case. Defer discussion for a bit (In GR, the cosmological constant force if it exists is part of gravity.)

Electromagnetism $\underset{\sim}{E}$ and $\underset{\sim}{B}$ which, of course, mix their identities via special relativity

a) But does the B-field have PE?

Yes $\qquad U = -\underset{\sim}{\mu} \cdot \underset{\sim}{B}$ (Google A.I.)

magnetic dipole moment

b) Yes Inductor $\qquad U = \frac{1}{2} L I^2$ (Google A.I.)

other cases too.

c) But Lorentz force $\underset{\sim}{F} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

Say $\underset{\sim}{E} = 0$, B uniform

$$dW = q\,\underset{\sim}{v} \times \underset{\sim}{B} \cdot d\underset{\sim}{s} = q\,\underset{\sim}{v} \times \underset{\sim}{B} \cdot \underset{\sim}{v}\,dt$$

$$= q[\epsilon_{ijk} v_j B_k v_i]\,dt$$

$$= q[(\epsilon_{kij} v_i v_j) B_k]\,dt$$

$$= q[\underset{\sim}{v} \times \underset{\sim}{v} \cdot \underset{\sim}{B}]\,dt = 0$$

The $\underset{\sim}{B}$-field does <u>no work</u> on charge particle moving thru it.

It does something, but <u>no work</u>.

In sense, the work between any two points in space is the same no matter the path,
But No potential energy scalar field to derive $\underset{\sim}{B}$ from.

So <u>No potential</u> energy in this case as most would agree.

d) <u>Electric field</u> <u>a conservative force?</u>

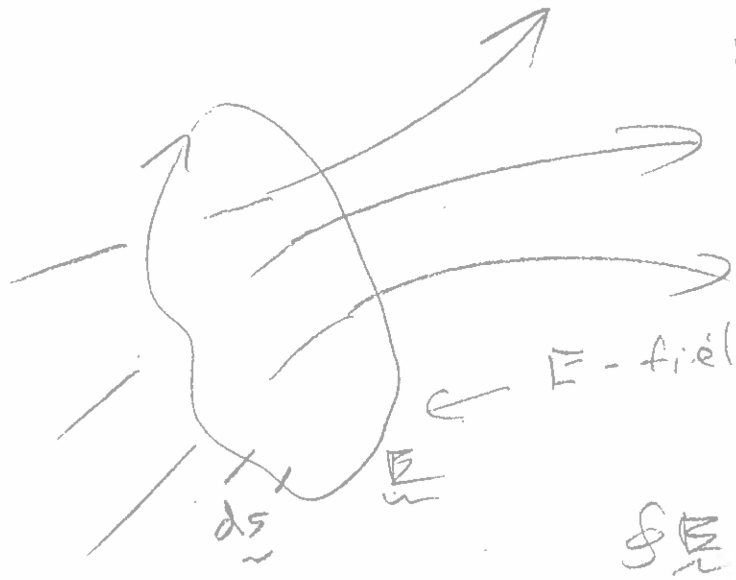
For <u>static charges</u> yes. They are like the central forces discussed on p. 3020.

Test particle.

But the induced magnetic field? (Wiki: Faraday law of induction)

$$\mathcal{E}_{emf} = \oint \underset{\sim}{E} \cdot d\underset{\sim}{s} = -\frac{d}{dt} \int \underset{\sim}{B} \cdot d\underset{\sim}{A}$$

work per unit charge; closed loop; Area enclosed by loop.



B time varying B-field

E-field line forms a loop.

$\oint \underline{E} \cdot d\underline{s} \neq 0$,

and so no PE ~~of~~ ~~for~~ for particle can be defined.

B) But in all cases, there is an E & B field energy whether or Not it can be characterized as a PE or Not.

Note the energy is localized. There is something that is energy at every point and it can be integrated for total energy

$$\mathcal{E}_{density} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \quad \text{(Google AI)}$$

energy per unit volume

and by $E = mc^2$ ~~the~~ energy has mass, both inertial and gravitational

VI don't know the general proof but a proof just for $\underline{E}$ due to charge (Not Faraday law of induction) is useful

and $\mathcal{E}_{density} \geq 0$ always.

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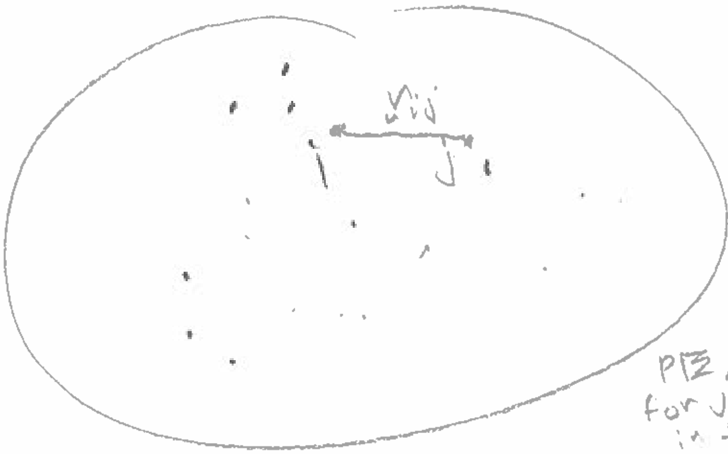
Localized Distribution of point charges. (One can draw finite surface around them)



From our PE for gravity (P. 3020-3021) mut atis mutaudis

PE for j in field ϕ_i

$$U_{ij} = \frac{k q_i q_j}{r_{ij}} = q_j \phi_{ij}$$

Potential $\phi_{ij} = \frac{k q_i}{r_{ij}}$

total PE

$$U = \frac{1}{2} \sum_{\substack{i j \\ i \neq j}} \frac{k q_i q_j}{r_{ij}}$$

Corrects for double counting

$$= \frac{1}{2} \sum_j q_j \sum_i \phi_{ij}$$

$$= \frac{1}{2} \sum_j q_j \phi_j$$

total Potential for particle j.

Note we are Not counting the PE of assembling the point charges.

This PE is a tricky question for classical point particles that is can't be solved unambiguously since they don't exist.

Quantum mechanical point particles are spread out in space existing a superposition positions determined by the wave function ψ.

quantum field theory may have different understanding, but I don't know QFT.

[2025 jan 05] ~~3023~~ 3027

e) What is Potential Energy Really and where is it?

In ~~Potential theory~~ in the generic way we've discussed, it only what we say it is: an energy of position in a generic field of force and we do not say and cannot say where it is.

Not specified where.

But what about real fields of force? Whether they have definable Potential energy or not.

Gravitational, Electromagnetic, nuclear (strong & weak) other?

The Cosmological Constant Λ field in cosmology as we'll see later. But the cut line can. Needs to be introduced.

Note you cannot assign a potential energy to a field of force. Depends on the structure of a particular field.

Eg., magnetic dipole

$U = -\underset{\sim}{\mu} \cdot \underset{\sim}{B}$

dipole moment, external B-field

So a potential can be defined in this case and some B-field cases but not in general,

Lorentz force (with)

$\underset{\sim}{F} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

$dW_B = \underset{\sim}{v} \times \underset{\sim}{B} \cdot d\underset{\sim}{r}$

$= \underset{\sim}{v} \times B \cdot \underset{\sim}{v}\, dt$

$= \epsilon_{ijk} v_j B_k v_i$

$= 0$

So potential energy always zero — but B-field force still affects motion.

Google AI tells us the Magnetic force is cons

B fields in general don't. In special cases yes.

3028

Let's consider
the electromagnetic force
and gravity.

Other forces are beyond our
scope & my knowledge.
(Except ~~the~~ cosmological ~~constant~~ force → which is a very special case)

1) Electromagnetic force & field

Someone has proven in general (for vacuum)

$$\mathcal{E} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

energy density of the EM field
↳ Never Negative
↳ No Negative mon ~~and no negative~~ from $E=mc^2$: Negative contribution to mass energy

vacuum permittivity (of free space) (Wik)

E-field strength

vacuum permeability (of free space) (Wik)

B-field strength

So an energy but can't in general be considered a potential energy

But total always positive

QM sees it. Electron is a point particle in a sense but it's smeared out in ~~a~~ a continuous charge density by the the wave function ($|\Psi|^2$ probability of existence), but QFT beyond our scope

So time & space varying
and in general for Special Relativity and non-relativistic QM, but (above) (below) level of ~~QFT~~ quantum field theory (QFT)

~~except for point charges~~
~~and near~~
~~which is a big exception.~~

I think classical point particles are not really treated by this since $E \to \infty$ at the point. But classical point particles do Not exist

Someone in quantum field ~~theory~~ (QFT) has dealt with this.

So $U = \frac{1}{2}\sum_j q_j \phi_j \to \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})\,dV$

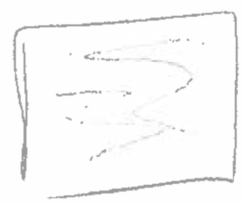
is fine?

I'm still uncertain as to why this is a good idea even we did it all the time in intro physics without challenging the instructor.

Maybe the very local contributions are not important.

cell j

cell j smoothed

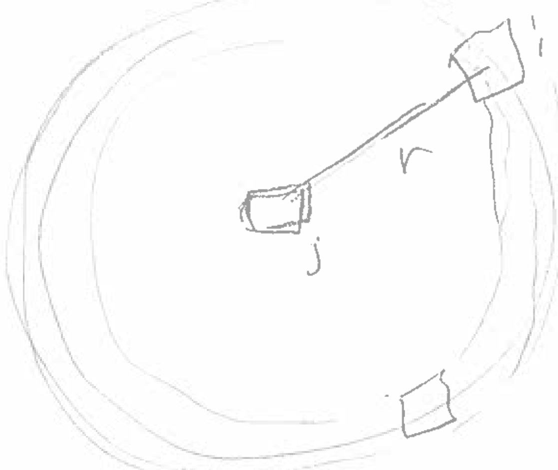


small charge density hills and valleys in small cell

Now totally smoothed out.

But can the local PE be the same really in both cases?

No. I think. But that may be this negligible error macroscopically.



Proof

$\phi_{ij} = \frac{k q_i}{r}$

sum over shell

$\phi_{shell}(r) = \frac{k\,4\pi r^2 \rho \Delta r}{r}$

$\propto r$

So shell contributions grow like r.

So it seems plausible that long range contributions to U dominate and local contributions are negligible (if they are NOT infinite) QED

3030)

In 3-d?
In 2-d?
In 1-d?

Seems plausible, but is this really quantitatively of high accuracy?

Instructors blandly passing over this fine point suggest yes/no.

But $U = \frac{1}{2} \int \rho(\underset{\sim}{r}) \phi(\underset{\sim}{r})\, dV$

Now Gauss' law differential form

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$$ (see p. 3011)

$$= \frac{\partial E_i}{\partial x_i}$$ in Einstein summation

No divergence where $\rho = 0$, so integrand zero where $\rho = 0$

Clairvoyance needed

$$U = \frac{\varepsilon_0}{2} \int \frac{\partial E_i}{\partial x_i} \phi\, dV = \frac{\varepsilon_0}{2} \int \left[\frac{\partial (E_i \phi)}{\partial x_i} - E_i \frac{\partial \phi}{\partial x_i} \right] dV$$

$$= \frac{\varepsilon_0}{2} \left[\int \nabla \cdot (\underset{\sim}{E} \phi)\, dV - \int \underset{\sim}{E} \cdot \nabla \phi\, dV \right]$$

by Gauss thm p. 3009

Recall $\underset{\sim}{E} = -\nabla \phi$

But these integrands do not have to be zero where $\rho = 0$.

$$= \frac{\varepsilon_0}{2} \left[\oint \underset{\sim}{E} \phi \cdot d\underset{\sim}{A} + \int E^2\, dV \right]$$

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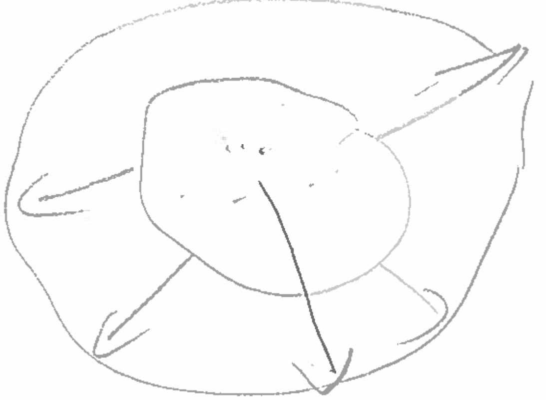


We take the Gaussian surface that encloses all charge to infinity

$$E\phi \sim \frac{1}{r^2}\frac{1}{r} \propto \frac{1}{r}$$

$$\therefore \quad E\phi\, 4\pi r^2 \propto \frac{1}{r}$$

and the 1st integral must vanish asymptotically

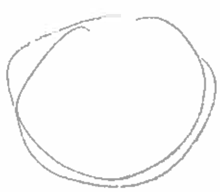
$$\therefore \quad U\left(\substack{\text{integrating} \\ \text{to } r\to\infty}\right) = \frac{\varepsilon_0}{2}\int_{\substack{\text{all} \\ \text{volume}}} E^2\, dV$$

Plausible to assume, but Not proven as we show in a moment.

$$\mathcal{E}_{\substack{\text{energy} \\ \text{density}}} = \frac{\varepsilon_0}{2} E^2 \geq 0$$

and only zero if $\rho = 0$ everywhere, so that $\underset{\sim}{E} = 0$ everywhere (macroscopically).

Note say we had two charge distributions



1

2

$$U_1 = \tfrac{1}{2}\int \rho_1 \phi_1\, dV \qquad\qquad U_2 = \tfrac{1}{2}\int \rho_2 \phi_2\, dV$$

But $U_{total} \neq U_1 + U_2$ because of inter-clump E-fields.

$$U_{total} = \frac{\varepsilon_0}{2}\int (\underset{\sim}{E}_1 + \underset{\sim}{E}_2)^2\, dV$$

$\underset{\sim}{E}$'s add Not $\underset{\sim}{E}^2$'s.

So reasonable to interpret

$$\mathcal{E}_{\substack{\text{energy}\\ \text{density}\\ \text{for static}\\ \text{charge}}} = \frac{\varepsilon_0}{2} E^2$$

and this is correct, but a real proof need to consider changing fields and moving charge and at least implicitly special relativity to get general formula $\mathcal{E}_{\text{density}} = \frac{\varepsilon_0}{2} E^2 + \frac{B^2}{2\mu_0}$

Need Maxwell's equations.

above level of QFT

9) But what if we'd done the same proof for gravity.

$$\phi_{ij} = \frac{kq_i}{r_{ij}} \rightarrow \phi_{ij} = -\frac{GM_i}{r_{ij}}$$

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0} \rightarrow \nabla \cdot \underset{\sim}{g} = -4\pi G\rho \quad \left\{ \text{See p. 3011} \right.$$

$$\underset{\sim}{E} \rightarrow \underset{\sim}{g} \quad \text{and} \quad \varepsilon_0 \rightarrow \frac{1}{-4\pi G}$$

$$\therefore \; U = -\frac{1}{8\pi G} \int_{\text{all volume}} g^2 \, dV$$ (see p. 3027)

This is the correct Potential energy, but it's just an amazing transformation of the original

$$U = \frac{1}{2} \sum_{ij} m_j \phi_j$$ (see p. 3025)

$$\mathcal{E}_{\text{density gravity field energy}} \stackrel{?}{=} -\frac{g^2}{8\pi G}$$

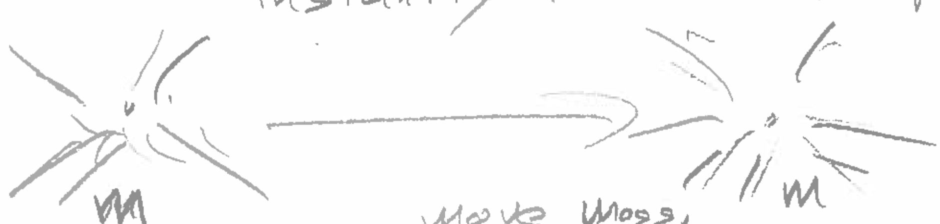
What's wrong?

By $E = mc^2$, this implies negative mass-energy which implies negative inertial mass and negative gravitational mass and that a field of gravity had its own gravity which has its own gravity ad infinitum.

$-\frac{g^2}{8\pi G}$ is a valid integrand for the integral on p. 3028, but it's not gravitational field energy.

One aspect of wrongness is Newtonian gravity is NOT relativistic like classical E&M.

The Newtonian gravitational field propagates instantly in common assumption



move mass,
— field everywhere move in step with mass.

Einstein in developing GR believed this had to be wrong.

And he believed this had to be wrong.

2026 jan 24

3.9 General Relativity and Gravitational Field Energy

So in GR there is No explicit gravitational field energy

4x4 tensor Einstein Field Equations (1915), the GR replacement for Newton's law of universal gravitation (in $\nabla^2 \phi = 4\pi G\rho$ Poisson equation form (wiki) most closely)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

set of differential equations, so true at every point in spacetime

Einstein tensor

Cosmological constant

metric tensor

energy-momentum tensor

Invented 1917 for Einstein universe

- effect of "antigravity", No effect on small scales

$\rho_{\text{dark energy equivalent}} = \frac{\Lambda}{8\pi G}$, Just move $\Lambda g_{\mu\nu}$ to the right-hand side, Constant dark energy often called Λ,

geometry of space-time

energy + momentum tell spacetime how to curve

then curvature tells mass-energy how to move under gravity via geodesic equation ($F = ma$ analogue but just for gravity

Where is gravity field energy.

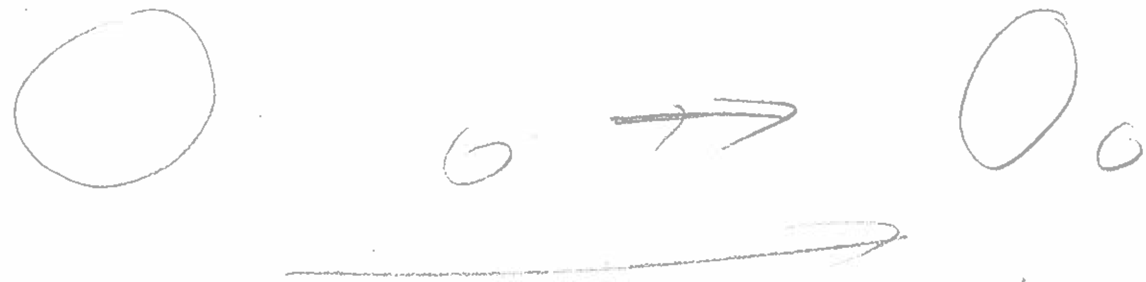
It's nowhere explicitly.

It's not in $T_{\mu\nu}$.

The understanding is that gravitational field energy is encoded non-locally (Penrose 464 – 469) (Wald 84 – 88)

You cannot say there is so much here or there unlike the electromagnetic field (see p. 3024e)

Say you take two spherically symmetric masses



move them closer together

The $T_{\mu\nu}$ tensor contents integrate don't change ideally.

<u>But</u> g in far field decreases just as if the classical P.E calculation (in the classical limit) as if mass had decreased and if the change was due to gravitational waves, they travel across empty space (ideally) where $T_{\mu\nu} = 0$ and then arive at LIGO and deposit kinetic energy (a tiny amount)

Another aspect of the funniness of GR is that ordinary energy conservation is NOT guaranteed (Carroll-120)

2026 jan 24

There is $\underbrace{\nabla_\mu}_{\text{covariant derivative}} T^{\mu\nu} = 0$ the energy-momentum conservation equation, another differential equation true at every point in spacetime

Carroll-120

But it seems less useful than non-GR conservation of energy.

In fact, GR guarantees itself, not conservation of energy.

And, in fact, in expanding universe models energy in an ordinary sense does not ~~seem~~ to be conserved.

Radiation energy just seems to redshift away due to the cosmological redshift. (Carroll-120)

Maybe it goes somewhere, but where. But maybe the expanding universe models that follow from the Friedmann equation are somehow over-idealized.

3.1U Supplementary Notes on Gravity p. 3037—~~3053~~

(2024 jan 09) ~~3031~~ 3037

Analogy it's a bit like a car

Drive to Reno or

{ Non-local gravitational energy

~~It~~ you take a car apart in Las Vegas, and ship the pieces and reassemble in Reno does the car exist?

Sort of non-locally.

~~Y~~You can't say it's here or there.

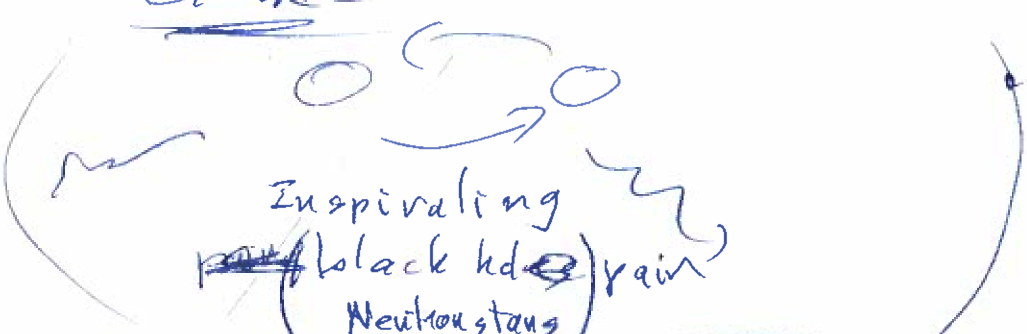
To illustrate the point a bit more, gravitational waves travel ~~through~~ empty space (in ideal limit with no Quantum field theory stuff,) where

$$T_{\mu\nu} = \emptyset$$

No localized gravitational field energy

But they carry energy away from a source.

example



$g_{\mu\nu}$ { gravity field far away (decreases by the mass-energy loss from the pair

Proven since the 1970s by the binary pulsar PSR J~~0737-3039~~ B1913+16

Penrose p. 464-468 says the total mass-energy contribution from $T_{\mu\nu}$ unchanged ideally (aside from shape changes)

~~then~~ stays ~~to~~ the same amount. So energy was encoded in the field and it escaped — but doesn't this mean the energy change as measured by Newtonian field is not local in case

And ~~then the~~ gravitational waves non-locally

~~3032~~
3033

deposit energy when they are absorbed by encountered objects like LIGO.

So the disassemble car is transported in pieces to Reno. But maybe the car isn't complete, Or maybe there are extra parts

There is a theorem (Bondi - Sachs theorem) which shows in a special case ~~that there~~ is an energy-momentum conservation law (Penrose 467-468) for grav. waves from a binary system.

But with NO localizable energy between the two endpoints

But still not clear if ~~all this~~ this energy is deposited in a conserved fashion anywhere (to me from ~~the~~ descriptions) in a localizable form.

GR just guarantees itself actually.

In fact, general relativity does not guarantee conservation of energy in an ordinary sense.

Carroll-120

It does give us the $\nabla_\mu T^{\mu\nu} = 0$ the energy-momentum conservation equation (Carroll-~~1~~.20) where ∇_μ is the covariant Derivative (Carroll-95,97)

2025 jan 05

~~3033~~ 3039

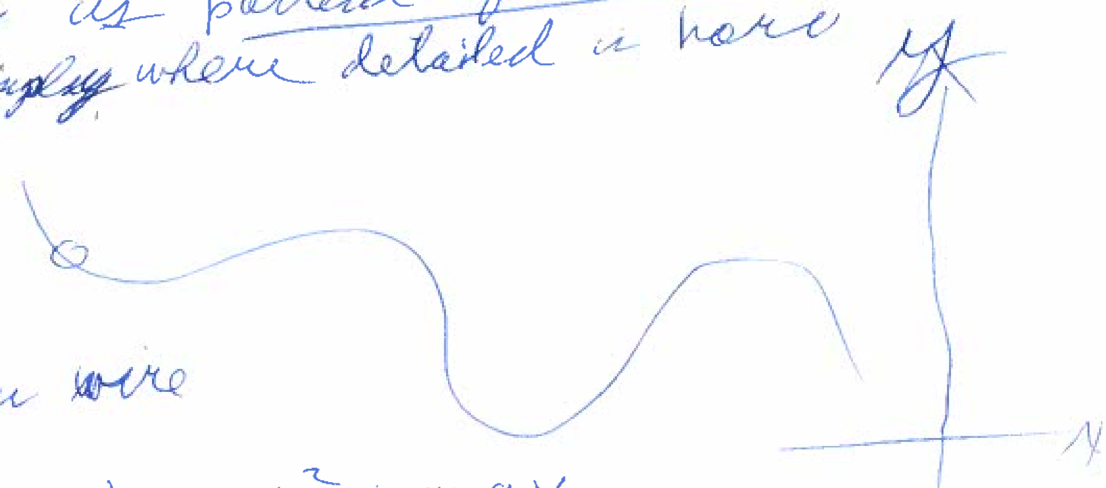
~~∇_μ of the covariant derivative (Carroll 0th 1997)~~

and ∇_μ ~~which~~ we will not go into.

But $\nabla_\mu T^{\mu\nu} = 0$ is not as useful as ordinary conservation of energy which is recovered in the ~~classical~~ non-GR limit

Recall energy conservation often gives us partial information ~~simply~~ where detailed is hard

e.g.,



bead on a frictionless wire

$$E_{mec} = \frac{1}{2} m v^2 + m g y$$

$$v = \sqrt{\frac{2E_{mec}}{m} - 2gy}$$

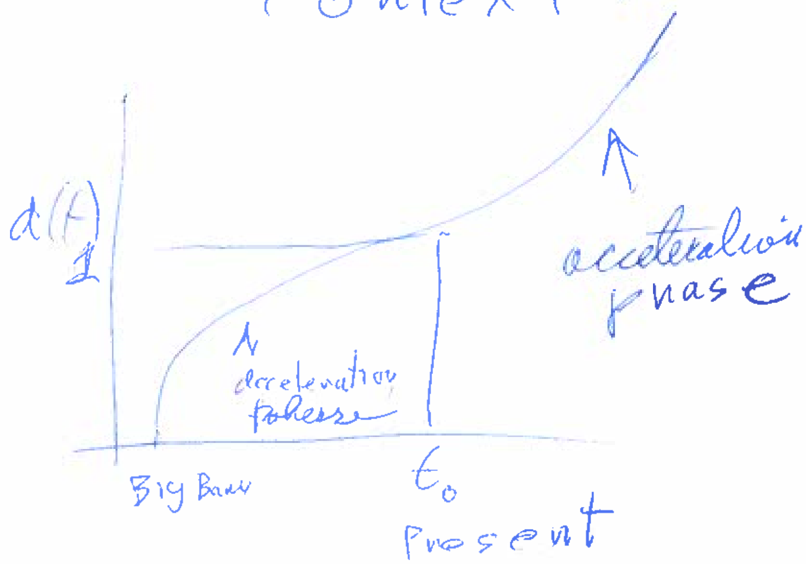
speed for any y easily obtained

whereas $x(t)$ is hard.

You need to specify the ~~wire~~ path and there ~~may well be~~ will be NO analytic solution in general.

3040 (old 2025 Jan 05 notes no v. 3034-3039)

To give an example of GR non-conservation of energy in the cosmological context:



$a(t)$ (which we will discuss later) is the scale factor of Friedmann equation:

- All distance that participate in the mean expansion of the universe scale with $a(t)$

where t is cosmic time

and by convention $\underbrace{a(t_{present}) = 1}_{\text{Because we live now.}}$ usually.

Cosmological redshift (which is not Doppler shift though it is related) gives $\lambda \propto a(t)$ (which we will show later) for photons just travelling through cosmic space like the Cosmic microwave background photons.

$\therefore$ $E_{photon} \propto \frac{1}{\lambda} \propto \frac{1}{a}$

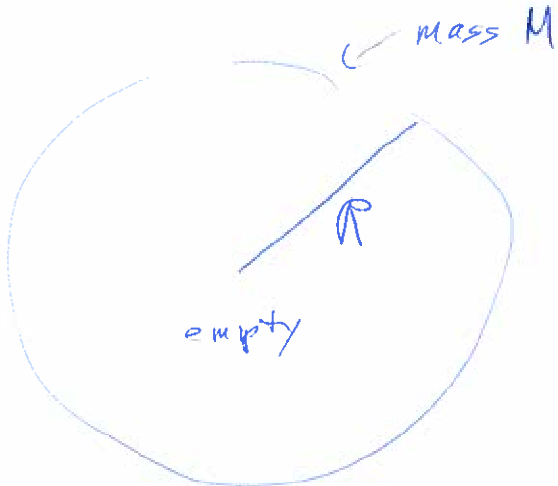
Where does that energy go? Some books say it goes into the expansion of universe, but that statement has no definite meaning as far as I know

That energy is just gone (Carroll - 120) as far as we know.

i) To Return to Gravitational Potential Energy and some more Insights

3092 | 2025ja405

Say we have a thin (i.e., infinitely thin) spherical shell.



~~From Gauss~~

From Shell theorem p. 3011 (in ~~Intro~~ Gauss law Shell theorem)

$$\underset{\sim}{g} = \begin{cases} -\frac{GM}{r^2}\hat{r} & r > R \\ 0 & r < R \\ ? & r = R \end{cases}$$

we'll leave this for ~~short~~ ~~lect~~ see p. 3014 d 2025jan05 notes

$$V = \begin{cases} -\frac{GM}{r} & r > R \\ -\frac{GM}{R} & r = R \text{ best guess for continuity} \\ -\frac{GM}{R} & r < R \end{cases}$$

p. 3020

No gravity force inside and so V is constant.

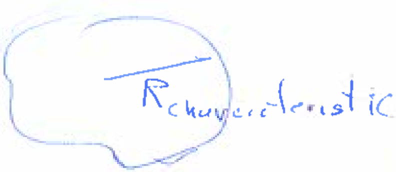
$$U = \cancel{V(R)M} = -\frac{1}{2}\frac{GM^2}{R}$$

Follows from integral assembling

$$dU = -\frac{Gm}{R}\,dm \quad \text{dm brought from infinity}$$

$$U = \int_0^M dU = -\frac{GM^2}{2R}$$

[For any localized mass distribution,



it follow that $U_{characteristic} = -\frac{GM}{R_{characteristic}}$ or $U \approx U_{characteristic}$]

[2025 Jun 05] 3043

What do we get from our ~~free~~ gravitational field energy density result on p. 3028

(Which result is wrong as GR tells since the gravitational field energy cannot be localized)

$$U = \int_R^\infty \left(-\frac{1}{2}\frac{g^2}{4\pi G}\right) 4\pi r^2 dr$$

$g = 0$ inside the shell recall

$$= -\frac{1}{2}\frac{G^2M^2}{4\pi G} 4\pi \int_R^\infty r^{-4} r^2 dr$$

$$= -\frac{1}{2}\frac{GM^2}{R}\int_1^\infty x^{-2} dx = -\frac{1}{2}\frac{GM^2}{R}\left(-x^{-1}\right)\Big|_1^\infty$$

$$= -\frac{1}{2}\frac{GM^2}{R}$$ which is exactly right.

Why? Recall we got the analogus formula for potential energy integral for ~~the~~ an electrostatic system of charge

$$\frac{1}{2}\int \rho(\mathbf{r})\phi(\mathbf{r})dV = U = \frac{\varepsilon_0}{2}\int E^2 dV \quad \text{and} \quad U \overset{\frac{1}{2}\int \rho\,\phi_g(r)\,dV}{=} -\frac{1}{2}\frac{1}{4\pi G}\int g^2 dV$$

ρ charge density

Coulomb force only. No B-field, No Maxwell eqn

gravity

This result follow as exactly correct by the same proof.

3044 (2029 jan 05)

The Magic of an inverse-square law means the integral with integrand ~~is an~~ $\left\{ \begin{matrix} \rho \phi \\ \rho V \end{matrix} \right\}$ rewrites ~~to~~ exactly

~~the~~ ~~in~~ integral with integrand

$$\left\{ \begin{matrix} \frac{\varepsilon_0}{2} E_0^2 \\ -\frac{1}{2} \frac{g^2}{4\pi G} \end{matrix} \right\}$$ but

interpreting the new integrand as energy density correct for E&M as we know from more general proof from Maxwell's equations

but incorrect for gravity as we know from GR.

The $U = \int \rho \phi_g \, dV \Rightarrow -\frac{1}{2} \frac{1}{4\pi G} \int g^2 \, dV$

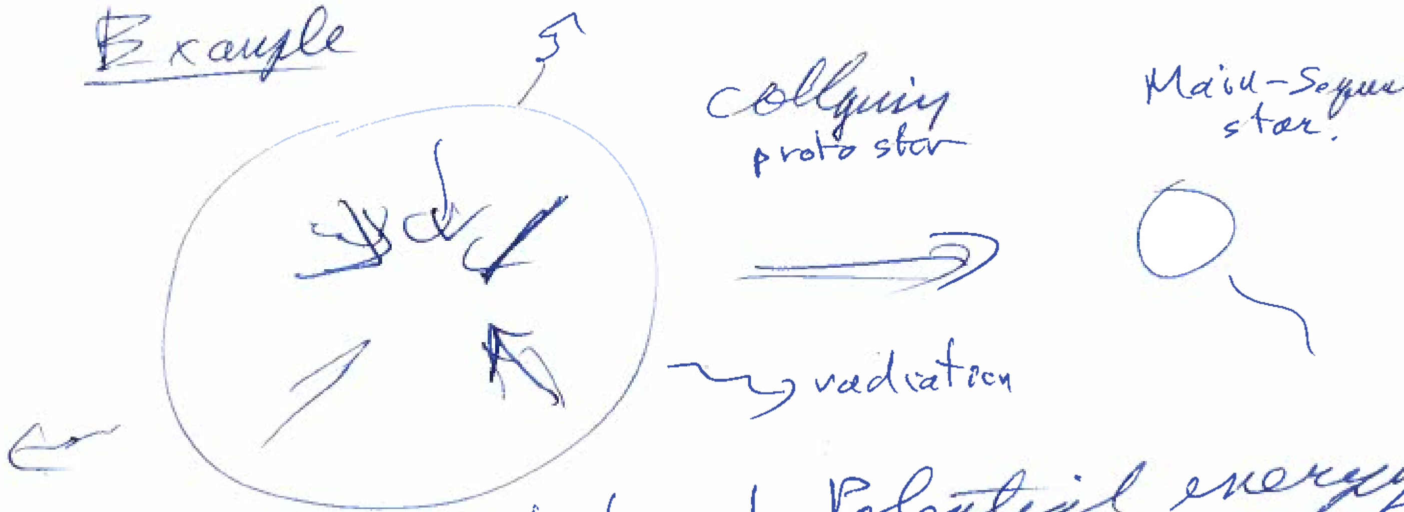
Just a magic equivalence with $-\frac{1}{2} \frac{1}{4\pi G} g^2$ having no clear meaning apart from the integral, except that it is ~~what~~ it is the Newtonian ~~the~~ gravitational squared times $-\frac{1}{2}\frac{1}{4\pi G}$.

3045

But certainly the non-GR limit calculation of potential energy

$$U = \int \rho \, \phi_g \, dV$$ is correct. in the classical limit.

Example



The gravitational Potential energy gets turned into kinetic energy and ~~that~~ turns to heat energy and the protostar radiates radio → infrared → visible as it heats up and then H burning turns on in the core and stabilizes the star on the Main sequence.

Now if the collapse continued into the GR region a GR calculation would be needed.

3046) [2025 jun 04]

But an interesting factoid is say you magicly turned off the Sun's pressure support + rotation and it collapsed in exact spherical symmetry

Without losing any gravitating ~~effect~~ of Pressure

$$R_{Schwarschild} = \frac{2GM}{c^2} = 2.95318 \text{ km} \left(\frac{M}{M_\odot}\right)$$

So ~~the~~ sun BH would have $R_{Schwarschild} \cong 3$ km (event horizon)

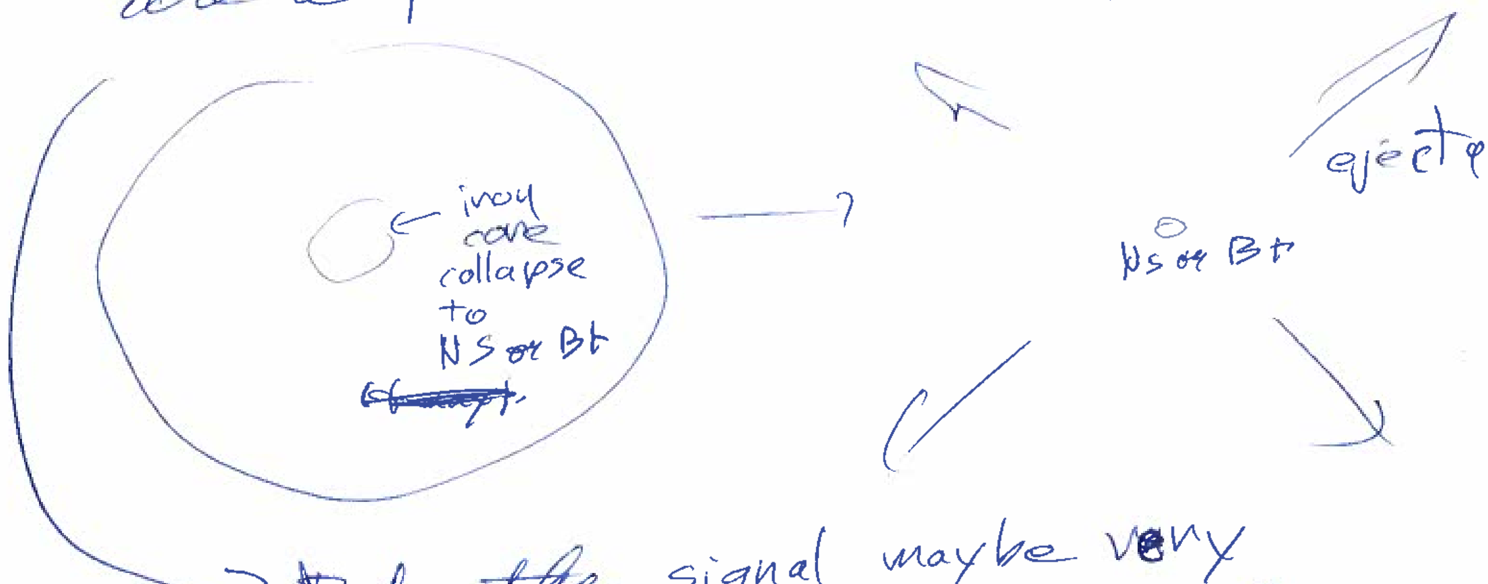
But would the Sun lose mass in the collapse.

It might lose a bit in radiation and neutrinos, but spherical collapse ~~radiates~~ gives no gravitational waves.

So the Sun mass ~~would~~ stay almost the same.

This is unlike ~~the~~ inspiralling {NS/BH} pair on p. 3031 where the asymmetry does allow strong gravitational waves.

Core-Collapse Supernovae are a potential gravitational wave events.



But the signal maybe very weak if the collapse is nearly spherically symmetric which is the most common ~~idea~~ theory, but there may be strong jets due to complex B-fields, and so maybe there will be a strong signal.

But to return to the collapsing Sun to a BH.

What would happen to Earth's orbit?

Ideally nothing. To explicate:

Birkhoff theorem — the GR analogue to the Newtonian shell theorem

One corollary is outside of the spherical symmetric mass distribution

3048) Antigravity
the $\underset{\sim}{g}$ field is the same no matter how the mass is distributed as long as spherical symmetry is maintained;

$$\underset{\sim}{g} = -\frac{GM}{r^2}\hat{r}$$

It could be a classical point mass.

In GR, outside of spherically symmetric mass distribution one has the Schwarzschild metric

Weinberg -180
Carroll -204

$$d\tau^2 = c^2\left[1 - \frac{2GM/c^2}{r}\right]dt^2 - \left[1 - \frac{2MG/c^2}{r}\right]^{-1}dr^2 - r^2 d\Omega^2$$

invariant proper time between events in space time in any coordinate system;

No change No matter M is concentrated (as long as you are outside) and Birkhoff theorem addition; it is static no matter how $\rho(r)$ changes as long as spherical symmetry is maintained (Weinberg 337-338)

Which depends on how you set things up and should be chosen for understanding but different coordinate systems give different information.

Of course this solution leads to Schwarzschild Blackholes and the Schwarzschild radius (see p. 3046).

But Black holes are off our topic.

~~But the~~ No GW emission from spherically symmetric motions as $\frac{2GM/c^2}{r}$ gets small the metric approaches the flat metric of special relativity spacetime

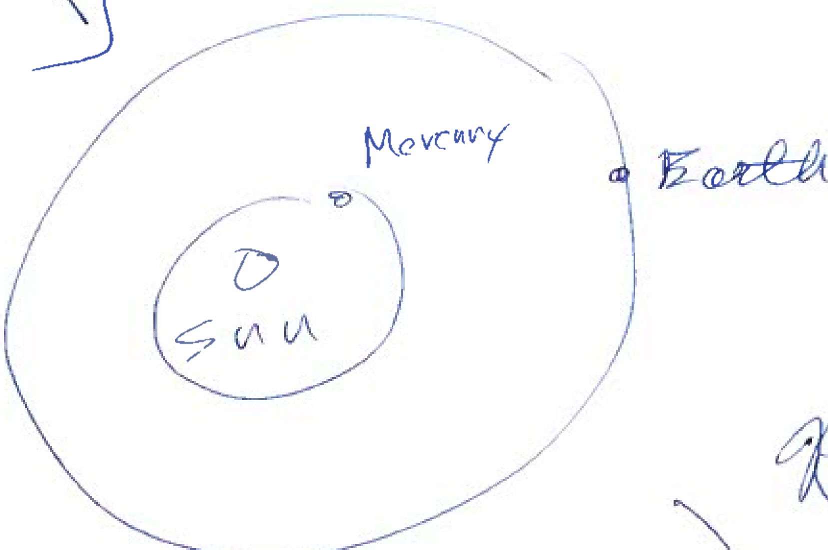
$$d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\Omega^2$$

for spherical coordinates

A 4-d pseudo Euclidean space

$$d\tau^2 = \lambda_i^2 dx_i^2$$

would be a 4-d Euclidean space metric.



but this is the case for the Earth

$$\mathcal{R} = \frac{2GM/c^2}{r}$$

$$\simeq \frac{3\ \text{km}}{(1.495\ldots)\times 10^8\ \text{km}}$$

" 1 AU

So the Earth is in essentially a non-GR orbit and because

$$\frac{v_{orbit}}{c} \simeq \frac{30\ \text{km/s}}{3\times 10^5\ \text{km/s}},$$

also a essentially a Newtonian orbit.

7050)

As is well known the perihelion shift of Mercury is partially explained by GR effects and this was Einstein's first confirmation of a ~~precise~~ GR effect (1916, Wik: Tests of GR)

So if you collapse the sun spherically symmetrically ~~and~~ and ~~don't~~ let any mass energy escape by photons or neutrinos or solar wind

and GWs won't escape by spherical symmetry, the orbit of the Earth and everything else (e.g., Mercury) is unchanged → nearly Newtonian with just a bit of GR noticeable only for Mercury, I think. //

j)

~~in a local sense~~

~~but~~

$$U = \int \left(- \frac{1}{2} \frac{g^2}{4\pi G} \right) dV$$

Another Example Omit in class.

still gives the classical limit change in ~~tot~~ over all mass-energy of a system

Example

uniform sphere of mass

Ordinary calculation of over U

$U \sim \frac{GM^2}{R}$ order of calculation

exact calculation

in dm

imagine assembling by bring mass in from infinity

$$dU = - \frac{G m(r)}{r} dm$$

potential at radius r

$$m(r) = M\left(\frac{r}{R}\right)^3$$

$$dm = M\left(3\frac{r^2}{R^3}\right) dr$$

$$\therefore U = -3G M^2 \int_0^R \frac{1}{R}\left(\frac{r}{R}\right)^5 \frac{1}{r}\, dr$$

$$= -\frac{3GM^2}{R} \int_0^1 x^4 dx$$

$$= -\frac{3}{5} \frac{GM^2}{R}$$ and so order of calculation not so bad

gravity field calculation

$$U = -\frac{1}{2} \frac{1}{4\pi G} \int g^2 dV$$

From Gauss law (p. 3007)

$$g = -\frac{G\,M(r)}{r^2} \qquad m(r) = \begin{cases} M\left(\frac{r}{R}\right)^3 & r < R \\ M & r \geq R \end{cases}$$

$$dV = 4\pi r^2 dr$$

$$U = -\frac{1}{2} GM^2 \left[\int_0^R \frac{1}{r^4}\left(\frac{r}{R}\right)^6 r^2 dr + \int_R^\infty \frac{1}{r^4} r^2 dr \right]$$

$$= -\frac{1}{2}\frac{GM^2}{R}\left[\int_0^1 x^4 dx + \int_1^\infty x^{-2} dx \right]$$

$$= -\frac{1}{2}\frac{GM^2}{R}\left[\frac{1}{5} + \left(-\frac{1}{x}\right)\Big|_1^\infty \right]$$

$$= -\frac{1}{2}\frac{GM^2}{R}\left[\frac{1}{5} + 1 \right]$$

$$= -\frac{1}{2}\frac{GM^2}{R}\left[\frac{6}{5} \right]$$

$$= -\frac{3}{5} \frac{GM^2}{R}$$ and this is the same QED

But we knew it would be by discussion on p. 3043-3044

30a

So in ~~fact in the classical limit the~~

assembled sphere would have mass less than M ~~the total of the~~ bits by U/c^2

~~But for~~ most masses this isn't much

~~Ratio~~ = $\dfrac{U/c^2}{M} = -C\,\dfrac{GM/c^2}{R} = -\dfrac{C}{2}\,\dfrac{R_{sch}}{R}$

constant of order 1 and $\frac{3}{5}$ for the uniform sphere

where recall ~~the~~ Schwarzschild radius

$0 = \frac{1}{2} m v^2 - \dfrac{GMm}{R_{sch}}$ from classical escape velocity formula

$r = \dfrac{2GM}{c^2}$

Redundant to earlier pages

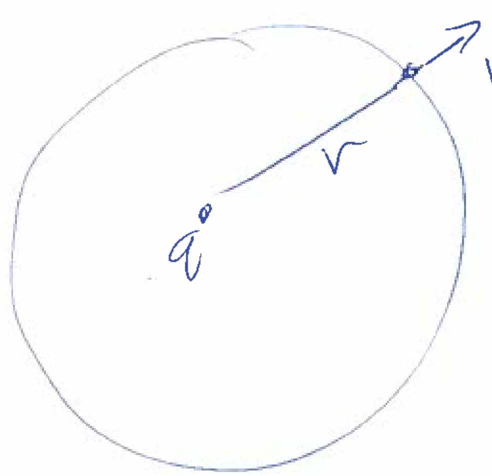
(2025 Jan 05) 3053

~~30 d)~~ 3.11 Linear Force and orbits as a Prelude ~~to~~ To The Cosmological Constant (Cos Constant force Λ or Lambda)

(Just Skim in Class)

a) A generic linear force field law for a point linear force "charge"

$$\underset{\sim}{f} = q r \hat{r}, \quad V = -\frac{q}{2} r^2$$



a) If $q < 0$, you have Hooke's law and the solution in 1-d is simple harmonic motion.

(Super important about ~~any~~ almost all stable equilibria the 1st order restoring force is the Hooke's law force)

~~This is of cours~~

b) What if 2-d? simple Harmonic motion

$$V = \frac{|q|}{2}(x^2 + y^2)$$

$\therefore f_i = -|q| x_i$ simpler to solve in Cartesian coordinates

$$\frac{d^2x}{dt^2} = -\frac{|q|}{m} x = -\omega^2 x$$

$$\frac{d^2y}{dt^2} = -\frac{|q|}{m} y = -\omega^2 y$$

(Conservation of angular momentum dictates there are no ~~two~~ ~~act~~ 3-d solutions for spherically symmetric central forces. (Goldstein p 72)

Also initial velocity and central force direction define a plane of motion that can't be changed $\underset{\sim}{r} \rightarrow \underset{\sim}{v}$: Simple perspective.

~~where 2ϕ is the phase difference~~ ~~$\cos(\omega t - \phi)$~~

3054 | 2025 Jan 05

General solutions

$$x = C\cos(\omega t + \phi_C)$$
$$y = D\cos(\omega t + \phi_D)$$

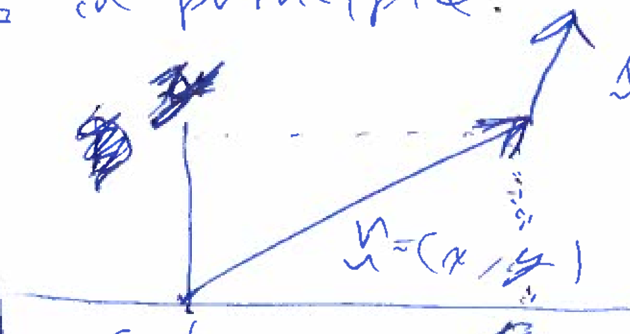
For arbitrary x and y axes at arbitrary time zero.

Where ϕ_C, ϕ_D is the phase difference

Note if C or D is negative, we just add π to ϕ_C or ϕ_D and we get C and D positive

Can we constrain the parameters? Yes in principle.

Odd that 4 initial conditions set 5 parameters. But ϕ_C and ϕ_D are actually NOT independent. So really only 4 parameters. See p. 3060



4 initial conditions ~~the~~ in 2-d space.

The 4 initial conditions allow us somehow to determine C, D, ϕ_C, ϕ_D, and the sign of $\omega = \pm\frac{|\omega|}{\omega}$

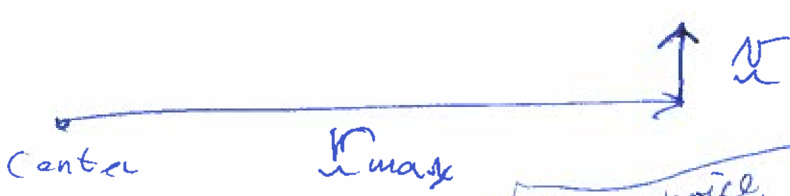
~~But there is a hell.~~

Again $\underset{\sim}{r}$ and $\underset{\sim}{v}$ define the plane all motion is confined too by conservation of angular momentum $\underset{\sim}{L}$ (which we won't nev[?] here)

Note the orbit is closed since $(x(t), y(t))$ repeat every period $T = \frac{2\pi}{\omega}$.

Bound since force goes ~~to infinity~~ as $r \to \infty$

But a better way to write the general solutions



$\underset{\sim}{v}$ must perpendicular to $\underset{\sim}{r}_{max}$.

Our choice that this is r_{max} rather than r_{min}

(2025 jan 05) 3055

Let $t = 0$ be when $\underset{\sim}{v} = \underset{\sim}{v}_{max}$ and let that direction be the x - axis.

Then $x = v_{max} \cos(\omega t)$ $\Big\{$ $v = v_{max}$ at $t = 0$

$\dot{x} = -\omega v_{max} \sin(\omega t)$

$y = \Big\{ v_{min} \sin(\omega t)$ ~~which~~ is 0 at $= t$ as required

but is v_{min} really the minimum v

$r = \sqrt{x^2 + y^2} = \sqrt{v_{max}^2 \cos^2\omega t + v_{min}^2 \sin^2\omega t}$

$$\frac{\partial v}{\partial t} = \left\{ \frac{\left(-2 v_{max}^2 \cos\omega t \sin\omega t + 2 v_{min}^2 \sin\omega t \cos\omega t\right)\omega}{\sqrt{\;\cdots\cdots}} \right.$$

0 for $\omega t = 0, \pi$

which is $v = v_{max}$

0 for $\omega t = \frac{\pi}{2}, \frac{3\pi}{2}$

and this is not v_{max} and so must v_{min}

but this is where

$x = 0$

$y = v_{min}$,

and so v_{min} is the minimum v

where $\dot{y} = v_{min} \cos(\omega t) = 0$

So

$x = v_{max} \cos(\omega t)$

$y = v_{min} \sin(\omega t)$

is the general solution with ~~the~~ 4 parameters

v_{max}, v_{min}, sign of ω, and the orientation which we implicitly fixed

for $\omega < 0$ we run time backward

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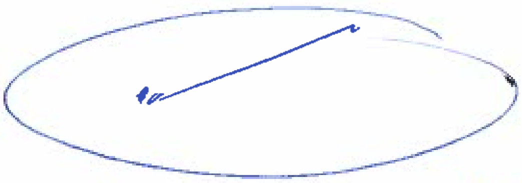
⇒ Now finally a relevant point

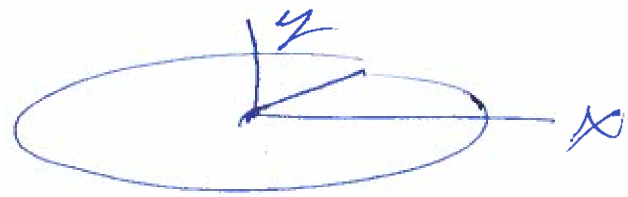
The linear force orbits are elliptical orbits

Inverse-Square Law orbits



elliptical with center of force at one focus. The other focus is just an empty point in space without much descriptive relevance.

linear force attractor



$$1 \stackrel{?}{=} \left(\frac{x}{r_{max}}\right)^2 + \left(\frac{y}{r_{min}}\right)^2$$

ellipse formula satisfied

$$= \cos^2\omega t + \sin^2\omega t$$

$$= 1 \quad \text{QED.}$$

So There is an odd symmetry between the inverse square-law force and the linear force.

Perhaps just because they are SO simple, there had to be some relation, but what couldn't have predicted.

In flat 2-d Euclidean space

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Actually, Newton himself proved ~~both~~ orbits for both inverse-square and linear forces were elliptical in Principia (1687) (see Ian Bruce English translation) of 3rd edition 1726

Book I, Sec 2 Bruce's intro note

But it was beyond Newton's math technique to prove that only these two central forces give closed orbits in general; i.e., orbits where ~~the~~ always repeat its path on every revolution.

Other central forces ~~do~~ give closed orbits in special cases: e.g., they all do for circular orbits of course.

The proof is Bertrand's theorem (1873)

[I once saw a paper that gave a detailed discussion of the symmetry between the inverse-square law & linear force law but I lost track of it and can't find it now]

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d) Detail Omit in ~~Lecture~~ Class

On p. 3055 we give

$$x = v_{max} \cos \omega t$$
$$y = v_{min} \sin \omega t$$

as general solutions.

Conservation of Energy + Angular momentum — And different representations

Now general theorem (we will Not prove here) show mechanical energy and angular momentum are conserved.

But we can verify

$$\frac{E}{m} = \frac{1}{2}(v_x^2 + v_y^2) + \frac{1}{2}\omega^2(x^2+y^2)$$

$\left[\frac{L^2}{T^2}\right]$ dimensions

$$= \frac{1}{2}\omega^2\left[v_{max}^2 \sin^2\omega t + v_{min}^2 \cos^2\omega t + v_{max}^2 \cos^2\omega t + v_{min}^2 \sin^2\omega t\right]$$

$$= \frac{1}{2}\omega^2\left[v_{max}^2 + v_{min}^2\right] \quad \text{QED}$$

$$\frac{L_z}{m} = \left.\underline{r} \times \underline{v}\right|_z = r_x v_y - v_x r_y$$

$\left[\frac{L^2}{T}\right]$

$$= \omega\left[v_{max}v_{min}\cos^2\omega t + v_{max}v_{min}\sin^2\omega t\right]$$

$$= \omega v_{max} v_{min} \quad \text{QED}$$

You can solve for v_{max} and v_{min} in terms of E ~~and~~, L_z, and ω

L dimension of ~~length~~ squared [2025 Jun 05] [3058]

$$\hat{E} = \frac{2E}{mw^2} = v_{max}^2 + v_{min}^2$$

L dimension of length squared

$$\hat{L} = \frac{2L}{mw} = 2 v_{min} v_{max}$$

$$\sqrt{\hat{E} + \hat{L}} = v_{max} + v_{min}$$

$$\sqrt{\hat{E} - \hat{L}} = v_{max} - v_{min}$$

$$v_{max} = \frac{1}{2}\left[\sqrt{\hat{E}+\hat{L}} + \sqrt{\hat{E}-\hat{L}}\right]$$

$$v_{min} = \frac{1}{2}\left[\sqrt{\hat{E}+\hat{L}} - \sqrt{\hat{E}-\hat{L}}\right]$$

Note a constraint $v_{max} \geq v_{min}$ or in fact we have built in. If $v_{max} = v_{min}$, the orbit is circular

So E, L, and w do determine ~~the orbit with the~~ semi-major axis chosen to be ~~[illegible]~~ aligned with the x-axis

Another detail.

Can we recover $x' = C\cos(wt + \phi_C)$
$y' = D\cos(wt + \phi_D)$

from $x = v_{max}\cos(wt) = A\cos(wt)$
$y = v_{min}\sin(wt) = B\sin(wt)$

Yes but it's rather tedious.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

see Arf-8-9 for θ a counterclockwise rotation of axes.

$\theta \in [0, 2\pi]$

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$$x' = A\cos\omega t\cos\theta + B\sin\omega t\sin\theta$$

Let $A\cos\theta = C\cos\phi_C$ and $B\sin\theta = -C\sin\phi_C$

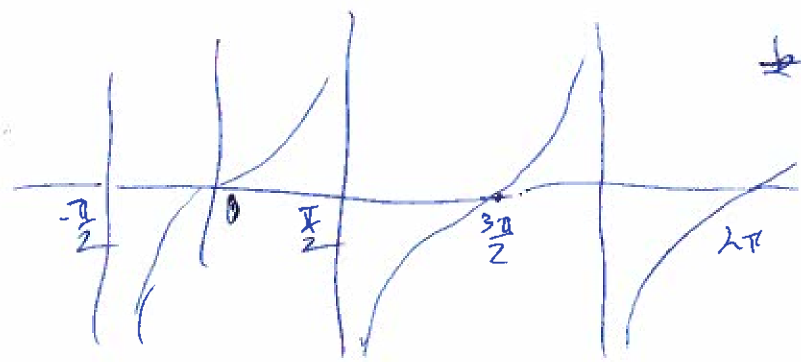
then $\tan\phi = \frac{B}{A}\tan\theta$ $\quad \phi_C = \tan^{-1}\left(\frac{B}{A}\tan\theta\right)$

$$\phi = \tan^{-1}\left(\frac{B}{A}\tan\theta\right) + n\pi$$

~~but the tan~~ where

$n = 0, 1$

$n = 0$



But $\sin(\phi_C + \pi) = \cancel{B\sin\phi}$ $-\sin\phi_C$

& $\cos(\phi_C + \pi) = -\cos\phi_C$

and so would just change the sign of C.

and we might as well let $\phi_C \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$x' = C\cos(\omega t + \phi_C)$

transformation

$$C = \frac{A\cos\theta}{\cos\phi_C} = \frac{B\sin\theta}{\sin\phi_C}$$ ~~and $|C| = \sqrt{A}$~~

and $|C| = \sqrt{A^2\cos^2\theta + B^2\sin^2\theta}$

Similarly $y' = -A\cos\omega t\sin\theta + B\sin\omega t\cos\theta$

Let $-A\sin\theta = -D\cos\phi_D$ and $B\cos\theta = -D\sin\phi_D$

$$\phi_D = \tan^{-1}\left(-\frac{B}{A}\frac{1}{\tan\theta}\right) = \tan^{-1}\left(-\left(\frac{B}{A}\right)^2\frac{1}{\tan\phi_C}\right)$$

$$D = -\frac{B\cos\theta}{\sin\phi_D} = \frac{A\sin\theta}{\cos\phi_D}$$

$$|D| = \sqrt{A^2\sin^2\theta + B^2\cos^2\theta}$$

$y' = D\cos(\omega t + \phi_D)$

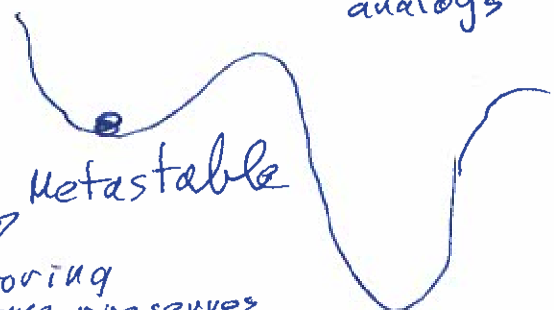
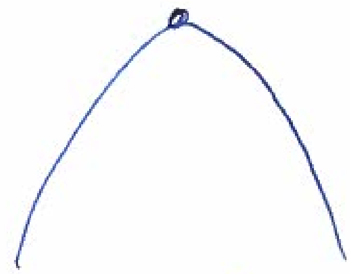
So 4 conditions (i.e. A, [illegible], B, sign of ω, and θ) allow us to determine C, D, ϕ_C, ϕ_D and the sign of ω just as on p. 3054 ~~from~~ 4 initial conditions set 5 ~~things~~ parameters. But ϕ_C and ϕ_D are NOT independent and so only 4 parameters.

e) Stability of orbits

only forces to give closed orbits → Inverse-square + linear forces

stability Cases for equilibria

Mechanical system analogs

stable

restoring force preserves against all displacements

Metastable

restoring force preserves for small enough displacements

unstable

No restoring force. equilibrium is an ideal limit.

Neutral

No restoring force. ~~small d.~~ displacements give comparable changes in state Not runaway changes

A complete stable equilibrium ~~system~~ is an ideal limit. All stable equilibria are metastable if the perturbation is big enough

similarly neutral stability is an ideal limit. Metastable or unstable with big enough perturbation

Orbits are neutrally stable if only one orbiting body and one central force.

Example The elliptical linear force orbit See p. 3055

$$x = r_{max} \cos(\omega t)$$
$$y = r_{min} \sin(\omega t)$$

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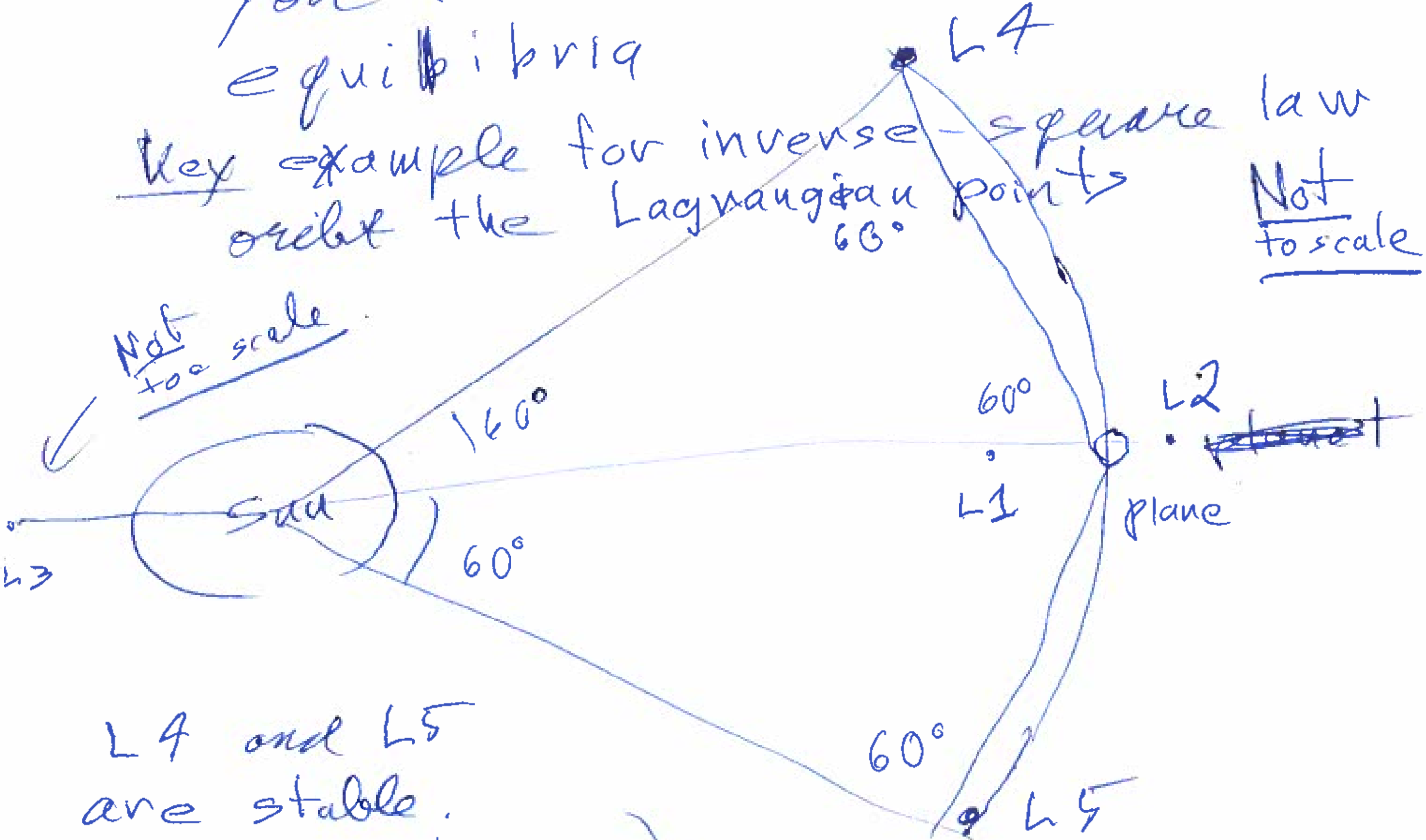
If you change v_{max} to $\tilde{v}_{max}$, then the orbit changes permanently comparably. Some for inverse-square orbits.

But if you have multiple bodies and multiple centers of force you can have stable & unstable equilibria

Key example for inverse-square law orbit the Lagrangian points



L4 and L5 are stable.

In the rotating frame of Planet; The two gravity forces and the centrifugal force give stable equilibria

Jupiter has the strongest L4 & L5 in solar system and the Trojan asteroids have accumulated there and stay there mostly over solar system history

The L1, L2, L3 are unstable equilibria.

But L2 ~~is used~~ for Earth is used for spacecraft ~~where orbit~~ ~~od~~ where strange orbits about the exact equilibrium plus occasional corrections keep them in place (station keeping thrusters)

Gaia, Euclid, JWST.

Lissajous orbit

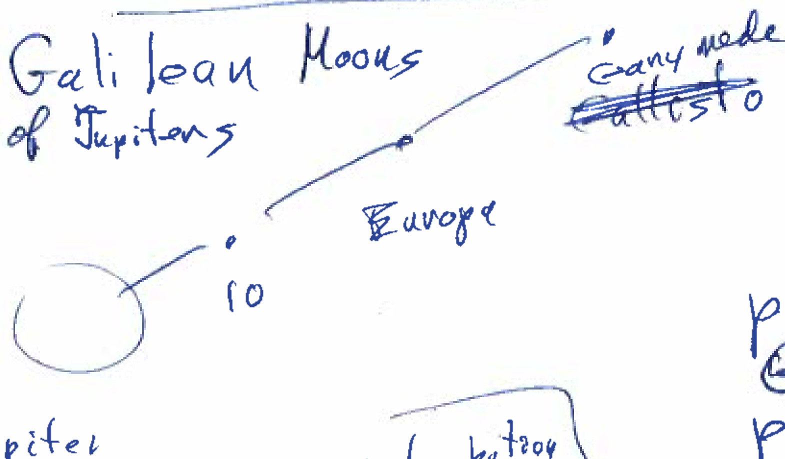
halo orbits.

Resonances can create stable orbits

Or at least stable ratios of orbital parameters

Laplace resonances

Ex. Galilean Moons of Jupiters



$$P_G = 4 P_{Io}$$

$$P_{Eu} = 2 P_{Io}$$

Perturbation always make the ratio a bit unexact, but the stabilizing resonances always drives them too exactness.

$$P_{Ca} \simeq 9.4 P_{Io}$$

Callisto is to remote from the other moons ~~to~~ for the resonance to work for its orbit.

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Resonances can also make unstable.

The Kirkwood gaps of the asteriod belt where asteriods are cleared out.

↳ where periods of orbit are an integer fraction of Jupiters orbit

$$\frac{P_K}{P_J} = \frac{3}{1}, \frac{5}{2}, \frac{7}{3} \quad \frac{2}{1}$$

3 2.5 2.33... 2

In ~~these~~ kirkwood gap orbits, Jupiter keeps perturbing ~~the an as~~ an asteroid until it moves out of that orbit.

Of course, you know how to drive a natural frequencies before you knew the name of torque.

You drive a ~~swin~~ playground swing at its resonance frequency as the oscillations got bigger and bigger until you chickened out.