Classical Mechanics Problems

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Introduction

Classical Mechanics Problems (CMP) is a source book for instructors of advanced classical mechanics at the Goldstein level. The book is available in electronic form to instructors by request to the author. It is free courseware and can be freely used and distributed, but not used for commercial purposes.

The problems are grouped by topics in chapters: see Contents below. The chapter ordering follows the Goldstein chapter/topic ordering. For each chapter there are two classes of problems: in order of appearance in a chapter they are: (1) multiple-choice problems and (2) full-answer problems. Almost all the problems have complete suggested answers. The answers may be the greatest benefit of CMP. The problems and answers can be posted on the web in pdf format.

The problems have been suggested mainly by Goldstein problems, but have all been written by me. Given that the ideas for problems are the common coin of the realm, I prefer to call them redactions. Instructors, however, might well wish to find solutions to particular problems from well known texts. Therefore, I give the suggesting source (when there is one or when I recall what it was) by a reference code on the extra keyword line: e.g., (Go3-29.1) stands for Goldstein (3rd Edition), p. 29, problem 1. Caveat: my redaction and the suggesting source problem will not in general correspond perfectly or even closely in some cases. The references for the source texts and other references follow the contents. A general citation is usually, e.g., Ar-400 for Arfken, p. 400.

At the end of the book are two appendices. The first is an equation sheet suitable to give to students as a test aid and a review sheet. The second is a set of answer tables for multiple choice questions.

Classical Mechanics Problems is a book in progress. There are gaps in the coverage and the ordering of the problems by chapters is not yet final. User instructors can, of course, add and modify as they list.

Everything is written in plain TeX in my own idiosyncratic style. The questions are all have codes and keywords for easy selection electronically or by hand. A fortran program for selecting the problems and outputting them in quiz, assignment, and test formats is also available. Note the quiz, etc. creation procedure is a bit clonky, but it works. User instructors could easily construct their own programs for problem selection.

I would like to thank the Physics Department of New Mexico Tech for its support for this work. Thanks also to the students who helped flight-test the problems.

Contents

Appendices

References

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Chapt. 1 Elementary Problems and the Lagrangian Formulation

Multiple-Choice Problems

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001 qmult 00100 1 1 1 easy memory: holonomic constraint defined
Extra keywords: (Go3-12)

1. Constraints that can be expressed as equations of coordinates and time, i.e., by an expression of the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, t) = 0 \ ,$$

are said to be:

a) holonomic. b) nonholonomic. c) scleronomous. d) Hieronymus.
e) cruciform.

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001 qmult 00200 1 4 2 easy deducto-memory: scleronomous constraint
Extra keywords: (Go3-13)

2. Scleronomous constraints have:

a) explicit time dependence.
b) no explicit time dependence.
c) both explicit time dependence and no explicit time dependence.
d) neither explicit time dependence nor no explicit time dependence.
e) a sclerous time dependence.

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001 qmult 00300 2 5 3 moderate thinking: non-Lagrangianable forces
Extra keywords: (Go3-21,23)

3. The Lagrange equations can be written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \ :$$

a) never.
b) **ONLY** when there are frictional forces.
c) when some of the generalized forces (expressed by Q_j) cannot be included in the Lagrangian (since they can't be derived from a generalized potential) or are not included in the Lagrangian for some reason.
d) when all the generalized forces can be and must be included in the Lagrangian.
e) when all the generalized forces **CANNOT** be included in the Lagrangian, but **ARE** so included nevertheless.

---

001 qmult 00400 1 4 4 easy deducto-memory: Lorentz force
Extra keywords: (Go3-22)

4. An important force that can be included in a Lagrangian via a **GENERALIZED POTENTIAL** is:

a) the force of nature. b) the normal force. c) the kinetic friction force.
d) the Lorentz force. e) the passion force.

Full Answer Problems

001 qfull 00100 1 5 0 easy thinking: force and kinetic energy

Extra keywords: (Go3-29.1), force related to time derivative of kinetic energy

5. Starting from Newton's 2nd law show that

$$\mathbf{F}\cdot\mathbf{p} = \frac{d(mT)}{dt} \ ,$$

where $\mathbf{p}$ is momentum, of course, and T is kinetic energy. Do **NOT** assume that mass is constant. What is the special-case formula when mass is constant?

Who knows: these formulae may have an Earthly use.

001 qfull 00200 2 5 0 moderate thinking: magnitude of CM vector

Extra keywords: (Go3-29.2)

6. Let $\mathbf{R}$ be the center of mass (CM) vector for a system of total mass M. Now

$$\mathbf{R} \equiv \frac{\sum_i m_i \mathbf{r}_i}{M} \ ,$$

where i indexes the particles making up the system, m_i is a particle mass, and $\mathbf{r}_i$ is a particle displacement. The origin is arbitrary. Show that

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2 \ .$$

where $\mathbf{r}_{ij} \equiv \mathbf{r}_j - \mathbf{r}_i$.

HINT: There are three tricks available one can foresee using: (1) using the center of mass definition, (2) relabeling indices, and (3) exploiting the symmetry of required formula in the indices i and j since $r_{ij} = r_{ji}$. Just to expand on the 3rd trick: if a final formula exhibits certain symmetries, it is often a good idea to derive the formula maintaining those symmetries in all steps. Following the path of symmetry often guides you to the answer and looks elegant and intelligible. If you leave the path of symmetry, you may still get to the result by guess and by golly, but the step can seem unaccountable.

Who knows: the formula to be proven may have an Earthly use, but I can't imagine what it is.

001 qfull 00700 2 5 0 moderate thinking: Nielsen Lagrange equation

Extra keywords: (Go3-30.7, see p. 23 too)

7. The Lagrange equation with some generalized force Q_j not encorporated into the Lagrangian $L = L(\{q_j\},\{\dot{q}_j\},t)$ (where the curly brackets mean "complete set of") is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \ .$$

Now I know what you are thinking: what would we get if we expanded $d/dt(\partial L/\partial \dot{q}_j)$ using the multi-variable chain rule? Could we then rearrange things to get another potentially useful version of the Lagrange equation? We can.

Prove that the Lagrange equation is equivalent to

$$\frac{\partial \dot{L}}{\partial \dot{q}_j} - 2\frac{\partial L}{\partial q_j} = Q_j \ .$$

Then find the Nielsen form of the Lagrange equation which is the equation above in the case where $L = T - U$ has no generalized potential (i.e., where $U = 0$).

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001 qfull 00800 1 5 0 easy thinking: non-standard Lagrangians
Extra keywords: (Go3-30.8)

8. The standard Lagrangian has a unique specification for a system

$$L = T - U \ .$$

But a non-standard Lagrangian L_* can be created by adding suitable functions to L that cancel out of the Lagrange equations. These non-standard forms might be more convenient to use sometimes: they may for instance have fewer terms than the standard Lagrangian. If L is the standard Lagrangian for a system, show that the Lagrange equations are also satisfied by

$$L_* = L + \frac{dG(\{q_j\}, t)}{dt} \ ,$$

where $\{q_j\}$ stands for the set of generalized coordinates.

Give some obvious simple examples of possible G and dG/dt functions.

What can you do as a simplification if you see a dG/dt function in a Lagrangian and all you want immediately is the equations of motion? Why can you do this?

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001 qfull 01200 1 5 0 easy thinking: escape velocity
Extra keywords: (Go3-31.12)

9. Using the conservation of mechanical energy theorem (or the work-energy theorem for conservative forces) find the expression for the escape velocity from the surface of an isolated planet sans air resistance. The escape velocity is defined as the mininum velocity for the escaping object to reach infinity. What is the escape velocity from the Earth from this formula?

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001 qfull 01300 2 5 0 moderate thinking: rocket launch
Extra keywords: (Go3-31.13)

10. Show that the equation of motion for a launching rocket sans air resistance and assuming g is constant is

$$m\frac{dv}{dt} = -mg - v_{\text{ex}}\frac{dm}{dt} \ ,$$

where v_{ex} is the exhaust speed of the ejecting fuel. Consider a launching rocket with initial mass m_0. Given $v_{\text{ex}} = 2.1\,\text{km/s}$ and $dm/dt = -m_0/(60\,\text{s})$, find the ratio of initial mass to final mass m_0/m when the rocket has reached the Earth's escape velocity $11.2\,\text{km/s}$.

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001 qfull 01600 2 5 0 moderate thinking: Weber force
Extra keywords: (Go3-32.16) generalized potential, velocity-dependent potential

11. Long, long ago on a continent far, far away, Weber proposed in some half-baked electrodynamics the following central force

$$F = -\frac{k}{r^2}\left[1 - \left(\frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right)\right]$$

where k is some constant representing the coupling strength of the force between charges and r is the radius from the center of force. Find the generalized potential U for F and then the Lagrangian for motion in a plane.

Recall the expression for the generalized force is

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \ .$$

Here F is the generalized force and r is the generalized coordinate.

HINT: I don't know of any nifty way of finding this potential. Just start with some very simple guess at U and see how it turns out and then iterate to the solution. The formula for U is very simple.

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001 qfull 01700 1 5 0 easy thinking: conservation of momentum
Extra keywords: (Go3-32:17) nucleus, non-relativistic mechanics

12. A nucleus of mass 3.90×10^{-22} g undergoes a beta decay emitting an electron of momentum $1.73\,\mathrm{MeV}/c$ and a neutrino of momentum $1.00\,\mathrm{MeV}/c$. (If direction isn't specified, we mean just the magnitude of momentum of course.) The two emitted particles are at right angles to each other. What is the nucleus' direction of recoil relative to the emitted particles, its momentum, and its kinetic energy (in MeV)? Do you need to consider relativistic effects in answering this question? Why or why not?

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001 qfull 02000 2 5 0 moderate thinking: weird Lagrangian
Extra keywords: (Go3-33.20)

13. Say for a one particle system you have the rather complex one-dimensional Lagrangian

$$L = f(x)m\dot{x}^2 + \frac{1}{24T_{\rm ch}}m^2\dot{x}^4 - V(x) \ ,$$

where $f(x)$ is some function of x, $T_{\rm ch} > 0$ is a constant, $V(x)$ is a potential, and x is just the ordinary spatial Cartesian x spatial coordinate with no explicit time dependence. For convenience, let's call the terms containing $T_{\rm ch}$ weird terms.

a) What must $f(x)$ actually be? Why?

b) Can the weird term in the Lagrangian give rise to an inertial force? Why or why not?

c) Find the explicit equation of motion for this Lagrangian and then write it so that the $m\ddot{x}$ is alone on one side of the equal sign with no $m\ddot{x}$ quantities on the other side of the equal sign. Simplify as much as reasonably possible.

d) If $T_{\rm ch} < \infty$, how would you describe the effect of the weird term in the equation of motion on the system? Consider explicitly the cases where kinetic energy T is much less and much greater than $T_{\rm ch}$.

e) Solve the equation of motion for x as a function of t for the case of $V(x)$ constant.

f) Recall the general formula for canonical momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \ .$$

What is the canonical momentum for our system?

g) Recall the general formula for energy function

$$h = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

and the Beltrami identity

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} \ ,$$

which is actually a differential equation satisfied by the extremal path constituted by the set of generalized coordinates $\{q_j\}$.

Find the energy function for our system and simplify it as much as reasonably possible. Is the energy function value conserved: i.e, is it a constant of the motion? Is the energy function the energy of the system in the sense of kinetic energy plus potential energy (i.e., an energy that depends on position alone).

h) Say

$$V(x) = \frac{1}{2}kx^2 ,$$

where k is constant greater than zero. Qualitatively describe the motion of the particle.

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001 qfull 02100 2 5 0 moderate thinking: rotational Atwood-like machine.

Extra keywords: (Go3-33.21)

14. You have a frictionless surface with a tiny hole with a frictionless edge. There are two point masses, m_1 and m_2, linked by a taut, ideal rope of length ℓ. One mass is down the hole the other is sliding on the table. What are the two generalized coordinates? Write down the Lagrangian and find the Lagrange equations. Reduce the problem to one 2nd order differential equation and obtain the first integral of this equation.

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001 qfull 02110 2 5 0 moderate thinking: Atwood-like machine

Extra keywords: (Go3-33.21) double incline problem, Atwoodoid machine

15. A taut light string connects two masses sliding on the opposing, frictionless inclines of an upright wedge. On each side the rope is parallel to the surface and the rope passes over a frictionless, massless pulley at the apex of the wedge. You've got the picture? The mass 1 is on the incline of angle θ_1 at position x_1 measured up the incline and mass 2 is on the incline of angle θ_2 at position x_2 measured up the incline. Formulate the Lagrangian for the system and solve the Lagrange equations for the accelerations of both masses.

Chapt. 2 Hamilton's Principle and More on the Lagrangian Formulation

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Multiple-Choice Problems

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002 qmult 00100 1 4 5 easy deducto-memory: Hamilton's principle
Extra keywords: (Go3-34)

16. Hamilton's principle is an example of a:

    a) force.
    b) Hamiltonian.
    c) Lagrange multiplier.
    d) stationary point.
    e) variational principle.

---

002 qmult 00130 1 4 5 easy deducto-memory: Euler-Lagrange equations

17. "Let's play *Jeopardy*! For \$100, the answer is: it is the set of differential equations

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_j}\right) - \frac{\partial f}{\partial q_j} = 0 ,$$

where t is some parameter, q_j are functions of t, $\dot{q}_j$ are the total derivatives of q_j with respect to t, and f is a general function of the sets of q_j and $\dot{q}_j$, and in general of t explicitly. The solution of this set of differential equations gives one the the set of q_j that make the functional

$$J = \int_{t_1}^{t_2} f(\{q_j\}, \{\dot{q}_j\}, t)\, dt$$

stationary with respect to varied q_j, except that the endpoints at t_1 and t_2 of the q_j are fixed."

What is ____________, Alex?

a) Lagrange equations b) D'Alembert equations c) Bernoulli equations
d) Leibniz equations e) Euler-Lagrange equations

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002 qmult 00150 1 1 2 easy memory: Lagrange multipliers

18. In the Lagrangian formulation, one can determine constraint forces for semi-holonomic constraints and:

a) Euler undetermined multipliers. b) Lagrange undetermined multipliers.
c) D'Alembert undetermined multipliers. d) Bernoulli undetermined multipliers.
e) Goldstein undetermined multipliers.

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002 qmult 00200 1 5 3 easy thinking: canonical or conjugate momentum
Extra keywords: (Go3-55)

19. The general expression for canonical or conjugate momentum is

$$p_j = \frac{\partial L}{\partial \dot{q}_j} .$$

Given the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \ ,$$

what is x's conjugate momentum?

a) $\frac{1}{2}m\dot{x}^2$.
b) mx.
c) $m\dot{x}$.
d) $-(\partial V/\partial x)$.
e) $-(\partial V/\partial \dot{x})$.

---

002 qmult 00300 1 1 2 easy memory: cyclic coordinate
Extra keywords: (Go3-55)

20. If the Lagrangian is cyclic in q_j, then:

a) p_j is not conserved.
b) p_j is conserved.
c) q_j appears in the Lagrangian.
d) $\dot{q}_j$ (i.e., dq_j/dt) does not appear in the Lagrangian.
e) the Lagrangian is circular.

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002 qmult 00400 1 1 3 easy memory: Beltrami identity, energy function

21. Given a Lagrangian $L(\{q_j\},\{q_j\},t)$ where the set of generalized coordinates $\{q_j\}$ are the functions of time that make the action integral extremal (i.e., they constitute the extremal path), one can show that

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} \ .$$

This differential equation is called the ____________. The h function that appears in the differential equation is

$$h = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

and it is called the ____________.

The differential equation can be used to replace one of the Lagrange equations in a solution for the extremal path. The replacement is particularly useful if $\partial L/\partial t = 0$ which makes h a conserved quantity (i.e., a constant of motion). In this case, the differential equation reduces to a reduced differential equation

$$h = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \ ,$$

where h is constant as aforesaid. In some cases, for example when there is only one generalized coordinate, the reduced differential is very useful in solving for the extremal path.

a) D'Alembert's principle; D'Alembertian b) D'Alembert's principle; Hamiltonian
c) Beltrami identity; energy function d) Pastrami identity; energy function
e) Hamilton's principle; Hamiltonian

Full-Answer Problems

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002 qfull 00200 2 5 0 moderate thinking: conjugate rotation momentum
Extra keywords: (Go3-63.2)

22. Show that the conjugate momentum for a representative rotation coordinate ϕ about an axis in the $\hat{\mathbf{z}}$ direction for a system of particles with an ordinary potential V and a velocity-dependent potential U is

$$p_\phi = J_z - \sum_i \hat{\mathbf{z}} \cdot \mathbf{r}_i \times \nabla_{\mathbf{v}_i} U \ ,$$

where J_z is the ordinary mechanical angular momentum. (Note L is usually used for angular momentum, but here we need L for Lagrangian.) If the potential is the Lorentz force potential,

$$U = q(\varphi - \mathbf{A} \cdot \mathbf{v}) \ ,$$

show that the expression specializes to

$$p_\phi = J_z + \sum_i \hat{\mathbf{n}} \cdot \mathbf{r}_i \times q_i \mathbf{A}_i \ ,$$

where q_i is the charge on particle i.

Since ϕ is a representative rotational coordinate, we could specify the ϕ_i by

$$\phi_i = \phi_i' + \phi \ ,$$

where ϕ_i' is a relative coordinate that we never need to mention again in this problem. Note that varying ϕ is effectively like shifting the origin of the ϕ_i coordinates or like moving all the ϕ_i coordinates in formation.

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002 qfull 00300 1 5 0 easy thinking: geodesic in 3-d Euclidean space
Extra keywords: (Go3-64.3)

23. Using the Euler-Lagrange equation of variational calculus find what is the geodesic in 3-dimensional Euclidean space: i.e., what is the shortest distance between two points?

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002 qfull 00400 2 5 0 moderate thinking: geodesics of a sphere
Extra keywords: (Go3-64.4)

24. Consider a spherical surface.

a) Give short argument as to why there must be a global minimum path length between two different points on the spherical surface.

b) Give a short argument as to why there is no global maximum path length between two different points on the spherical surface.

c) The geodesics of spherical surfaces (i.e., the extremum paths) between two different points are great circles (i.e., circles with the centers at the origin). Show this. Remember that you have to deal with two Euler-Lagrange equations. **HINT:** Do it the easy way or as Dr. Einstein said "the quick way"—I mean Dr. Herman Einstein (Heidelberg 1919).

d) There actually two great circle paths between any two points. Give an argument why the shorter one must be the global minium path.

e) How would the longer great path be characterized? Is it a global maximum, local maximum, local mininum, or none of those (i.e., it is just a path with a stationary length). **HINT:** Consider two new points anywhere on the longer path that are closer together than half the great circle. If you went along the longer path, except that you deviated between these new two points would your path length be longer or shorter than if you'd stayed on the great circle path all the way?

f) Consider the two antipodal points on the surface: i.e., points on the same line that passes through the sphere's center. How many paths of global minimum length connect the two points?

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002 qfull 00500 2 5 0 moderate thinking: minimizing action hard way
Extra keywords: (Go3-64.5)

25. A particle subject to potential $V(x) = -Fx$ (where F is a constant) goes from $x = 0$ to $x = x_0$ in a time t_0. Assuming $x(t) = a + bt + ct^2$ determine a, b, and c by minimizing the action using ordinary calculus (not the action integral using variational calculus) subject to the constraints provided by the endpoints. Impose the $x = 0$ constraint immediately to simplify the problem somewhat, but use the Lagrange multipliers with the other constraint. We're doing this problem the hard way, of course.

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002 qfull 01100 2 5 0 moderate thinking: collision and Lagrangian
Extra keywords: (Go3-65.11)

26. Collisions are events in which strong interaction forces act between particles over a comparatively short time. Let's consider a collision in Lagrangian formalism. Let L be a Lagrangian that contains all the forces, except for the collision forces. Thus L is all that is needed to determine the equation of motion before and after the collision. In principle some of the collision forces might be included in L, but we want to isolate the collision event from the rest of the evolution and not complicate L. Moreover, the collision forces will include dissipation processes if the collision is inelastic and forces with dissipation cannot readily be included in the Lagrangian. Let the generalized collision force for the jth generalized coordinate q_j be $Q_j^{\rm col}$. The Lagrange equation for the jth generalized coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{\rm col} \ .$$

Show that the change in the conjugate momentum p_j in a collision event is

$$\Delta p_j = \Delta p_j^{\rm col} \ ,$$

where $\Delta p_j^{\rm col}$ is the collision impulse for the jth generalized coordinate. (Since the Lagrangian does not contain the collision forces, p_j is kind of an weird quantity through the collision, but before and after it is a normal conjugate momentum.) The result is exactly true if q_j is cyclic for L and is approximately true in the collision approximation where over the collision time Δt the collision forces are much stronger than any other forces. In fact, in the ideal collision, $\Delta t \to 0$ and the collision forces become Dirac Delta function-like and other forces stay finite and their impulse vanishes.

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002 qfull 01300 2 5 0 moderate thinking: cat flying off hoop
Extra keywords: (Go3-66.13) Catullus

27. Catullus the cat is at the top of a fixed frictionless hoop of radius a. His location on the hoop is specified by the angle measured from the vertical θ. At time zero he starts sliding down from rest.

    a) In hand-waving terms why must Catullus fly off the hoop before he reaches $\theta = 90°$.

    b) Using an intro-physics approach find the θ at which he flys off. **HINT:** A free-body diagram might help.

    c) Now using the Lagrange equations with a Lagrange multiplier for the constraint, find the θ at which Catullus flys off.

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002 qfull 01600 2 5 0 moderate thinking: damped harmonic oscillator
Extra keywords: (Go3-66.16)

28. Sometimes it is possible to include dissipation effects in the Lagrangian by other means than the dissipation function. It all seems a bit ad hoc to me: constructing a special sort of Lagrangian,

not according to the general rule, but only according to what one already knows gives the equation of motion. But I suppose there must be some point. Anyway consider the special Lagrangian

$$L = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right) ,$$

where $\gamma \geq 0$ and $k \geq 0$.

a) Find the equation of motion, identify the system, and the solve the equation of motion.

b) Does the system present any obvious constants of the motion?

c) Make the point transformation (coordinate transformation)

$$q = e^{-\gamma t/2}s$$

for the Lagrangian. Find the equation of motion in terms of s. Identify a constant of the motion.

---

002 qfull 01700 1 5 0 easy thinking: separable Lagrangian case

Extra keywords: (Go3-66.17) with simple reduction to quadrature

29. If kinetic and potential energy can be written

$$T = \frac{1}{2}\sum_i f_i(q_i)\dot{q}_i^2 \qquad \text{and} \qquad V = \sum_i V_i(q_i) ,$$

show that the Lagrange equations separate into independent equations and that the solutions formally can be obtained by quadrature (i.e., a single integration each for each).

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002 qfull 02000 2 5 0 moderate thinking: block and wedge

Extra keywords: (Go3-67.20) double constraint problem

30. A block of mass m is sliding on wedge of mass M and inclination angle θ. The surface of the wedge is frictionless. The wedge is on a horizontal plane that is also frictionless.

NOTE: There are parts a,b,c,d,e,f,g.

a) Impose the constraints on the system via the Lagrange multiplier approach and determine the equations of motion of the block and wedge, the accelerations of all the coordinates, and the constraint forces. Note imposing the constraints directly is somewhat easier, but using Lagrange multipliers does give you a direct path to finding the constraint forces. Just so we all do this problem the same way, use only horizontal and vertical coordinates and make the wedge slope down in the negative direction of the horizontal coordinates.

b) Check that your answer to part (a) is correct by showing that acceleration of the block down the wedge is $g\sin\theta$ in the limit that $m/M \to 0$.

c) Find the most obvious constants of the motion.

Chapt. 3 The Central Force Problem

Multiple-Choice Problems

003 qmult 00100 1 4 2 easy deducto-memory: reduced mass
Extra keywords: (Go3-71)

31. The reduced mass:

    a) has lost weight.
    b) is the funny way we account for the inertia properties in a 2-body problem reduced to an equivalent 1-body problem.
    c) is $m_1 + m_2$ for 2-body system with the two bodies having masses m_1 and m_2.
    d) the smaller of the two masses in a 2-body system.
    e) the larger of the two masses in a 2-body system.

003 qmult 00200 1 4 1 easy deducto-memory: angular momentum
Extra keywords: (Go3-73)

32. A usual expression for the conserved angular momentum in a central force problem is:

    a) $\ell = mr^2\dot{\theta}$.
    b) $\ell = m/(r^2\dot{\theta})$.
    c) $\ell = \dot{\ell}/k$.
    d) $\ell = (1/2)mr^2\dot{\theta}$.
    e) $\ell = m_1 m_2/(m_1 + m_2)$.

003 qmult 00800 1 4 5 easy deducto-memory: Bertrand's theorem.
Extra keywords: (Go3-92)

33. "Let's play *Jeopardy*! For \$100, the answer is: The only central forces that give closed non-circular orbits for all bound particles are the inverse-square law force and the linear force (AKA Hooke's law force or the harmonic oscillator force). All central forces give closed circular orbits."

    What is ____________, Alex?

    a) the virial theorem b) Euler's theogonic proof c) the brachistochrone problem
    d) Shubert's unfinished symphony e) Bertrand's theorem

Full-Answer Problems

003 qfull 00100 1 5 0 easy thinking: virial theorem, drag force
Extra keywords: (Go3-126.1)

34. Consider a system of particles (with particles labeled by i) in which the only forces are conservative forces F'_i and linear drag forces of the form

$$\mathbf{f}_i = -k_i r_i^{2n} \mathbf{v}_i$$

for particle i where n is an integer. The drag force dependence on r_i^{2n} may be a unimportant generalization, but it is a easily treated one. Show that the virial theorem,

$$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} \ ,$$

for our case specializes to

$$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}'_i \cdot \mathbf{r}_i} \ ,$$

where the two sides are only non-zero if some energy is continuously pumped into the system in such a way as to compensate on average for the dissipation of the drag force.

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003 qfull 00300 2 5 0 moderate thinking: solving Kepler's equation

Extra keywords: (Go3-126.3) small ϵ solution, Kepler equation

35. In the solution of the Kepler problem for a bound orbit one encounters three Medieval anomalies: i.e., angular positions for the Kepler particle (hereafter the planet). The true anomaly θ (the honest, good anomaly) is the angular coordinate measured from the force center focus with zero position at perihelion: mentally I always think of this as counterclockwise rotation with perihelion at the left side of the ellipse. The true anomaly determines the radial position via the ellipse function

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} = \frac{p(1+\epsilon)}{1+\epsilon\cos\theta}$$

where p is perihelion distance.

The mean anomaly ωt (the most cruelest anomaly) is the mean angular measured conventionally from the same zero position. The ω follows from the Kepler bound orbit period:

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ma^3}{k}} \ .$$

The eccentric anomaly (the wild and crazy anomally is an auxilary angle defined by

$$r = a(1-\epsilon\cos\psi) \ .$$

The r is, of course, the radial position determined by the ellipse function

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} = \frac{p(1+\epsilon)}{1+\epsilon\cos\theta}$$

where p is perihelion distance. The ψ need not be thought of as position in space angle at all. However, to 1st order in small ϵ, r determined by ψ definition agrees with r determined by the ellipse function. As we will show in this problem $\cos\theta$ and $\cos\psi$ agree only to zeroth order in small epsilon. Kepler's equation is a transcendental equation that relates mean and eccentric anomally

$$\omega t = \psi - \epsilon\sin\psi \ .$$

It is derived by e.g., Go100–102.

The historical method for solving for true anomaly for a given time t is to specity mean anomaly for t, solve Kepler's equation for ψ by some approximate or numerical means, and then use an analytic expression (which we derive in part (a) of this problem) to obtain θ from ψ. By symmetry one only has to treat the problem of solving for θ for $\theta \in [0,\pi]$, and this turns out to mean for $\omega t \in [0,\pi]$ and $\psi \in [0,\pi]$ too as we will show in this problem.

In addition to the Medieval anomalies, I've defined a fourth anomaly, equant anomaly (the most equable anomaly). This is the angular position measured from the empty focus of the elliptical orbit with the zero at the perihelion (the real perihelion). The equant anomaly is taken up in a later problem in this chapter of CMP.

a) Find the relation between $\cos\theta$ and $\cos\psi$ and find its 1st order in small ϵ form.
b) Show that that the three Medieval anomalies all have the same values at perihelion and aphelion.
c) Show that ψ and ωt have a one-to-one relation in the ψ range $[0,\pi]$: i.e., show that ωt increases or decreases strictly with ψ in the range $[0,\pi]$. What is the size of ψ relative to ωt in the ψ range $[0,\pi]$?
d) Show that ψ and θ have a one-to-one relation in ψ range $[0,\pi]$.
e) Show that ωt and θ have a one-to-one relation in the ωt range $[0,\pi]$. Should there be any ambiguity in principle in determining a unique θ in the range $[0,\pi]$ from ωt in the range $[0,\pi]$? Does dealing with $\omega t \in [0,\pi]$ and $\psi \in [0,\pi]$ suffice to determine *theta* for all time?
f) In solving Kepler's equation it is sometimes useful to define

$$\rho = \psi - \omega t \ .$$

In terms of ρ Kepler's equation becomes

$$\rho = \epsilon \sin(\omega t + \rho) \ .$$

Show that ρ is always greater than zero, except at $\omega t = 0$ and π (i.e., perihelion and aphelion). **HINT:** Reflect on the original Kepler's equation.

g) Show that ρ must in general be treated as a quantity of order ϵ: e.g., by showing that ρ can equal *epsilon*. **HINT:** Reflect on the original Kepler's equation.

h) What is the iterative approach to finding and expression for ρ in terms of ωt to any order n in small ϵ. Such an expression, of course, is an *nth* order in small ϵ solution to Kepler's equation. Using this expression, one can solve for θ to *nth* order in small ϵ as well.

i) Derive 0th, 1st, 2nd, and 3rd order in small ϵ solution to Kepler's equation.

---

003 qfull 01000 2 5 0 moderate thinking: comet-planet collision
Extra keywords: (Go3-128.10) and escape

36. A bound planet of mass m and comet of mass $m_{\rm co}$ collide completely inelastically at radius r from the Sun. The post-collision system of mass $m_{\rm total} = m + m_{\rm co}$ has just enough energy to escape the solar system.

a) For simplicity assume the planet-comet collision occurs while the two objects are both moving on the same line: allow that they may be either colliding head-on or that one overtakes the other. Find the comet kinetic energy $KE_{\rm co}$ in terms of k/a, $m_{\rm total}/m_{\rm co}$, $m/m_{\rm co}$, and a/r. Try to find a reasonably simple and meaningful final form. Which of head-on or overtaking requires more energy to effect and ejection?
b) Find the specialized expression for $KE_{\rm co}$ in the limit that $m_{\rm co} << m$ and $r = a$: the former condition is very likely and the latter typical. Why must the comet kinetic energy be so much larger than the binding energy of the planet $k/(2a)$.
c) What must be the rough average comet kinetic energy be to effect an escape for any collision angle and radius r?

Find the specialized expression for $KE_{\rm co}$

---

003 qfull 01100 2 5 0 moderate thinking: collapsing Kepler orbit
Extra keywords: (Go3-128.11)

37. Say you had a bound Kepler orbital system. In an instant at time t_0 you suddenly take away all of the planet's (i.e., the Kepler particle's) kinetic energy. If the pre-t_0 orbit was a circle, show that the time $t_{\rm col}$ from t_0 to the planet's collision with the center of force is

$$t_{\rm col} = \frac{\tau_{\rm pre}}{4\sqrt{2}} \ ,$$

where $\tau_{\rm pre}$ is the pre-t_0 orbital period of the planet. **HINT:** Before fussing around with equations think about what kind of post-t_0 orbit the planet must have.

---

003 qfull 00111 2 5 0 moderate thinking: Gauss' law and inner Earth

38. You-all remember Gauss' law for the gravitational field (i.e., force per unit mass)

$$\oint \mathbf{f}\cdot d\mathbf{A} = -4\pi G M_{\rm enc} \ ,$$

where $\mathbf{f}$ is the gravitational field, $d\mathbf{A}$ differential surface vector, and $M_{\rm enc}$ is the mass enclosed by the surface integral.

a) Find the gravitational field $\mathbf{f}$ for a spherically symmetric mass distribution. Don't forget to give an argument for crucial step.

b) Assume that the spherical distribution is a sphere with radius R and a constant density ρ. What is the field **INSIDE** the sphere? What is the potential (not potential energy in this case) **INSIDE** the sphere? For simplicity set the zero of potential at $r = 0$.

c) Write down the zero-angular-momentum Lagrangian (i.e., the Lagrangian for a purely radial motion) for a particle of mass m moving inside the sphere. Assume the only force the particle feels is gravity.

d) From the Lagrangian what is the equation of motion of the particle?

e) What is the general solution of the equation of motion? Give an expression for the period of the motion. **HINT:** You don't need to solve for the solution although you can if you like. Recognition of what it has to be is enough.

f) Given $G = 6.673\times10^{-8}$ in cgs and the Earth mean density $5.5\,\mathrm{g/cm^3}$, what is the period you obtain from the answer in part (e). **NOTE:** If you are in a test mise en scène and forgot your calculator one digit accuracy will suffice.

---

003 qfull 01600 2 5 0 moderate thinking: equant problem

Extra keywords: (Go3-129.16)

39. The crime of Ptolemy (circa 100–170 AD) was that he proclaimed the divinity of the uniform circular motions the planets. Thus compounded uniform circular motion was his physical principle. But in constructing his planetary models he violated this principle by introducing the equant. The equant was a point at the same distance from the center of the main planetary circles (i.e., the deferents) as the Earth—the Earth was not actually at the center in his "geocentric" model—and directly opposite the Earth. The angular motion of the epicycle on the deferent was uniform as seen from the equant point.

Now as it turns out, to 1st order in small eccentricity ϵ the angular motion of a planet as viewed from the empty focus of an elliptical orbit is uniform. Thus Ptolemy was gropping toward a better geometry and kinematics for the planetary that would allow better fits. Unbeknowst to him these were elliptical orbits and true anomaly motion in time. Of course, with a geocentric starting point he could not arrive at the real orbits sans a leap of imagination. In fact he was aware of the heliocentric models, but rejected them as physically absurd: he was basically Aristotlean in physics. Nevertheless, it is interesting that he was willing to abandon uniform circular motion even though he refused to admit it.

Succeeding astronomers viewed Ptolemy's equant as a defect of his models which it is certainly is on the basis of the principle of uniform circular motion. Medieval Islamic astronomers went to work to eliminate the equant. And of course they succeeded—compounded uniform circular motions can pretty well account for all observed planetary motions if one is tricky enough and one's data isn't of modern accuracy. Ibn al-Shatir (1304–1375), the muwaqquit of the Damascus mosque (???), used triple circle orbits to account for planetary motions adhering strictly to uniform circular motion. Copernicus too adhered to uniform circular motion in his heliocentric models: it is almost certain that he knew something of the work of al-Shatir or related astronomers, but probably very indirectly through anonymous manuscripts circulating in the universities of 15th century Italy.

Your mission Mr. Phelps, if you decide to accept it, is to demonstate that the angular motion of a planet relative to the empty focus in an elliptical orbit is indeed uniform to first order in small ϵ. Let's call the empty focus the equant and the angular coordinate measured from the equant, the equant anomaly $\theta_{\rm eq}(t)$. In any case, the State Department will disavow any knowledge of your activities. **HINTS:** First find a 1st order expression for true anomaly $\theta(t)$ as a function of mean anomaly ωt. Then find the relation of equant anomaly to true anomally.

---

003 qfull 02100 1 5 0 tough thinking: precessing Kepler orbit

Extra keywords: (Go3-130.21)

40. Say you are given the potential

$$V(r) = -\frac{k}{r} + \frac{h}{r^2} \ .$$

a) Show that the bound orbital solution for r (radius from the force center) is an ellipse function of pseudo angle parameter θ_*. The θ_* parameter isn't really an angle or a time, but is some function of either. Express ellipse semimajor axis, eccentricity, perihelion distance, and period τ (i.e., time to go from θ_* to $\theta_* + 2\pi$ as functions of energy E (the real energy actually) and a pseudo angular momentum ℓ_*. **HINTS:** The second term in the potential has the same r dependence as the centrifugal potential. You don't have to re-solve the Kepler problem, just reduce the current problem to Kepler problem.

b) Determine θ as a function of θ_*. **HINTS:** Remember $\ell = mr^2\dot{\theta}$ where ℓ is real momentum is true for all central force problems not just inverse-square law ones.

c) Find the expression for angular shift $\Delta\theta = \theta(t+\tau) - \theta(t) - 2\pi$: i.e., the difference in the advance of θ from a complete orbit relative to an inertial frame that occurs in one complete period of θ_*. What is the angular frequency $\omega_{\rm prec}$ of this shift? Argue that this is the precession frequency of the orbit. What is the sense of the precession relative to the orbital sense of the planet.

d) Show that expression for $\omega_{\rm prec}$ to 1st order in small mh/ℓ^2 is

$$\omega_{\rm prec} = -\frac{2\pi mh}{\tau\ell^2} \ .$$

e) The perihelion of Mercury advances by $40''$ of arc per century in the sense of Mercury's motion above what can be accounted for by planetary perturbations. Find the ratio of $h/(ka)$ that gives this precession. Mercury's orbital eccentricity and period are 0.206 and 0.24 years, respectively. **NOTE:** The extra perihelion advance was first explained by general relativity and for many years was the best verification of that theory.

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003 qfull 02300 1 5 0 easy thinking: Sun mass/Earth mass

Extra keywords: (Go3-130.23)

41. Deduce to good approximation the ratio of the Sun's mass to the Earth's mass (i.e., $M_\odot/M_\oplus$) using only the following data: Earth sidereal year 365.25636556 days, mean Sun-Earth distance

1.4959787×10^{13} cm, lunar sidereal period 27.321661 days, and mean Earth-Moon distance 3.84400×10^{10} cm.

Chapt. 4 Rigid Body Kinematics

Multiple-Choice Problems

Full-Answer Problems

Chapt. 5 Rigid Body Equations of Motion

Multiple-Choice Problems

Full-Answer Problems

Chapt. 6 Oscillations

Multiple-Choice Problems

Full-Answer Problems

Chapt. 7 The Classical Mechanics of Special Relativity

Multiple-Choice Problems

Full Answer Problems

Chapt. 8 The Hamiltonian Formulation

Multiple-Choice Problems

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008 qmult 00100 1 4 5 easy deducto-memory: Hamiltonian formulation

42. "Let's play *Jeopardy*! For \$100, the answer is: An alternative formulation of advanced classical mechanics that in some respects is superior to the Lagrangian formulation."

   a) What is the Aristotelian formulation, Alex?
   b) What is the Chaucerian formulation, Alex?
   c) What is the Harriotian formulation, Alex?
   d) What is the Lovelacian formulation, Alex?
   e) What is the Hamiltonian formulation, Alex?

---

008 qmult 00200 2 1 2 moderate memory: Hamiltonian formula

43. The Hamiltonian can be constructed from the Lagrangian using the formula:

   a) $H = \dot{p}_i\dot{q}_i - L$.
   b) $H = p_i\dot{q}_i - L$.
   c) $H = \partial L/\partial\dot{q}_i$.
   d) $H = 1/L$.
   e) $H = L + dF(q,t)/dt$.

---

008 qmult 00300 1 4 3 easy deducto-memory: simplest Hamiltonian rule

44. If (1) defining relations of the generalized coordinates do not depend explicitly on time and (2) the potential is ordinary (i.e., not velocity-dependent), then the Hamiltonian is given by:

   a) $H = T - V$.
   b) $H = T/V$.
   c) $H = T + V$.
   d) $H = V - T$.
   e) $H = VT$.

---

008 qmult 00400 2 1 5 moderate memory: Hamiltonian constant of motion

45. If

$$\frac{\partial H}{\partial t} = 0 \ ,$$

then

   a) nothing further can be inferred about the system.
   b) all other aspects of the system follow from simple algebra.
   c) $dH/dt = 0$ and the Hamiltonian **IS NOT NECESSARILY** a constant of the motion, but it **IS** necessarily the total energy of the system.
   d) $dH/dt = 0$ and the Hamiltonian **IS NOT** a constant of the motion, but it **IS** necessarily the total energy of the system.
   e) $dH/dt = 0$ and the Hamiltonian **IS** a constant of the motion, but it **IS NOT** necessarily the total energy of the system.

Full Answer Problems

008 qfull 00100 2 5 0 moderate thinking: general action integrand
Extra keywords: (Go3-362.1) or maybe "modified Lagrangian"

46. The action integrand of the modified Hamiltonian isn't in general a Lagrangian ($L = T - V$) nor a non-standard Lagrangian ($L = T - V + dF(q,t)/dt$), but

$$f = p_i \dot{q}_i - H(q,p,t) + \frac{dF(q,p,t)}{dt} ,$$

where Einstein summation is used and unsubscripted canonical variables means the set of them. Perhaps it could be called the modified Lagrangian, but maybe no one would recognize that term. The general action integrand is the Lagrangian (standard or non-standard) whenever F is not a function of the p_i.

a) Show that the Lagrange equations follow from the general action integrand in the case where $F = 0$. In this case f is the Lagrangian L. This is a reverse Legendre transformation.

b) Now find the equations analogous to the Lagrange equations for a general action integrand in the case where $F = -p_i q_i$.

c) Show that the parts (a) and (b) results would be the same if one added $F = F(y,t)$ to the general action integrand where $y = q$ for part (a) and $y = p$ for part (b).

008 qfull 00200 1 5 0 easy thinking: non-standard Lagrangian-Hamiltonian
Extra keywords: (Go3-362.2)

47. Recall the non-standard Lagrangian (my own idiosyncratic name) is related to the standard Lagrangian $L = T - V$ (or any other non-standard Lagrangian) by

$$L' = L + \frac{dF(q,t)}{dt} ,$$

where $F(q,t)$ is arbitrary function of the generalized coordinates and time.

a) Find the relationship between the Hamiltonian H' corresponding to L' and the Hamiltonian H corresponding to L.

b) Show that the Hamilton's equations for H' reduce to those for H. **NOTE:** This is hard than it looks because one has to be careful about what is held constant partial differentiation.

008 qfull 00300 2 5 0 moderate thinking: Gamiltonian
Extra keywords: (Go3-362.3)

48. The Gamiltonian $G(\dot{q},\dot{p},t)$ is like the Hamiltonian, except the total time derivatives of the q_i and p_i replace q_i and p_i. Show how to contruct the Gamiltonian from the Lagrangian and derive Gamilton's equations:

$$q_i = \frac{\partial G}{\partial \dot{p}_i} \qquad \text{and} \qquad p_i = -\frac{\partial G}{\partial \dot{q}_i} .$$

NOTE: Gamilton was Hamilton's smarter twin sister, but she died young and Hamilton suppressed almost all of her brilliant work in classical mechanics. Thus the world has lost Gamilton's principle and the Gamilton-Jacobi theory—we are all the poorer for it.

008 qfull 00400 1 5 0 easy thinking: inverse eigenvalues
Extra keywords: (Go3-362.4)

49. If square matrix $\mathbf{A}$ has eigenvalues λ_i eigenvectors $\mathbf{a}_i$ and an inverse $\mathbf{A}^{-1}$, show that the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$ are λ_i^{-1} and $\mathbf{a}_i$, respectively.

---

008 qfull 01200 2 5 0 moderate thinking: Hamiltonian 2-body central force

Extra keywords: (Go3-363.12)

50. Let's tackle the 2-body central force problem ab initio until we have reduced it to quadratures (i.e., to integrations).

a) You are given two particles acting only under a central force that each exerts on the other. Write down the Lagrangian for the system using coordinates from a fixed origin.

b) Let's use new generalized coordinates: relative and center of mass (CM) coordinates. Find the transformation relations and their inverses.

c) Determine the kinetic energy expression in the CM-relative coordinates. Define a reduced mass to simplify the expression.

d) In the CM-relative frame what is the role of the reduced mass?

e) What inequality does m obey with respect to the two masses? What are two important limiting behaviors of the reduced mass?

f) Write down the Lagrangian in the CM-relative coordinates and then determine the Hamiltonian in these coordinates. Make sure you write the Hamiltonian in terms of the correct canonical variables. Is the Hamiltonian the total energy in this case? Is it a constant of the motion?

g) Argue that the CM and relative parts of the Hamiltonian are separately constants of the motion. Then argue that the two parts can be treated as independent Hamiltonians for a separate solution of the CM and relative problems.

h) Solve the CM Hamiltonian problem.

i) Show that the relative motion must be confined to a plane. Then change the relative coordinates to polar coordinates in a legitimate way using a point transformation of the relative Lagrangian and determine the polar coordinate canonical variables. Then write out the relative Hamiltonian problem in terms of polar coordinate canonical variables.

j) Show that one canonical space variable is cyclic in the relative Hamiltonian, and thus its corresponding conjugate momentum is conserved. Use this information to reduce the radial problem to an equivalent one-dimensional problem.

k) Now reduce the relative problem to quandratures (i.e., integrals) for the canonical space variables. Show how the canonical momenta are obtained.

---

008 qfull 01400 2 5 0 moderate thinking: Hamiltonian construction 3rd rule

Extra keywords: (Go3-364.14) contruction from the 3rd easiest rule

51. You are given the Lagrangian

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - h\sqrt{x^2+y^2} \ ,$$

where a, b, c, f, g, and h are constants.

a) Determine the canonical momenta for the system.

b) Determine Hamiltonian. Is the Hamiltonian the total energy?

c) What quantities are obvious constants of the motion?

d) Write out Hamilton's equations for the Hamiltonian. Are there any other identifiable constants of the motion? Write out the solutions of the equations. You don't have to be too explicit if that creates a mess.

---

008 qfull 01500 3 5 0 hard thinking: messy Hamiltonian
Extra keywords: (Go3-364.15)

52. Given the Lagrangian

$$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k\dot{q}_1\dot{q}_2 - cq_1^2 ,$$

find the Hamiltonian and the particular Hamilton's equations.

---

008 qfull 01600 2 5 0 moderate thinking: disguised SHO
Extra keywords: (Go3-364.16)

53. You are given a Hamiltonian of one degree of freedom:

$$H = \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{b\alpha}{2}q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2} .$$

a) Find the corresponding Lagrangian in terms of q and $\dot{q}$.

b) Find an equivalent Lagrangian L' that is explicitly time-independent.

c) What is the equivalent Hamiltonian H' in terms of its proper canonical variables p' and q' corresponding to L'? What as a matter of category are p' and q' with respect to p and q? **HINT:** I'm just looking for a sentence as an answer to the last question.

Chapt. 9 Canonical Transformations

Multiple-Choice Problems

009 qmult 00100 1 4 1 easy deducto-memory: canonical transformation

54. "Let's play *Jeopardy*! For \$100, the answer is:

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF(Q,P,t)}{dt} \text{ ."}$$

a) What equation must a canonical transformation satisfy, Alex?
b) What is the modified Hamilton's principle, Alex?
c) What is the plain, old Hamilton's principle, Alex?
d) What is a canonical transformation, Alex?
e) What is to mind one's P's and Q's, Alex?

009 qmult 00200 1 1 2 easy memory: generating function use

55. Canonical transformations can often be conveniently found or verified by using a:

a) generation gap.
b) generating function.
c) degenerating function.
d) separation tensor.
e) desperation matrix.

Full-Answer Problems

009 qfull 00100 2 5 0 moderate thinking: complex canonical trans.
Extra keywords: (Go3-421.1)

56. The one-degree-of-freedom complex transformation

$$Q = q + ip \ ; \qquad P = Q^*$$

is not canonical: the Sacred College did not approve. But can an appropriate scale transformation of this transformation be made canonical?

009 qfull 00400 2 5 0 moderate thinking: canonicality verified
Extra keywords: (Go3-422.4)

57. Given the transformation

$$Q = \ln\left(\frac{\sin p}{q}\right) \ ; \qquad P = q \cot p \ ,$$

verify directly that it is a canonical transformation.

---

009 qfull 01000 2 5 0 moderate thinking: canonical transformation
Extra keywords: (Go3-423.10) constructing a generating function

58. You are given the transformation

$$Q = \frac{\alpha p}{q} \; ; \qquad\qquad P = \beta q^2 \; .$$

a) Find the inverse transformation. You should take care of any ± cases explicitly.

b) Find the partial derivatives of the old variables q and p with respect to the new ones Q and P.

c) Find the condition under which the new variables satisfy Hamilton's equations: i.e., the condition which makes the transformation canonical. Write down the transformations and inverse transformations with that condition applied.

d) Find a generating function of one of the four basic types that leads to the transformation. **HINT:** Reflect on the relations with the old and new variables that the generating function must yield. It can help to find an old variable as a simple function of a mixture of old and new variables and integrate.

---

009 qfull 02200 2 5 0 moderate thinking: canonical transformation
Extra keywords: (Go3-425.22) 2-degree of freedom cyclic canonical transformation

59. Consider the Hamiltonian

$$H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2 \; ,$$

where the dynamical quantities are all dimensionless.

a) Find a very general canonical transformation consistent with the point transformation

$$Q_1 = q_1^2 \; ; \qquad\qquad Q_2 = q_1 + q_2 \; .$$

Find also the inverse transformations.

b) Give the Hamiltonian in terms of the new canonical variables. Be carefull there is an easily overlooked generality.

c) Now find the particular canonical transformation that makes the transformed Hamiltonian cyclic in both Q_1 and Q_2. If you can't find it, your answer to part (a) wasn't sufficiently general.

d) Given the Hamiltonian of the part (c) answer, solve for the Q_i's and P_i's in terms of the given initial conditions $q_{1,0}$, $q_{2,0}$, $p_{1,0}$ and $p_{2,0}$. You don't have to be completely explicit if that creates a mess.

e) Now find the expressions for the q_i and p_i as functions of time. Again you don't have to be completely explicit if that creates a mess.

Chapt. 10 Hamilton-Jacobi Theory and Action-Angle Variables

Multiple-Choice Problems

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010 qmult 00100 1 1 1 easy memory: Hamilton-Jacobi equation

60. The solution of the Hamilton-Jacobi equation is:

    a) Hamilton's principle function which is the generating function of a canonical transformation to canonical variables that are all **CONSTANTS**.
    b) Hamilton's principle.
    c) Hamilton's principle function which is the generating function of a canonical transformation to canonical variables that are all **NOT CONSTANT**.
    d) the time-dependent quantum mechanical wave function.
    e) Hamilton's principle function which is the generating function of a canonical transformation to canonical variables that are all **CANONICAL MOMENTA**.

---

010 qmult 00200 1 4 3 easy deducto-memory: Hamilton's characteristic function

61. When the Hamiltonian does not depend explicitly on time, Hamilton's principle function can be written

$$S(q,\alpha,t) = W(q,\alpha) - \alpha_t t \ ,$$

    where

    a) W is Hamilton's principle and α_t is the fine structure constant.
    b) W is wabbit function and α_t is the Fudd constant.
    c) W is Hamilton's characteristic function and α_t the constant the Hamiltonian equals.
    d) W is Hamilton's principle function and α_t the constant the Hamiltonian equals.
    e) W is Hamilton's unprincipled function and α_t the constant the Hamiltonian does not equal.

---

010 qmult 00300 1 4 2 easy deducto memory: condition for separability

62. If it is possible to completely segregate a canonical coordinate q_j and its conjugate momentum p_j into some function $f(q_j, p_j)$ such that the restricted Hamilton-Jacobi equation can be written

$$H\left[q', \frac{\partial W'}{q'}, f\left(q_j, \frac{\partial W_j}{q_j}\right)\right] = \alpha_t \ ,$$

    where the primes refer to all canonical pairs that are not the j-pair, α_t is the constant the Hamiltonian equals, and the Hamilton characteristic function is prescribed to be

$$W(q,\alpha) = W'(q',\alpha) + W_j(q_j,\alpha) \ ,$$

    then coordinate q_j is:

    a) inseparable.
    b) separable.
    c) a constant.
    d) a conjugate momentum itself.
    e) righteous.

010 qmult 00400 1 4 5 easy deducto-memory: gravitational torque

63. "Let's play *Jeopardy*! For \$100, the answer is: that elegant classical mechanics equation that is almost only of any practical value in solving specific problems when it is completely separable."

a) What is the Schrödinger equation, Alex?
b) What is $F = ma$, Alex?
c) What is the wave equation, Alex?
d) What is the Hamilton-Jacobi equation, Alex?
e) What is the equilibrium equation, Alex?

Full-Answer Problems

Chapt. 11 Classical Chaos

Multiple-Choice Problems

Full Answer Problems

Chapt. 12 Canonical Perturbation Theory

Multiple-Choice Problems

Full-Answer Problems

Chapt. 13 Lagrangian and Hamiltonian Formulations for Continuous Systems and

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Multiple-Choice Problems

Full-Answer Problems

Appendix 1 Classical Mechanics Equation Sheet

Note: This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things.

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64 **Some Operator Expressions**

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

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65 **Binomial Theorem and Biderivative Formula**

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \qquad \text{binomial theorem}$$

$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k f}{dx^k}\frac{d^{n-k}g}{dx^{n-k}} \qquad \text{biderivative formula}$$

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66 **Trigonometric Identities**

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin\theta = 2\sin(\theta/2)\cos(\theta/2)$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

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67 **Kronecker Delta and Levi-Civita Symbol**

$$\delta_{ij} = \begin{cases} 1, & i=j; \\ 0, & \text{otherwise} \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} 1, & ijk \text{ cyclic}; \\ -1, & ijk \text{ anticyclic}; \\ 0, & \text{if two indices the same.} \end{cases}$$

$$\varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell} \qquad \text{(Einstein summation on } i\text{)}$$

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68 **Lagrangians and Lagrange Equation Versions**

$$L = T - V \qquad L = T - U \qquad L_F = L + \frac{dF(q_j,t)}{dt} \qquad L_{\text{ext}} = L + \sum_k \lambda_k f_k(q_j,\dot{q}_j,t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j$$

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69 **Forces and Potentials**

$$f(x)=-kx \qquad V(x)=\frac{1}{2}kx^2$$ linear force

$$\mathbf{F}(\mathbf{r})=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \qquad U=q(\phi-\mathbf{A}\cdot\mathbf{v})$$ Lorentz force

$$F=-\sum_i k_i v_i \qquad \mathcal{F}=\frac{1}{2}\sum_i k_i v_i^2$$ Rayleigh's dissipation function

$$\mathbf{F}_{12}=-\frac{k}{r_{12}^2}\hat{\mathbf{r}}_{12} \qquad V_{12}=-\frac{k}{r_{12}}$$ inverse-square law

$$\mathbf{F}_{12}=-\frac{Gm_1m_2}{r_{12}^2}\hat{\mathbf{r}}_{12} \qquad V_{12}=-\frac{Gm_1m_2}{r_{12}} \qquad \oint \mathbf{f}\cdot d\mathbf{A}=-4\pi G M_{\text{enc}}$$ gravitation

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70 **Central Force Problem**

$$\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1 \qquad \mathbf{R}=\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2} \qquad \mathbf{r}_2=\mathbf{R}+\frac{m_1}{M}\mathbf{r} \qquad \mathbf{r}_1=\mathbf{R}-\frac{m_2}{M}\mathbf{r}$$

$$M=m_1+m_2 \qquad m=\frac{m_1m_2}{m_1+m_2}$$

$$\ell=mr^2\dot{\theta} \qquad \frac{dA}{dt}=\frac{1}{2}\frac{\ell}{m}$$

$$m\ddot{r}-mr\dot{\theta}^2=f(r) \qquad m\ddot{r}-\frac{\ell^2}{mr^3}=f(r)$$

$$\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\dot{\theta}^2+V=E \qquad \frac{1}{2}m\dot{r}^2+\frac{1}{2}\frac{\ell^2}{mr^2}+V=E$$

$$\overline{T}=-\frac{1}{2}\overline{\sum_i \mathbf{F}_i\cdot\mathbf{r}_i} \qquad \overline{T}=-\frac{(\ell-1)}{2}\overline{V} \qquad \overline{T}=-\frac{1}{2}\overline{V}$$

$$\left(\frac{x}{a}\right)^2\pm\left(\frac{y}{b}\right)^2=1 \qquad r=\frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}=\frac{p(1+\epsilon)}{1+\epsilon\cos\theta} \qquad r_{\text{focus }1}+r_{\text{focus }2}=2a$$

$$\frac{\ell}{mk}=a(1-\epsilon^2)=p(1+\epsilon) \qquad E=-\frac{k}{2a} \qquad \epsilon=\sqrt{1+\frac{2E\ell^2}{mk^2}}=\sqrt{1-\frac{\ell^2}{mka}}$$

$$\tau=2\pi\sqrt{\frac{ma^3}{k}} \qquad \theta_{\text{mean}}=\omega t \qquad \omega=\sqrt{\frac{k}{ma^3}} \qquad \theta_{\text{mean}}=\omega t$$

$$r = a(1 - \epsilon\cos\psi) \qquad \omega t = \psi - \epsilon\sin(\psi)$$

$$\cos\theta = \frac{\cos\psi - \epsilon}{1 - \cos\psi} \qquad \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\left(\frac{\psi}{2}\right)$$

71 Hamiltonian Formulation

$$H = p_i\dot{q}_i - L \qquad \dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{q} = \frac{\partial H}{\partial p} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\text{if} \qquad L = L_0(q,t) + L_{1,j}(q,t)\dot{q}_j + L_{2,jk}(q,t)\dot{q}_j\dot{q}_k \ ,$$

$$\text{then} \qquad p_i = L_{1,i} + L_{2,ik}(q,t)\dot{q}_k + L_{2,ji}(q,t)\dot{q}_j$$

$$\text{and} \qquad H = p_i\dot{q}_i - L = L_{2,j,k}(q,t)\dot{q}_j\dot{q}_k - L_0$$

$$H = T + V$$

$$\text{if} \qquad L(q,\dot{q},t) = L_0(q,t) + \dot{\mathbf{q}}^T\mathbf{a}(q,t) + \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{T}(q,t)\dot{\mathbf{q}} \ , \qquad \text{then} \qquad \mathbf{p} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{a}$$

$$\text{and} \qquad H = \frac{1}{2}(\mathbf{p}^T - \mathbf{a}^T)\mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}) - L_0(q,t)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad I = \int_{t_1}^{t_2}\left[p_i\dot{q}_i - H(q,p,t) + \frac{dF(q,p,t)}{dt}\right]dt$$

72 Canonical Transformations

$$\lambda\left[p_i\dot{q}_i - H(q,p,t)\right] = P_i\dot{Q}_i - K(Q,P,t) \qquad \lambda = uv \qquad K = \lambda H \qquad Q_i = uq_i \qquad P_i = vp_i$$

$$p_i\dot{q}_i - H(q,p,t) = P_i\dot{Q}_i - K(Q,P,t) + \frac{dF}{dt} \qquad K = H + \frac{\partial F_i}{\partial t}$$

$$F = F_1(q,Q,t) \qquad p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$F = F_2(q,P,t) - Q_iP_i \qquad p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$F = F_3(p,Q,t) + q_ip_i \qquad q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F = F_4(p,P,t) + q_ip_i - Q_iP_i \qquad q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i}$$

$$F_2 = f_i(q,t)P_i + g(q,t) \qquad Q_j = f_j(q,t) \qquad p_j = \frac{\partial f_i}{\partial q_j}P_i + \frac{\partial g}{\partial q_j}$$

$$\mathbf{p} = \mathbf{P}\mathbf{f_q} + \mathbf{g_q} \qquad \mathbf{P} = (\mathbf{p} - \mathbf{g_q})\mathbf{f_q}^{-1}$$

73 **Hamilton-Jacobi Theory**

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \qquad S = S(q, \alpha, t)$$

$$P_i = \alpha_i \qquad p_i = \frac{\partial S}{\partial q_i} \qquad \beta_i = \frac{\partial S}{\partial \alpha_i} \qquad q_i = q(\alpha, \beta, t) \qquad p_i = p(\alpha, \beta, t)$$

$$S(q, \alpha, t) = W(q, \alpha) - \alpha_t t \qquad p_i = \frac{\partial W}{\partial \alpha_i} \qquad H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_t$$

$$P_i = \alpha_i \qquad \dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = \begin{cases} 1\ , & i = t \\ 0\ , & i \neq t \end{cases} \qquad Q_i = \frac{\partial W}{\partial \alpha_i} = \begin{cases} t + \beta_t\ , & i = t \\ \beta_i\ , & i \neq t \end{cases}$$

$$H\left[q', \frac{\partial W'}{q'}, f\left(q_j, \frac{\partial W_j}{q_j}\right)\right] = \alpha_t \qquad W(q, \alpha) = W'(q', \alpha) + W_j(q_j, \alpha)$$

$$f\left(q_j, \frac{\partial W_j}{q_j}\right) = \alpha_j$$

Appendix 2 Multiple-Choice Problem Answer Tables

Note: For those who find scantrons frequently inaccurate and prefer to have their own table and marking template, the following are provided. I got the template trick from Neil Huffacker at University of Oklahoma. One just punches out the right answer places on an answer table and overlays it on student answer tables and quickly identifies and marks the wrong answers

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
74.	O	O	O	O	O	6.	O	O	O	O	O
75.	O	O	O	O	O	7.	O	O	O	O	O
76.	O	O	O	O	O	8.	O	O	O	O	O
77.	O	O	O	O	O	9.	O	O	O	O	O
78.	O	O	O	O	O	10.	O	O	O	O	O

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e
79.	O	O	O	O	O
80.	O	O	O	O	O
81.	O	O	O	O	O
82.	O	O	O	O	O
83.	O	O	O	O	O
84.	O	O	O	O	O
85.	O	O	O	O	O
86.	O	O	O	O	O
87.	O	O	O	O	O
88.	O	O	O	O	O

	a	b	c	d	e
11.	O	O	O	O	O
12.	O	O	O	O	O
13.	O	O	O	O	O
14.	O	O	O	O	O
15.	O	O	O	O	O
16.	O	O	O	O	O
17.	O	O	O	O	O
18.	O	O	O	O	O
19.	O	O	O	O	O
20.	O	O	O	O	O

NAME: