2019 jul 18

3001

3 Newtonian Gravity & Physics & The Friedmann Equations

3.0a Historical Intro

⇒ followed by ~~older intros~~ — overview — curious quirk

Notes drawn from
Liddle (2015)
Wik
Bondi (1960) for historical tidbits
Carroll (2004)
Coles & Lucchin (2002)
Penrose (2002)
Weinberg (1972)
Lunden (2002)

The Friedmann equations for a(t) scale factor

DE key of motion of universal expansion 1st order ODE nonlinear ⟹ can't add solutions

(i.e., the ~~Friedmann equation~~ plus & the Friedmann acceleration equation) governs the overall dynamics of the observable universe — provided General Relativity (GR) is a correct theory well a correct emergent theory since we believe it is ~~the~~ macroscopic limit of quantum gravity — for which there is no established theory.

ts
f
ariations
but
allows
complex
one
have
required
useful
for held
theorist
to
develop
their
power

al
$r = r$
convention $a_0 = 1$
$r - r_0 a$
$H_0 = \frac{\dot{a}}{a}\Big|_0$ velocity rate of expansion of universe inverse time in km/s/Mpc

Though fairly simple

Because nonlinear — in general Solutions of Friedmann eqn will not add to make a solution

In fact, the Friedmann equations can be derived from Newtonian physics with plausible/natural ad hoc assumptions { If only ordinary matter is considered — Radiation, & ~~Cosmological~~ constant — ~~Constant~~ Dark energy

However, One needs the GR perspective to understand that they include the constant or other same as Λ & ...

At universal level Λ and constant energy same & best but differ in theory of microscopically

002

effects of the curvature of space → and radiation + Λ (constant dark energy)

but he derived them in Russian

Historically, the Friedmann equations ~~was~~ first derived by Alexander Friedmann (1888–1925) in 1922 in Russian (Wiki; Friedmann eqs.)
Georges Lemaître derived ~~that~~ it independently a bit later in the 1920s

ith
R
ldiation
ut
l'hoc
latu

The Newtonian derivation came later remarkably by Milne & McCrea in 1934. (Bondi-75

[illegible]

It's Remarkable that a lot of progress in cosmology could have been made before GR (1915), but that didn't happen.

Two main hold-ups

1) People seemed to belive the universe was static on average
— Newton thought this
— odd since the universe is not in thermodynamical equlibrium
↳ as manifested by Olbers' Paradox
— a dark night sky is inconsistent with an eternal unchanging static universe — homogeneous & isotropic too
↳ if true in every direction you should see a star (homewk problem ___)

Many
people
ut it's hard
to know, no
pinion polls
from the past
before ~1950

2) People didn't yet know there were other galaxies until 1923/4 when Hubble established that the Andromeda nebula was the Andromeda galaxy (Wiki: Hubble) — so then all spiral nebulae were spiral galaxies and all elliptical nebulae too since they came clustered with spirals

people had suspected this for a long time — since the 18th century (Wiki: galaxy)

Even the 17th Chr. Wren in ~1660 suggested before leaving astr. & physics for "rubbish'

In any case, it was hard to ~~get this~~ get an expanding universe idea without knowing galaxies and observing their redshifts which came along with Slipher ~~after~~ starting in 1912 (but very slowly)

Of course a ~~vigorous~~ vigorous derivation of the Friedman equations must be from GR — but we leave that to another course — one on GR

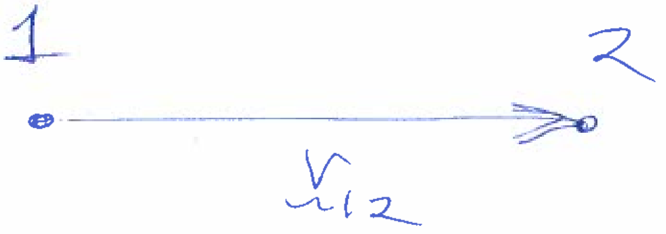
A vigorous demonstration as to why the GR Friedman eqn ~~GR~~ and Newtonian analogs ~~as the same~~ should be the same is given by Wells (2014)

To become an expert in cosmology, you should study GR (not me, just a specialist in teaching intro cosmology)

3004

3.06 Newtonian Gravity, Gauss' Law, & The Shell Theorem

Newton's law of universal gravitation



$$\hat{r}_{12} = \frac{\underset{\sim}{r}_{12}}{r_{12}}$$

$$\underset{\sim}{F}_{12} = -\frac{G m_1 m_2}{r_{12}} \hat{r}_{12}$$

— formally it's for point masses

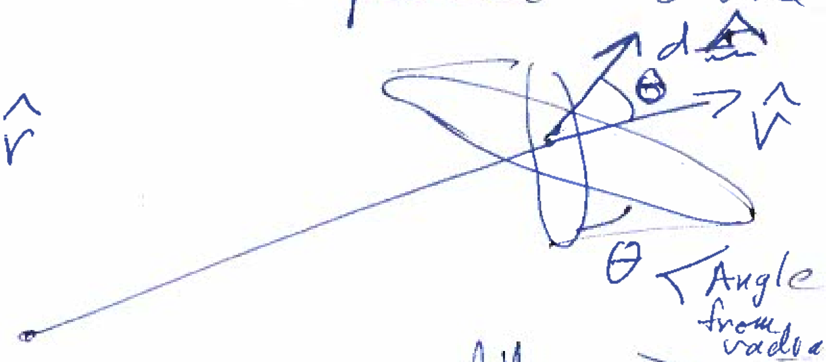
Newton never wrote it down like this. He used an obscure formalism in ~~the~~ Principia (1687) and a few year later Pierre Varignon translated Newton's results into Leibniz calculus formalism ore ecognizable to us, but vector notation didn't come until the later 19th century (Wik: Euclidean Vector)

We can derive Gauss' law from it now — nonrigorously

Gauss' Law

For a force field (force per unit charge) from a point source

$$\underset{\sim}{f} = \frac{Q}{r^2}\hat{r}$$



source

θ ← Angle from radial directio

Very special results due to inverse square-law nature

$d\underset{\sim}{A}$ is differential vector area

Consider ~~Now~~

$$dA_{R} = dA\cos\theta$$

$$\underset{\sim}{f}\cdot d\underset{\sim}{A} = \frac{Q\hat{r}\cdot d\underset{\sim}{A}}{r^2}$$

We are actually assuming a lot about 3-d Euclid geometry here but geomet is another course

$$= Q\,\frac{dA\cos\theta}{r^2}$$

$$= Q\,(\pm\underbrace{d\Omega}_{\text{differential solid angle}})$$

$\oint \underset{\sim}{f}_i\cdot d\underset{\sim}{A} = Q_i 4\pi$ (one charge

$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = 4\pi Q$

only 4π for $\oint\Omega$ because of cancelation See p. 3006

+ for ~~outward~~ $(\theta<\frac{\pi}{2})$ radial

− for antiradial $(\theta>\frac{\pi}{2})$

since $d\Omega$ is conventionally always +ve

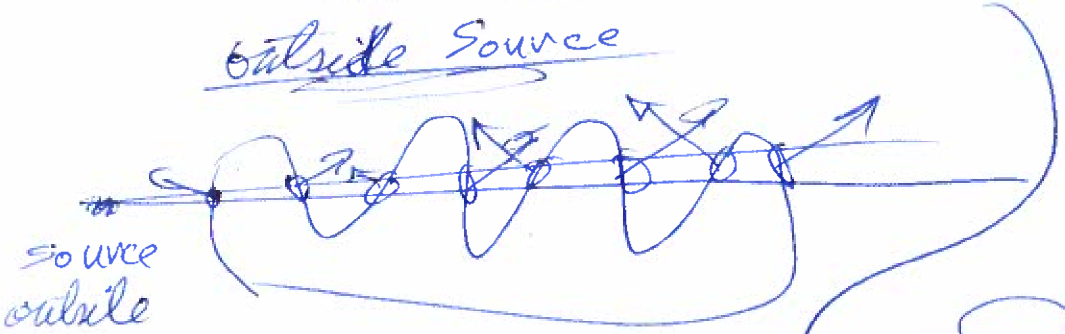
outside

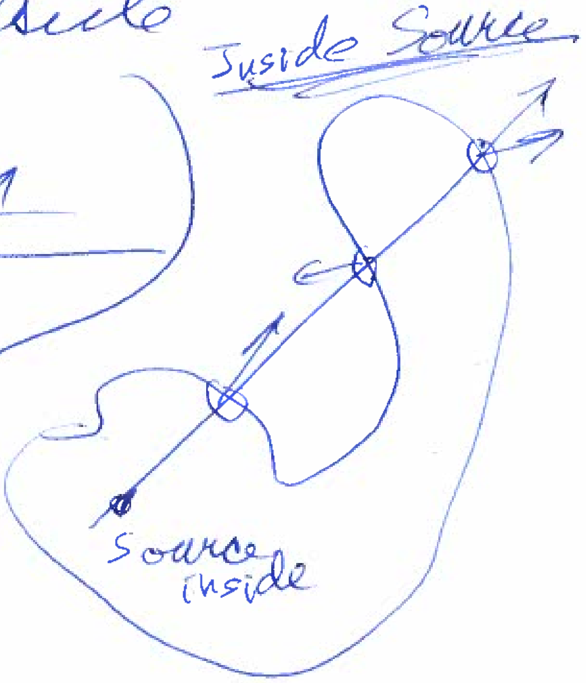
Consider now a closed surface, It has a definite inside and o

3006

Question
Is topology among ~~other~~ things, a way to get Yes/No answers?
Just musing

Any radial cone from an inside source or outside source too.
~~that goes thru the~~ surface must pass from inside to outside
or vice versa
~~and must ultimately pass to the outside~~

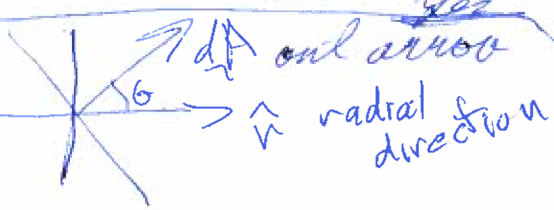




"in" always -ve ?
"out" always +ve

and so must ~~cancel~~? all differentiated solid angle bits cancel? yes

Always one more "out" than "in" and so one solid angle bit uncanceled? yes



field
$= 4\pi \sum_i Q_{i\,inside}$

$\theta \in [0, 90°]$ always an "out" and always positive
$\theta \in [90°, 180°)$ always an "in" and always negative

Gauss' law generic form $\oint \underset{\sim}{f} \cdot dA = 4\pi Q_{encl}$ Freq. 300 s 300

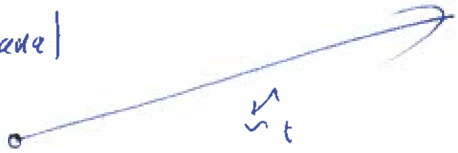
Gauss' law (wik)

~~integral~~ form

Gravity

$$\underset{\sim}{g}_i = -\frac{GM_i}{r_i^2}\hat{r}_i \qquad Q = -GM_{in}$$

gravitational field



M_i point source i

$\oint \underset{\sim}{g} \cdot dA$

No Einstein summation

$$= \sum_i \oint \underset{\sim}{g}_i \cdot dA = -G \sum_{\underset{inside}{i}} M_i 4\pi$$

$$= -4\pi G M_{inside}$$

only -ve

in total, can have -ve contributions

And mass is always -ve unlike electric charge

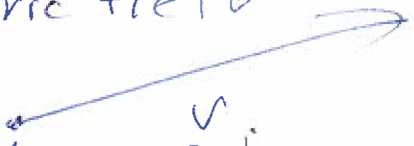
minus because gravity always attractive

Coulomb force

$$\underset{\sim}{E} = \frac{kq_i}{r_i^2}\hat{r}_i$$

k = coulomb constant $\doteq 8.99 \times 10^{\ldots}$ $\doteq 10^{10}$ ... (wik

Electric field



q_i

$\oint E \cdot dA$

No Einstein summation

$$= \sum_i \oint \underset{\sim}{E}_i \cdot dA = k \sum_i q_i 4\pi$$

$Q = kq$

$$= 4\pi k q_{inside}$$

$$= \frac{q_{inside}}{\varepsilon_0}$$

Vacuum permitivity

$\frac{1}{r^2} r^2 d\Omega$

Very special results due to special nature of inverse-square law forces

For high symmetry cases, One can use the integral Gauss' law to obtain solutions

1) spherical sym,
2) cylindrical sym
3) planar sym

and that's all I think

We'll look at this case in a moment

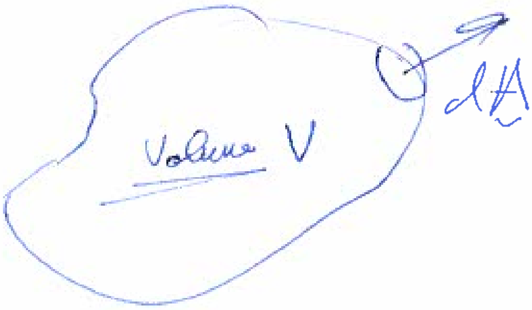
3008

Gauss' theorem (AkA divergence theorem)

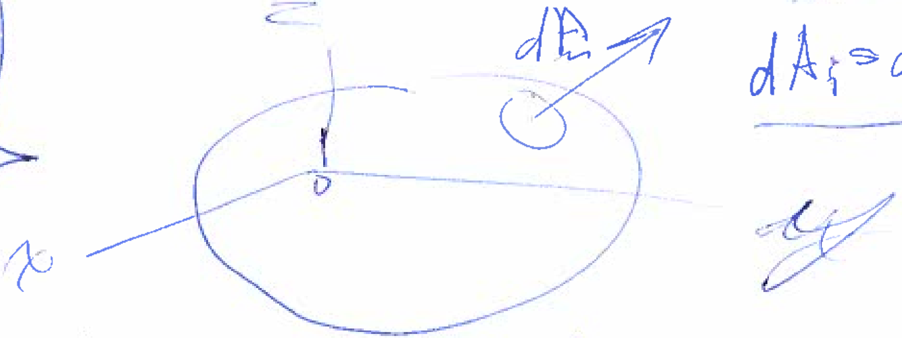
For differential form of Gauss

vector field

$$\int \nabla\cdot\underset{\sim}{F}\, dV = \oint \underset{\sim}{F}\cdot d\underset{\sim}{A}$$



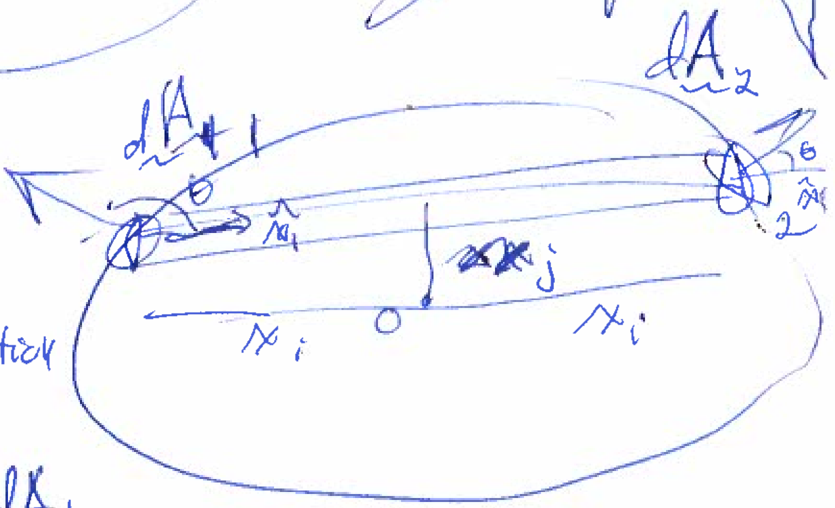
Proof Consider a sufficiently small volume to allow 1st order expansion to be accurate



$dA_i = dA\cdot\hat{x}_i$

unny shape

.ways
n-out
pair



$$\oint \underset{\sim}{F}\cdot d\underset{\sim}{A}$$

$$= \oint F_i\, dA_i$$ with Einstein summation on i

Taylor expand

$$= \oint \Big(\underbrace{F_{i0}}_{} + x_j \frac{\partial F_i}{\partial x_j}\Big|_0\Big)\, dA_i$$

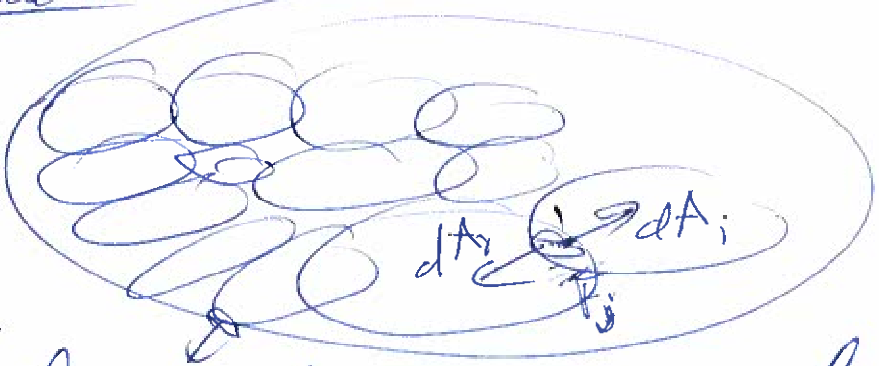
This term cancels pairwise – along and x_i line $dA_{i_1} = -dA_{i_2}$

3010

Finite Volume

Not a rigorous proof

Now just add but small volume.



the internal surfaces cancel pairwise

and only the outer surfaces contribute and you get Gauss' theorem QED.

outer cancel surface but cancel pairwise

$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{A} = \int \nabla \cdot \underset{\sim}{F}\, dV$$

Gauss' Law Differential form

$$\underbrace{\oint \underset{\sim}{f} \cdot dA}_{\text{total field per unit charge}} = 4\pi \underbrace{\sum_{i\ \text{inside}} Q_i}_{\text{sum of inside ~~internal~~ charges}}$$

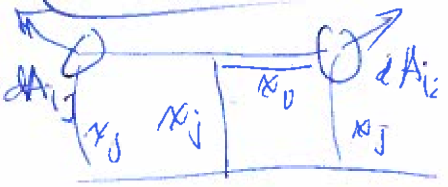
Sep. 30 07

Say these form a continuum

$$\sum_i Q_i = 4\pi \int \rho\, dV$$

ρ is a density

3009



$$= \oint x_j \left.\frac{\partial F_i}{\partial x_j}\right|_0 dA_i$$

~~$x_j dA_{i1} + x_j dA_{i2}$~~ but if $i \neq j$ $x_{j1} = x_{j2}$

~~$= \delta_{ij} dV$~~

dA_{i1}

dA_{i2}

x_{j1} x_j x_{i2}

pairwise cancellation

No Einstein sum

$A_{i1} x_{i1} + dA_{i2} x_{i2}$

$dA_{i2}(x_{i2} - x_{i1})$

dV

cancels pairwise also if $i \neq j$

$$x_i dA_j = \delta_{ij} dV$$

~~No sum over~~ i

$dA_{i1} x_{i1} + x_{i2} dA_{i2}$

$= |dA_{i2}|(x_{i2} - x_{i1})$

But $dA_{i1} x_j + dA_{2i} x_j$

$= x_j (dA_{i2} - |dA_{i1}|)$

No Einstein sum

a volume element ~~tube-like~~ $= 0$

$$= \oint \left.\frac{\partial F_i}{\partial x_i}\right|_0 d\cancel{V}$$

$$= \oint \nabla \cdot F \, dV$$

~~$\nabla \cdot F \, dV$~~

$$= \nabla \cdot F \; V$$

for a sufficiently small volume

d/ Einstein summation clouds the issue

$$\sum_{ij} \oint x_j \frac{\partial F_i}{\partial x_j} dA_j$$

$$= \sum_i \oint \frac{\partial F_i}{\partial x_i} dV_i$$

$$= \left(\sum_i \frac{\partial F_i}{\partial x_i}\right) V$$

$$= \nabla \cdot F \; U$$

$dA_{j1} x_{j1} + dA_{j2} x_{j2}$

~~if $i \neq j$ then $x_{j1} = x_{j2}$ and $x_j (dA_{i2} + dA_{i2})$~~

~~if $i = j$ then $dA_{i2}(x_{i2} - x_{i1}) = dV$~~

$$\therefore \oint \underset{\sim}{f} \cdot d\underset{\sim}{A} = 4\pi \int \rho \, dV$$

$$\oint \nabla \underset{\sim}{f} \, dV = 4\pi \int \rho \, dV$$

using divergence Thm p.3008 ~~[illegible]~~ Gauss Theo (p.3008)

but since the shape is general

$$\nabla \cdot \underset{\sim}{f} = 4\pi \rho$$

{ General Gauss' law differential form

Gravity ⟷ Coulomb's force

see p.3007

$$\nabla \cdot g = -4\pi G \rho_{mass}$$

$$\nabla \cdot E = 4\pi k \rho_{charge} = \frac{\rho_{charge}}{\varepsilon_0}$$

(Wiki (Gauss law

~~the Mass distribution~~ ~~[illegible]~~ ~~velocity~~

See p. 3012 for general statement c

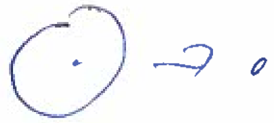
a & b are corollary

~~Shell Theorem~~

Shell Theorem: Gravity Case

a/b/c are 3 perspectives

a) A spherically symmetric body acts as if all mass were concentrated at a point; i.e., it acts like a point mass at the center of symmetry



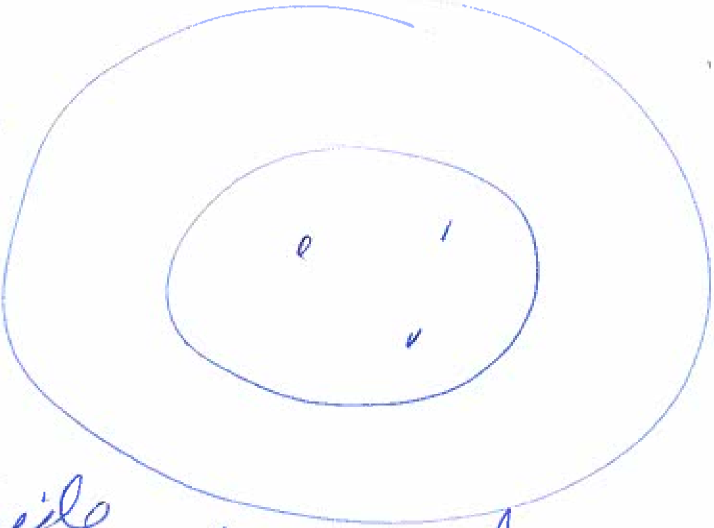
7012

) General statement Given a spherical symmetric distribution

at r only that interior if all very concentric at center

ss
stem
u
o
ving
z
ay
~~the~~
hune
m. is
intained

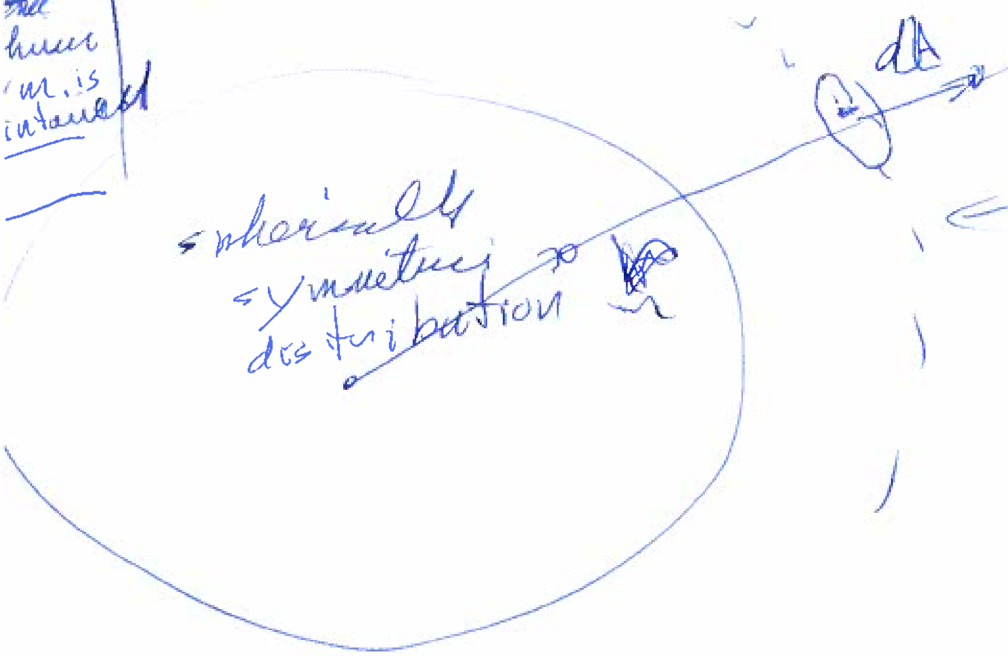
b) A spherically symmetric shell exerts no force on a body inside no matter where inside
((b) is really a corollary of (a).)



Proof from Gauss' law ~~for~~ gravity integral form

g & dA exact opposite

$$\oint \underset{\sim}{g} \cdot d\underset{\sim}{A} = - 4\pi G M_{enclosed}$$



could extend to here

Gaussian surface.

By symmetry $\underset{\sim}{g}$ must be radial and have equal magnitude at $r = |\underset{\sim}{r}|$

∴ ~~$\oint \underset{\sim}{g} \cdot d\underset{\sim}{A} = g(r)(4\pi r^2) = -4\pi G M_{encl}$~~

~~$|g(r)| =$~~

3013

That minus sign means $\underset{\sim}{g}$ and $d\underset{\sim}{A}$ point in opposite directions: $\underset{\sim}{g} \cdot d\underset{\sim}{A} < 0$

$$\therefore \; -g(r)\, 4\pi r^2 = -4\pi G M_{encl}$$

minus signs cancel and

$$\therefore \; \underset{\sim}{g}(\underset{\sim}{r}) = -\frac{4\pi G M_{intern}}{r^2}\hat{r}$$

so nothing which is ~~positive~~ $g(r)$, M both the they should be,

Which proves part (c) QED

Note: M could be moving - pulsing - anything

Note since if $M_{enclosed} = 0$, $\underset{\sim}{g}(\underset{\sim}{r}) = 0$ as in cavity

~~$\underset{\sim}{g}_{ext}(r' < r) = 0$~~

~~$\therefore \; \underset{\sim}{g}(r' < r) = 0$~~ if there is no mass for $r' < r$

~~Which proves part (c)~~

QED. Shell theorem. see p. 3012 (c), ab 3011

as long as they maintain spherical symmetry the shell thm k

Do two spherically symmetric masses interact like point masses if they do NOT touch?

Yes, but I think it's not quite obvious and needs a proof

- The analogous GR Theorem Birkhoff theory is the same
- Motion that doesn't break spherical symmetry changes nothing

See p. 3051

3014) Proof

$\underset{\sim}{F}_{12} = \underset{\sim}{F}_{1\,\underline{\text{point}}\;2} = -\underset{\sim}{F}_{2\;1\,\text{point}}$

by shell theorem — by 3rd law

$= -\underset{\sim}{F}_{2_{\underline{\text{point}}}\;1\,\text{point}}$

by shell thm again

$= \underset{\sim}{F}_{1\,\text{point}\;2\,\text{point}}$

By 3rd law again

<u>QED.</u>

Newton had to work really hard to prove this and the shell thm using his klutzy formalism.

<u>But</u> he needed these proofs in order to solve solar system motions tractibley (i.e., celestial mechanics)

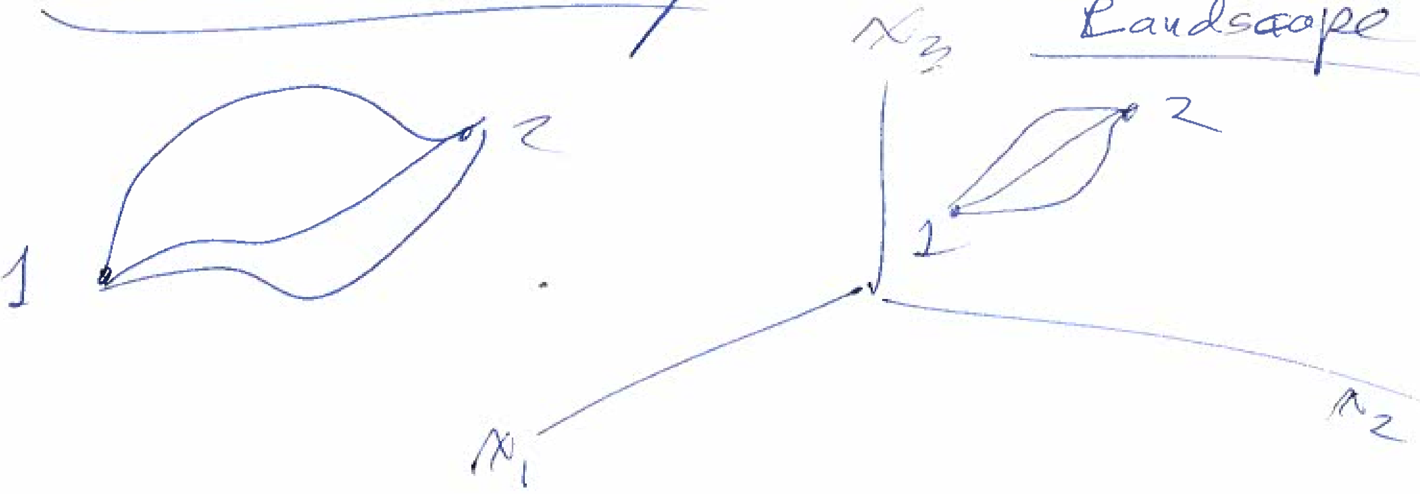
→ Exact solution for the 2-body system

→ his early perturbation theory for general solar system <u>motions</u>

Potential Theory

L3015

Landscape



Say $W_{12} = \int_1^2 \underset{\sim}{F} \cdot d\underset{\sim}{s}$ is path independent, for ~~the~~ space or some subspace

Then we are free to define a potential energy.

$$PE = U$$

by $U_{12} = -W_{12} = -\int_1^2 \underset{\sim}{F} \cdot d\underset{\sim}{s}$

$$\therefore \quad dU = -\underset{\sim}{F} \cdot d\underset{\sim}{s} = -F_i dx_i$$

$$\frac{\partial U}{\partial x_i} dx_i = -\underset{\sim}{F}_i dx_i$$

using Einstein summation

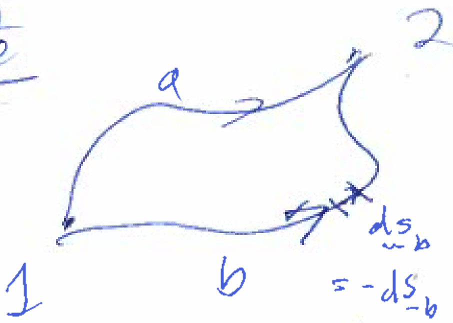
$$\therefore \quad F_i = -\frac{\partial U}{\partial x_i}$$

$$\underset{\sim}{F} = -\nabla U$$

actually I like these element forms best and think ~~[illegible]~~ we should use them all the time and teach them first to students

3016

Note



$= -ds_{-b}$

U_{12} path a $=$ U_{12} path b

by hypothesis of PE theory

$= -\int_1^2 \underset{\sim}{F}\cdot d\underset{\sim}{s}_b$

$= +\int_1^2 \underset{\sim}{F}\cdot(-d\underset{\sim}{s}_b)$

and add minus

$= \int_2^1 \underset{\sim}{F}\cdot d\underset{\sim}{s}_{-b}$

path -b

~~$= \int_2^1 \underset{\sim}{F}\cdot ds_{-b}$~~

$= -U_{21}$ path (-b)

$\therefore \Delta U_{\text{closed path}} = 0$

The reverse proof (converse) is easily proven too

So $U_{12} + U_{21} = 0$

For gravity with a spherically symmetric mass distribution (outside of it)

$$\underset{\sim}{g} = -\frac{GM}{r^2}\hat{r}$$

By Clairvoyance

$$V = -\frac{GM}{r}$$

with $V(\infty) = 0$ by usual convention.

~~$\nabla V = -\left(\frac{GM}{r^2}\right)$~~

$$-\nabla V = -\left(G\frac{M}{r^2}\right) = \underset{\sim}{g}$$

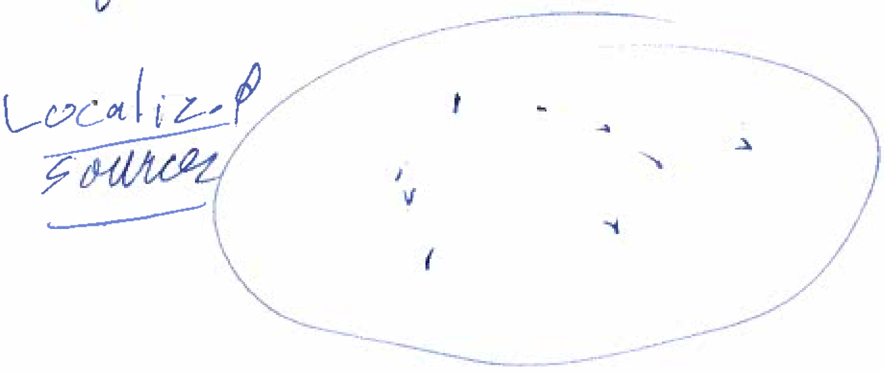
Potential not potential energy. do we call potential energy? potential by convention

$$\nabla\cdot V = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2V_r)}{\partial r} + r\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + r\frac{\partial V_\phi}{\partial\phi}\right]$$ (Arf-104)

Zero-Point PE

In PE theory alone, you never need to define a zero-point PE

However for a localized set of sources (i.e., ~~they~~ can be put in a finite closed surface)



$U(\infty) = 0$ is conventional

What is PE anyway?

What else is there that is interact

Gravity is a special case, it turns out

For abstract PE theory, PE is just itself.

However for real forces (Except gravity!!), I believe the answer is field energy (except gravity)

What else can it be? I wonder? → Gravity shows som e

Let's consider Electromagnetic first

Not contributions to E+B but total otherwise because u square

$E_1^2+E_2^2=E^2$

$E_1+E_2=E$

$$E_{density} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

energy density

see p. 3020

(Wik: energy energy density)

We can sort of prove this result. E Reduction proves it!!

~~too~~ ~~ene~~ Loca ener -sc muo at ea poi

Someone has derived this — with the ambiguity what about point sources?

- point charges ⇒ electron, quarks
- point magnetic dipoles → electron ele.

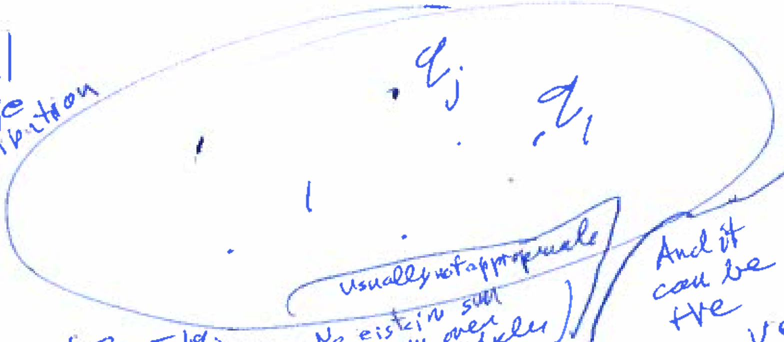
⇒ somebody in field theory has dealt with this. (maybe by trickoux renormalization whatever it really is)

Let's ~~consider~~ Derive the E^2 part; consider electrostatics with charge smeared out into a continuum ⇒ as we usually do for macroscopic treatments. (Wiki: Electrostatics)

$$U = \sum_{\substack{i \\ j<i}} \frac{k q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{ij} \frac{k q_i q_j}{r_{ij}}$$

to correct for double counting

Note the U is nowhere here Just an abstract accounting so far



Usually not appropriate

$U = \frac{1}{2}\sum_i q_i \sum_j \frac{k q_j}{r_{ij}}$ No einstein sum ~ sum over particles

$$U = \frac{1}{2}\sum_i q_i \phi_i$$

And it can be +ve or -ve like +ve ulike -ve

electric potential due to all other charges.

continuous limit gets rid of ities

$$U = \frac{1}{2}\int \rho(\underset{\sim}{r})\,\phi(\underset{\sim}{r})\,dV$$

Integral over all space for our localized distribution

works

But also -ve potential energy → we are counting all PE in assembly of the charge distribution

3019

Now recall p. 3011 and Gauss' law differential form.

$$\nabla\cdot\underset{\sim}{E} = \rho/\varepsilon_0$$

N~~ote the~~ ~~$\nabla\cdot\underset{\sim}{E}$~~ total not ~~just~~ a ~~[illegible]~~

and note

(with Einstein summation)

$$\frac{\partial}{\partial x_i}(E_i\phi) = \frac{\partial E_i}{\partial x_i}\phi + E_i\frac{\partial\phi}{\partial x_i}$$

$$= (\nabla\cdot\underset{\sim}{E})\phi + \underset{\sim}{E}\cdot\nabla\phi$$

QED

So $(\nabla\cdot\underset{\sim}{E})\phi = \nabla\cdot(\underset{\sim}{E}\phi) - \underset{\sim}{E}\cdot\nabla\phi$

~~[illegible]~~

$$\therefore U = \tfrac{1}{2}\int(\rho\phi)\,dV = \frac{\varepsilon_0}{2}\int(\nabla\cdot\underset{\sim}{E})\phi\,dV$$

$$= \frac{\varepsilon_0}{2}\int\left[\nabla\cdot(\underset{\sim}{E}\phi) - \underset{\sim}{E}\cdot\nabla\phi\right]dV$$

$$= \frac{\varepsilon_0}{2}\oint\underset{\sim}{E}\phi\cdot d\underset{\sim}{A} + \frac{\varepsilon_0}{2}\int\underset{\sim}{E}\cdot\underset{\sim}{E}\,dV = \frac{\varepsilon_0}{2}\int E$$

by Gauss' theorem p. 3008

this term must be +ve

using $\underset{\sim}{E} = -\nabla\phi$

From potential energy theo

Always positive due to our smearing out of charge — which we assume is a valid macroscopic limit of dealing with point charges — which all of classical E&M verifies / QED may do it by renormalization — we [illegible] mother [illegible]

We assume a localized set of charge and just set the enclosing surface to ∞.

$\underset{\sim}{E}\phi \Rightarrow 0$ at infinity $\Rightarrow$ a reasonable assumptio

3020 | So

'o distributions


U_2

$U_1 + U_2$
$= U_{total}$
..ce not
..antity the
E
..between
..teraction

$$U = 0 + \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 \, dV$$

$$U = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 \, dV$$

$\mathcal{E}_{density} = \frac{\varepsilon_0}{2} E^2$

Seems absolutely reasonable to define ~~[illegible]~~ it and it is right in classical E&M.

For E and B fields it is right

But can be do this for gravity.

$U_g = \int \quad dV$

$\mathcal{E}_g = - ...$

Just would integral sign

The answer is No.

E^2
total
a
..ov..
E_2
t + total

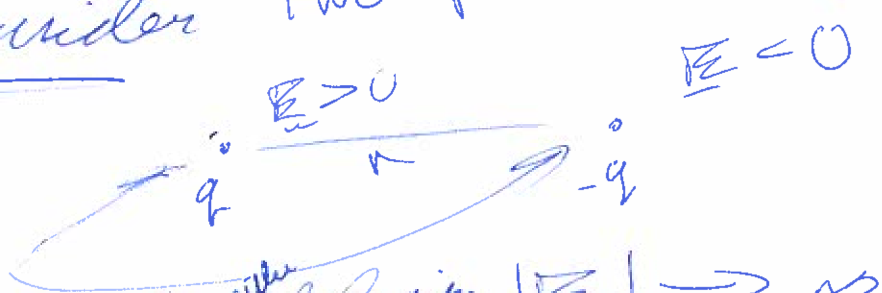
So this seems concrete PE is field energy in this case

or 3018 on ~~[illegible]~~ continuous of charge

Oddity this formula says $U \geq 0$ always.

but we do have $\Delta U < 0$ Delta U

Consider two point charges



close to either one $|E| \to \infty$

But ~~lets~~ just omit the ~~infinity~~ close ~~to~~ regions them together

Then as we bring

$$U(r = \infty) > U(r \text{ finite} > r_{close})$$

So $\Delta U < 0$ { So U total always +ve avoiding infinities
But ΔU's can be -ve.

How you define r_{close} to cut out the infinities might be tricky.

However, I think just smearing charge out to be continuous at the macro scale as we often do works.

In any case, we seldom calculate ΔU from fields ~~but from~~ integrated over all space, but by some ~~other~~ line integral

$$U_{ab} = -\int_a^b q\underset{\sim}{E} \cdot d\underset{\sim}{s} = -W_{done}$$

which is easier and avoids infinities

and also adding the energy of assembling the continuo charge which in E field prob we may not want but we do with capacitors $E = \frac{1}{2}CV^2 = \frac{1}{2}VQ$

~~What About Gravity?~~

~~Gravitational PE~~

~~Let's consider the classical limit first where Gravity is Newtonian and is an inverse-square law like the coulomb force~~

~~and where we define gravitational field $\underset{\sim}{g}(\underset{\sim}{r})$:~~

~~$\vec{F}_{\text{grav force on mass } m} = m\,\underset{\sim}{g}(\underset{\sim}{r})$~~

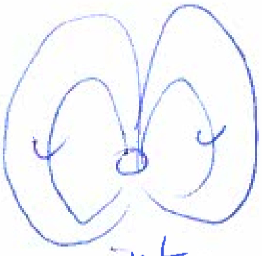
$U_B = \frac{1}{2\mu_0}\int B^2 dV$
see p. 3026

I don't know of similar ~~formula~~ derivation for B-fields, but probably exists.

Certainly potential energy for B-fields can be defined in some cases; e.g.,

a) $U = -\underset{\sim}{\mu} \cdot \underset{\sim}{B}$ ~~the~~ PE of a magnetic dipole.
(Wik: PE: Magnetic PE)

current loop
(Wik Magnetic dipole)



intrinsic like an electron

b) Omit
~~Ampère's force law~~

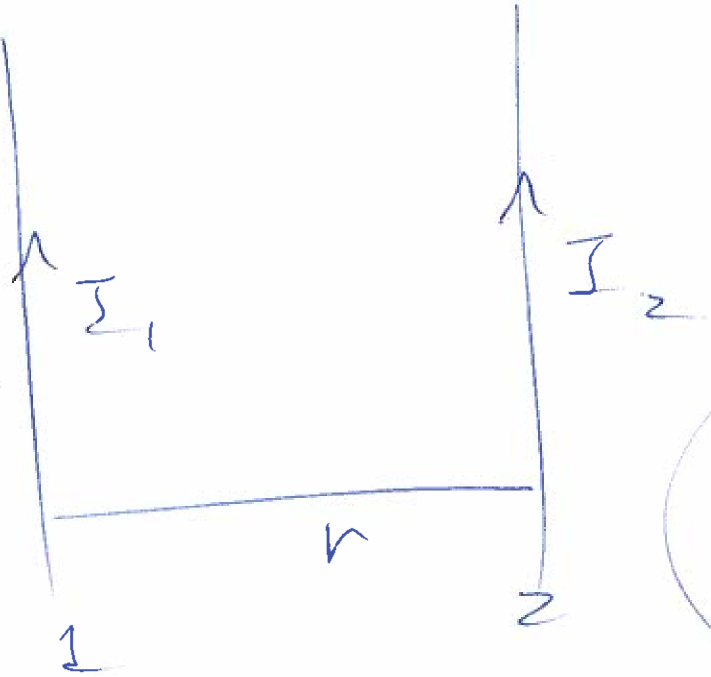
3023

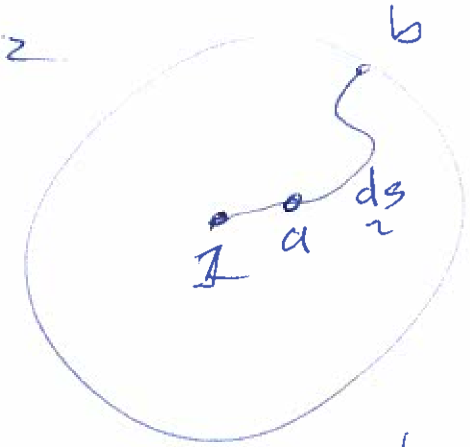
Omit (?)

Parallel wires

$$\frac{F_{12}}{L} = \frac{\mu_0 I_1 I_2}{2\pi r} * \hat{r}$$

attractive for $I_1 I_2 > 0$
repulsive if $I_1 I_2 < 0$





cross section

$$\Delta U = \int_a^b \frac{\mu_0 I_1 I_2}{r} \hat{r} \cdot d\underset{\sim}{s}$$

$$= \int_a^b \frac{\mu_0 I_1 I_2}{r} dr$$

$$= \mu_0 I_1 I_2 \ln(r_b/r_a)$$

~~Gravitational PE~~

$E = mc^2$

PE ~~must~~ obey this law
both E+M and probably nuclear fields

We'd say anyway

So having PE means having mass ⇒ both its inertial and gravitational effect

⇒ spread out $E = \frac{\varepsilon_0}{2} E^2 + \frac{B^2}{2\mu_0} = mc^2$

3024

— and if there is a field energy density, then there is a mass energy density too. ⇒ a mass energy density.

For $E = mc^2$ see Lawden p. 4 ~~B~~ note (i) for "derivation" → with physics micropostulates along the way.

Omit = done on p.3029 ⇒ {so redundant and/or off path

Gravitational Potential Energy

For gravity in the classical limit we do define a gravitational field $\underset{\sim}{g}$

and $\underset{\sim}{F}_{grav} = m \underset{\sim}{g}$ is the gravitational force on mass m.

We can now repeat the derivation for $U_E = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 \, dV$ (see p. 3018–3020) for localized charges for the gravitational case *mutatis mutandis*

$$E \rightarrow g \; , \quad \varepsilon_0 \rightarrow -\frac{1}{4\pi G} \quad \text{see p. 3011}$$

$$U_G = -\frac{1}{2} \frac{1}{4\pi G} \int_{\text{all space}} g^2 \, dV \qquad \rho_{grav} = -\frac{1}{2} \frac{g^2}{4\pi G}$$

t is formula in fact true in a sense derivation fine But can't call the energy of field mass $E = mc^2$ mass since energy density doesn't gravitate.

3025

Where does $E = mc^2$ come from?

Recall Special Relativity is derived from 2 axiom

a) Relativity principle
— all physical laws should have the same formulae in all inertial frames

b) Vacuum light speed is ~~the~~ maximum physical speed
→ ~~inad~~ is invariant for all inertial frames

exa
inert
fra
an
fre
fa
frame
[illegible]

Principle
of equi
~~of~~ [illegible]

The derivation is physicsy with all kinds of micro axioms along the way: e.g., it is natural

Einstein was guided (1905) by ~~the fact that the~~ classical limit low relative velocity limit must be Newtonian physics

In trying to maintain
conservation of mass
and conservation of energy
→ including PE + heat energy

he found it almost unavoidable or naturally

3026) to say $E = mc^2$

which means ~~that there~~ must be rest mass-energy

$$E_0 = m_0 c^2$$

$\Rightarrow$ ~~the~~ energy just for existing

~~Now as was well~~

→ at first this is first established for inertial mass for SR

but experimentally

$$m_{inertial} = m_{grav}$$

So it was "natural" to assume all Energy had a gravitational effect.

~~Of course when~~ Einstein ~~wrote~~ famously decided $m_{in} = m_{grav}$ should be a principle

$\Rightarrow$ principle of equivalence (POE) one of the axioms of general relativity (GR)

xioms
f GR } a) POE was one motivation for

Einstein to ~~[illegible]~~ pursue GR

b) Another was the Newtonian gravitational field responds instantly everywhere to motion of mass

⇒ He felt that had to be wrong since that violated the light-speed axiom

c) Actually another aspect must have turned up in his thinking. ⇒ A Paradox

The Electromagnetic field has energy ⇒ mass ⇒ gravitational mass

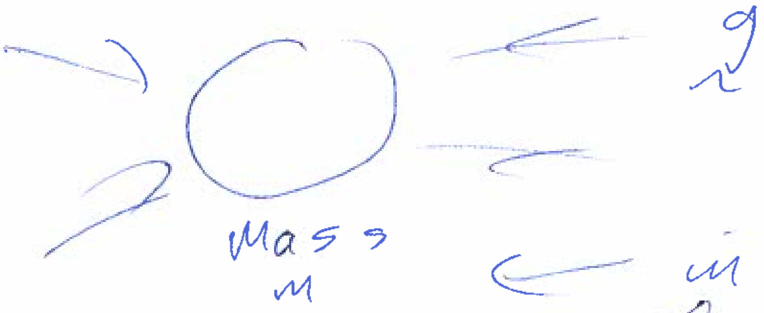
This OK — causes no problem.

Except for those point masses — Leave to Quantum field theory

But if the gravitational field has energy (i.e., gravitational potential energy) then the field itself ~~other~~ has mass-energy and its own

3028) gravitational field

→ But this leads to a tricky situation?



← in Newtonian gravity the mass creates the field.

— Maybe the field is the mass But it's negative see ~~p.30~~

Squirrels chasing their tails situation

From a SR-mixed-with Newtonian point of view (p.3029) there may be no consistent approach,

But GR does give the consistent approach but with mysteries still (Carroll-120) (Penrose 464-469)

Not gravitat[illegible] mass does [illegible]

R, gravitational field energy; if gravitating mass energy, curvature LHS of RHS of Einstein field eqn

But before going on to the GR fix let's still What gravity PE probably is in use?

Gravitational Potential Energy (Yes)
& Gravitational Field Energy (only soul exist)

In classical limit $\underset{\sim}{g}$ is the gravitational field and

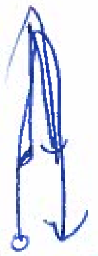
$\underset{\sim}{F} = m\underset{\sim}{g}$ { $\sim$ under means vector.

And of course, we do use Grav. PE cus for near E and

$$U = mgy$$

(derived from F=m

and energy is conserved in these classical limit cos



throw ball

$\Delta KE = W$
work-kinetic energy Th

But where is that energy?

$W = W_{non} + W_{co}$
$= W_{non} - \Delta U$
$\Delta KE + \Delta U = W_{non}$
work-energy Theorem

Well for E-field the PE is in the field as we derived p.3018-3020

$$U_E = \frac{\varepsilon_0}{2}\int_{\text{all space for localized energy}} E^2 dV \rightarrow \rho_E = \frac{\varepsilon_0}{2}E^2$$

$\hookrightarrow$ the field energy density

and this is true.

Similarily $\rho_B = \frac{1}{2\mu_B}B^2$

and I assume true for the Nuclear forces (strong & weak too)

mean E=... the f... has ine... and gra... effect Tru...

But can be do this for gravity

mutatis mutandis

$\varepsilon_0 \rightarrow -\frac{1}{4\pi G}$ (p. 3011) and $\underset{\sim}{E} \rightarrow \underset{\sim}{g}$

and some derivation $\underline{U_g = -\frac{1}{2}\frac{1}{4\pi G}\int_{\text{all space}} g^2 dV}$ [illegible]

smearing ~~out~~ mass into a conti distribu again implied.

$U_g = -\frac{1}{2}\frac{1}{4\pi G}\oint g^2 dV$ < Perfectly correct account of grav. PE

But can we localize $\rho_G = -\frac{1}{2}\frac{1}{4\pi G} g^2$ — True? No

as an energy density which then implies it has a negative inertial & negative ~~mass~~ grav. mass?

pockets of negative mass-energy — No!

(Not all contributions are negative => so no way to make sum positive)

The answer is no,

In ~~setting~~ setting up GR Einstein effectively had to dispense with the idea

Grav PE being anywhere exactly

…an not say so much here and so much there,
…i can say maybe this …spect of ~~energy~~ PE here,
…is aspect there — but
…at is a tricky job in GR
…maybe useless = unmeaningful

it exists, but nonlocal
→ no grav field energy density can be defined fundamentally but you may do it as an approximation somehow

in our classical way.

$\Delta U < 0$ but calculated same way we do

Too ~~E~~ elucidate

Let's consider a binary pulsar case → very strong gravity field close to

far enough to sort of localize

r

grav field far off

$g_2 = -G\frac{(m_1+m_2+\Delta U)}{r^2}$

~~then~~ true in ~~Newtonian~~ classical limit

…ne element …ric $= g_{\mu\nu}dx^\mu dx^\nu$ …c tensor ~~Carroll~~-71

…ass-energy-momentum tells space-time how to curve and that tell mass-energy - how to move under gravity not other forces

But close too you need GR in strong gravity and the only mass-energy in GR is m_1 and m_2

~~Einstein~~ Einstein Field Equation (to anticipate — ~~4~~ 4 x 4 Tensor ~~PE~~ → true at every point (above QM level)

~~gravity~~

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$ < mass-energy momentum

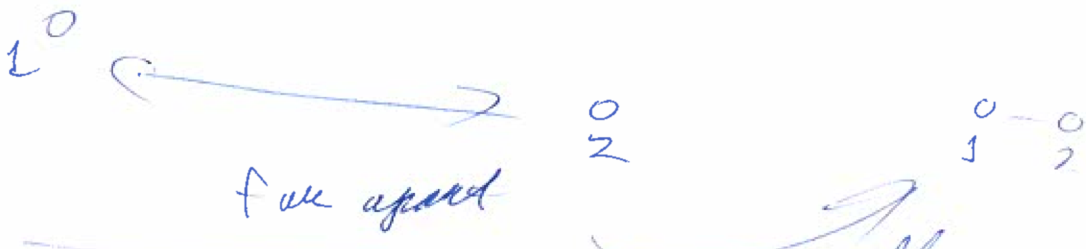
…ik Einstein tensor — geometry al — Cosmological constant — metric tensor — stress-energy tensor energy-momentum tensor Wik

Not going to really do GR. Talk around it

There is no grav. potential energy in it
the mass in the $T_{\mu\nu}$ tensor

Grav. PE
is not in GR
It can be
superimposed
as our
classical
way of
understanding
GR.

classical
limit
accounts
for
GR
effects
that
is
Va



close together
— but rigidly held rigid
and at rest.

All the
difference
is in the LHS
the geometry
of space-time

— Their contribution to
the Net $T_{\mu\nu}$ are
unchanged (Penrose 464
-465)

What happens as the pulsars
in spiral under GR effects
(an unperturbed Newtonian
2-body system is
perpetual, eternal,
no inspiral

Well $\underset{\sim\text{far field}}{g} = -\frac{G(m_1+m_2+\Delta U)}{r^2}$ gets smaller

— in ~~classical~~/SR terms mass energy
is lost
due to increase
$|\Delta U|$

equivalent
calculation
in a
sense
but
GR justifies
the first

— But the GR
calculation gives the
same-field g
but without any changes found mass

So there is ΔU, but in GR terms
it can't be said there is so much
here or there except when far enough

3030b } away in far field
that
you can
say it's
localized relative to you.

¿?

Of course, where does the lost
non-localized energy
got a
ΔM decrease
KM increase

We say carried off by Grav. waves

True but GWaves
travel across the universe
and make no contribution
to the local $T_{\mu\nu}$ as they
travel → unless they
deposit some by shaking
and not regaining
→ in true empty space
$T_{\mu\nu} = 0$
where the waves
travel. (Penrose – 466 caption
– 467 last paragraphs

Do the Gwaves
eventually deposit
exactly the energy they
carry off?
→ There is a GR proof that they do
in a special case Bondi-Sachs
mass/energy conservation
law
(Penrose 467–468)
But this not a general proof.

In fact, GR does not guarantee ordinary conservation of energy as we'll discuss in a bit

— See p. 3042

And, in fact, in cosmology it fails

But all is not lost,

as far as we can tell unless you say it is conserve in some ~~way~~ ~~to satisf~~ unspecified way to satisfy your physical intuition

Ex PE of spherically symmetric mass-distribution → localiz

So

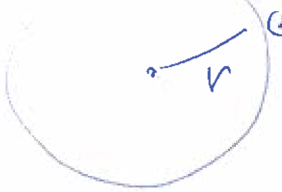
$$U \simeq \frac{1}{2} \frac{GM^2}{R}$$

characteristic — characteristic size

$\left(U = \frac{1}{2}\sum_{ij} \frac{Gm_i m_j}{r_{ij}} \right)$ similar to p. 3018 for electric charge

↑ one can drop the $\frac{1}{2}$ as insignificant

Exact calculation for a uniform sphere (a) by assembly, (b) from field energy



a)

$$dU = -\frac{Gm(r)}{r}\,dm = -\frac{Gm}{R}\frac{d}{x}$$

uniform dens

$$m(r) = \int_0^r \rho\, 4\pi r^2 dr$$

$$m(r) = M\left(\frac{r}{R}\right)^3 = Mx^3$$

$$dm = \frac{3M}{R^3} r^2 dr$$

$$= 3Mx^2 dx$$

$$U = -\int_0^1 \frac{3GM^2}{R} x^4 dx$$

$$= -\frac{3}{5}\frac{GM^2}{R}$$

– so the crude estimate won't be bad.

b) Now from the g-field

using P field density (p. 3029) $g = -\frac{GM}{r^2}$ outside ~~$g = \frac{GM\left(\frac{r}{R}\right)^3}{r^2}$ inside~~

$$U = \int_R^\infty \frac{1}{2}\frac{1}{4\pi G}\frac{G^2M^2}{r^4} 4\pi r^2 dr$$

$$= \frac{1}{2} GM^2 \int_R^\infty \frac{dr}{r^2} = \frac{1}{2} GM^2\left(-\frac{1}{r}\right)\Big|_R^\infty = \frac{1}{2}\frac{GM^2}{R}$$

3632

Inside from Shell theorem

$$g = -\frac{Gm(r)}{r^2}$$

$$= -\frac{GM\left(\frac{r}{R}\right)^3}{r^2}$$

$$= -\frac{GM}{R^2}x^2$$

use p. 3029 Pdensit

But does
it really mean
so much
PE is inside
and so much
outside.
I don't
think so.
ce again
le classical
estimate
somehow
generate
ll the
?
estimate
here
I
wasn't
sure
p.

$$U_{inside} = -\frac{1}{2}\frac{1}{4\pi G}\int_0^1 \frac{G^2M^2}{R^4}x^2\, 4\pi R^3 x^2 dx$$

$$= -\frac{1}{2}\frac{GM^2}{R}\int_0^1 x^4 dx$$

$$= -\frac{1}{2}\frac{GM^2}{R}\frac{1}{5}$$

This expression
can only be
true ~~near~~
classical limit
I think.
So $R >> R_{sch}$.
But how much
for field mass
loss occurs in
collapse to
Black hole!
~~wh~~
Can't
find
answer
NOW

$$U = U_{outside} + U_{inside}$$

$$= -\frac{GM^2}{R}\left(\frac{1}{2} + \frac{1}{10}\right) = -\frac{3}{5}\frac{GM^2}{R}$$

So we get the same answer.

What is the equivalent mass?

$$M_{equivalent} = \frac{U}{c^2} = -\frac{3}{5}\frac{GM^2}{Rc^2} = -\frac{3}{5}\cdot\frac{1}{2}\cdot\frac{R_{sch}}{R}M$$

$$= -\frac{3}{10}\frac{R_{sch}}{R}M$$

$M_{eq} \to \infty$ if $R \to 0$

Nah.

Recall Schwarzschild radius

$$0 = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

Classical derivation

$$R_{sch} = \frac{2GM}{c^2}$$

Is this right? If you collapse Sun to BH magically does it mass decrease by $-\frac{3}{10}M_\odot$? ~~seem so.~~

Nah

303

Not surprisingly, we find the mass equivalent M_{eq} only becomes comparable to the mass M if the sphere has radius of order ~~the~~ Schwarzschild radius. But does this formula mean anything? I don't think so as $R \to 0$, $M_{eq} \to -\infty$.

Line Element = Metric
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
Einstein summation
Carroll -71

$g_{\mu\nu}$ is metric tensor

Gravitational Field Energy in General relativity?

Let us Not let gravitat field grav

It's sort of a tricky subject.

Einstein Field Equations → Which gives ~~the~~ curvature of spacetime and dynamics ~~of energy~~ of mass-ener

(recapitulated of p.3030) Universal true → Like all physics DEs true at every point in space but have to solve for BCs → peculiar

4x4 Tensor of Differential equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Features

9) Einstein tensor
Wik GR: Einstein field equation
– embodies curvature effects

Wik: GR: Cosmology
Λ is the cosmological constant

Space-time-metric tensor

3034) It's local → i.e., it applies at each point in spacetime → above the quantum gravity level → wherever that takes over at small scale.

b) The left-hand side ~~curvature~~ of space-time geometry

It's a DE → (determine)

and the right-hand side is mass-energy and momentum ~~effec~~ effect.

$T_{\mu\nu}$ is the energy-momentum tensor.

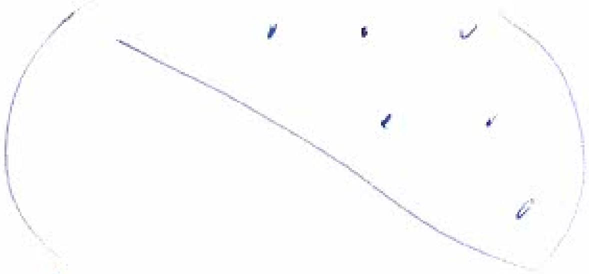
In brief $T_{\mu\nu}$ tells space-time how to ~~curvature~~ curve and then the curvature tells mass-energy how to move due to gravity

) The tensors correspond to 4 × 4 matrices but are symmetric

and

4 + 3 + 2 + 1

= 10



independent partial differential eqn. (Wik) Einstein field equations)

physical law → what holds point-by-point eternally.

4 ~~is~~ for 3 space and 1 - time dimension of spacetime.

d) Conservation of Mass-energy and Perfect fluids

Follows from the demand ~~that~~

$$\nabla^{\mu} T_{\mu\nu} = 0$$

with Einstein summation

Energy-Momentum Conservation Eq (Carroll-120) Carroll-156

in the form shown here

Covariant differentiation (Carroll-97) (Weinberg-46 -105-10 -15

Relativistic kind of differentiation

~~This~~ → This conservation law is a much glorified form of of mass & momentum conservations

in the NR case the continuity equation for mass is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\underset{\sim}{v}\rho) = 0$$

(Wik)

But $T_{\mu\nu}$ includes field~~s~~ energy too ~~so in~~ except the gravity field

~~So~~ it has forces included except the gravity force. (see Weinberg 45–46 & 360ff)

ik
ild
ict,
vitational
tential
ld
ix
ld)
ine
with the
IS of
eld
eqns.
Weinberg p.77–79
in that limit grav. field energy
eligned?
~~[illegible]~~
P
~~[illegible]~~
o
's ~~not~~
~~[illegible]~~h!
Not same.

For a perfect fluid in the SR limit

Weinberg -48

$$T^{\mu\nu} = p\eta^{\mu\nu} + (p+\rho)U^\mu U^\nu$$

Right?

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

the SR metric tensor
Weinberg 26

If it was there'd be negative mass see p. 3630

Perfect fluid is used in several ways

a) a general way is a fluid that is isotropic if you are moving with a fluid element.

b) In cosmology: No turbulence, no turbule

Weinberg - 47

~~In cosmology we use a more restricted definition~~

Wik characterized only by rest frame mass-energy density $\rho = \rho_m c^2$ and pressure $P = P(\rho$

$u^\mu u^\nu = g_{\mu\nu}$

⇒ EOS, no temp on $T(\rho)$ v

$$T^{\mu\nu} = \begin{bmatrix} \rho & & & 0 \\ & P & & \\ & & P & \\ 0 & & & P \end{bmatrix} = \mathrm{diag}(\rho, P, P, P)$$

~~(Wik)~~

Wik ÷ perfect fluid

Liddle - 123

This is for $u^\mu =$

"moving

fra

In cosmology one often (always) uses parameterization $P = w \rho c^2$ ⟨ Can make c^2 disappear $\rho c^2 \to \rho$

where w seems to have no special none EOS parameter?

$P = P(\rho)$ is general equation of state (EOS)

Temperature T heat energy just contribute to P for this perfect fluid.

w = 0 for "dust" = pressureless matter (Li - 40)

Essentially in cos. Stars, Galaxies, cosmic dust/gas → even if it sort of has pressure but not cosmically

3038)

$w = 1/3$ for radiation
⇒ extreme relativistic stuff

$w = -1$ for cosmological constant
equivalent Dark Energy

Li - 40
Li - 105, 57

$w =$ a function of a scalar field viewed as a perfect fluid (Wiki: EOS cosmological)

Quintessence (Wiki: Quintessence) is a particular theory of dark energy to cause acceleration of the universe

$w = -1/3$
or
= ct
universe
$\dot{\lambda} = 0$
Silvio Melia

and w can just be used as a free parameter in cosmology

ΛCDM has $w = -1$

wCDM has w as a ~~constant~~ free parameter

e) Conservation of Mass-Energy

energy-momentum conservation eqn (Carroll - 120, 156 in this form)

$$\nabla^{\mu} T_{\mu\nu} = 0$$

(Carroll - 97
Weinberg - 96, 105-107, 153, 156)

is what GR gives us for conservation of energy —— that's it.

There is no "integral of the motion corresponding to energy (Carrol-120) in general

Perhaps conservati
of energy
applies
only
when an
energy
density
can be
specif
bu
wha
of li
mo

Recall the gravitation field energy is excluded from $T_{\mu\nu}$

Where is it?

Well in a sense it Grav. field energy and the gravitational force is encoded in the Left-hand side of Einstein field equations

encoded then in curvature of spacetime

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

(Wiki: Einstein field equations)

In developing GR Einstein was guided by the idea that since

$$\nabla^\mu T_{\mu\nu} = 0$$

to give en-mo-conservation in a relativistic sense.

then $\nabla^\mu(G_{\mu\nu} + \Lambda g_{\mu\nu}) = 0$

I so he had to find the right tensor to determine the geometry of spacetime $G_{\mu\nu}$

the Einstein tensor (Viz Einstein tensor)

The $\Lambda g_{\mu\nu}$ was added on as an after thought for cosmology for reasons we'll discuss later

For damn good reasons, only terms that are linear in 2nd derivative of $g_{\mu\nu}$ or quadratic in 1st derivatives of $g_{\mu\nu}$ can appear in $G_{\mu\nu}$.

Penrose 464-467
esp. 467-469

Replaced by p. 3030-3030c

f) ~~Gravitational Field energy is non-local~~

i's
.OT
cal

~~As before, it is encoded in the Left hand side of the Einstein field equations, in the curvature of spacetime~~

~~So you can't write down a density for it: e.g., for electromagnetic field~~

In genera
But
can
in
Newton
limi

3029

$\rho = -\frac{1}{2}\frac{1}{4\pi G}g^2$ See p. 3017 $\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 B^2$

or $\nabla^{\mu}T_{\mu\nu} = 0$ itself.

~~There is a theorem~~

Is energy-momentum conserved by GR? → if not moment by moment at least in start to end of process.

It can't be proven generally (Penrose-467)

but it can in special cases.

e.g. asymptotically flat systems

– remote from the gravitating mass energy one has flat space-time

Then the Bondi-Sachs mass-energy conservation law can be proven from GR

e.g.

Binary Pulsar case

The energy-momentum carried by grav. waves equals the energy lost by some accounting of local Grav PE.

It gets damped somewhere

3042)

grav waves ~~make~~ No contribution to T as they travel thru space

~~Note the grav waves travel through ~~matter~~ empty space where $T_{\mu\nu} = 0$) or can in principle~~

Every term in $T_{\mu\nu}$ is ~~zero~~

But RHS ≠ 0 but Not term by term

al we still y dently the RHS = 0 nd $J_{\mu\nu}$ HS = 0 > gives geometry

9) Does GR conserve energy generally?

- as said above it can't be proven generally
- some folks hope this will be restored somehow.

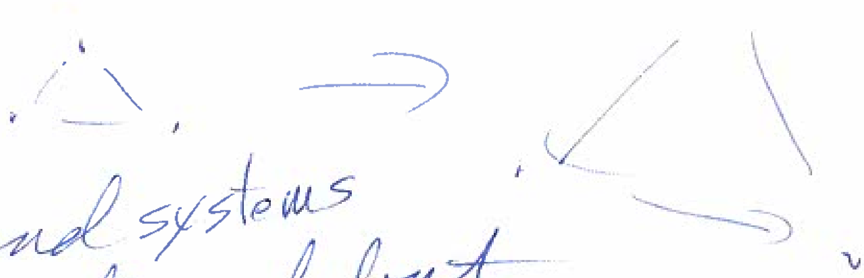
But others think we may have to live without it (Carroll - 120).

A good reason for this is cosmological models don't conserve energy.

- to explicate P_{matter} or dust which only moves w expansion of universe.

$P_{matter} \propto \frac{1}{a^3}$ where a is the cosmological scale factor

Space expands as $a(t)$ 30 43



— Bound systems do ~~not~~ expand but the space between them grows

Now $\rho_{matter} \propto \frac{1}{a^3}$ can be argued to conserve mass-energy (Li-4)

all ER particles photons neutrinos

But $\rho_{radiation} \propto \frac{1}{a^4}$ (Li-42

where does the "radiation" energy go?

→ I used to say it goes into the expansion of space, but ~~and~~ what does that mean?

— It is just gone from the description.

~~It was no gravita~~ It's gone as gravitating mass

And then there is the cosmological constant to account for the dark energy. to account for the acceleration of the universe

$\rho_\Lambda = \rho_{DE} = \text{Constant}$ in the simplest theory

emit → absorb photon

does photon loss due to redshift just become DE in some desired relation sense?

3044

But this means as space expands it grows.

Maybe there is some ~~any~~ way to save single conservation of energy

but maybe not.

There's a theorem → Noether's thm which shows energy should conserved under time invariance

→ but an expanding universe doesn't have time (Carroll-120) invariance ~~and so~~ ~~Noether~~ in any obvious way.

← Carroll-120 146

So upshot GR gives us $\nabla^{\mu} T_{\mu\nu} = 0$ as our energy-momentum conservation law

rdinary
Energy
onservation
ly in classical
n SR
imit

and ~~if~~ ~~insofar~~ insofar as GR is true we may have to live with that.

Of course, we expect GR or whatever is the true macroscopic theory of gravity to emerge as the macroscopic limit of quantum gravity whatever that is.

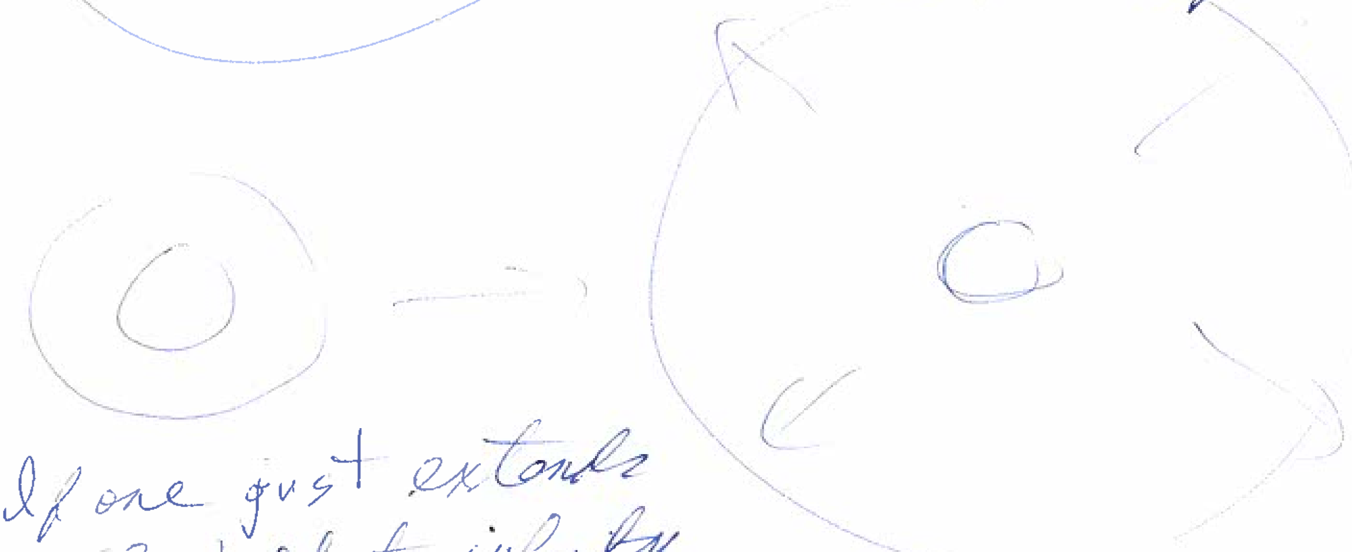
Shell Theorem ~~& Birkhoff Theorem~~

→ recall p. 3011-3014

One aspect is that a spherical shell has no grav. effects on the cavity.

But what if the shell is infinite?

→ A consideration for Newtonian cosmology.

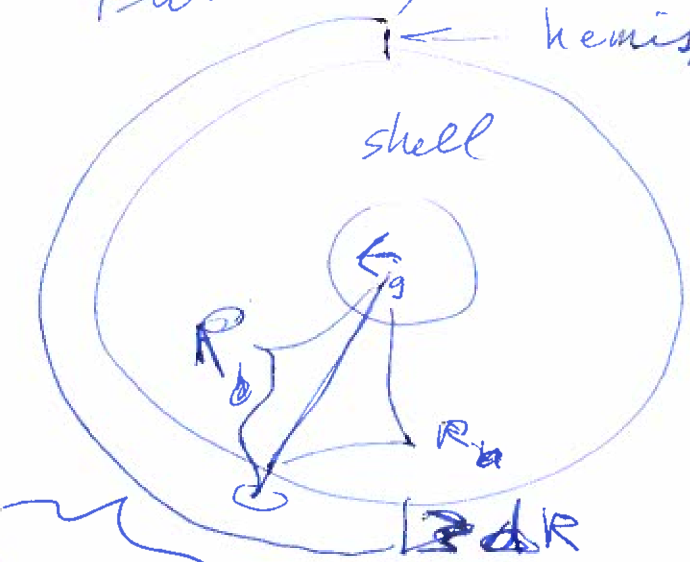


If one just extends the shell to infinity, it seems like shell theorem should still hold since it does

3046

for any step along the way.

But say we had



← hemisphere then

– the hemisphere will create a gravity field in the cavity.

– Now what if one increases shell and and the ~~[illegible]~~ hemisphere

$M_h = 4\pi R^2 \sigma$

(holding σ fixed ~~holding ΔR fixed~~.

~~– each layer of the shell grows as R_h^2 in area which cancels the $1/R_h^2$ decreases of Newtonian gravity. $\oint_{\text{hemisph}} = \int_R^{R_H} GM$~~ $2\pi r^2 dr$

So the

$$g_h^{(\text{at center})} \simeq f \frac{GM_h}{R^2} = f \frac{G(4\pi R^2 \sigma)}{R^2}$$

$$= f\, G (4\pi \sigma)$$

$$= \text{constant}.$$

geometric factor $\sim \frac{1}{2}$

In this case ~~the load~~ is always a net force in the cavity.

and that would be the limit as $R_{*} \to \infty$.

So, the limit of extending a ~~mass~~ shell around the spherical cavity to infinity depends on the ~~mass shell~~ shell's mass distribution even if an ~~infin~~ going to infinity part of it is spherically symmetric,

Upshot there is no Newtonian physics solution to an infinite universe full of infinite mass <u>without</u> extra

<u>ad hoc</u> hypotheses — ad hoc relative to Newtonian physics <u>alone</u>.

⟷ Which is what Milne & McCrea did in 1934 (p. 3002 & Bondi-7

Relative to GR these hypotheses are <u>Valid</u>.

However the most natural ad hoc hyphosis is that the Shell Theorem

extension to infinity is lost.

However Newton himself in unpublished work gave up on finding a static infinite uniform density on average universe (No - 376)

It doesn't sound like he tried very hard, but one must always remember his math tools were ~~primitive~~ primitive

He was also aware of Olbers paradox — A static eternal ~~full univers should~~ universe full of stars should have a sky star surface brightness.

Thomas Digges & Kepler even earlier (WiK)

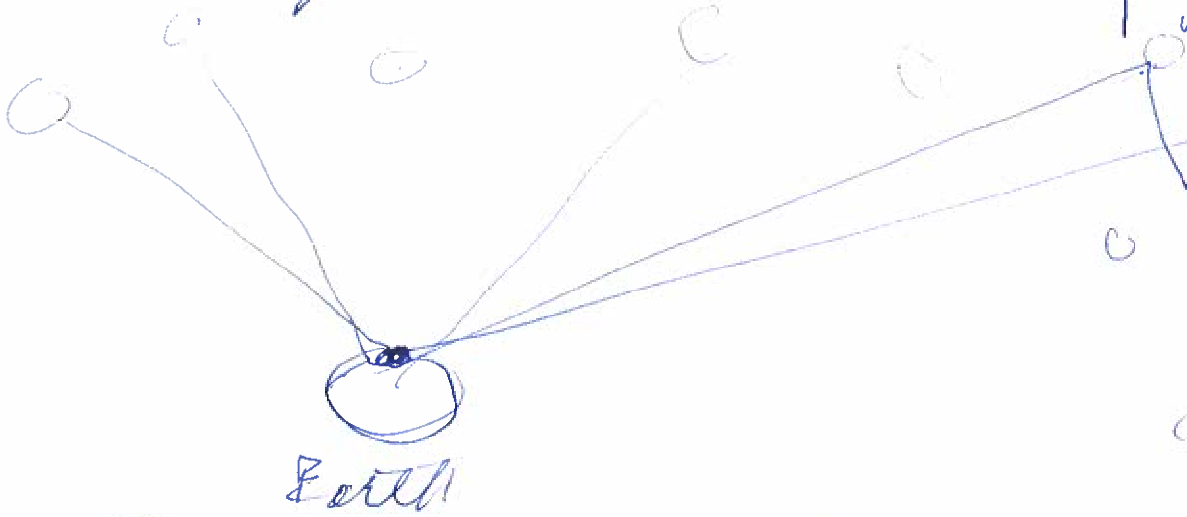
3047

~~Ed~~ Edmund Halley in primitive terms discussed it before the Royal Society — in 1721 with 79 year old Newton presiding as president (No-377)

But he attributed it to anonymous David Gregory No-377

Note pre-thermodynami one it doesn seem it worried anyone much till 19 when Edgar Poe 1848 but Lord Kelvin 1901 first reall found resolu anticipat by Poe

From a casual perspective, it is easy to understand



- In every direction, you observe a pencil beam of light from a star surface ⟹ all the photons coming your direction — so you are fried.

In modern radiative transfer term

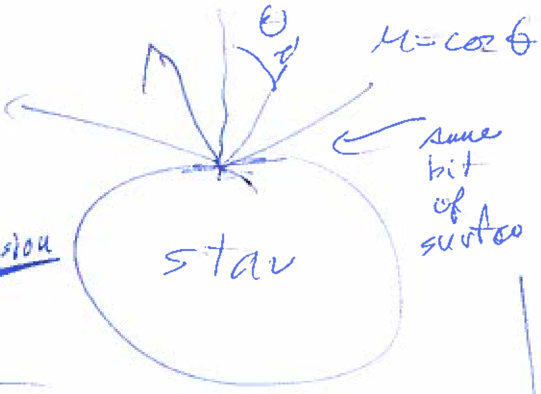
$$I = \frac{E}{\text{time} * \text{Area} * \text{freq} * \text{Solid Angle}}$$

No spreading out in this picture — geometrical opt

3050

~~stor~~

assume I is isotropic on emission

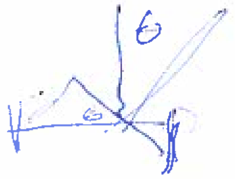


Flux from a surface of a star $= \int_{2\pi} I \mu \, d\Omega$

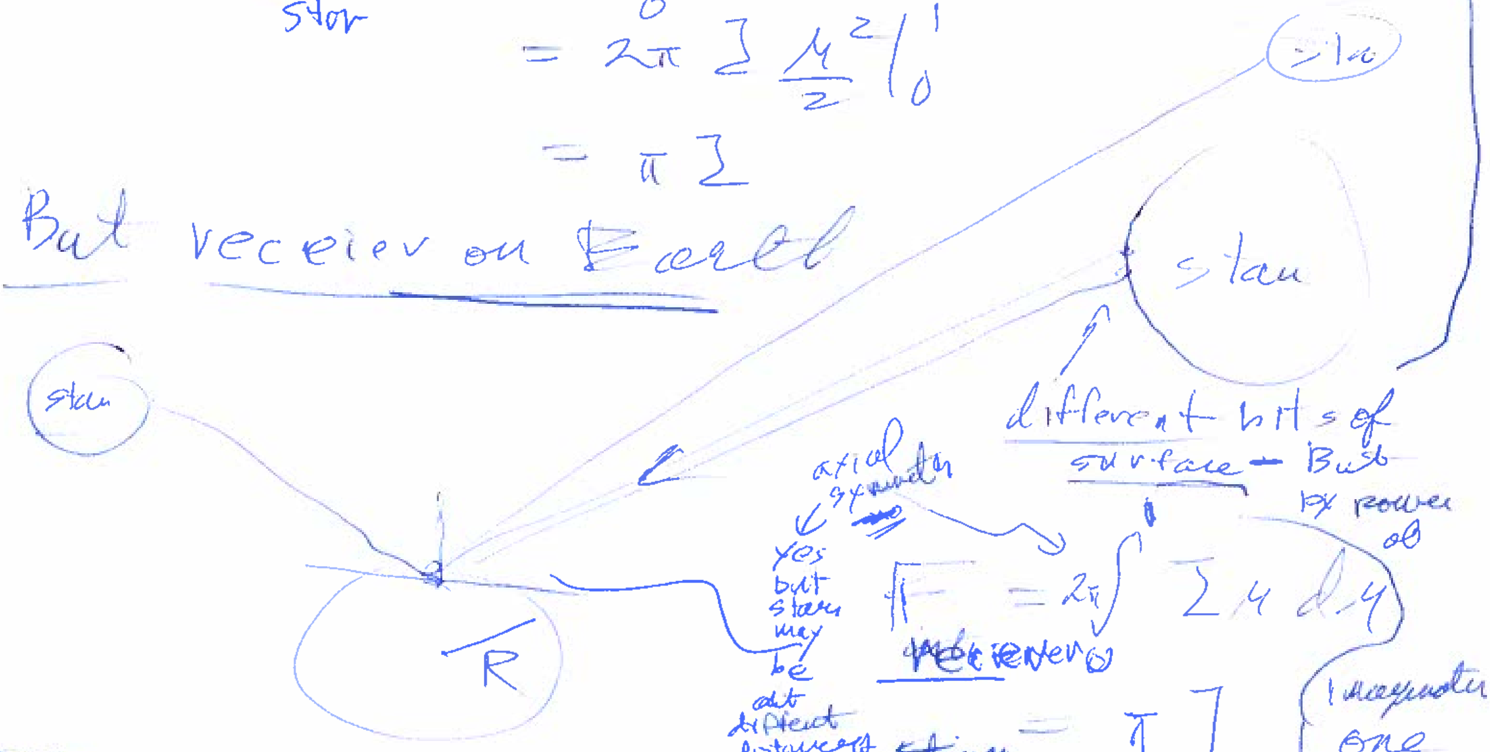
$$= 2\pi \int_0^1 I \mu \, d\mu$$

$$= 2\pi I \left.\frac{\mu^2}{2}\right|_0^1$$

$$= \pi I$$



But receiver on Earth



different bits of surface — But fix power of

axial symmetry

yes but stars may be at different distances and orientations

$$F = 2\pi \int_0^1 I \mu \, d\mu$$

wherever

$$= \pi I$$

irregardless one sees, it makes no difference

Total input $L = 4\pi R^2 F_{int}$

$$= 4\pi R^2 \frac{L_{star}}{4\pi R_{star}^2}$$

$$= \left(\frac{R}{R_{star}}\right)^2 L_{star}$$

Birkhoff's Theorem

3051

So Newtonian Physics is ambiguous for an infinite universe

→ homogeneous & static or not.

GR rescues us with Birkhoff's Thm which among other things says

given → spherically symmetric surrounding

— static or not

— finite or infinite (

A key point for applying to an expanding universe

leaves the cavity)

flat Minkowski space

Weinberg -338

-474

$R_c << R$ Gaussian curvature radius CL-12-13 which in the ΛCDM model is infinite — but we don't know if that model can be applied to infini

So frames that don't rotate ~~or accelerate~~ relative to the surroundings and are in free fall are inertial frames (without inertial forces) Weinberg-474

~~free~~ fall ~~frames~~

With inertial forces you can make any local frame an inertial frame

So as long R_c is sufficiently small and the ~~motion~~ contents are in the classical limit and we can use Newtonian Physics, {which ~~does~~ span velocity in inertial frame → c limit → between mo

so they do not perturb the surround

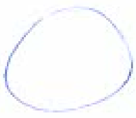
3052

and remarkably we can derive the Friedmann equations → ~~But really~~ Not ~~even~~ then if you have radiation and $\Lambda \neq 0$ → then yet another special hypothesis is needed arguably natural

— Is this just a lucky accident?

— Wells (2014) shows that it should be so → but it takes a lot of work — so we won't do that.

By the way Birkhoff's theorem is analogous to the shell theorem in another respect



exterior core

outside of a spherical symmetric mass-energy distribution of any kind static or moving one gets the Schwarzschild metric

Weinberg - 337

— so the mass could be replaced by a point mass or black hole.

If you are far enough away, you are in the classical limit and Newtonian ~~gravity~~ & dynamic physics applies,

Why does it not matter exterior ~~other~~ case for cavity and the ~~other~~ case that the spherically symmetric mass-energy can be moving.

→ I can't give a full answer.

↳ But no gravitational waves to infinity can escape ~~a~~ from inside a pulsing spherically symmetric mass-distribution and so they can't affect the spacetime geometry of the cavity or exterior case.

Omit Discuss p. 3063ff

— in pure Newtonian physics ~~velocity~~ ~~between~~ in. only one true mythical absolute space inertial frame

– in GR ~~only~~ true inertial frames are free fall frames

→ So it emerges that Newtonian physics does spare velocities in ~~inertial~~ a local inertial frame and between them

In an local inertial ~~frame~~ v < c between them the v > c can do happen

3064

Linear Force

$$\underset{\sim}{F} = q r \hat{r}$$

q is the "point charge" +ve repulsive -ve and you have the SHO force

Hooke's law force

This sort of turns up in cosmology because you can derive in the Newtonian derivation the cosmological constant effect. But it may just be an ad hoc fudge. I don't know, but it's worth a bit of look at

Radial harmonic oscillator RHO

Weird aspect

r↑ |F|↑ which means shouldn't let r→∞ but well anyway

then storm

to ... one a E. ... but even ... down GR.

i) First curious point (Wik)

Bertrand's ~~theorem~~ (19th century) shows that only two central force laws give all bound orbits as closed.

these force laws allow one closed orbit

a) $\underset{\sim}{F} = -\frac{k}{r^2}\hat{r}$ the well known inverse-square law like gravity & coulomb's law

b) $\underset{\sim}{F} = -k r \hat{r}$

$-k = q < 0$

$q<0$ -k case

the RHO force.

Of ~~for~~ course any central force law gives bound circular orbits

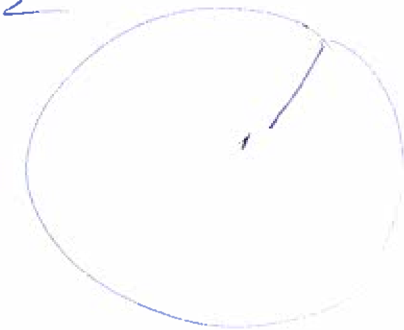
3055

Really
ma for
F=ma
specially
for
uniform
circular
motion

$F_{\text{centripetal}} - \frac{mv^2}{r}$

$\therefore \quad F_{\text{central}}(r) = -\frac{mv^2}{r}$

anything.



But (a) & (b) cases
(wik Bertrand's
Thm: Radii
Hamm
osc
RH

- elliptical
orbits
with ~~force~~
source at
one focus

- also elliptical
orbits
but the
force center
at
the geometrical
center

I recall that Newton
when investigating
central forces laws in
Book I of the principia (1687)
showed this — in his
old-fashioned klutzy
formalism

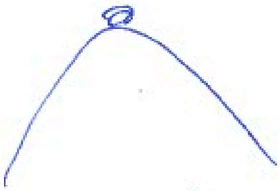
And showed
that other
central force
laws ~~that~~
he looked at
didn't

But it was beyond
his techniques to prove there were only
(a) & (b)

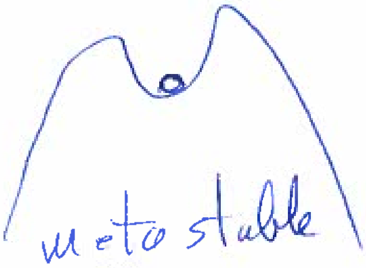
3056

Both kinds of orbit
have neutral stability

stable



unstable



metastable

?
really all
stability
is metastable
since with
enough perturbation
anything disrupts

Neutral
stable

the particle will
be moved by perturbation displacement
but won't go off to infinity

In case of (a) & (b) orbits, a perturbation
changes them but only "proportional"
to the perturbation.

There was a paper I looked at
once that discussed the
deep meaning of odd symmetry between
the inverse-square force &
the RHO force, but
I forgot to make a note of it

NASA/ADS
draws a blank
too

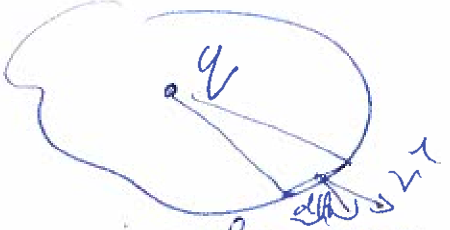
ii) Shell Theorem Analog for the linear force | 3057

$$\underset{\sim}{F} = q\,r\,\hat{r}$$

{ Assuming ideal ~~that it~~ extends to infinity and doesn't turn off }

"charge" could be +ve or -ve

Consider closed surfaces and a point charge



inside case

outside case

Divergence theorem again (Wiki: Divergence Theorem)

$$\underbrace{\oint \underset{\sim}{F}\cdot d\underset{\sim}{A}}_{\text{over whole surface}} = \underbrace{\int \nabla\cdot\underset{\sim}{F}\,dV}_{\text{over whole enclosed volume}}$$

$$\nabla\cdot\underset{\sim}{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right)$$

for spherical polar coordinates (Wiki: Divergence)

$$\nabla\cdot\underset{\sim}{F} = \nabla\cdot\left(q\,r\,\hat{r}\right)$$

$$= \frac{q}{r^2}\frac{\partial r^3}{\partial r} = 3q$$

It's constant everywhere from $r=0$ to $r=\infty$,

$$\therefore \oint \underset{\sim}{F}\cdot d\underset{\sim}{A} = 3qV$$

q anywhere in space inside or outside

Volume enclosed by the surface

In general $\oint \underset{\sim}{F}\cdot d\underset{\sim}{A} = 3VQ$ ← sum of all point charges everywhere in universe

3058

"Gauss law" analog for linear force.

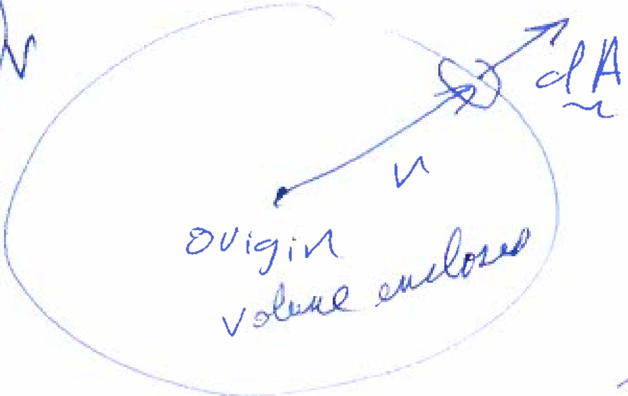
$$\oint \underset{\sim}{F}\cdot d\underset{\sim}{A} = 3QV$$

surface encloses volume V

~~Total~~ Total charge anywhere
- sum of all point charge
- inside and outside.

for the Shell Thm analogy
assume Q is distributed with spherical symmetry about the origin.

) is Not enclosed in V
) it all ~~throughout~~ ~~[illegible]~~



$\oint \underset{\sim}{F}\cdot d\underset{\sim}{A}$ for a spherical Gaussian surface

$$= \oint F(\pm\hat{r})\cdot d\underset{\sim}{A} = \pm 4\pi r^2 F$$

magnitude / upper case outward, lower case inward

So $\pm 4\pi r^2 F = 3VQ = 3Q\,\frac{4\pi}{3}r^3 = Q\,4\pi r^3$

+ for $Q > 0$
− for $Q < 0$ } must be for equation.

$\therefore\ \underset{\sim}{F} = Q r \hat{r}$ shell Theorem analog.

$Q>0$, point out
$Q<0$ points in

The net force is just as if the spherical distribution were compacted to a point, But what if the charge distribution were spread evenly thru infinite space

$$Q \Rightarrow \infty \ ?$$

We're flummoxed

But there's the old renormalization trick. If a quantity goes to ∞, just make it a free parameter and set it to what you want.

Justification = fudge

or your theory is an emergent theory - a correct limit of a more general theory you don't know, but if you did you could take the limit.

What is the cosmological constant force?

→ If you write it in the classical limit as a force

is $F_{\Lambda} = \frac{\Lambda}{3} m r \hat{r}$ where

m is mass of test particle

Λ is Einstein's cosmological constant

$Q \rightarrow$ our Q

$V = -\frac{1}{2} \frac{\Lambda}{3} m r^2$

But perhaps this linear force perspective is NOT a fundamental one and it is unfruitful for understanding at a deeper level.[2] a) For one thing, how does Birkhoff's ~~theorem~~ theorem apply to such a linear force? Not clear to me. So another POV will be used. → We Just assume we can use the linear force. Maybe renormalization trick or on p. 308

b) For another thing − $F_\wedge$ a m like gravity − so maybe it really is gravity in a way we'll see!

Work-Energy Theorem

& Conservation of Mech. Energy

− ~~Classical result~~.

− but we can use it since Birkhoff's theorem permits. & apparently GR in the weak field low velocity limit.

Start with where one always starts in classical mechanics

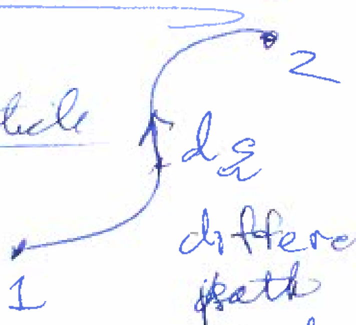
$$\underset{\sim}{F}_{net} = m\underset{\sim}{a} \quad \text{for a particle}$$

$$dW = \underset{\sim}{F}_{net} \cdot d\underset{\sim}{s} = m\underset{\sim}{a} \cdot d\underset{\sim}{s}$$

$$= m \frac{d\underset{\sim}{v}}{dt} \cdot \underset{\sim}{v}\,dt$$

$$= \frac{1}{2} m \frac{dv^2}{dt}\,dt$$

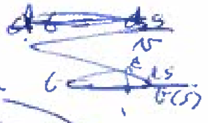
$$dW = \frac{1}{2} m\,dv^2$$



differential path vector

$W = \Delta KE$ Work-Kinetic energy theorem.

Sort of time indefinite secure



s(t) for init condit v_1

$W = W_{con} + W_{non}$

nonconservative force work

Can calcula $W = \int \underset{\sim}{F} \cdot d\underset{\sim}{s}$ and know ΔKE but need to solve F=ma to get s(t) and Δt_{12}

A conservative force

$\underset{\sim}{F} = -\nabla \underset{\sim}{U}$

a ~~potential~~ energy — an energy of position in some sense as discussed p. 304 ff

$\int \underset{\sim}{F} \cdot ds$

$\underset{\sim}{F} \cdot d\underset{\sim}{s} = -\nabla \underset{\sim}{U} \cdot ds = -\Delta U$ along path

$W_{con} = -\Delta U$

Can be much ~~[illegible]~~

$W = -\Delta U + W_{non}$

L mechanical energy

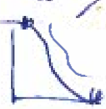
$\therefore \Delta E = \Delta KE + \Delta U = W_{non}$

Work-Energy theorem

If $W_{non} = 0$, $\Delta KE + \Delta U = 0$, $E_{mech} = KE + U$

or $\Delta KE = -\Delta U$

— Recall Intro physics Mechanics energy conservation given speed ~~[illegible]~~ bodies but not s(t)



→ Need to solve for this from F = ma –

3062a

3.1 Friedmann Equation

We've had a long warm up with various bits of physics and considerations

Jump to p. 3062b

Consider an in~~finite~~ Unbounded, homogeneous isotropic Universe

We only know of curvature of 3-d space from GR — but ~~that~~ means the space can be finite — like the 3-d surface of a sphere in 4-d Euclidean space

→ Called a 3-sphere (Wik: n-sphere or hypersphere)

inite
ut
oundless
ike
urface
phere

An ordinary sphere is a 2-sphere.

We consider a ~~3 sphere~~ sm small enough region ~~that~~ 3-d space is flat or Euclidean.

Inertial Frames

are frames with respect to which all physical laws are defined — except in our modern view, GR which tells us what inertial frames are.

— You could quibble about this → e.g., what of the 2nd law of thermodynamics → but after quibbling...

GR tells us they are free-fall frames ⟷ under mass-dependent forces
— gravity
— cosmological constant force → important on cosmic scale
— inertial forces in non-inertial frames

But all ~~inert~~ such de Not rota with respect observ

let's just concentrate on this case not to go down the rabbit hole of generality

Harder in classical limit absolute rotation

- No Absolute space
- But absolute rotation

I invented the nonce term COMFFI frames
Center-of-mass-free fall inertial frames Just in NR lim
(which planet people know more about than me

classical limit

System of particles i
— could be anything
~~of~~ a grav. bound eg. planetary system
— a pressure supported system (eg. planet, star)

$$① \quad \sum_j F_{ji} + F_{ei} + m_i g_i = m_i a_i$$

force of particle j on particle i

grav and all other internal force

all external forces except gravity

Just external gravity

mass of particle i

acceleration relative to some extra ... defined inertial frame ↳ we will make it disappear

$\boxed{3064}$

sum over i

② $$\underbrace{\sum_{ij} F_{ij}}_{} + \underbrace{\sum_i F_{ei}}_{F_e} + \underbrace{\sum_i m_i g_i}_{M\left(\frac{\sum m_i g_i}{M}\right)} = \underbrace{\sum_i m_i a_i}_{M a_{cm}}$$

0 by pairwise cancellation of 3rd law (recall in classical limit)

$$M\left(\frac{\sum m_i g_i}{M}\right) \equiv g_{ave}$$

M total mass

$$F_e + M g_{ave} = M a_c$$

if $F_e = 0$, $a_{cm} = g_{ave}$

Now $\frac{①}{m_i} - \frac{②}{M}$

$$\sum_j \frac{F_{ji}}{m_i} + \frac{F_{ei}}{m_i} - \frac{F_e}{M}$$

$$+ \underbrace{g_i - g_{ave}}_{} = a_i - a_{cm} = \underbrace{a_i'}_{\text{relative to CM}}$$

$$\underbrace{g_{ti}}_{\text{tidal force}} = g_i - g_{ave} = \underbrace{(g_i - g)}_{\text{gravitational field at CM}} - \underbrace{(g_{ave} - g)}_{= \frac{\sum_i m_i (g_{ave} - g)}{M}}$$

$$g_{ti} \sim -\frac{GM}{r_{cm}^2}\left(\frac{1}{(1+\frac{\Delta r_i}{r_{cm}})^2} - 1\right)$$

$$\simeq -\frac{GM}{r_{cm}^2}\left(2\frac{\Delta r_i}{r_{cm}}\right) \doteq -\frac{GM}{r_{cm}^3}\Delta r_i$$

$$\sim \frac{1}{r_{cm}^3}$$

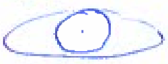
Note worst case for ~~grav~~ tidal force

$$\int_{r_1} \frac{GR}{r_{cm}^3} 4\pi r_{cm}^2 \, dr_{cm} \doteq \ln\left(\frac{r_2}{r_1}\right)$$

logarithmic divergence

it actually averages to zero RT

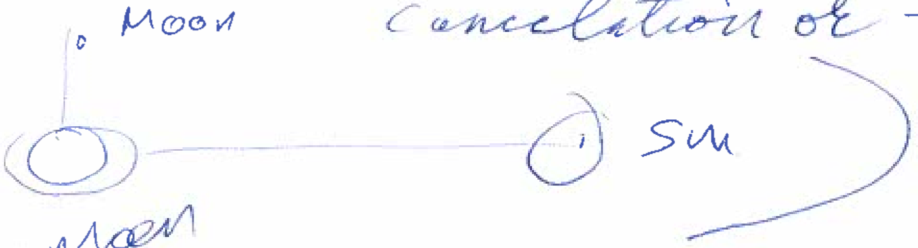
This can be used to calculate tidal bulges and the like (make a homework question someday)



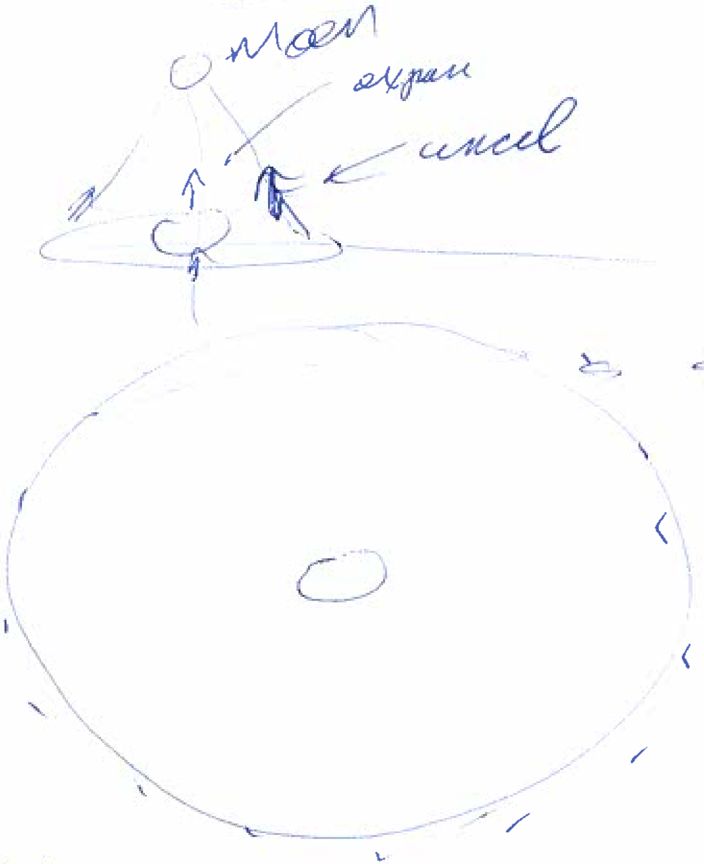
often very small for spherically symmetric bodies and can be set to zero to 1st order for stars planets

But does tidal force average to zero? | 3065

springtide, Not clear



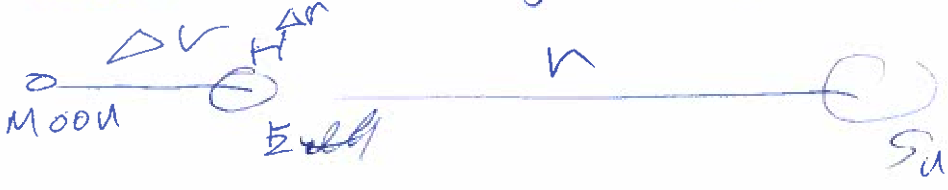
cancelation or → average
expan



= shell thm / Birkhoff thm

handwave
to cancelation
of gravity /
ergo longitudinal
increased tidal for

When is tidal force important



$$g_{t\,\mathrm{axial}} \sim \frac{M_\odot \Delta r}{r^3} = \frac{2\times10^{30}\,\mathrm{kg}}{(1.5\times10^{N}\,\mathrm{m})^3}\Delta r_{\oplus}$$

≈ scaled tidal force

$\sim .5 \times 10^{-3}$ or $\sim .5\times1$

$\Delta r_E \approx$

3066

for $\Delta r_E = 6370$ km $\simeq 10^7$ m

g_{ti} gives tides ~ 1 m in open ocean

counting Sun & Moon
Sun alone ~ $\frac{1}{3}$ m

Could calculate this to order of
(Make a problem some day)

$\Delta r \simeq 60 \times \Delta r$ ~~Earth~~ orbit

— bigger → probably a significant perturbation, but there are others (oblateness of Earth & Moon, other planets

$\ell_\odot$
$C \simeq 1.5 \times 10^5$

What of ~~Earth~~ star in Milky ~~way~~ and cosmology

$\Delta r \int_{r_1}^{r_2} \frac{\rho}{r^3} 4\pi r^2 dr$ { upper limit — no cancellation

$C = \Delta r \; \rho \; 4\pi \ln(r_2/r_1)$

~~$3 \log x$ ... $\ln 30$~~

~~$\Delta r\, C \; 30 \log(r_2/r_1)$~~

~~Δr_{Earth} $2.4 \times 10^3 \cdot \rho \cdot 30$~~

~~Δr_{Earth} ... $0^{-16} \log(r_2/r_1)$~~

9×10^{-20} $\frac{kg}{m^3}$

Milky Way

→ in solar Milky (with stellar dominated and Dark Matter included)

By which location already kills it

Even for $r_1 = 1$ lk, $r_2 = 10^9$ ly

[Cluster st

By which ~~[illegible]~~

$C \simeq 1.5 \times$ ... 10^{-8}

$$C = \Delta V \;\; \rho \cdot \underbrace{4\pi \cdot 2.3}_{\sim 30} \log(V_2/V_1)$$

[306

$\ln x \simeq 2.3 \log$

$30 \cdot 4 \times 10^{-20}$
$= 10^{-18}$

$\rho_{crit} = 10^{-26}\ \mathrm{kg/m^3}$

$\underbrace{\rho_{MW} = 4 \times 10^{-21}\ \mathrm{kg/m^3}}_{\text{star density}} \times \underbrace{10}_{\text{Dark matter fallow}} = 4\times1$

$C^{\text{upper limit}} = \Delta V \cdot 10^{-18} \cdot \log(V_2/V_1)$

say $\frac{1\ \mathrm{Gly}}{1\ \mathrm{ly}} \sim 10^2$

$= \Delta V \cdot 10^{-17}$ < so even with No cancellation due to spherical averaging (shell / Birkhoff theorem)

Recall Sun tidal force at Earth

$C \sim .5 \times 10^{-3} \Delta V$

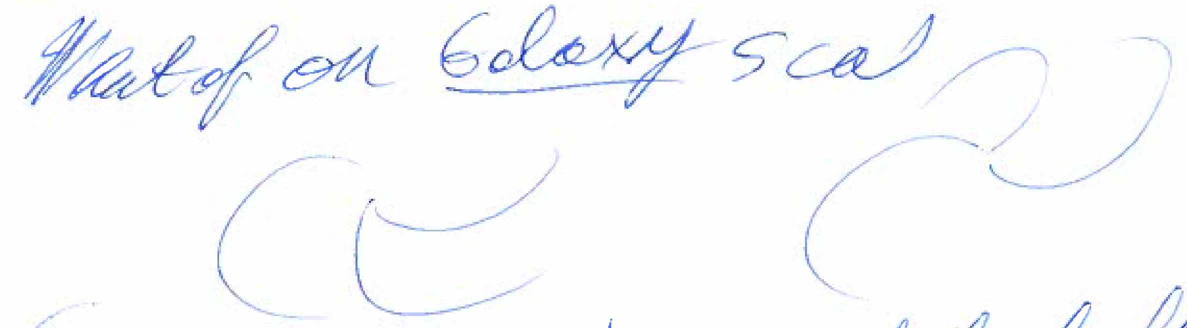
So tidal force of rest of universe on whole solar << $\underbrace{\Delta V}_{\Delta V \sim 100 \cdot 1\,\mathrm{AU}} \cdot 10^{-17}$

$\sim 100 \cdot 1.5 \times 10^{11}$
$\sim 1.5 \times 10^{13}$ Tu

is minute even in vast overestimate

166b1

What of on Galaxy scale



Actually we know tidal forces are significant in these cases since interacting galaxies show tidal tails.

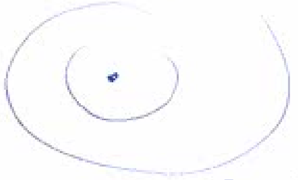
(e.g., Mice Galaxies)

So tidal force of sun & planets in planetary system important but not that of rest of universe.

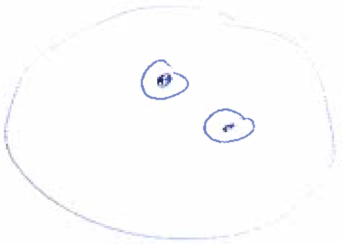
Same true of all isolated planetary system

4) Planetary System isolated

COMF] frame each



single star



Binary

free fall in average $\underset{\sim}{g}$ of universe

but no tidal force from outside significant

So and no other F_{e_i} significant → pressure — magnetic field

So equation

$$\sum_j \frac{\underset{\sim}{F}_{ij}}{m_i} = a_i'$$

No need to consider $\underset{\sim}{a}_i$ at all.

gravity + internal pressure support

Earth in free fall — so can inertial frame defined by CM → but we have ~~[illegible]~~ rotation on surface

3063)

This was known to celestial mechanics people from Newton on

on they wouldn't have got the right answer

But up until GR in 1915 people thought of Newton's absolute space

↳ One unique inertial frame → those not accelerated with respect to it.

For COMFFI frames → they thought of as noninertial frame

↳ they said $g_{cm} = g_{non}$

minus acceleration $\cancel{=}$ $-a_{cm}$, $0 = -a_{cm} + g_{non}$ as an inertial force to cancel gravity field.

ovies
~~names~~
CMs of
ld galaxies
d galaxy
usters
pproximately

Both perspectives ~~At~~ give some classical answer

But GR perspective is the true one (not just a classical limit since there is no absolute space but absolute rotation of observable universe)

ee
fall
frames
unrotating
with respect
to observable
universe with
o inertial forces
nvoked are the basic inertial frames of the basic → and the basic are the comoving frames participating in expansion of universe

b) Specialize to 2-objects in ~~that~~ case (a)

$$\frac{\underset{\sim}{F}_{12}}{m_2} = \underset{\sim}{a}_2' , \quad \frac{\underset{\sim}{F}_{21}}{m_1} = \underset{\sim}{a}_1'$$

Relate $\underset{\sim}{\Delta a}$

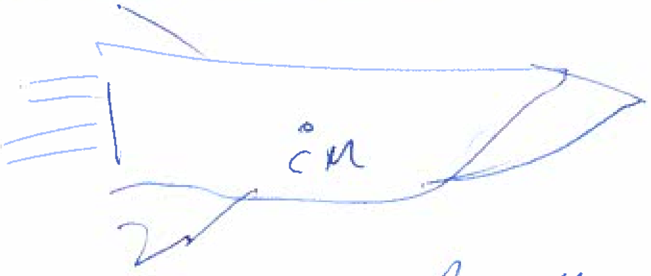
$$\underset{\sim}{a}_2' - \underset{\sim}{a}_1' = \frac{\underset{\sim}{F}_{12}}{m_2} - \frac{\underset{\sim}{F}_{21}}{m_1}$$

$$= \underset{\sim}{F}_{12}\left(\frac{1}{m_2} + \frac{1}{m_1}\right)$$

$$= \underset{\sim}{F}_{12} \frac{1}{\underbrace{m_{reduced}}_{\text{old friend}}}$$

c) Rocket in space

Here we want



$$\underset{\sim}{F}_e + m\underbrace{\underset{\sim}{g}_{ave}}_{\text{field so uniform that just } \underset{\sim}{g}_{ave} = \underset{\sim}{g}_{cm} = \underset{\sim}{g}} = m\underset{\sim}{a}_{cm}$$ from p. 306A – rigid object

For a rocket, the the ~~explos~~ ejected burning fuel is ~~the~~ provides the external force on the rocket.

070)

Actually in this case the mass is actually being lost and you have to use the more general $\frac{dP}{dt} = F_{ext}^{+m}$ since rocket mass changes.

Set to zero

In 1-dimension

$$\frac{dP}{dt} \quad \frac{dm}{dt} v + m\frac{dv}{dt} = v_{ex}\frac{dm}{dt}$$

$$F_{\cancel{ex}} = v_{ex}\frac{dm}{dt}$$

~~By conservation of momentum~~

~~$F_{ex} = v_{exhaust}$~~ $\frac{dm}{dt}$

~~rocket~~ parameter ~~from rest~~

In motion

~~$F = (v_{ex} - v)\frac{dm}{dt}$~~

~~$(v_{ex} - v)$ is relative velocity ejected fuel in~~ inertial reference frame

~~force~~ $-F_e$ must be exert on fuel to make it ~~more at~~ $-v_{ex}$ relative to ~~rocket~~,

So F_e exerted on Rocket and force is reference frame invariant.

$$\therefore \quad m\frac{dv}{dt} = (v_{ex} - v)\frac{dm}{dt}$$

$$\frac{dv}{v_{ex} - v} = \frac{dm}{m}$$

$$-\ln(v_{ex} - v)\Big|_{v_0=0}^{v} = \ln(m)\Big|_{m_0}^{m}$$

$$\ln\left(\frac{v_{ex} - v}{v_{ex}}\right) = \ln\left(\frac{m_0}{m}\right)$$

$$v = v_{ex} -$$

Let's set $\underset{\sim}{mg} = 0$, so

zero gravity or we measuring with respect to a local free-fall inertial frame.

$F_{ext} = ?$

Now momentum flow is ~~time~~ frame-dependent but not force

Thus ~~this~~ is also force of rocket on ejecta

momentum flow from ejected fuel

$$= -v_{exhaust}\left|\frac{dm}{dt}\right|$$

$$\frac{dm}{dt} < 0$$

note $\underset{\sim}{v}$ relative to rocket

$$\therefore \quad F_{ex} = -\left(-v_{ex}\left|\frac{dm}{dt}\right|\right)$$

$$= v_{ex}\left|\frac{dm}{dt}\right|$$

$$= -v_{ex}\frac{dm}{dt} \text{ since } \frac{dm}{dt} < 0$$

The mass m is of the rocket which decreases.

so

$$\frac{d\underset{\sim}{p}}{dt} = \frac{dm}{dt}v + m\frac{dv}{dt} = -v_{ex}\frac{dm}{dt}$$

$$m\frac{dv}{dt} = -(v + v_{ex})\frac{dm}{dt}$$

$$\frac{dv}{v + v_{ex}} = -\frac{dm}{m}$$

$$\ln\left(\frac{v + v_{ex}}{v_0 + v_{ex}}\right) = \ln\left(m_0/m\right)$$

$$\therefore \quad v = -v_{ex} + (v_0 + v_{ex})\frac{m_0}{m}$$

$$v(m = m_0) = v_0 \quad \text{and} \quad v(m \to 0) = \infty$$

Note this is a classical calculation and so $v(m \to 0) \to \infty$ is OK,

3072)

3.1 Friedmann Equation

We will derive semi-classically the way a 19th century physicist could have but none did

With some clairvoya
↓
which they could have had
↓
after thought to generaliz

The semi-classical derivation could've been done in the 19th century but actually was done in 1934 by Milne & McCrea (1934) (Wik), Bondi - 75

After the GR derivation of Friedmann 1922 (Wik) and, independently, LeMaître 1920s (Wik)

Wells (2014) gives a more rigorous demonstration why the GR and Newtonian derivations are equivalent (heavy going).

a) First we assume all Free fall frames (unrotating with respect to whole universe) are true inertial frames without need of inertial forces (19th century could have done this but didn't)

b) Assume cosmological principle — whole universe homogeneous and isotropic → can be scaling up or down under gravity

on large enough scale