Mathematical Physics Problems

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2008 January 1

Introduction

Mathematical Physics Problems (MPP) is a source book for a mathematical physics course. The book is available in electronic form to instructors by request to the author. It is free courseware and can be freely used and distributed, but not used for commercial purposes.

The problems are grouped by topics in chapters: see Contents below. The chapters correspond to the chapters of Weber & Arfken (2004). For each chapter there are two classes of problems: in order of appearance in a chapter they are: (1) multiple-choice problems and (2) full-answer problems. All the problems have will have complete suggested answers eventually. The answers may be the greatest benefit of MPP. The questions and answers can be posted on the web in pdf format.

The problems have been suggested by mainy by Weber & Arfken (2004) and Arfken (1970), they all been written by me. Given that the ideas for problems are the common coin of the realm, I prefer to call my versions of the problems redactions.

At the end of the book are three appendices. The first is an equation sheet suitable to give to students as a test aid and a review sheet. The next is a table of integrals. The last one is a set of answer tables for multiple choice questions. The first two appendices need to be updated for the mathematical physics course. They are still for an intro physics course.

MPP is currently under construction and whether it will grow to adequate size depends on whether I have any chance to teach mathematical physics again.

Everything is written in plain TEX in my own idiosyncratic style. The questions are all have codes and keywords for easy selection electronically or by hand. The keywords will be on the question code line with additional ones on the extra keyword line which may also have a reference for the problem A fortran program for selecting the problems and outputting them in quiz, assignment, and test formats is also available. Note the quiz, etc. creation procedure is a bit clonky, but it works. User instructors could easily construct their own programs for problem selection.

I would like to thank the Department of Physics & Astronomy of the University of Nevada, Las Vegas for its support for this work. Thanks also to the students who helped flight-test the problems.

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Chapt. 1 Vector Analysis

Multiple-Choice Problems

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001 qmult 00100 1 4 5 easy deducto-memory: seven samurai

Extra keywords: not a serious question

1. "Let's play *Jeopardy*! For \$100, the answer is: In Akira Kurosawa's film *The Seven Samurai* in the misremembering of popular memory, what the samurai leader said when one of the seven asked why they were going to defend this miserable village from a horde of marauding bandits."

   What is "____________," Alex?

   a) For honor. b) It is the way of the samurai. c) It is the Tao. d) For a few dollars more. e) For the fun of it.

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001 qmult 00202 1 4 4 easy deducto-memory: Einstein summation

Extra keywords: mathematical physics

2. "Let's play *Jeopardy*! For \$100, the answer is: The person who made the remark to his/her friend Louis Kollros (1878–1959): 'I made a great discovery in mathematics: I suppressed the summation sign every time that the summation has to be done on an index which appears twice in the general term.' "

   Who is ____________, Alex?

   a) Saint Gall (circa 550–646) b) Wilhelm Tell (fl. circa 1300) c) Henri Dunant (1828–1910) d) Albert Einstein (1879–1955) e) Friedrich Dürrenmatt (1921–1990)

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001 qmult 00350 1 4 2 easy deducto-memory: Levi-Civita symbol identity

Extra keywords: WA-156-2.9.4 gives this identity

3. "Let's play *Jeopardy*! For \$100, the answer is:

$$\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \ .$$

   What is ____________, Alex?

   a) $\vec{A} \times (\vec{B} \times \vec{C})$ b) $\varepsilon_{kij}\varepsilon_{k\ell m}$ c) ε_{ijk} d) the Levi-Civita symbol e) the Synge-Yeats symbol

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001 qmult 00410 1 1 3 easy memory: triple scalar product

4. The **ABSOLUTE VALUE** of the triple scalar product of three vectors has a geometrical interpretation as:

   a) the plane spanned by the three vectors.
   b) a fourth vector orthogonal to the other three.
   c) the volume of a parallelepiped defined by the three vectors.
   d) the area of a left-hand rule quadrilateral.
   e) the direction of steepest descent from the vector peak.

---

001 qmult 00530 1 4 1 easy deducto-memory: gradient does what
Extra keywords: mathematical physics

5. "Let's play *Jeopardy*! For \$100, the answer is: It is a vector field whose direction at every point in space gives the direction of maximum space rate of increase of a scalar function $f(\vec{r})$."

   What is the ____________ of $f(\vec{r})$, Alex?

   a) gradient b) divergence c) curl d) Gauss e) ceorl

---

001 qmult 00610 1 4 2 easy deducto-memory: divergence
Extra keywords: mathematical physics WA-46

6. "Let's play *Jeopardy*! For \$100, the answer is: For a current density vector field, it is the net outflow rate per unit volume of whatever quantity makes up the current (e.g., mass or charge)."

   What is the ____________, Alex?

   a) gradient b) divergence c) curl d) Laplacian e) Gaussian

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001 qmult 01110 1 1 1 easy memory: Gauss's theorem stated

7. The formula

$$\oint_{\mathrm{S}} \vec{F} \cdot d\vec{\sigma} = \int_V \nabla \cdot \vec{F}\, dV$$

   (where $\vec{F}$ is a general vector field, the first integral is over a closed surface S, and the second over the volume V enclosed by the closed surface) is

   a) Gauss's theorem. b) Green's theorem. c) Stokes's theorem.
   d) Gauss's law. e) Noether's theorem.

---

001 qmult 01220 2 1 5 mod memory: potential implications

8. The expression

$$\nabla \times \vec{F} = 0$$

   for vector field $\vec{F}$, implies

   a) $\vec{F} = -\nabla\phi$ where ϕ is some scalar field.
   b) $\oint_C F \cdot d\vec{r} = 0$ for all closed contours C.
   c) $\int_{\vec{a}}^{\vec{b}} F \cdot d\vec{r}$ is independent of the path from $\vec{a}$ to $\vec{b}$.
   d) $\vec{F}$ is curlless or irrotational.
   e) all of the above.

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001 qmult 01310 1 4 4 easy deducto-memory: Gauss's law ingredient

9. Gauss's law in integral form is for electromagnetism

$$\oint_{\mathrm{S}} \vec{E} \cdot d\vec{\sigma} = \frac{q}{\varepsilon_0}$$

   (where S is a closed surface, $\vec{E}$ is the electric field, q is the charge enclosed inside the closed surface, and ε_0 is the vacuum permittivity or permittivity of free space) and for gravity is

$$\oint_{\mathrm{S}} \vec{g} \cdot d\vec{\sigma} = -4\pi GM$$

   (where S is a closed surface, $\vec{g}$ is the gravitational field, M is the mass enclosed inside the closed surface, and G is the gravitational constant). The key ingredient in deriving the integral form of Gauss's law is:

a) the fact that the charge or mass is entirely contained inside the closed surface. Gauss's law does **NOT** apply in cases where there is charge or mass outside of the closed surface.
b) the fact that the closed surface has spherical, cylindrical, or planar symmetry.
c) the use of Stokes's theorem.
d) the inverse-square-law nature of the electric and gravitational forces.
e) all of the above.

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001 qmult 01320 1 4 5 easy deducto-memory: Gauss's law symmetries
Extra keywords: three geometries

10. "Let's play *Jeopardy*! For $100, the answer is: They are the three cases of high symmetry to which the integral form of Gauss's law can be applied to obtain directly analytic solutions for the electric or gravitational field."

What are ____________ symmetries, Alex?

a) cubic, cylindrical, planar b) simple cubic, faced-centered cubic, body-centered cubic c) tubular, muscular, jugular d) spherical, elliptical, hyperbolical e) spherical, cylindrical, planar

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000 qmult 01430 1 4 2 easy deducto-memory: Dirac delta use
Extra keywords: mathematical physics

11. "Let's play *Jeopardy*! For $100, the answer is: It can be considered as a means for modeling the behavior inside of an integral of a normalized function whose characteristic width is small compared to all other physical scales in the system for which the integral is invoked."

What is the ____________, Alex?

a) Kronecker delta function b) Dirac delta function c) Gaussian d) Laurentzian e) Heaviside step function

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Full-Answer Problems

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001 qfull 00210 1 3 0 easy math: Schwarz inequality
Extra keywords: Schwarz inequality for simple vectors: WA-513

12. Given vectors $\vec{A}$ and $\vec{B}$ show that

$$|\vec{A}\cdot\vec{B}| \leq AB \ ,$$

where A and B are, respectively, the magnitudes of $\vec{A}$ and $\vec{B}$. The inequality is the Schwarz inequality for the special case of simple vectors.

---

001 qfull 00220 1 3 0 easy math: Einstein summation rule
Extra keywords: and Kronecker delta

13. The Einstein summation rule (or Einstein summation) is to sum on a repeated dummy index and suppress the explicit summation sign: e.g.,

$$A_iB_i \qquad \text{means} \qquad \sum_i A_iB_i \ .$$

This rule is very useful in compactifying vector and tensor expressions and it adds a great deal of mental clarity. Of course, in some cases it cannot be used. For instance if a dummy index is repeated more than once for some reason (i.e., the dummy index appears 3 or more times in a term). The Kronecker delta often turns up when using the Einstein summation rule and in other contexts. It has the definition

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

The Kronecker delta is obviously symmetric under interchange of its indices: i.e., $\delta_{ij} = \delta_{ji}$. In this question, the Einstein summation rule is **TURNED ON** unless otherwise noted. **NOTE:** The indices run over 1, 2, and 3, unless otherwise noted.

a) Given that index runs over 1 and 2 (note this), expand $A_i B_i$ and $A_i B_i C_j D_j$ in the values of the indices. Verify that the latter expression is **NOT** the same as

$$\sum_i A_i B_i C_i D_i \ ,$$

where the Einstein summation rule has been turned off since the index is repeated more than once.

b) Prove by inspection (i.e., by staring at it) that

$$A_{ij}\delta_{ij} = A_{ii} \ .$$

Verify this result explicitly for the indices running over 1, 2, and 3 (as goes without saying). This result shows that the Kronecker delta will kill a summation. Note that A_{ij} could be $B_i C_j$.

c) What is δ_{ii} equal to? **HINT:** This is so easy.

d) What is $\delta_{ij}\delta_{ij}$ equal to?

e) What is $\delta_{ij}\delta_{jk}$ equal to? Give a word argument to prove the identity.

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001 qfull 00302 1 3 0 easy math: general symmetry identity

Extra keywords: Levi-Civita symbol general symmetry identity

14. Say you have a mathematical object whose components are identified by specifying two indices and which is symmetric under the interchange of the index values: i.e.,

$$H_{jk} = H_{kj} \ ,$$

where the indices can take on three values which without loss of generality we can call 1, 2, and 3. Such an object may be a set of second partial derivatives (provided they are continuous [WA-55]): e.g.,

$$\frac{\partial^2 \phi}{\partial x_j \partial x_k} = \frac{\partial^2 \phi}{\partial x_k \partial x_j} \ .$$

Or the object's components could each be a product of two components of one vector multiplied with each other: i.e.,

$$A_j A_k = A_k A_j \ ,$$

where the vector is $\vec{A}$. Or the object could be a tensor symmetric in two indices. Note it is common practice to refer to such object, particularly tensors, by just specifying a typical component: e.g., tensor H_{jk} (WA-146).

Whatever the object is, show that

$$\varepsilon_{ijk} H_{jk} = 0 \ ,$$

where ε_{ijk} is the Levi-Civita symbol and we have used the Einstein summation rule. Since this identity has no particular name it seems, I just call it the Levi-Civita symbol general symmetry identity or the general symmetry identity for short.

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001 qfull 00315 1 3 0 easy math: dot product identity
Extra keywords: dot product of the same cross product (WA-28-1.3.6)

15. Do the following.

a) Using the Levi-Civita symbol and the Einstein summation rule prove

$$(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = (AB)^2 - (\vec{A} \cdot \vec{B})^2 \ ,$$

where A and B are, respectively the magnitudes of $\vec{A}$ and $\vec{B}$.

b) Prove that the last result implies

$$\sin^2\theta + \cos^2\theta = 1 \ ,$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

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001 qfull 00320 2 3 0 moderate math: law of sines
Extra keywords: based on WA-28-1.3.9

16. Prove the law of sines for a general triangle whose sides are composed of vectors $\vec{A}$ (opposite angle α), $\vec{B}$ (opposite angle β), and $\vec{C}$ (opposite angle γ). With these labels the law of sines is

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma} \ ,$$

where the italic letters are the magnitudes of the corresponding vectors. **HINT:** Draw a diagram and exploit the vector cross product and vector addition.

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001 qfull 00390 3 3 0 tough math: Levi-Civita symbol identity
Extra keywords: WA-156-2.9.4-2.9.3 proof, Wik says varepsilon

17. In this problem, we consider the identity

$$\varepsilon_{kij}\varepsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \ .$$

This identity seems to have no conventional name despite its great utility. The author just calls it the Levi-Civita symbol identity.

a) Show that a general component i of

$$\vec{A} \times (\vec{B} \times \vec{C})$$

is given by

$$\left[\vec{A} \times (\vec{B} \times \vec{C})\right]_i = \varepsilon_{kij}\varepsilon_{k\ell m} A_j B_\ell C_m \ .$$

This expression shows why the entity $\varepsilon_{kij}\varepsilon_{k\ell m}$ is of great interest. It turns up whenever one has successive cross products or successive curl operators or combinations of cross products and curl operators.

b) Explicitly expand the k summation of $\varepsilon_{kij}\varepsilon_{k\ell m}$: i.e., write the summation explicitly in 3 terms.

c) Argue that $\varepsilon_{kij}\varepsilon_{k\ell m}$ is zero if the indices $ij\ell m$ span 1 value only: i.e., if i, j, ℓ, and m all have the same value.

d) Argue that $\varepsilon_{kij}\varepsilon_{k\ell m}$ is zero if the indices $ij\ell m$ span 3 values: i.e., if all of 1, 2, and 3 occur among i, j, ℓ, and m.

e) Having proven that $\varepsilon_{kij}\varepsilon_{k\ell m}$ is zero in the cases where $ij\ell m$ span 3 and 1 values, only the case where $ij\ell m$ span 2 values is left as a possibility for a non-zero result. Say that p and q are the 2 distinct values that $ij\ell m$ span. How many possible ways are there to choose $ij\ell m$ for a non-zero result for $\varepsilon_{kij}\varepsilon_{k\ell m}$? What are the non-zero results?

f) Now consider the Kronecker delta expression

$$\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$$

in the case where the indices $ij\ell m$ span 2 values. Say that p and q are the 2 distinct values that $ij\ell m$ span. There are clearly $2^4 - 2 = 16 - 2 = 14$ ways of choosing $ij\ell m$: the -2 prevents overcounting from the all p and all q choices included in the $2^4 = 16$ possibilities. But the situation is made simpler by proving that if either $i = j$ or $\ell = m$, the Kronecker delta expression is zero. Make that argument and find and evaluate the non-zero cases.

g) Show $\varepsilon_{kij}\varepsilon_{k\ell m}$ and the Kronecker delta expression $\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$ are equal in the case where $ij\ell m$ span 2 values.

h) Now argue that

$$\varepsilon_{kij}\varepsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$$

holds generally. This completes the proof of the Levi-Civita symbol identity.

i) Show that

$$\varepsilon_{kij}\varepsilon_{kim} = \varepsilon_{kji}\varepsilon_{kmi} = \varepsilon_{jki}\varepsilon_{mki} = 2\delta_{jm} \ .$$

j) Show that

$$\varepsilon_{kij}\varepsilon_{kij} = 6 \ .$$

k) Show that

$$\delta_{jk}\varepsilon_{ijk} = 0 \ .$$

l) This last part is strictly voluntary and is unmarked on homeworks and tests. It is only recommended for students whose obstinacy knows no bounds. There is actually an alternative derivation of the Levi-Civita identity. One must turn the Einstein summation rule off for the whole derivation, except one turns it on for the last step to get the identity in standard form. Without the Einstein summation rule, note that the Levi-Civita identity becomes

$$\sum_k \varepsilon_{kij}\varepsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \ .$$

First, one defines the cycle function

$$f(i) = \begin{cases} i & \text{for } i = 1,2,3; \\ 1 & \text{for } i = 4; \\ 2 & \text{for } i = 5; \\ 3 & \text{for } i = 6; \\ \text{mod}(i-1,3)+1 & \text{for integer } i \geq 1 \text{ in general.} \end{cases}$$

Note that $f(i+3) = f(i)$ and this identity (along with its versions $f[i+2] = f[i-1]$, etc.) must be used in the proof. Note that the mod function has the following property:

$$\text{mod}(kn+j,k) = j \qquad \text{for } j = 0,1,2,\ldots,k-1 \text{ and integer } n \geq 0 \ .$$

If one wants a similar function that runs over the range $j = 1, 2, 3, \ldots, k$, then somewhat clearly one needs

$$\text{mod}(kn + j - 1, k) + 1 = j \ .$$

Now

$$\text{mod}(kn + j - 1 + k, k) + 1 = \text{mod}[k(n + 1) + j - 1, k] + 1 = \text{mod}(kn + j - 1, k) + 1 \ .$$

Thus, our function $f(i)$ has the properties we claim for it.
Now after some head shaking, one sees that

$$\varepsilon_{kij} = \delta_{f(k+1)i}\delta_{f(i+1)j} - \delta_{f(k+2)i}\delta_{f(i+2)j} \ .$$

This is sort of clear. The Levi-Civita symbol gives 1 for cyclic ordering of indices and -1 for an anticyclic ordering of the indices. For a cyclic ordering, for each index i, $f(i+1)$ equals the next index in the ordering. For an anticyclic ordering, for each index i, $f(i+2)$ equals the next index in the ordering. The expression is, indeed, equivalent to the Levi-Civita symbol. Now complete the proof.

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001 qfull 00392 1 3 0 hard math math: Levi-Civita symbol, epsilon identity

18. The Levi-Civita symbol is defined

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } ijk \text{ cyclic;} \\ -1 & \text{for } ijk \text{ anticyclic;} \\ 0 & \text{for any pair of } ijk \text{ having the same value,} \end{cases}$$

where indicies ijk can take on values 1, 2, and 3. Note the Levi-Civita symbol indices in this problem are taken to span all values unless explicitly stated otherwise.

NOTE: There are parts a,b,c,d,e,f,g. The parts can all be done independently. So don't stop if you can't do a part.

a) Prove

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{cases} 1 & \text{for } ijk \text{ and } lmn \text{ having the same cyclicity;} \\ -1 & \text{for } ijk \text{ and } lmn \text{ having the different cyclicity;} \\ 0 & \text{for any pair of } ijk \text{ or } lmn \text{ having the same value.} \end{cases}$$

HINT: Recall the phrase "by inspection."

b) The epsilon identity is

$$\varepsilon_{ijk}\varepsilon_{lmn} = f(i, j, k; l, m, n) \ ,$$

where

$$\begin{aligned} f(i, j, k; l, m, n) &= \delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{in}\delta_{jm}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} \\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \ , \end{aligned}$$

where lmn run through all possible $3! = 6$ permutations in the terms as one can see and the δ's are Kronecker deltas. Recall the Kronecker delta is 1 if the indices are equal and zero otherwise. Prove the epsilon identity for the special case that both index sets ijk and lmn have all distinct values. **HINT:** This is not hard.

c) **On a test, do NOT to this part and assume the result.** Prove the epsilon identity when at least one pair of indices in ijk or in lmn have the same value. **HINT:** To be cogent and concise, and exploit symmetries.

d) Now prove the epsilon identity in general. **HINT:** This is easy.

e) Prove the contracted epsilon identity

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \ ,$$

where the Einstein sum rule for repeated indices is used: i.e., there is a sum over all values of i.

f) Prove the doubly contracted epsilon identity

$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn} \ ,$$

where the Einstein sum rule for repeated indices is used again.

g) Prove the triply contracted epsilon identity

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \ ,$$

where the Einstein sum rule for repeated indices is used again.

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001 qfull 00410 1 3 0 easy math: BAC-CAB rule (e.g., WA-32)
Extra keywords: This is a better question than WA-33-1.4.1

19. Prove the BAC-CAB rule,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \ ,$$

using the Levi-Civita symbol ε_{ijk} and the Einstein summation rule.

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001 qfull 00430 2 3 0 moderate math: ang. mon. and rotational inertia
Extra keywords: WA-33-1.4.3

20. A particle has angular momentum $\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$, where $\vec{r}$ is particle position, $\vec{p}$ is particle momentum, m is the particle mass, and $\vec{v}$ is particle velocity. Now $\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{\omega}$ is the angular velocity.

a) Show using the Levi-Civita symbol that

$$\vec{L} = mr^2[\vec{\omega} - \hat{r}(\hat{r} \cdot \vec{\omega})] \ ,$$

where $\hat{r}$ is the unit vector in the direction of $\vec{r}$ and r is the magnitude of $\vec{r}$.

b) If $\hat{r} \cdot \vec{\omega} = 0$, the expression for angular momentum in part (a) reduces to $\vec{L} = I\vec{\omega}$, where $I = mr^2$ is the moment of inertia or rotational inertia. Argue that the rotational inertia of a rigid body rotating with $\vec{\omega}$ is

$$I = \int_{\text{body}} \rho r^2 \, dV \ ,$$

where ρ is the density, r here is the cylindrical coordinate radius from the axis of rotation (i.e., the radius measured perpendicular from the axis of rotation), and V is volume.

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001 qfull 00504 2 3 0 moderate math: undetermined Lagrange multipliers
Extra keywords: WA-38-ex-1.5.3 Dull question.

21. Do the following problems:

a) The equation of an ellipse not aligned with the x-y coordinate system is:

$$x^2 + y^2 + xy = 1 \ .$$

To find the ellipse in an aligned coordinate system use the transformations

$$x = x' \cos\theta + y' \sin\theta$$

and

$$y = -x' \sin\theta + y' \cos\theta \ ,$$

where going from the primed to the unprimed coordinates rotates the system counterclockwise by an angle θ and going from the unprimed to the primed coordiantes rotates the system clockwise by θ (WA-137–138). In this case, $\theta = \pi/4 = 45°$. Find the equation of the ellipse in the primed system and determine its semimajor and semiminor axes.

b) Consider the squared-distance-from-the-origin function

$$F(x,y) = x^2 + y^2 \ .$$

Find the locations of F's extrema and the extremal values subject to the constraint of x and y lying on the ellipse of part (a): i.e., subject to the constraint function $G(x,y) = x^2 + y^2 + xy - 1 = 0$. Use (undetermined) Lagrange multipliers and determine the values of the multipliers. What is the relation of the extremal values to the semimajor and semiminor axes?

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001 qfull 00530 1 3 0 easy math: full derivative of a vector function
Extra keywords: WA-44-1.5.3

22. We are given $\vec{F}$ as an explicit function of position $\vec{r}$ and time t: i.e.,

$$\vec{F} = \vec{F}(\vec{r},t) \ .$$

Show that

$$d\vec{F} = (d\vec{r}\cdot\nabla)\vec{F} + \frac{\partial \vec{F}}{\partial t} dt \ ,$$

where we interpret $\nabla\vec{F}$ as a nine-component quantity consisting of the gradient of each component of $\vec{F}$. How is $(d\vec{r}\cdot\nabla)\vec{F}$ to be interpreted? **HINT:** This is really easy. All that is needed is to show that ordinary scalar calculus operations on a component of $\vec{F}$ lead to the given expression with the right interpretation.

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001 qfull 00540 2 3 0 moderate math: parallel gradients
Extra keywords: WA-44-1.5.4

23. Do the following:

a) Given differentiable scalar functions $u(\vec{r})$ and $v(\vec{r})$, show that

$$\nabla(uv) = v\nabla u + u\nabla v \ .$$

The product-rule expression to be proven in this problem part looks pretty obviously true. But it does involve a vector operator, and thus one needs to prove that the ordinary scalar product rule leads to this vector product rule.

b) What does $\nabla u \times \nabla v = 0$ imply geometrically speaking about u and v? Assume here and in subsequent parts of this question that ∇u and ∇v are non-zero.

c) Show that $\nabla u \times \nabla v = 0$ is a necessary condition for u and v to be related by some function $f(u,v) = 0$. Translating the last sentence into plain English, show that $f(u,v) = 0$ implies $\nabla u \times \nabla v = 0$. Assume that $f(u,v)$ is non-trivial: i.e., a variation in u implies a variation in v and vice versa.

d) Show that $\nabla u \times \nabla v = 0$ is a sufficient condition for u and v to be related by some function $f(u,v) = 0$. In other words, show that $\nabla u \times \nabla v = 0$ implies $f(u,v) = 0$.

---

001 qfull 00620 1 3 0 easy math: vector identities
Extra keywords: WA-47-1.6.2 which isn't much about divergence

24. The product-rule expressions to be proven in this question look pretty obviously true. But they do involve vector objects (i.e., vector quantities and operators), and thus one needs to prove that the ordinary scalar product rule leads to these vectorial product rules.

    a) Show that
    $$\nabla \cdot (f\vec{V}) = \nabla f \cdot \vec{V} + f\nabla \cdot \vec{V} ,$$
    using the Einstein summation rule.

    b) Show that
    $$\frac{d(\vec{A} \cdot \vec{B})}{dt} = \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt} ,$$
    using the Einstein summation rule.

    c) Show that
    $$\frac{d(\vec{A} \times \vec{B})}{dt} = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt} ,$$
    using the Einstein summation rule and the Levi-Civita symbol.

---

001 qfull 00709 2 3 0 moderate math: magnetic moment
Extra keywords: WA-52-1.7.9

25. The force on a **CONSTANT** magnetic moment $\vec{M}$ (which the non-physics majors can just regard as an arbitrary vector) in an external magnetic field $\vec{B}$ is
    $$\vec{F} = \nabla \times (\vec{B} \times \vec{M}) .$$
    From Maxwell's equations, we have $\nabla \cdot \vec{B} = 0$ and for time constant fields (which we assume here) $\nabla \times \vec{B} = 0$ (e.g., WA-56).

    a) Given $\nabla \times \vec{B} = 0$, show that
    $$\frac{\partial B_j}{\partial x_i} = \frac{\partial B_i}{\partial x_j} ,$$
    where i and j are general indices. Remember to consider the case of $i = j$ without any Einstein summation being implied.

    b) Now show that
    $$\vec{F} = \nabla(\vec{B} \cdot \vec{M}) ,$$
    where recall that $\vec{M}$ is a constant. **HINT:** You need the part (a) result, but you don't have to have done part (a) to be able use the part (a) result.

---

001 qfull 00712 3 3 0 tough math: QM ang. mom.
Extra keywords: WA-53-1.7.12–13

26. Classical angular momentum is given by the dynamical variable expression
    $$\vec{L} = \vec{r} \times \vec{p} ,$$
    where $\vec{r}$ is position and $\vec{p}$ is momentum. The same expression holds in quantum mechanics, except that the variables are replaced by the corresponding operators. The $\vec{r}$ operator is just the displacement vector and the momentum operator is
    $$\vec{p} = \frac{\hbar}{i}\nabla ,$$
    where $\hbar$ is Planck's constant divided by 2π and $i = \sqrt{-1}$ is the imaginary unit.

a) What is the expression for a general component L_k of the angular momentum operator in Cartesian components using the Einstein summation rule and the Levi-Civita symbol. Note that the index i and the imaginary unit i are not the same thing.

b) Given that

$$H_{\ell pmq} = H_{p\ell mq} = H_{\ell pqm} = H_{p\ell qm} \ ,$$

or in other words that $H_{\ell pmq}$ is symmetric for the interchange of the 1st and 2nd indices and for the 3rd and 4th indices, show that

$$\varepsilon_{i\ell m}\varepsilon_{jpq}H_{\ell pmq} = \varepsilon_{j\ell m}\varepsilon_{ipq}H_{\ell pmq} \ .$$

The entity $H_{\ell pmq}$ is general except for the symmetry propertites we attribute to it. It could be a tensor or tensor operator for example.

c) Show that

$$\epsilon_{kij}\varepsilon_{i\ell m}\varepsilon_{jpq}H_{\ell pmq} = 0 \ .$$

It might help to knote—er note—that $\varepsilon_{i\ell m}\varepsilon_{jpq}H_{\ell pmq}$ only depends on indices i and j: all the other index dependences are eliminated by the summations. Thus, one could define

$$G_{ij} \equiv \varepsilon_{i\ell m}\varepsilon_{jpq}H_{\ell pmq} \ .$$

This definition may make the identity to be proven more identifiable.

d) Prove the operator expression

$$\vec{L} \times \vec{L} = i\hbar\vec{L}$$

using the part (c) answer, the Einstein summation rule, and the Levi-Civita symbol. Remember that quantum mechanical operators are understood to act on everything to the right including an understood invisible general function. Note again that the index i and the imaginary unit i are not the same thing.

e) Given the operator commutator formula

$$[L_i, L_j] = L_iL_j - L_jL_i \ ,$$

show that

$$i\hbar\varepsilon_{ijk}L_k = L_iL_j - L_jL_i = [L_i, L_j] \ .$$

HINT: Using the expressions from part (d) answer makes this rather easy.

---

001 qfull 00802 1 3 0 easy math: double curl identity

Extra keywords: WA-58-1.8.2

27. Using the Einstein summation rule and the Levi-Civita symbol show that

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - (\nabla \cdot \nabla)\vec{A} \ ,$$

where $(\nabla\cdot\nabla)\vec{A}$ is a vector made up of the Laplacians of each component of $\vec{A}$: i.e., for component i, one has $\nabla \cdot \nabla A_i = \nabla^2 A_i$.

---

001 qfull 00905 2 3 0 moderate math: surface integral

Extra keywords: WA-64 surface integral of saddle shape.

28. Consider a surface in three-dimensional Euclidean space defined by

$$z = f(x, y) \ .$$

a) What is the formula for a vector field that is normal to the surface everywhere on the surface and has a positive z-component?

b) What is the formula for a unit vector $\hat{n}$ normal to the surface with the normal vector having a positive z-component?

c) Given that differential area in the x-y plane $dx\,dy$, what is the differential area dA (in terms of $dx\,dy$ and the partial derivatives of the function $f(x,y)$) of the surface $z = f(x,y)$ that overlies $dx\,dy$ and projects onto it? **HINT:** It might be helpful to draw a diagram showing the two differential areas and the normal vector to the surface and the vector $\hat{z}$.

d) Consider the special case of

$$z = xy \ .$$

Describe this surface. **HINT:** Consider it along lines of $y = 0$ (i.e., the x-axis), $x = 0$ (i.e., the y-axis), $y = x$, and $y = -x$.

e) Find the surface area of $z = xy$ over the circular area defined by a radius R. **HINT:** Given the symmetry of the system, a conversion to polar coordinates looks expedient.

---

001 qfull 01010 1 3 0 easy math: closed surface B-field integral
Extra keywords: WA-70-1.10.1

29. Given $\vec{B} = \nabla \times \vec{A}$, show that

$$\oint \vec{B} \cdot d\vec{\sigma} = 0 \ ,$$

where the integral is over a closed surface S with differential surface area vector $d\vec{\sigma}$. The surface bounds volume V. The vector field $\vec{B}$ could be the magnetic field. In this case, $\vec{A}$ is the vector potential.

---

001 qfull 01110 2 3 0 moderate math: area in a plane
Extra keywords: WA-75-1.11.1

30. The area bounded by a contour C on a plane is given by the line integral

$$A = \frac{1}{2}\left|\oint_C \vec{r} \times d\vec{r}\right| \ ,$$

where we are assuming that the contour does not cross itself and $\vec{r}$ is measured from an origin that could be inside or outside the contour. Take counterclockwise integration to be positive. This means that the $\hat{z}$ direction is the direction for positive contributions.

a) Argue that the area formula is correct using vectors and parallelograms and triangles for the case where the origin is inside the contour and the contour shape is such that $\vec{r}$ sweeping around the origin never crosses the contour. **HINT:** A diagram might help.

b) The area formula is more general than the argument of the part (a) answer implies. The area formula is valid for any contour shape that does not cross itself and for any origin. Give the argument for this. **HINT:** You might think of a long wormy bounded region. Consider any sub-region bounded by the contour and two rays radiating from the origin. The sub-region may be bounded by inner and outer contour pieces or there may just be an outer contour piece and the inner boundary of the sub-region just being the origin (which will happen sometimes if the origin is inside the contour). The whole bounded region can be considered as made up of these sub-regions that are as small as you wish. Also remember that $\vec{r} \times d\vec{r}$ is a vector.

c) The perimeter of an ellipse is described by

$$\vec{r} = \hat{x} a \cos\eta + \hat{y} b \sin\eta \ ,$$

where η is not the polar coordinate of $\vec{r}$, but an angular path parameter, and a and b are positive constants. Using the result from the parts (a) and (b) answers show that the area of an ellipse is πab.

c) Verify that the contour equation of part (b) describes an ellipse and find the ellipse formula in x and y coordinates in standard form: i.e., x over one semi-axis all squared plus y over the other semi-axis all squared equal to 1. What are the semi-axes and which is the semimajor axis and which the semiminor axis?

d) Find the relationship between η and polar coordinate θ.

---

001 qfull 01203 2 3 0 moderate math: cross-producted gradients
Extra keywords: WA-80-1.12.3

31. Vector $\vec{B}$ is given by

$$\vec{B} = \nabla u \times \nabla v \ ,$$

where u and v are scalar functions. Recall the general symmetry identity for this problem:

$$\varepsilon_{ijk} H_{jk} = 0$$

if $H_{jk} = H_{kj}$.

a) Show that $\vec{B}$ is solenoidal (i.e., divergenceless).

b) Given

$$\vec{A} = \frac{1}{2}\left(u\nabla v - v\nabla u\right) \ ,$$

show that

$$\vec{B} = \nabla \times \vec{A} \ .$$

---

001 qfull 01302 2 3 0 moderate math: Gauss's law with symmetry
Extra keywords: WA-85, but the it resembles WA-67-1.91.

32. Gauss's law for electrostatics is

$$\oint_S \vec{E} \cdot d\vec{\sigma} = \frac{q}{\epsilon_0} \ ,$$

where $\vec{E}$ is the electric field, S is any simply-connected closed surface (i.e., one without holes), $d\vec{\sigma}$ is the differential surface vector (which points outward from the surface), q is the total charge enclosed, and ϵ_0 is vacuum permittivity (or permittivity of free space).

a) Gauss's law can be used directly to obtain simple, exact, analytic formulae for the electric field and electric potential for three cases of very high symmetry. What are those symmetry cases?

b) What is the electric field everywhere outside of a spherically symmetric body of charge q and radius R? Use spherical coordinates with the origin centered on the center of symmetry.

c) For the electric field from the part (b) answer find the potential with the potential at infinity set to zero. Recall

$$\vec{E} = -\nabla\phi \ ,$$

where ϕ is the potential and in spherical coordinates for a spherically symmetric scalar field

$$\nabla = \hat{r}\frac{\partial}{\partial r} \ .$$

d) What is the electric field everywhere outside of a cylindrically symmetric body of linear charge density λ and radius R? Use cylindrical coordinates with the axis on the axis of symmetry.

e) For the electric field from the part (d) answer find the potential with zero potential at a fiducial radius $R_{\rm fid}$. Recall

$$\vec{E} = -\nabla\phi \ ,$$

where ϕ is the potential and in cylindrical coordinates for a cylindrically symmetric scalar field

$$\nabla = \hat{r}\frac{\partial}{\partial r} \ .$$

Why can't the potential be set to zero at infinity in cylindrical symmetry?

f) What is the electric field everywhere outside of a planar symmetric body of area charge density η and thinkness Z? Use Cartesian coordinates with the z-axis perpendicular to the plane of symmetry and the origin on the central plane of the body.

g) For the electric field from the part (f) answer find the potential with zero potential at a fiducial distance $\pm Z_{\rm fid}$ from the plane of symmetry. Recall

$$\vec{E} = -\nabla\phi \ ,$$

where ϕ is the potential and in planar coordinates for a planar symmetric scalar field with the plane of symmetry being the $z = 0$ plane

$$\nabla = \hat{z}\frac{\partial}{\partial z} \ .$$

Why can't the potential be set to zero at infinity in planar symmetry?

---

001 qfull 01410 2 3 0 moderate math: Dirac delta function
Extra keywords: WA-90

33. The Dirac delta function $\delta(x)$ is incompletely defined by the following two properties:

$$\delta(x) = 0 \qquad \text{for } x \neq 0 \qquad \text{and} \qquad f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx \ .$$

Note that the two properties imply

$$\int_{-A}^{B} f(x)\delta(x)\,dx = \begin{cases} f(0) & \text{for } 0 \in (A,B); \\ \text{undefined} & \text{for } A = 0 \text{ and/or } B = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The definition is incomplete because the Dirac delta function as defined is not a real function because of the divergence at $x = 0$. The Dirac delta function is actually a short-hand for the limit of a sequence of integrals with a sharply peaked normalized function $\delta_n(x)$ (with peak at $x = 0$ and parameter n controlling width and height) as a factor in the integrand. In the limit, as $n \to \infty$, the width of $\delta_n(x)$ vanishes, but function does not really have a limit as $n \to \infty$ since all forms of $\delta_n(x)$ diverge at $x = 0$ in this limit. (At least all forms that are commonly cited.) The sequence of integrals has the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\delta_n(x)\,dx = f(0) \ .$$

Now having said enough to satisfy mathematical rigor, we note that in many cases in physics, the Dirac delta function is used to model a real normalized function whose region of significant non-zero behavior around its fiducial geometrical central point (which is characterized by its characteristic width around its fiducial geometrical central point) is much smaller than any other scale of variation in the system. This real normalized function has all the ordinary mathematical properties. Thus, any proofs with the Dirac delta function treating it as an ordinary function had better lead to correct results or else the Dirac delta function does not have the behavior that is desired for many physical applications. In such proofs, one may need to use one of the sequence of integrals with functions $\delta_n(x)$ in all the steps for mental clarity or to remove any ambiguity about what is to be done and take the limit of sequence of integrals in the last step. One assumes in such proofs that $\delta_n(x)$'s width is much smaller than the scale of variation of anything else in the system.

NOTE: There are parts a,b,c. The parts can be done independently. So don't stop if you can't do a part.

a) The Heaviside step function has the following definition:

$$H(x) = \begin{cases} 0 & \text{for } x < 0; \\ 1/2 & \text{for } x = 0; \\ 1 & \text{for } x > 0, \end{cases}$$

where $H(0)$ is sometimes left undefined. The Heaviside step function is a real function, but with an undefined derivative at $x = 0$. In physics, the Heaviside step function is often used to model functions that rise rapidly from zero to 1 over a scale small compared to anything else in a system. Prove that

$$\frac{dH}{dx} = \delta(x)$$

is a reasonable way to define the derivative of the Heaviside step function (so that it correctly models functions that rise rapidly from zero to 1 ...) when it occurs in integrals (e.g., after integration by parts) which do not include 0 as an endpoint. **HINT:** Start from

$$\int_a^b f(x)H(x)\,dx = \int_a^b f(x)H(x)\,dx \ ,$$

where f has antiderivative F, and work left-hand side and right-hand side until you get the equality you want.

b) Show that

$$\int_{-\infty}^{\infty} f(x)\frac{d\delta(x)}{dx}\,dx = -f'(0)$$

defines the derivative of $\delta(x)$ in the only way consistent with $\delta(x)$ acting like a real narrow-width normalized function. We assume $f(x)$ is finite everywhere.

c) Say that $g(x)$ has a set $\{x_i\}$ of simple zeros (i.e., zeros or roots where the derivative of $g(x)$ is not zero and $g(x)$ can be expanded in a Taylor's series). Show that

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g_i'|} \ ,$$

where $g_i' \equiv g'(x_i)$, is the reasonable model for the behavior of a narrow-width function $\delta_n[g(x)]$ in an integral in the limit that $n \to \infty$. We assume $f(x)$ is finite everywhere. **HINT:** This is a case where working with the $\delta_n(x)$ functions and the limit of the sequences of integrals helps give guidance to the steps and credibility to the result.

Chapt. 2 Vector Analysis in Curved Coordinates and Tensors

Multiple-Choice Problems

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002 qmult 00100 1 4 5 easy deducto-memory: curvilinear coordinates
Extra keywords: mathematical physics

34. "Let's play *Jeopardy*! For $100, the answer is: In these coordinate systems, unit vectors are not in general constant in direction as a function of position."

    What are ____________ coordinates, Alex?

    a) Cartesian b) Galilean c) parallax d) appalling e) curvilinear

---

002 qmult 01002 1 1 2 easy memory: standard coordinate systems

35. For 3-dimensional Euclidean space, the 3 most standard coordinate systems are the Cartesian, spherical, and:

    a) elliptical. b) cylindrical. c) hyperbolical. d) tetrahedral. e) cathedral.

---

002 qmult 03002 1 1 4 easy memory: orthogonal coordinates

36. If the unit vectors of a coordinate system are perpendicular to each other, the coordinates are:

    a) Cartesian and only Cartesian. b) spherical and only spherical.
    c) orthorhombic and only orthorhombic. d) orthogonal. e) orthodox.

---

002 qmult 05002 1 4 1 easy deducto-memory: spherical coord. scale factors
Extra keywords: mathematical physics

37. "Let's play *Jeopardy*! For $100, the answer is:

$$h_r = 1 \ , \qquad h_\theta = r \ , \qquad h_\phi = r\sin\theta \ ."$$

    What are the ____________, Alex?

    a) spherical-coordinate scale factors b) cylindrical-coordinate scale factors
    c) complex conjugates d) spherical-coordinate gradient components
    e) spherical-coordinate unit vectors

Full-Answer Problems

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002 qfull 03020 2 3 0 moderate math: spherical coord. scale factors
Extra keywords: WA-120-2.3.2

38. A fairly general expression for the metric elements for a Riemannian space is

$$g_{ij} = \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j} \ ,$$

where Einstein summation has been used and the x_k are Cartesian coordinates and the q_i are general coordinates. The squared distance element in the q_i coordinates is

$$ds^2 = g_{ij}\,dq_i\,dq_j \ .$$

We now turn Einstein summation **OFF** for the rest of the problem since it turns out to be a notational burden in dealing the commonest orthogonal curvilinear coordinates: i.e., polar, spherical, and cylindrical coordinates. If we specialize to orthogonal coordinates, the expression for the metric elements becomes

$$g_{ij} = \delta_{ij} \sum_k \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_i} \ .$$

We define the orthogonal coordinate scale factors by

$$h_i = \sqrt{g_{ii}} = \sqrt{\sum_k \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_i}} = \sqrt{\sum_k \left(\frac{\partial x_k}{\partial q_i}\right)^2} \ .$$

The squared distance elemtent in the q_i coordinates is now

$$ds^2 = \sum_i h_i^2 dq_i^2 \ .$$

We recognize that

$$ds_i = h_i\,dq_i \ ,$$

is a length in the ith direction in space space. ("Space space" makes sense. The first word is adjective meaning ordinary space and the second word is a noun meaning general mathematical space). It follows (with a bit of trepidation) that

$$d\vec{r} = \sum_i h_i\,dq_i\hat{q}_i \ , \qquad \frac{\partial \vec{r}}{\partial q_i} = h_i\hat{q}_i \ , \qquad \hat{q}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial q_i} \ .$$

We see that

$$h_i = |h_i\hat{q}_i| = \left|\frac{\partial \vec{r}}{\partial q_i}\right| = \sqrt{\sum_k \left(\frac{\partial x_k}{\partial q_i}\right)^2}$$

which agrees with the general formula given above.

The transformation equations, from spherical to Cartesian coordinates are

$$x = r\sin\theta\cos\phi \ , \qquad y = r\sin\theta\sin\phi \ , \qquad z = r\cos\theta \ ,$$

where r is the radial coordinate, θ the polar angle coordinate, and ϕ the azimuthal angle coordinate. The position vector $\vec{r}$ expressed using Cartesian unit vectors and the components expressed in spherical coordinates is

$$\vec{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z} \ .$$

a) Determine the scale factors h_r, h_θ, and h_ϕ for spherical coordinates. How can the scales factors can be obtained less rigorously, but more concretely?

b) For orthogonal coordinates the differential vector areas are just given by the orthogonal cross product

$$d\vec{\sigma}_k = d\vec{r}_i \times d\vec{r}_j = h_i\,dq_i\hat{q}_i \times h_j\,dq_j\hat{q}_j = h_ih_jdq_idq_j\hat{q}_k \ ,$$

where the ijk are in cyclic ordering (WA-119). Find the differential vector areas for spherical coordinates.

c) For orthogonal coordinates the differential volume element is just given by the orthogonal triple scalar product

$$dV = d\vec{r}_i \cdot d\vec{r}_j \times d\vec{r}_k = h_1 h_2 h_3 dq_1 dq_2 dq_3 \ ,$$

where the ijk are in cyclic ordering (WA-119). Find the differential volume element for spherical coordinates.

d) For curvilinear coordinates, the unit vectors are functions of the coordinates. Find the formulae for the unit vectors for spherical coordinates in terms of the spherical coordinates and the Cartesian unit vectors.

---

002 qfull 04010 1 3 0 easy math: curvilinear dot and cross product

39. The dot and cross product formulae for orthogonal curvilinear coordinates involve no scale factors and don't look odd by comparison to the corresponding Cartesian coordinate formulae. Einstein summation is turned **OFF** for this problem.

a) Find the formula for the dot product of vectors

$$\vec{A} = \sum_i A_i \hat{q}_i \qquad \text{and} \qquad \vec{B} = \sum_i B_i \hat{q}_i$$

which are expressed in the orthogonal curvilinear coordinates q_i. Note that

$$\hat{q}_i \cdot \hat{q}_j = \delta_{ij}$$

since the coordinates are orthogonal.

b) Find the formula for the cross product of vectors

$$\vec{A} = \sum_i A_i \hat{q}_i \qquad \text{and} \qquad \vec{B} = \sum_i B_i \hat{q}_i$$

which are expressed in the orthogonal curvilinear coordinates q_i. Note that

$$\hat{q}_i \times \hat{q}_j = \hat{q}_k$$

(where ijk are in cyclic ordering) and

$$\hat{q}_i \times \hat{q}_i = 0$$

since the coordinates are orthogonal. Does the Levi-Civita symbol formula for the cross-product components hold?

---

002 qfull 05060 2 3 0 moderate math: motion in a plane

Extra keywords: WA-134-2.5.6,2.5.7

40. Motion under a central force is one of the key problems in physics and it is best analyzed in polar coordinates or spherical coordinates limited to the $\theta = \pi/2$ (or x-y) plane. Here we investigate this motion.

a) The unit vectors in spherical coordinates are

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \ ,$$
$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \ ,$$
$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \ .$$

Specialize them to the $\theta = \pi/2$ plane.

b) Now determine $d\hat{r}/dt$ and $d\hat{\phi}/dt$ for motion **CONFINED** to the $\theta = \pi/2$ plane (i.e., in the case where $\theta = \pi/2$) in terms of spherical coordinate quantities: i.e., elminate $\hat{x}$ and $\hat{y}$ in favor of $\hat{r}$ and $\hat{\phi}$. Since it is traditional in this context, use Newton's own dot-over notation (e.g., WA-134) for the time derivatives of r and ϕ where they are needed here and below: i.e., use

$$\dot{r} = \frac{dr}{dt} \qquad \text{and} \qquad \dot{\phi} = \frac{d\phi}{dt}$$

where needed. Note no particular motion is being specified yet other than motion confined to the $\theta = \pi/2$ plane.

c) In spherical coordinates for motion confined to the $\theta = \pi/2$ plane obtain the expressions for velocity and acceleration: i.e., for

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \text{and} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} \ .$$

Simplify the latter by expressing it in $\hat{r}$ and $\hat{\phi}$ components. Note the expression for $\vec{r}$ is

$$\vec{r} = r\hat{r} \ .$$

d) A central force has $\vec{F}(\vec{r}) = F(r)\hat{r}$: i.e., the force is purely radial and its magnitude just depends on r. Thus, the net torque on a body acted on by a central force alone is

$$\vec{\tau}_{\text{net}} = \vec{r} \times \vec{F}(\vec{r}) = \vec{r} \times F(r)\hat{r} = 0 \ .$$

The rotational version of Newton's 2nd law is

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \ ,$$

where $\vec{L}$ is angular momentum of the body. So for a central force $d\vec{L}/dt = 0$ and $\vec{L}$ is a constant. If $\vec{L}$ is a constant, its direction in space is a constant and we can define that direction as the z direction. All the motion in this case is confined to the $\theta = \pi/2$ plane.

The expression for angular momemtum for a point mass is

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \ ,$$

where $\vec{p}$ is momentum and m is mass. For the central-force case outlined above, **SHOW** that

$$\vec{L} = mr^2\dot{\phi}\hat{z} = \text{constant}$$

making use of the results from the part (c) answer. The constancy, of course, just follows from the preamble of this part of the problem.

e) Evaluate $d\vec{L}/dt$ from the second part (d) expression for $\vec{L}$ and show explicitly that $d\vec{L}/dt$ along with the condition of $\vec{L}$ constant implies $\vec{a}$ is purely radial using the $\vec{a}$ expression from the part (c) answer.

f) Kepler's 2nd law is that a planet sweeps out equal areas in equal times: i.e., the planet-Sun line sweeps over equal areas in equal times as the planet orbits the Sun. Kepler's 2nd law can be expressed in the formula

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \text{constant} \ ,$$

where dA/dt is called the areal velocity (Go3-73). The integration of dA/dt over equal times gives equal areas since dA/dt is a constant. Note that

$$dA = \frac{1}{2} r^2 \, d\phi = \frac{1}{2} r^2 \dot{\phi} \, dt$$

is the differential bit of area swept out in dt.

Prove dA/dt is a constant (i.e., prove Kepler's 2nd law) using the results developed in this central-force problem. **HINT:** This is really easy.

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002 qfull 05140 1 3 0 easy math: the spherical coord. Laplacian

41. Consider the Laplacian formula for orthogonal curvilinear coordinates:

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial q_3} \right) \right] .$$

a) Specialize the Laplacian formula to spherical coordinates and simplify the formula as much as possible.

b) Specialize the part (a) answer to the case where the functions the Laplacian acts on have only radial dependence and expand the expression using the product rule.

c) Show that the part (b) answer can also be written

$$\nabla^2 = \frac{1}{r} \frac{\partial^2}{\partial r^2} r ,$$

where the operator is still understood to be acting on an unspecified function to the right. This version is actually of great use in quantum mechanics and probably elsewhere in reducing 3-dimensional systems to 1-dimensional systems. Say the operator acts ψ in 3-dimensional differential equation You define a new function $r\psi$ which in some cases is then the solution of 1-dimensional differential equation. **HINT:** The general Leibniz rule for the derivative of a product (Ar-558; also called the biderivative theorem by me)

$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$$

can be used albeit more for the sake of using it than for any great help in this case.

---

002 qfull 05142 2 5 0 moderate thinking: Leibniz formula

Extra keywords: also called by the biderivative theorem by

42. Leibniz's formula for the derivative of a product (Ar-558; also called the biderivative theorem by me) is

$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}} .$$

Leibniz's formula is analogous in form to the binomial theorem.

a) Prove Leibniz's formula by induction.

b) Prove the binomial theorem,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} ,$$

starting from Leibniz's formula. **HINT:** Note for any constant a that

$$a^k = \frac{1}{e^{ax}} \frac{d^k(e^{ax})}{dx^k} \ .$$

c) One could also prove the binomial theorem by induction, but that proof is exactly analogous to the proof of the biderative theorem. But there is another proof making use of calculus. Fairly clearly

$$(x+y)^n = \sum_{k=0}^{n} C_{n,k} x^k y^{n-k} \ ,$$

where $C_{n,k}$ is a constant coefficient depending on n and k. We note when you expand $(x+y)^n$, you will always get terms where the sum of the powers of x and y is n. Use multiple differentiation to find $C_{n,k}$. The binomial theorem is then proven.

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002 qfull 06002 2 3 0 moderate math: orthogonality relation
Extra keywords: WA-140

43. Let us limit ourselves throughout this problem to orthogonal Cartesian coordinates where we do not need to make use of the contravariant-covariant index distinction and can use subscripts (as is conventional) for all coordinate indices. In these coordinates, the coordinate transformation coefficients (or partial derivatives) obey the following inverse relation:

$$\frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'}$$

where the left-hand side is for transformation from an unprimed to primed system and the right-hand side for the reverse transformation and i and j are general indices (as is usually just understood). The inverse relation relation is valid for rotations simply because the transformation coefficients are just cosines of the angles between the axes and those are the same for transformation and inverse transformation. The inverse relation is also clearly valid for coordinate inversions (or reflections), where

$$\frac{\partial x_i'}{\partial x_j} = \pm\delta_{ij} \ ,$$

where the upper case is for no inversion and the lower case for inversion.

The general formula for a tensor transformation of tensor B_i from unprimed to primed coordinates is

$$B'_{\ldots i \ldots} = \ldots \frac{\partial x_i'}{\partial x_j} \ldots B_{\ldots j \ldots} \ ,$$

where Einstein summation is used (as throughout this problem unless otherwise stated) and the ellipses stand for all possible other tensor indices and transformation coefficients.

a) Prove orthogonality relations

$$\frac{\partial x_i'}{\partial x_j}\frac{\partial x_i'}{\partial x_k} = \delta_{jk} \qquad \text{and} \qquad \frac{\partial x_j'}{\partial x_i}\frac{\partial x_k'}{\partial x_i} = \delta_{jk} \ ,$$

where δ_{jk} is the Kronecker delta and recall that the coordinates of one coordinates system are independent of each other. Do the orthogonality relations hold for inversions as well as rotations?

b) Prove that the scalar product is a scalar: i.e., an invariant under coordinate transformations. (Here the proof is limited, of course, to orthogonal coordinate transformations, but the result can be generalized.) **HINT:** Show that $\vec{B}\cdot\vec{C}$ transforms like a scalar by relating the dot product in the primed system (i.e., $\vec{B}'\cdot\vec{C}'$) to its value in the unprimed system. The vectors $\vec{B}$ and $\vec{C}$ are general.

c) Prove that the magnitude of a vector is a scalar.

d) There is a rather obscure identity that we need to prove for 3-dimensional space. We limit the proof to orthogonal Cartesian coordinates, of course, but the identity probably generalizes to general coordinates somehow. The identity is

$$\det(a)\frac{\partial x_i'}{\partial x_p}\varepsilon_{p\ell m} = \varepsilon_{ijk}\frac{\partial x_j'}{\partial x_\ell}\frac{\partial x_k'}{\partial x_m} \qquad \text{or} \qquad \det(a)a_{ip}\varepsilon_{p\ell m} = \varepsilon_{ijk}a_{j\ell}a_{km} \ ,$$

where the second form just uses a more compact notation for the transformation coefficients and $\det(a)$ is the determinant of the matrix a made of coefficients a_{ij}. For a general vector $\vec{B}$, the transformation

$$B_i' = a_{ij}B_j$$

can also be written in explicit vector form with a matrix multiplication: i.e.,

$$\vec{B}' = a\vec{B} \ .$$

We will not prove this, but

$$\det(a) = \pm 1 \ ,$$

where the upper case is for rotations and even inversions and lower case for odd inversions (WA-197; Ar-131). Odd inversions are when 1 or 3 coordinates are inverted: the case of 1 inverted coordinate is a reflection. Such inversions change right-handed coordinate systems into left-handed coordinate systems. Even inversions are when 2 coordinates are inverted and the third is not. Even inversions are actually identical to a rotations when you think about it. Think about it. Hereafter, we will just subsume even inversions under rotations and when we say an inversion mean an odd inversion which cannot be created by a rotation.

Now from one general expression for determinants (WA-164,166; Ar-156), we know

$$\pm\det(a) = \varepsilon_{qjk}a_{pq}a_{\ell j}a_{mk} = \varepsilon_{qjk}a_{qp}a_{j\ell}a_{km} \ ,$$

where there are no repeated values among $p\ell m$ and the upper case is for $p\ell m$ cyclic and the lower case is for $p\ell m$ anticyclic. The 2nd and 3rd members of the last equation are zero if there is a repeated value among $p\ell m$ and in this case the equality with first member does not hold. The zeros follows from the general symmetry identity for the Levi-Civita symbol: i.e.,

$$\varepsilon_{ijk}G_{ij} = 0 \ ,$$

for $G_{ij} = G_{ji}$. Say $p = \ell$, then $a_{pq}a_{\ell j}$ and $a_{qp}a_{j\ell}$ are symmetric entities under the interchange of q and j and the 2nd and 3rd members of the penultimate equation are zero.

Now prove the obscure identity using the penultimate equation. Show explicitly that it holds for the case when $\ell = m$. **HINT:** It's easy. First find a new version of penultimate equation that holds in all cases without the $\pm$ sign.

e) In Cartesian coordinates, prove that the Levi-Civita symbol is an isotropic 3rd rank pseudovector: i.e., prove that it transforms according to the formula

$$\varepsilon_{ijk}' = \det(a)a_{i\ell}a_{jm}a_{kn}\varepsilon_{\ell mn}$$

and that its components are the same in all coordinate systems (which is what isotropic means [WA-146, 154]). Probably the simplest way to do the proof is to define the Levi-Civita in the unprimed system by the usual prescription

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } ijk \text{ cyclic;} \\ -1 & \text{for } ijk \text{ cyclic;} \\ 0 & \text{for a repeated index,} \end{cases}$$

and then simply define the Levi-Civita symbol as a 3rd rank pseudovector. With the latter definition, one already has the transformation rule

$$\varepsilon'_{ijk} = \det(a) a_{i\ell} a_{jm} a_{kn} \varepsilon_{\ell mn} \ .$$

All that remains is to prove that the Levi-Civita symbol is isotropic: i.e., that

$$\varepsilon'_{ijk} = \varepsilon_{ijk} \ .$$

f) Find the transformation formula for the cross product by considering the general case

$$\vec{T} = \vec{R} \times \vec{S}$$

in unprimed and primed coordinates, where $\vec{R}$ and $\vec{S}$ are general vectors. When does the cross product transform like a vector and when not? What do we call the cross product?

g) Say that we invert all three coordinates in 3-dimensional space. What are the transformation relations for the general vectors $\vec{R}$ and $\vec{S}$ and the cross product

$$\vec{T} = \vec{R} \times \vec{S}$$

of part (e)? Does $\vec{T}$ transform like a vector in this case? Is the result consistent with the general cross product transformation rule of the part (e) answer?

Chapt. 3 Scalars and Vectors

Multiple-Choice Problems

Full-Answer Problems

Chapt. 4 Two- and Three-Dimensional Kinematics

Multiple-Choice Problems

Full-Answer Problems

Chapt. 5 Infinite Series

Multiple-Choice Problems

005 qmult 00100 1 4 5 easy deducto-memory: infinite series value
Extra keywords: mathematical physics

44. "Let's play *Jeopardy*! For \$100, the answer is:

$$\lim_{L\to\infty}\sum_{\ell=0}^{L} u_\ell \ .\text{"}$$

What is the ____________, Alex?

a) Leibniz criterion b) Weierstrass M test c) uniqueness of power series theorem
d) mean value theorem e) definition of the value of an infinite series

005 qmult 00110 1 1 3 easy memory: necessary conditon for convergence

45. A necessary, but not sufficient, condition for the convergence of an infinite series

$$\sum_{\ell=0}^{\infty} u_\ell$$

is:

a) $u_\ell \geq 0$. b) $u_\ell = 0$ for ℓ sufficiently large. c) $\lim_{\ell\to\infty} u_\ell = 0$. d) $u_\ell \leq 0$.
e) $\lim_{\ell\to\infty}(1/u_\ell) = 0$.

005 qmult 00120 1 1 1 easy memory: geometric series

46. The series

$$\frac{1}{1-r} = \sum_{\ell=0}^{\infty} r^\ell$$

which is absolutely convergent for $|r| < 1$ and otherwise divergent, is called the:

a) geometric series. b) harmonic series. c) alternating harmonic series.
d) power series. e) Hamilton series.

005 qmult 00220 1 1 3 easy memory: ratio test

47. For an infinite series

$$\sum_{\ell=0}^{\infty} u_\ell \ ,$$

the limiting procedure

$$\lim_{\ell\to\infty}\left|\frac{u_{\ell+1}}{u_\ell}\right| = r = \begin{cases} r < 1 & \text{for convergence;} \\ r = 1 & \text{for indeterminate;} \\ r > 1 & \text{for divergence} \end{cases}$$

is called the:

a) square root test. b) root test. c) ratio test. d) Leibniz criterion test.
e) final test.

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005 qmult 00550 1 1 3 easy memory: absolute and uniform convergence

48. Absolute and uniform convergence:

a) are the same thing. b) imply each other. c) do **NOT** imply each other.
d) imply conditional convergence. e) do **NOT** apply to infinite series.

Full-Answer Problems

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005 qfull 00130 1 3 0 easy math: limit of sum is sum of limits

49. Say u_ℓ is a sequence of real numbers indexed by ℓ: i.e., $u_0, u_1, u_2, u_3, \ldots$. The sequence has a limit u as $\ell \to \infty$ if for general (or arbitrary or every) real number $\epsilon > 0$, there exits L such then when $\ell > L$, we have $|u_\ell - u| < \epsilon$ (Wikipedia: Limit (mathematics)).

a) Before we make use of the limit definition to do an interesting proof, we need another result. Prove the triangle inequality for 1-dimensional vectors: i.e., prove

$$|x + y| \leq |x| + |y|$$

for general real numbers x and y.

b) Prove

$$\lim_{\ell\to\infty} (u_\ell + v_\ell) = \lim_{\ell\to\infty} u_\ell + \lim_{\ell\to\infty} v_\ell$$

given the limits

$$\lim_{\ell\to\infty} u_\ell = u \qquad \text{and} \qquad \lim_{\ell\to\infty} v_\ell = v \ .$$

HINT: Make use of the triangle inequality.

---

005 qfull 00146 2 3 0 moderate math: prove n! greater/less than 2**n

50. Given n is an integer greater than **OR** equal to 0, prove that

$$n! \leq 2^n \qquad \text{for } n \leq 3 \qquad \text{and} \qquad n! > 2^n \qquad \text{for } n \geq 4 \ .$$

HINT: Divide by 2^n.

---

005 qfull 00156 2 3 0 mod. math: generalized Gauss trick

51. The story, possibly true, is that the schoolboy Johann Carl Friedrich Gauss (1777–1855) discovered formula for sum of integers in arithmetic progression (i.e., $1, 2, 3, \ldots, n$) within seconds of being challenged with adding up the integers from 1 to 100. His insight to see that one could add the arithmetic progression integers (starting from zero not 1) to their counterparts in the reverse arithmetic progression which always gave the same sum then multiply by the number of pairs and divide by 2. Thus, the formula is

$$S_1(n) = \sum_{\ell=0}^{n} \ell = \frac{n(n+1)}{2} \ ,$$

where the last expression is the evaluation formula for the sum.

One wonders if Gauss's trick can be generalized to general sums of powers

$$S_p(n) = \sum_{\ell=0}^{n} \ell^p$$

(where p is any integer greater than 0) to get evaluation formulae. Well maybe, but yours truly has found that it can only be done for odd powers p and one only gets a formula that is in terms of lower p formulae.

Find this formula. One starts from

$$S_p(n) = \sum_{\ell=0}^{n} \ell^p = \sum_{\ell=0}^{n} (n-\ell)^p$$

and makes use of the binomial theorem.

---

005 qfull 00158 1 3 0 easy math: series of powers

52. The general formula for a series of powers is

$$S_p(n) = \sum_{\ell=0}^{n} \ell^p \ ,$$

where p is any integer greater than 0. Simple evaluation formulae for S_p (whose number of terms is independent of n) can be found though for how high a p value I don't know. We will try to find some simple evaluation formulae. Note that as usual "find" implies giving a proof of the result to be found.

NOTE: There are parts a,b,c,d.

a) Find the explicit evalution formula for $p = 0$.

b) For **ODD** powers p, the formula

$$S_p(n) = \frac{1}{2} \sum_{k=0}^{p-1} \binom{p}{k} n^{p-k} (-1)^k S_k(n)$$

allows one to find simple evaluation formulae in terms of lower p formulae. Use this formula to find the simple evaluation formula for $p = 1$. Simplify your formula as much as possible.

c) Find the simple evaluation formula for $p = 2$. **HINT:** Convert each power of 2 into a column of numbers that you sum. Then instead of summing column by column, sum row by row. Simplify your formula as much as possible.

d) For **ODD** powers p, the formula

$$S_p(n) = \frac{1}{2} \sum_{k=0}^{p-1} \binom{p}{k} n^{p-k} (-1)^k S_k(n)$$

allows one to find simple evaluation formulae in terms of lower p formulae. Use this formula to find the simple evaluation formula for $p = 3$. Simplify your formula as much as possible.

---

005 qfull 00164 2 3 0 moderate math: harmonic series explored

53. The harmonic series

$$S = \sum_{\ell=1}^{\infty} \frac{1}{\ell} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

is divergent.

a) Show that is very plausible that the harmonic series is divergent using an integral.

b) Using brackets group the terms of the harmonic series starting from the 2nd term into additive groups of $p_i = 2^i$ terms where i is the group number that runs $i = 0, 1, 2, 3, \ldots$. What is the largest and smallest term in any group i? What is the sum $\Delta S'_i$ for any group i? What is the partial sum S'_I of the harmonic series up to group I.

c) Using the results of the part (b) answer show that the harmonic series diverges. Does the divergence depending on the grouping of terms? I.e., would one get different behavior of partial sums without the grouping?

d) First prove that

$$\Delta S'_{i-1} < \Delta S'_i$$

for all finite i. Second, prove that

$$\lim_{i\to\infty} \Delta S'_i = \ln(2) \ .$$

Third, prove

$$\Delta S'_i \leq \ln(2)$$

where the equality only holds for $i = \infty$. These are not terribly important results, but they are tedious to prove. **HINT:** For the first proof, a plausible proof using an integral approximation rather than definitive proof may be the best that can be done in a reasonable time. For the second proof, note that if the terms of a series $u_\ell = f(u_\ell)$ for $\ell = L$ to $\ell = n$, where $f(x)$ is monotonic decreasing function of x, then

$$\int_L^{n+1} f(x)\,dx \leq \sum_{\ell=L}^{n} u_\ell \leq u_L + \int_L^n f(x)\,dx$$

which is an inequality we will prove later. One can use the inequality to find the limit by squeeze.

e) Omit this part on tests. Write a computer code to evaluate the harmonic series partial sum to any index n. Evaluate the partial sum four ways: forward single precision, reverse single precision, forward double precision, and reverse double precision. By "forward", we mean add up the terms in their standard order largest to smallest and by reverse, we mean add up the terms from smallest to largest. Also estimate the summations using the lower and upper bound integrals given in the hint to part (d) and using the a priori best-guess integral. Test the code for $n = 10^p$ with $p = 0, 1, 2, 3, \ldots, 9$. Are the results for the four evaluation methods consistent? How well do the integral approximations work?

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005 qfull 00182 2 3 0 moderate math: inverse square-like sums

54. Partial sums of the form

$$S_n = \sum_{\ell=1}^{n} \frac{1}{(a\ell + b)(a'\ell + b')}$$

can be rewritten given a certain condition as term-count-independent-of-n formula (short formula for short)

$$S_n = \frac{1}{a'}\frac{a}{(c-b)}\left[\sum_{\ell=1}^{(c-b)/a} \frac{1}{a\ell + b} - \sum_{\ell=n+1}^{n+(c-b)/a} \frac{1}{a\ell + b}\right] ,$$

where $c = (a/a')b'$ and where, without loss of generality, we take $c > b$. Note $c > b$ implies

$$\frac{a}{a'}b' > b \qquad \text{or} \qquad \frac{b'}{a'} > \frac{b}{a} \ .$$

If it had turned out that

$$\frac{b'}{a'} < \frac{b}{a} \ ,$$

then we simply exchange which letters we prime and we get $c > b$ again.

a) What happens to the short formula if $c = b$? Why does the short formula have this bad case?

b) What is the certain condition on a, b, and c is needed for the rewrite to the short formula? Why is this condition needed?

c) What is the short formula for the infinite series that results for setting $n = \infty$? Are the infinite series always convergent?

d) Given

$$S_n = \sum_{\ell=1}^{n} \frac{1}{(\ell + 3/2)(2\ell + 5)} ,$$

what is the simple partial sum evaluation formula and what is the infinite series limit?

---

005 qfull 00210 1 3 0 easy math: root test proof

55. The ratio test (AKA the Cauchy ratio test) is the simplest and most memorable of all convergence tests. Here we prove and investigate this test. Note that part (d) can be done without having done the other parts, but not without having looked them over.

a) The ratio test is proven using a comparison test to the geometric series whose convergence properties are known by being able to explicitly find the limit of the partial sums. When the geometric series converges the explicit evaluation formula is

$$\frac{1}{1-r} ,$$

where r is the common ratio. Write down the geometric series say from memory (or by proof if you must) how the convergence depends on r.

b) Consider the general of all positive terms

$$S = \sum_{\ell=0}^{\infty} a_\ell .$$

The ratio test is

$$\lim_{\ell\to\infty} = \frac{a_{\ell+1}}{a_\ell} = f = \begin{cases} f < 1 & \text{for convegence;} \\ f > 1 & \text{for divergence;} \\ f = 1 & \text{for indeterminate.} \end{cases}$$

Prove the ratio test using the comparison test with the geometric series. **HINT:** Remember

$$\lim_{\ell\to\infty} \frac{a_{\ell+1}}{a_\ell} = f$$

means that for general small $\epsilon > 0$, there exists an n such that for all $\ell \geq n$ we have

$$\frac{a_{\ell+1}}{a_\ell} \in [f - \epsilon, f + \epsilon] .$$

This means if for example $f < 1$ that

$$\frac{a_{\ell+1}}{a_\ell} \leq f + \epsilon = r < 1$$

for sufficiently large ℓ where we define $r = f + \epsilon$

c) Consider the general of all positive terms

$$S = \sum_{\ell=0}^{\infty} a_\ell \ .$$

The inverse ratio test is

$$\lim_{\ell\to\infty} = \frac{a_\ell}{a_{\ell+1}} = f = \begin{cases} f > 1 & \text{for convegence;} \\ f < 1 & \text{for divergence;} \\ f = 1 & \text{for indeterminate.} \end{cases}$$

Prove the inverse ratio test. **HINT:** Just follow the path of the part (b) answer.

d) Consider the general power series

$$S = \sum_{\ell=0}^{\infty} a_\ell x^\ell \ ,$$

where the coefficents are a_ℓ are general: i.e., they can be positive, negative, or zero. The radius of convergence of the series R defined such that the series converges absolutely for $|x| < R$. Derive the formula for R. What can one say about the convergence properties for $|x| \geq R$? **HINT:** Make use of the part (c) inverse ratio test.

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005 qfull 00234 1 3 0 easy math: integral test of 1/(k*ln(k)**q) forms

56. Test for the convergence of

$$S = \sum_{\ell=2}^{\infty} \frac{1}{\ell[\ln(\ell)]^q} \ ,$$

where $q \geq 0$. **HINT:** Remember to test for all cases of $q \geq 0$. Also remember the integral test. If series

$$\sum_{\ell=L}^{\infty} a_\ell$$

has monotonically decreasing terms with ℓ and $a_\ell = f(\ell)$ where $f(x)$ is a monotonically decreasing function of x, then the series converges/diverges if

$$\int_L^\infty f(x)\,dx$$

converges/diverges.

---

001 qfull 00240 1 3 0 moderate math: elementary convergence tests of series

57. Consider the following three infinite series

$$S = \sum_{\ell=0}^{\infty} \ell^p \ , \qquad S = \sum_{\ell=0}^{\infty} r^\ell \ , \qquad S = \sum_{\ell=0}^{\infty} \frac{1}{\ell^q} \ ,$$

where p is an integer greater than or equal to zero, r is a constant greater than or equal to zero, q is an integer greater than or equal to 1.

a) Apply the ratio test to the three series and report the results.

b) Apply the root test to the three series and report the results.

c) Apply the integral test to the three series and report the results.

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005 qfull 00242 2 3 0 moderate math: limit test

58. The limit test for a series $\sum_\ell u_\ell$ with all $u_\ell \geq 0$ is

$$\lim_{\ell\to\infty} \ell^p u_\ell = \begin{cases} A < \infty & \text{for } p > 1 \text{ gives convergence;} \\ A = \infty & \text{for } p > 1 \text{ is indeterminate;} \\ A > 0 & \text{for } p = 1 \text{ gives divergence;} \\ A = 0 & \text{for } p = 1 \text{ is indeterminate.} \end{cases}$$

Note that for the first two cases, that the bigger p is, the more likely an indeterminate result since bigger p gives a greater tendency for the limit to go to infinity. So p should be chosen as small as one conveniently can. The limit test may not be all that useful in practice since other simple tests (e.g., the ratio, root, and integral test) may be as good or better. But proving the limit test is a good exercise for students.

Prove the limit test. **HINT:** One knows that

$$\sum_{\ell=1} \frac{1}{\ell^p}$$

converges for $p > 1$ and diverges for $p = 1$ by the integral test.

---

005 qfull 00280 2 3 0 easy math: a general intro infinite series

Extra keywords: WA-267-5.2.3 and other material

59. Infinite series are summations of infinitely many real numbers. Of course, what we really mean by an infinite series is that it is the limit of a sequence of partial sums. Say we have partial sum

$$S_N = \sum_{n=1}^{N} a_n \ .$$

The infinite series is

$$S = \lim_{n\to\infty} S_n$$

which we conventionally write

$$S = \sum_{n=1}^{\infty} a_n \ .$$

Note that the summation can also start from a zeroth term and does so in many cases.

If a sequence of sums approaches a finite limit (i.e., a finite, single value including zero), the series is convergent. The classic convergent series is the geometric series for $|x| < 1$: i.e.,

$$S = \sum_{n=0}^{\infty} x^n$$

(WA-258). The geometric series is a special case of power series which are defined by

$$S = \sum_{n=0}^{\infty} a_n x^n$$

(WA-291).

If a sequence of sums approaches an infinite limit or oscillates among finite or infinite values, the series is divergent. The geometric series with $|x| > 1$ and $x = 1$ is divergent (WA-258). The classic divergent series is the harmonic series:

$$S = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

(WA-259). However, it diverges very slowly:

$$S_{N=1,000,000} = \sum_{n=1}^{1,000,000} \frac{1}{n} = 14.392726\ldots$$

(WA-266). If a divergent series turns up in a physical analysis for a quantity that is actually finite (which is always/almost always the case), then the divergent series is not the correct result. Series that diverge to infinity have special uses. The most common one in physical analysis is to prove divergence of another series in the comparison test for convergence.

Oscillatory divergence is pretty common: e.g., the geometric series for $x = -1$:

$$S = 1 - 1 + 1 - 1 + 1 - \ldots$$

(WA-261). Oscillatory series have some mathematical interest, but have don't had much application in the empirical sciences (WA-261): if they turn up, one probably has the wrong result. There are what also what are called asymptotic or semiconvergent series which are very useful (WA-314).

If the absolute values of the terms of series converge, then the series is said to be absolutely convergent (WA-271). The terms of an absolutely convergent series can be summed in any order with the same result. A conditionally convergent series is one that is not absolutely convergent, but converges because there is a cancelation between positive and negative terms (WA-271). Unlike absolutely convergent series, conditionally convergent do not converge to a unique value independent of the order of summation. It can be shown that a conditionally convergent series will converge to any value desired or even diverge depending on the order of summation (WA-272). In this problem (which we are slowly converging to), we will not consider conditional covergence problems.

Con/divergence can be proven if one has an explicit formula for the partial sums of an infinite series. One just evaluates the limit of the partial sums and one has convergence if it is a single finite number. Often one doesn't have such an explicit formula and one must use convergence tests. There are many tests for convergence. Four simple ones are comparison test (WA-262, which makes use of a comparison series of known convergence behavior), the ratio test (WA-263), and the limit test (Ar-244). Actually, most convergence tests are derived from the comparison test or so WA-263 and Ar-243 imply.

There is is a necessary, but **NOT** sufficient, condition for convergence This condition is that the

$$\lim_{n\to\infty} a_n = 0$$

(WA-259). If this limit is not obtained, clearly the series won't sum to a single finite value.

Any convergence test can fail: i.e., give an indeterminate or no-test answer. If a test fails, one must look for a more sensitive test.

Here we consider convergence tests only for the cases of series of all positive terms.

The comparison test is just that given series of terms a_n and series of terms u_n, then if for $n > N$ where N is some finite integer

$$u_n \begin{cases} \leq a_n & \text{and series } a_n \text{ converges, then series } u_n \text{ converges;} \\ \geq a_n & \text{and series } a_n \text{ diverges, then series } u_n \text{ diverges;} \\ \leq a_n & \text{and series } a_n \text{ diverges, then there is no test;} \\ \geq a_n & \text{and series } a_n \text{ converges, then there is no test.} \end{cases}$$

Remember we are only considering all-positive-term series. It can be shown that there is no most slowly convergent and no mostly slowly divergent series (Ar-243). These means that comparison test can fail for any given comparison series.

The ratio test is

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \begin{cases} <1 & \text{for convergence;} \\ >1 & \text{for divergence;} \\ =1 & \text{for no test.} \end{cases}$$

The ratio test is easy to remember and apply, but often fails (i.e., gives a no-test result).

The limit test (or tests if you prefer) are as follows. If

$$\lim_{n\to\infty} na_n = \begin{cases} A > 0\ , & \text{then the series diverges.} \\ & A = \infty \text{ is allowed of course;} \\ 0\ , & \text{then there is no test.} \end{cases}$$

If for some $p > 1$

$$\lim_{n\to\infty} n^p a_n = \begin{cases} A < \infty\ , & \text{then the series converges.} \\ & A = 0 \text{ is allowed of course;} \\ \infty\ , & \text{then there is no test.} \end{cases}$$

The limit test is actually quite sensitive, but it can fail. This happens when the first part gives $A = 0$ and in the second part, one fails to find a $p > 0$ that gives a finite A.

The ratio test is pretty easy to memorize and I suggest everyone do that The limit test is a bit trickier to remember since there is tendency to get the no-test cases confused: it's zero for the divergence version and infinity for the convergence version. Perhaps the best way is just to say to oneself, how can zero NOT be a no-test case for divergence since zero is what one would get for a rapidly convergent series. Similarly just to say to oneself, how can infinity NOT be a no-test case for convergence since infinity is what one would get for a rapidly divergent series.

End of preamble.

a) Show that a partial sum for the geometric series evaluates to

$$S_N = \sum_{n=0}^{N} x^n = \begin{cases} \dfrac{1-x^{N+1}}{1-x} & \text{for } x \neq 1 \text{ and most useful for } |x| < 1; \\ \dfrac{x^{N+1}-1}{x-1} & \text{for } x \neq 1 \text{ and most useful for } |x| > 1; \\ N+1 & \text{for } x = 1. \end{cases}$$

b) Find the sum of the geometric series for $|x| < 1$. Is the geometric series convergent in this case?

c) Show that the geometric series is divergent or oscillatory for $|x| \geq 1$.

d) Consider the general power series

$$S = \sum_{n=0}^{\infty} a_n x^n \ .$$

In many cases of interest, one finds

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R} \ ,$$

where $R \in [0, \infty]$. (There may be no single limit if the coefficients ocscillate somehow: e.g., they run $1, 2, 1, 2, 1, 2, \ldots$, but such cases pathological.) For what values of x does the power series absolutely converge? For what values of x does the power series not absolutely converge? For what values of x can one not decide about absolute convergence?

e) Test for the convergence behavior of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} \ .$$

f) Test for the convergence behavior of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^q} \ ,$$

where $q \geq 0$.

g) Test for the convergence behavior of the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)} \ , \qquad \sum_{n=1}^{\infty} \frac{n!}{A^n} \text{ with } A > 0, \qquad \sum_{n=1}^{\infty} \frac{1}{2n(2n+1)} \ ,$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} \ , \qquad \sum_{n=0}^{\infty} \frac{1}{2n+1} \ .$$

---

005 qfull 00330 2 3 0 moderate math: ideal 2-hemisphere capacitor problem

Extra keywords: Ar-553

60. If you take an ideal, infinitely thin conducting spherical shell of radius a and divide into two hemispheres separated by an infinitely thin insulator, then you an ideal 2-hemisphere capacitor. Since the hemispheres are ideal conductors, in an electrostatic situation, they must each have a constant potential. Say the top one is at potential V_0 and the bottom one is at potential $-V_0$. It can be shown that the potential outside the hemispheres is

$$V(r,\theta) = V_0 \sum_{\ell=0}^{\infty} (-1)^{\ell} (4\ell+3) \frac{(2\ell-1)!!}{(2\ell+2)!!} \left(\frac{a}{r}\right)^{2\ell+2} P_{2\ell+1}(\cos\theta) \ ,$$

where r is the radial coordinate measured from the sphere center, θ is the polar coordinate measure the symmetry axis that passes through the top hemisphere, and $P_{2\ell+1}(\cos\theta)$ is the Legendre polynomial of order $2\ell+1$. Note that $|P_n(x) \leq 1$ for all $x \in [-1,1]$ (Ar-543). Oddly enough there is a potential discontinuity at the surface, but that seems to be a feature of the system (Ar-553).

The surface charge density is

$$\sigma(r,\theta) = \frac{\epsilon_0 V_0}{a} \sum_{\ell=0}^{\infty} (-1)^{\ell} (4\ell+3) \frac{(2\ell-1)!!}{(2\ell)!!} P_{2\ell+1}(\cos\theta) \ .$$

The !! symbols indicate the double factorial function. The definitions for the even and odd cases of this function are, respectively,

$$(2n)!! = 2n \cdot (2n-2) \cdot \ldots \cdot 6 \cdot 4 \cdot 2 \qquad \text{and} \qquad (2n+1)!! = (2n+1) \cdot (2n-1) \cdot \ldots \cdot 5 \cdot 3 \cdot 1$$

(e.g., Ar-457).

a) Given that

$$\lim_{\ell\to\infty} (4\ell+3) \frac{(2\ell-1)!!}{(2\ell+2)!!} = 0 \ ,$$

prove that the infinite series

$$\sum_{\ell}^{\infty}(-1)^{\ell}(4\ell+3)\frac{(2\ell-1)!!}{(2\ell+2)!!}$$

is convergent.

b) Determine the convergence status of the infinite series

$$\sum_{\ell=0}^{\infty}(-1)^{\ell}(4\ell+3)\frac{(2\ell-1)!!}{(2\ell)!!} .$$

---

005 qfull 00420 1 3 0 easy math: rearrangement of a double-sum series

61. We are given the absolutely convergent double-sum infinite series

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{n,m} .$$

One can picture adding up the terms of this series on a 2-dimensional table of rows and columns. Say the m index runs over the columns and the n index over the rows. A straightforward adding procedure is add up the terms in a row in an inner loop and then in an outer loop, add up the columns. Since the series is infinite, in numerical calculation one has to truncate each addition at a finite number of terms and check that the value obtained is close enough to the converged value for your purposes. Usually one stops adding when the addition makes no change to within some tolerance. Of course, if one has a simple exact evaluation formulae, one could add part or all of the series exactly.

But adding straightforwardly along rows then columns may not be the fastest procedure or may not be useful in some mathematical development. Since the series is absolutely convergent, the value is independent of the order of addition, and so any addition ordering can be done. An immediate possibility is add along diagonals—using the term diagonal in the loose sense of sloping line.

a) Picture starting at column m and adding terms along a diagonal that runs to the left. One starts at row 0 and adds to row p_{up} which is the upper limit on diagonal before going off the table. At each new row one moves left by k columns. One then adds up all the diagonals starting from the $m=0$ diagonal. A little thought with a diagram shows that all the terms will get added up. Write down the rearranged summation and determine p_{up}.

b) The finite series

$$S=\sum_{m=0}^{M}\sum_{n=0}^{N}u_{n,m}$$

can be rearranged similarly to the infinite series to add up along diagonals. The only difficulty is the extra problem of finding the limits on the indexes, except for the lower limit on m which is still zero. Find those limits and write the series formula with them. Note that as usual "find" implies giving a proof of the result to be found.

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005 qfull 00640 2 5 0 moderate thinking: exponential function 1

62. Let's define a function $E(x)$ by the powers series:

$$E(x)=\sum_{\ell=0}^{\infty}\frac{x^{\ell}}{\ell!} .$$

Note that the parts of this question are largely independent. So do **NOT** stop if you can't do a part.

a) Find the radius of convergence of the function using the power series radius of convergence formula

$$R = \lim_{\ell\to\infty}\left|\frac{a_\ell}{a_{\ell+1}}\right| ,$$

where the a_ℓ is the coefficient of x^ℓ and the $a_{\ell+1}$ is the coefficient of $x^{\ell+1}$. What does the resulting radius imply about the convergence properties of the series?

The power series converges absolutely for any $x \in (-R, R)$ and coverges uniformly for any interval $[-S, S]$ where $S < R$ and any sub-interval of $[-S, S]$. See Ar-267 for these properties.

A few more statements can be made that are needed to complete the rest of the parts of this problem rigorously.

Since all the functions in the series $E(x)$ are continuous, the function $E(x)$, where uniformly convergent, is continuous and

$$\int_a^b E(x)\,dx = \sum_{\ell=0}^{\infty}\int_a^b \frac{x^\ell}{\ell!}\,dx ,$$

where the a and b limits are in the region of uniform convergence See Ar-258 for these properties.

Furthermore, we note that

$$\sum_{\ell=0}^{\infty}\left(\frac{d}{dx}\right)\frac{x^\ell}{\ell!} = \sum_{\ell=1}^{\infty}\frac{x^{\ell-1}}{(\ell-1)!} = \sum_{\ell=0}^{\infty}\frac{x^\ell}{\ell!} = E(x) .$$

Thus, we see that all orders of derivatives of the series's functions are continuous and wherever $E(x)$ is uniformly convergent, the series formed from these derivatives are uniformly convergent since these series are all just $E(x)$ itself. Therefore, wherever the series $E(x)$ is uniformly convergent

$$\frac{d^n E(x)}{dx^n} = \sum_{\ell=0}^{\infty}\left(\frac{d^n}{dx^n}\right)\frac{x^\ell}{\ell!}$$

for any n. See Ar-258 for this property.

b) Evaluate $E(1)$ to 4th order in ℓ. The value $E(1)$ is assigned the special symbol e and is just called e. **HINT:** You can do the evaluation by hand.

c) Prove that

$$E(x)E(y) = E(x+y) ,$$

where x and y are general real numbers. You are **NOT** allowed to assume $E(x) = e^x$. That is something we prove/define below.

d) Prove that $E(-x) = 1/E(x)$ for general real number x. **HINT:** This is easy given the result of the part (c) question.

e) Prove that $E(x)^m = E(mx)$ for general integer m and general real number x. **HINT:** Do **NOT** forget to consider the case of $m \leq 0$.

f) Prove that $E(x)^{1/n} = E(x/n)$ for general integer n except $n \neq 0$ and general real number x. **HINT:** Start from $E(x/n)^n$.

g) Prove that $E(x)^{m/n} = E(mx/n)$ for general integers m and n except $n \neq 0$ and general real number x.

h) Prove that $e^{m/n} = E(m/n)$ for general integers m and n except $n \neq 0$.

i) From part (h) result, we know that

$$E(m/n) = e^{m/n} = E(m/n)$$

for general integers m and n except $n \neq 0$. Thus, $e^x = E(x)$ for general **RATIONAL NUMBER** x. Argue that the only natural way to define e^x for general real number x is by $e^x = E(x)$.

j) Given that $e^x = E(x)$ for all real numbers, find the derivatives of e^x and e^{ax}, where a is a general real number. As usual "find" means prove the required result.

k) Prove that the derivative of e^x is always greater than or equal to zero. Then describe the general nature of the function e^x: e.g., does is increase or decrease with x and does it have any stationary points and what are their natures (i.e., are they maxima, minima, or inflexion points).

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005 qfull 00642 1 3 0 easy math: exponential function 2

63. The exponential function is defined by the infinite power series

$$e^x = \sum_{z=0}^{\infty} \frac{x^z}{z!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots \; .$$

The series converges absolutely for any $x \in (-\infty, \infty)$ and coverges uniformly for any interval $[-S, S]$ where $S < \infty$ and any sub-interval of $[-S, S]$. See Ar-267 for these properties.

In this problem, we will only consider x for the range $[0, \infty)$. Note that the parts can be done nearly independently. So do **NOT** stop if you cannot do a part.

a) The summation for e^x can pictured as adding up the the columns of a **HISTOGRAM** where the horizontal axis is a continuous z variable. Each column has width 1 and is centered on an integer values of z: i.e., on $z = 0, 1, 2, 3, \ldots$. The histogram will for $x > 1$ rise from $z = 0$ to a maximum for some z and then decline as z goes to infinity. (For x in the interval $[0, 1]$ the histogram decreases monontonically with z.) Sketch this histogram for a general x value. The sketch is meant to be just qualitatively acceptable. Add a continuous curve that passes through the center of the top of each column. This curve is function

$$f(z) = \frac{x^z}{z!} \; ,$$

where z is regarded as a continuous variable. Note the factorial function does generalize to real and complex variables.

b) Show that the maximum of the curve

$$f(z) = \frac{x^z}{z!}$$

as a function of z for $x \geq 1$ occurs for

$$z \approx x - \frac{1}{2} \; .$$

HINT: Start from the ratio of $f(z)/f(z-1)$ confining z to integer values. Think about what this ratio means for terms in the series for e^x.

c) Actually, we can find the maximum of

$$f(z) = \frac{x^z}{z!}$$

from differentiation with respect to z as for any ordinary differentiable function if we have a differentiable expression for $z!$. No simple exact differentiable formula for $z!$ exists. But we do have Stirling's series

$$z! = \exp\left[z\ln(z) - z + \frac{1}{2}\ln(z) + \frac{1}{2}\ln(2\pi) + O\left(\frac{1}{z}\right)\right] ,$$

where $O(1/z)$ stands for terms of order 1 and higher in $1/z$: actually only terms with odd powers of $1/z$ occur: i.e., $1/z, 1/z^3, 1/z^5, \ldots$ (Ar-464). Sterling's series is an asymptotic series which means it is actually divergent, but if truncated to a finite number of terms gives a value whose accuracy increases as z increases and is exact in the limit $z \to \infty$ (Ar-293). The accuracy isn't so bad even for rather small z. The Sterling's series (omitting $O(1/z)$) is only about 8 % in error for $z = 1$ and improves rapidly as z increases. But as z decreases below 1, Stirling's series's accuracy rapidly declines.

Determine the approximate maximum point z for $f(z)$ using the Stirling's series omitting $O(1/z)$. **HINT:** Recall that

$$x^z = e^{z\ln(x)}$$

and note that

$$\frac{1}{2z} \approx \ln\left(1 + \frac{1}{2z}\right)$$

to 1st order in $1/(2z)$.

d) It is of some interest to know how much the maximum term in the series for e^x contributes to the e^x value. Find an approximate formula for relative contribution

$$g = \frac{f(z_{\rm max})}{e^x}$$

in as simplified form as possible. Note the $f(z_{\rm max})$ factor in the expression for g is multiplied by an implicit 1 in order to make the contribution of a histogram column to the value of e^x. **HINT:** You will need to use the result of part (b) or (c) for maximum position $z_{\rm max}$, the Stirling's series again omitting $O(1/z)$, and 1st order approximation

$$\frac{1}{2z} \approx \ln\left(1 + \frac{1}{2z}\right) .$$

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005 qfull 00650 1 3 0 easy math: natural logarithm function

64. The exponential function can be defined by the power series

$$e^x = \sum_{\ell=0}^{\infty} \frac{x^\ell}{\ell!} .$$

where

$$e = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} = 2.71828182845904523536\ldots .$$

The radius of convergence for power series is

$$R = \lim_{\ell\to\infty}\left|\frac{a_\ell}{a_{\ell+1}}\right| = \lim_{\ell\to\infty}\left|\frac{1/\ell!}{1/(\ell+1)!}\right| = \lim_{\ell\to\infty}|\ell+1| = \infty$$

which is good since the power series definition allows e^x to be evaluated for the x interval $(-\infty,\infty)$, and we know that $e^{-\infty}=0$ and $e^{\infty}=\infty$ by the nature of positive number raised to a power.

The inverse of the exponential function is the natural logarithm function $\ln(x)$, where "ln" is often vocalized as "lawn". So

$$\ln(e^x) = x$$

which implies

$$\ln(e) = 1 \ .$$

Let's investigate the natural logarithm function.

a) Say function f^{-1} is the inverse of function f: i.e.,

$$x = f^{-1}[f(x)] \ ,$$

where x is a general real number. Prove that f is the inverse of f^{-1} over the range (or in more modern jargon the image) of f at least? The image of a function is the set of all possible output values.

What is the image of e^x? What does the above result imply for the value of $e^{\ln(x)}$?

b) What are $\ln(0)$ and $\ln(\infty)$?

c) What is the derivative of a^x, where a is a general real number greater than or equal to 0.

d) Prove that

$$\ln(ab) = \ln(a) + \ln(b) \ ,$$

where a and b are general real numbers greater than or equal to 0. **HINT:** Let $a = e^x$ and $b = e^y$.

e) Prove that

$$\ln(b^a) = a\ln(b) \ ,$$

where a is a general real number and b is a general real numbers greater than or equal to 0.

f) Starting from $x = x$, prove that the derivative of

$$\frac{d\ln(x)}{dx} = \frac{1}{x} \ .$$

Now describe the behavior of the function $\ln(x)$: i.e., where are its stationary points and what are they and how does it rise or fall with x?

Now prove that

$$\frac{d^n\ln(x)}{dx^n} = (-1)^{n-1}\frac{(n-1)!}{x^n} \ .$$

g) The Mercator series is

$$\ln(1+x) = \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{x^\ell}{\ell} \ .$$

This series can be derived from Taylor's series—but I was unable to make the remainder term vanish in all the relevant cases—it's tricky. One can alternatively derive it from the finite geometric series:

$$\frac{1-(-x)^n}{1-(-x)} = \sum_{\ell=0}^{n-1}(-1)^\ell x^\ell \ .$$

Do the derivation and determine the convergence properties. Remember it is a necessary, but not sufficient, condition for convergence that the remainder vanish. **HINT:** You will need to do an integration and use the mean value theorem.

h) Show how the Mercator series can be used to evaluate $\ln(y)$ for any value of $y>0$.

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005 qfull 00652 1 3 0 easy math: numerical natural logarithm evaluation

65. Numerically the natural logarithm function can be evaluated using the Mercator series

$$\ln(1+x) = \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{x^\ell}{\ell}$$

which is absolutely convergent for interval $(-1,1)$ and conditionally convergent for $x=1$. Some tricks have to used for values of $x+1$ that are in not the interval $(0,2]$. But in any case evaluation using Mercator series is computationally inefficient compared to other methods. One of these methods is the Newton-Raphsom method which is an iteration method for evaluating the inverse of a known, evaluatable function.

a) In a Newton-Raphson method case, you have evaluatable function $f(x)$ and know a particular value y. What you want is the x input that yields y as an output. But there is no simple algebraic inverse function. The idea is that you expand $f(x)$ in a Taylor's series about some $(i-1)$th iteration value x_{i-1} to 1st order:

$$y = f(x) = f(x_{i-1}) + (x-x_{i-1})f'(x_{i-1}) \ .$$

You now solve for x:

$$x = x_{i-1} + \frac{y-f(x_{i-1})}{f'(x_{i-1})} \ .$$

If the function $f(x)$ were exactly linear, then the x obtained will be the solution. If x_{i-1} is in the region around x where the function is approximately linear, then the obtained x is only an approximate solution that we can call the ith iteration and denote x_i. Thus, the iteration formula is

$$x_i = x_{i-1} + \frac{y-f(x_{i-1})}{f'(x_{i-1})} \ .$$

In a Newton-Raphson method, you somehow obtain an initial iteration value x_0 that is in the linear region of the solution and then iterate until the iteration values stop changing to within some tolerance. Often one stops when the relative difference between iteration values is a numerical zero. The Newton-Raphson method converges very quickly if your initial iteration value is in the linear region.

Note that there is a problem if $f'(x_{i-1})=0$ since one gets an indefinite in the iteration process. Simple tricks can get around this. The Newton-Rapshon method will fail if solution x gives $f'(x)=0$. Something else has to be done in this case.

Actually, the Newton-Raphson method (combined with some numerical trickery) is guaranteed to converge no matter what the initial iteration value is if the function has a certain common property. What is that property? Explain why it guarantees convergence (when combined with some numerical trickery: e.g., a binary search algorithm).

b) Write down an algorithm for determing the natural logarithm of y given the exponential function $y=e^x$. If you don't know a programming language, use pseudo-code (i.e., a line by line set of steps in programmy jargon). If you know a programming language write the algorithm in that language and use it to compute the natural logarithm of integers 1 through 10.

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001 qfull 00654 1 3 0 easy math: rule of 70

66. The rule of 70 is simple approximate formula to calculate the doubling time of an amount that increases by a fixed fraction per unit time. One simply divides 70 by the percentage change percentage change and that gives the time in the time units. For example, say you owed 10 trillion dollars at 2 % interest compounded yearly. You'd owe 20 trillion dollars in $70/2 = 35$ years.

Prove the rule of 70. **HINT:** Start from

$$2 = (1+f)^n ,$$

and note that $\ln(2) = 0.6931\ldots$.

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005 qfull 00750 1 3 0 easy math: Leibniz's formula for pi

Extra keywords: From Ar-271, Need to tighten up the solution

67. One knows that

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\big|_0^1 = \frac{\pi}{4} .$$

But what's π? Something like 3. Expand the integrand in a series and determine the convergence properties of that integrand series. Then integrate the integrand series to get a result series expression for π. Prove the series for π converges and evaluate to 4th order.

Chapt. 6 Functions of a Complex Variable I

Multiple-Choice Problems

006 qmult 00130 1 4 3 easy deducto-memory: complex numbers
Extra keywords: mathematical physics

68. "Let's play *Jeopardy*! For \$100, the answer is: They are ordered pairs of real numbers (x, y) that in many respects are like 2-dimensional vectors, but they have a special multiplication law

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \ .\text{''}$$

What are ____________, Alex?

a) integers b) real numbers c) complex numbers d) complex conjugates
e) imaginary numbers

006 qmult 01004 1 1 3 easy memory: complex number magnitude

69. The magnitude (or modulus) of a complex number $z = x + iy$ is:

a) $\tan^{-1}(y/x)$. b) $x^2 + y^2$. c) $\sqrt{x^2 + y^2}$. d) $\tan^{-1}(x/y)$. e) $\tan(y/x)$.

Full-Answer Problems

006 qfull 10020 2 3 0 moderate math: complex numbers

70. A complex number z is actually an ordered pair of real numbers:

$$z = (x, y) \ ,$$

where x is called the real part and y is called the imaginary part. The names are conventional: the real and imaginary parts are both real and both real numbers. Addition/subtraction of complex numbers is straightforwardly defined and no different from that of 2-dimensional vectors. Given

$$z_1 = (x_1, y_1) \qquad \text{and} \qquad z_2 = (x_2, y_2) \ ,$$

we have

$$z_1 \pm z_2 = (x_1 \pm x_2, y_1 \pm y_2) \ .$$

Obviously, addition is commutative and associative given that it is for real numbers. The key distinction from 2-dimensional vectors is that a special complex number multiplication is defined. Given

$$z_1 = (x_1, y_1) \qquad \text{and} \qquad z_2 = (x_2, y_2) \ ,$$

we define

$$z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \ .$$

Obviously, multiplication is commutative given that it is for real numbers.

a) Prove that complex number multiplication has the distributive and associative properties using the ordered pair definition of complex multiplication: i.e., prove

$$z_1(z_2+z_3)=z_1z_2+z_1z_3 \qquad \text{and} \qquad z_1(z_2z_3)=(z_1z_2)z_3$$

for general complex numbers z_1, Z_2, and z_3. **HINT:** This a bit tedious, but straightforward. Just grind out the proofs.

b) Say z_1 is pure real (i.e $z_1=(x_1,0)$), what is z_1z_2 with z_2 general? Now say that z_1 is pure imaginary (i.e $z_1=(0,y_1)$), what is z_1z_2 with z_2 general?

c) From the part (a) answer, the product of a pure real complex number $(c,0)$ and a general complex number (x,y) is (cx,cy). It makes perfect sense to use the notation $c(x,y)$ for (cx,cy), and thus write

$$c(x,y)=(cx,cy)\ .$$

Thus using this notation, general complex numbers $z=(x,y)$ can be written

$$z=(x,y)=(x,0)+(0,y)=x(1,0)+y(0,1)\ ,$$

where $(1,0)$ is the real unit and $(0,1)$ is the imaginary unit. Find the rules for the coefficients of the sum of two general complex numbers using the new notation. Find the rules for the coefficients of the products of two general complex numbers using the new notation. For the latter, first find the all the possible products of two units? Are the rules what one expected?

d) Given the part (c) answer, one can write

$$z=rx+iy\ ,$$

where $r=(1,0)$ and $i=(0,1)$. For sums with this representation, one can just sum the coefficients of r and i like real numbers. The products of complex numbers in this representation are obtained by real-number-like multiplication and the terms collected into coefficients of r and i. The products of r and i with themselves and each other follow from the rules established in the part (c) answer:

$$r^2=r\ , \qquad ri=i\ , \qquad ,i^2=-r\ .$$

It makes sense to just replace r by an invisible 1 everywhere since r acts just like 1 in a real-number-like multiplication and since the i alone suffices to distinguish real and imaginary parts of the complex number. Thus, we write

$$z=x+iy$$

which is, in fact, conventional representation of a complex number. Show that z_1z_2 using the conventional representation agrees with our earlier multiplication rule for complex numbers.

e) $i=(0,1)$ is evidently the square root of $(-1,0)$ which in the conventional notation is $\sqrt{-1}$. But there is another square root for -1. What is it?

f) For several reasons, it turns out to be useful to be useful to define the complex conjugate of a complex number. The complex conjugate of z is symbolized z^* and is defined by

$$z^*=(x+iy)^*=x-iy\ .$$

The * symbol actually means a function that outputs the complex conjugate of the imput z. Now we define the magnitude (or modulus) of z (symbolized by $|z|$ where the vertical lines are a generalization of the absolute value sign of real numbers) to be the positive real number given by

$$|z| = \sqrt{zz^*}$$

Find the expression for $|z|$ in terms of the general x and y.

g) What about complex number division you ask? First, we define division of a general complex number z_1 by a general pure real number x_2 to be given by

$$\frac{z_1}{x_2} = \frac{x_1}{x_2} + i\frac{y_2}{x_1} \ .$$

No other definition would make much sense: i.e., lead to useful developments as far as one can see. Second, the only sensible definition for z/z is

$$\frac{z}{z} = 1 \ .$$

This is consistent with the real number definition which is certainly a sensible consistency to maintain. Third, it also seems seems sensible to define

$$\frac{z_1}{z_2}\frac{z_3}{z_4} = \frac{z_1 z_3}{z_2 z_4} \ .$$

(Remember as Leopold Kronecker [1823–1891] said "God made integers, all else is the work of man.") Given the above definitions, determine what z_1/z_2 equals in standard form: i.e., in the form which clearly consists of a real part plus an imaginary part.

h) What is $1/i$ in standard format?

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006 qfull 01030 1 3 0 easy math: complex conjugation proofs

71. Using general complex numbers $z = x + iy$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$, do the following:

a) Prove $(z^*)^* = z$. This result shows that the complex conjugation function is its own inverse. Give an example of another function that is its own inverse and an example of one that is not.

b) Prove $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$.

c) Prove $(z_1 z_2)^* = z_1^* z_2^*$.

d) Prove $(z_1 z_2^*)^* = z_1^* z_2$.

e) Prove $(1/z)^* = 1/z^*$ and then as a corollary that $(z_1/z_2)^* = z_1^*/z_2^*$.

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006 qfull 01020 3 3 0 hard math: triangle inequalities

Extra keywords: WA-328-6.1.2

72. The triangle inequalities for complex numbers are

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2| \ ,$$

where z_1 and z_2 are general complex numbers.

a) Prove the inequalities. **HINT:** In their addition and magnitude properties, complex numbers are just like 2-dimensional vectors. Let $\vec{c}$ stand for $z_1 + z_2$ and $\vec{a}$ and $\vec{b}$ for, respectively, z_1 and z_2. Find the dot product of $\vec{c}$ with itself and proceed as seems fit. There are at least two other ways to do the proof. The 2nd proof is to assume the inequalities are true and work toward obviously true statements and the proof is completed by just following in the steps in reverse (and this reversal of the steps can be left implicit). The 3rd proof is to start from

$$|z_1 + z_2|^2 = (z_1 + z_2)(z_1 + z_2)^*$$

and use explicitly complex number formalism.

b) Interpret the complex number triangle inequalities in terms of 2-dimensional vectors.

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006 qfull 01210 1 3 0 moderate math: RLC loop
Extra keywords: WA-331-6.1.21

73. In treating electrical circuits with potentials that are sinusoidal with time (i.e., AC circuits), it is expedient to use complex numbers. Say the potential drop across a single current loop is

$$V = V_0 \cos(\omega t) \ ,$$

where V_0 is a constant, ω is the angular frequency of the potential, and t is time. What one does is imagine an imaginary dual world in which Kirchoff's voltage and current laws also hold and where

$$V_{\rm im} = V_0 \sin(\omega t)$$

is the imaginary potential. The original potential formula is now

$$V_{\rm re} = V_0 \cos(\omega t) \ ,$$

where the subscript "re" stands for real. The complex potential is

$$V = V_{\rm re} + V_{\rm im} = V_0 e^{i\omega t} \ .$$

The advantage of using this complex potential along with the corresponding complex current in the current loop is just that it is easier to deal with the function $e^{i\omega t}$ in solving the current loop differential equation. One solves for a complex current and the real part is the solution to the original problem.

Kirchoff's voltage law states that the sum of emfs going around a current loop at one instant in time is zero. One often says "sum of potential changes" rather than "sum of emfs", but I think this is correct only stretching the meaning of potential. One can apply Kirchhoff's voltage when no potential can be defined in any normal meaning of the word as long as emfs are present. For example, consider a closed loop of wire with resistance only a Faraday law induced emf. No potential around the loop can be defined: there is only the Faraday law induced emf and the resistance emf. Kirchoff's law still works though when summing the emfs. Still potential changes trips off the tongue through long familiarity and as long as one knows what one means its OK.

Kirchoff's voltage law applies whenever two conditions hold I think. The first condition is that the total macroscopic kinetic energy changes of the electrons are negligible compared to energy inputs and outputs to the total macroscopic kinetic energy bank. In practice, I think this means the total macroscopic kinetic energy of the electrons is negligible which turns out be the case. The macroscopic kinetic energy is that associated with the drift velocity, not that of the random electron motions. The second condition is that energy of charge separation (or charge build-ups) in the circuit is negligible, except in capacitors which are treated external sources/sinks of energy. This second condition is actually virtually equivalent to Kirchoff's current law which states that charge build-ups never happen, except inside capacitors which stay neutral overall. (One wonders how true this is the dangling ends of an AC source. Maybe the ends regarded as a capacitor just have negligible capacitance usually.) Given the two conditions, Kirchoff's voltage law can be derived the work-kinetic energy theorem applied at one instant in time.

In simple circuit theory, we idealize a current loop as consisting of elements across which potential changes (really emfs) occur and ideal wires across which they don't. Real wires actually have some potential drops across them. In our case, V stands for a potential rise that is due to the whole circuit external to the current loop in question: it could be immensely complex in general, but all we know is V. The potential drops are in the current loop are due to the elements. The simplest are resistor (with resistance R), inductor (with inductance L), and capacitor (with capacitance C) which together in the current loop (which means they are in series in circuit jargon) form an RLC loop as it is called. The drops across them just add linearly as is consistent with Kirchhoff's voltage law. For a simple RLC circuit with one of each of resistor, inductor, and capacitor in series, one has

$$LI' + IR + \frac{Q}{C} = V = V_0 e^{i\omega t} ,$$

where Q is the charge on the capacitor and is given by

$$Q = \int_{-\infty}^{t} I(t')\, dt' + Q_0 ,$$

where Q_0 is value at time $t = -\infty$. For a periodic sinusoidal solution, the charge on the capacitor is periodic and averages to zero over a period.

a) Solve by inspection the given differential equation for the complex current I. The definition of impedance for a device is $Z = V/I$, where V is the potential drop across and I is the current through. From this definition obtain an explicit formula for Z in the present case. Write the solution for I in terms of Z.

b) What are the magnitude and argument (phase) θ of Z? What are the limits on the phase θ? Write Z in the polar representation and use that to simplify the expression for I.

c) Determine the real current $I_{\rm re}$. In this case, the real current is really real and not just mathematically real.

d) Describe the nature of the current amplitude as a function of ω. Is the function even or odd? Is the region $\omega < 0$ physically distinct? Where are the function's maxima and minima and its stationary points? Are the maxima and minima global or local? **HINT:** The behavior at $\omega = 0$ is tricky.

e) How would you characterize the maximum with respect to ω? Show how you know this characterization is correct. **HINT:** Set R and the driver term V to zero in the complex differential equation and solve for the current. At what angular frequency does it oscillate? You don't have to bother with $\omega < 0$ case as you should know from the part (d) answer.

f) What happens physically when $C = 0$ and $C = \infty$.

g) If you have multiple impedances Z_i in series (i.e., on the same loop) what is the net impedance Z of the loop? A proof is needed, not just the result. **HINT:** Remember Kirchhoff's voltage law. If you have multiple impedances Z_i in parallel (i.e., on parallel loops with the same V across the loops), what is the net impedance of the parallel loops collectively? **HINT:** Remember Kirchhoff's current law: the sum currents into a node, equals the sum of currents out a node in a steady state situation. What are the special cases for series and parallel for resistances, inductances, and capacitances alone?

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006 qfull 02004 2 3 0 moderate math: complex derivative rules

74. The derivative of a complex function (of a complex variable) $f(z)$ is defined analogously to the case of a real function (of real variable): i.e.,

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} .$$

This being the case, there are many differentiation rules for complex functions that are analogous and analogously proven to those for real function. But there are two special points. First, one has assume the theorem that the limit of a product of functions equals a product of the limits of complex functions. Even for real functions, proving this rigorously is tricky I think and needs compact sets and the like: I could be wrong. Someone has to proven it for complex numbers too. We will assume that has been done. Second, to complete the proofs one always has to say something like "since the factors and terms on the right-hand side are continuous, the left-hand side which is the derivative is continuous and exists."

a) Prove the chain rule for complex functions of a complex variable: i.e., prove

$$\frac{df[g(z)]}{dz} = \frac{df(g)}{dg}\frac{dg(z)}{dz} .$$

Assume f and g are continuous and analytic: i.e., they have existing or continuous derivatives.

b) Prove the product rule for complex function of a complex variable: i.e.,

$$\frac{d(fg)}{dz} = \frac{df}{dz}g + f\frac{dg}{dz} .$$

Assume f and g are continuous and analytic: i.e., they have existing or continuous derivatives.

c) The Cauchy-Riemann conditions for a general complex function

$$f(z) = u(x,y) + iv(x,y)$$

are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} .$$

If these conditions hold, the partial are themselves continuous, and the function itself exists (e.g., is not infinite), then $df(z)/dz$ exists (i.e., is continuous) and $f(x)$ is said to be analytic. Conversely if $df(z)/dz$ is continuous and $f(x)$ exists, then the Cauchy-Riemann conditions hold. These last two statements are proven by WA-332–333 (with some awkwardness I might add). Of course, a function is generally only analytic in some places and those are also the places the Cauchy-Riemann conditions hold.

There three rather special function of the complex variable: z^*, $\text{Re}(z) = x$, and $\text{Im}(z) = y$. Show that these three functions are **NOT** analytic anywhere. It follows from the chain rule that functions of these functions will not be analytic either except in special (but maybe not especially interesting) cases: e.g.,

$$f(z) = \text{Re}(z) + i\text{Im}(z) = z .$$

d) Actually real functions $f(x)$ generalize straightforwardly to being complex functions $f(z)$. However, in many cases it is non-trivial to separate $f(z)$ into real and imaginary parts: e.g.,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} ,$$

where we take WA-334's assurance that the series converges. Thus, applying the Cauchy-Riemann conditions to test for analyticity is **NOT** straightforward in many cases. On the other hand, a real function whose derivative exists at least somewhere on the real axis when generalized to a complex function will (at least in many interesting cases) have a derivative at least somewhere in the complex plane. Argue why this is so.

Note a generalized function might not exist at some points in the complex plane: e.g.,

$$\frac{1}{x^2+1} ,$$

which is defined everywhere on the real axis, generalizes to

$$\frac{1}{z^2+1}$$

which is undefined, and so non-analytic, at $z = \pm i$.

Chapt. 7 Sturm-Liouville Theory and Orthogonal Functions

Multiple-Choice Problems

009 qmult 01004 1 4 5 easy deducto-memory: Hermitian operator
Extra keywords: mathematical physics

75. "Let's play *Jeopardy*! For \$100, the answer is: A mathematical operator A defined by the **PROPERTY** $A = A^\dagger$, where the $A^\dagger$ in turn is defined by

$$\langle\alpha|A|\beta\rangle = \langle\beta|A^\dagger|\alpha\rangle^* ,$$

where $|\alpha\rangle$ and $|\beta\rangle$ are general vectors of a Hilbert (vector) space (which has, of course, an inner product defined for it)."

What is a/an ____________, Alex?

a) self-adjoint operator b) adjoint operator c) Henrician operator
d) Hermitian conjugate operator e) Hermitian operator

009 qmult 02002 1 1 5 easy memory: Hermitian operator properties

76. A Sturm-Liouville self-adjoint Hermitian operator has the following property/properties:

a) real eigenvalues.
b) orthogonal eigenfunctions, except for those eigenfunctions with degenerate eigenvalues. The degenerate eigenfunctions (as they are called), however, can always be orthogonalized.
c) a complete set of eigenfunctions.
d) a continuum of eigenvalues.
e) all of the above, except (d).

009 qmult 03002 1 4 2 easy deducto-memory: Gram-Schmidt procedure
Extra keywords: mathematical physics

77. "Let's play *Jeopardy*! For \$100, the answer is: It is a **PROCEDURE** for orthonormalizing a set of linearly independent vectors where the vectors are of very general sort, but with the inner-product property among other things. More exactly one can say that the procedure takes a linearly independent, but not orthonormal, set of vectors and constructs a new orthonormal set of vectors by linear combinations of the old set."

What is the ____________ procedure, Alex?

a) Sturm-Liouville b) Gram-Schmidt c) Hartree-Fock d) Euler-Lagrange
e) Heimlich

009 qmult 04002 1 4 3 easy deducto-memory: completeness
Extra keywords: mathematical physics

78. "Let's play *Jeopardy*! For \$100, the answer is: This **PROPERTY** possessed by a set of vectors $\{|n\rangle\}$ means that any vector $|\psi\rangle$ in the space of the set (which is called a Hilbert space which must have the inner-product property among other things) can be expanded in the set thusly

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle ,$$

where $\langle n|\psi\rangle$ is the inner product of $|n\rangle$ and $|\psi\rangle$ and we have also assumed the set $\{|n\rangle\}$ is orthonormalized (which can always be arranged)."

What is ____________, Alex?

a) orthonormality b) cleanness c) completeness d) degeneracy e) depravity

Full-Answer Problems

009 qfull 00520 1 3 0 easy math: normalizable

79. Say we have two general complex functions u and v that are normalizable (or square-integrable) over the interval $[a,b]$. This means that inner product of each function with itself exists (i.e., does not diverge). Expanding the bra-ket notation for the inner products for our function space, the self inner products are

$$\langle u|u\rangle = \int_a^b u^* u\,dx = \int_a^b |u|^2\,dx \qquad \text{and} \qquad \langle v|v\rangle = \int_a^b v^* v\,dx = \int_a^b |v|^2\,dx \ .$$

Prove that the inner product

$$\langle v|u\rangle = \int_a^b v^* u\,dx$$

exists. **HINT:** Write the functions in complex number polar form

$$u = r_u e^{\phi_u} \qquad \text{and} \qquad u = r_v e^{\phi_v} \ ,$$

where the r's stand for the magnitudes and the ϕ's for the phases of the functions. Both magnitudes and phases are functions of x in general.

009 qfull 01005 2 3 0 moderate math: Sturm-Liouville weight function

Extra keywords: WA-484–485

80. Recall the 2nd order linear operator

$$L = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2 \ ,$$

where p_0's first two deratives are continuous, p_1's first deratives is continuous, and p_0 cannot be zero except at the boundaries of the x interval of interest (Ar-424). This operator can be transformed to the self-adjoint form

$$L_{\text{self-adjoint}} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q(x)$$

by multiplying by a weight function

$$w = \frac{\exp\left[\int^x p_1(t)/p_0(t)\,dt\right]}{p_0} \ ,$$

where

$$p = wp_0 \qquad \text{and} \qquad q = wp_2 \ ,$$

and where the lower boundary of the integral for w can be left undefined since it just gives a constant scale factor and w can have any constant scale factor (e.g., 1 or some other value) one wants for whatever purpose (WA-484–485).

a) What differential equation must w satisfy in order for it to give the transformation? Write the differential equation out in terms of w and its derivative and known functions p_0, p_1, and p_2—whichever of these are needed. **HINT:** Expand $L_{\text{self-adjoint}}$, compare to wL, and equate what needs to be equated for the two expressions consistent. Remember both operators are understood to be operating on an unspecified function to the right. And do **NOT** assume you know what w is since that is what you solve for in part (b).

b) Solve the differential equation from the part (a) answer for w. **HINT:** Get w and its derivative alone on one side of the differential equation.

c) Show that the expression for w yields 1 if L is itself self-adjoint. **HINT:** What is p_1 if L is self-adjoint.

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009 qfull 01010 3 3 0 tough math: Laguerre equation
Extra keywords: WA-493-9.1.1

81. The Laguerre equation

$$xy'' + (1-x)y' + \alpha y = 0$$

and is a special case of its big brother the associated Laguerre equation

$$xy'' + (\nu + 1 + -x)y' + \alpha y = 0 \ .$$

(where α and ν are constants) The associated Laguerre equation is immensely important since it turns up as a transformed version of the radial Schrödinger equation for the hydrogen atom (and other hydrogenic atoms too) in quantum mechanics. Oddly enough the Laguerre equation doesn't turn up in the hydrogen atom solution since $\nu \geq 1$ in the transformed radial Schrödinger equation. Nevertheless it is interesting to study the Laguerre equation as simpler warm-up for the associated Laguerre equation.

We will restrict the x interval of interest to $[0, \infty]$, since this is the interval orver which the self-adjoint form of the Laguerre operator is a Hermitian operator for a set of its solutions. This set is the complete set for the Hilbert space of functions normalizable over $[0, \infty]$.

a) Find the asymptotic solution of the Laguerre equation equation for large x. **HINT:** Approximate the equation for very large x, solve, and then show that asymptotic solution validates the approximations made.

b) The asymptotic solution of the Laguerre equation for large x has the nasty property that it grows exponentially with x in magnitude. Solutions of the Laguerre equation that have this asymptotic form cannot be normalized with the weight function that turns the Laguerre operator into a self-adjoint operator, and so can't form part of the complete set of solution of solutions of the Laguerre equation for the Hilbert space defined by $[0, \infty]$. Remember functions in this Hilbert space are normalizable on this interval. But there are solutions that don't grow exponentially in magnitude for certain values of α. These solutions are polynomial solutions (i.e., finite power series solutions).

Find the α values that yield polynomial solutions by substituting the general power series

$$y = \sum_{\ell=0}^{\infty} c_\ell x^\ell$$

into the Laguerre equation and finding the recurrence relation for the coefficients c_ℓ. Is our asymptotic solution wrong for these values of α?

c) The polynomial solutions obtainable from the recurrence relation of the part (b) answer with $c_0 = 1$ are, in fact, the Laguerre polynomials. Calculate the first three Laguerre polynomials. There are easier ways to generate the Laguerre polynomials (Ar-616, WA-653).

d) Recall the 2nd order linear operator form

$$L = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$$

(where p_0, p_1, and p_0 are general functions of x, except that p_0 cannot be zero except at the boundaries of the x interval of interest) can be transformed to the self-adjoint form

$$L_{\rm self-adjoint} = \frac{d}{dx}\left[p\frac{d}{dx}\right] + q(x)$$

by multiplying by a weight function

$$w = \frac{\exp\left[\int^x p_1(x')/p_0(x')\,dx'\right]}{p_0} ,$$

where

$$p = wp_0 \qquad \text{and} \qquad q = wp_2 ,$$

and where the lower boundary of the integral for w can be left undefined since it just gives a constant scale factor and w can have any constant scale factor (e.g., 1 or some other value) one wants for whatever purpose (Ar-425, WA-484–485). Find the explicit w for the Laguerre (equation) operator and put the Laguerre operator in self-adjoint form.

e) A self-adjoint operator

$$L = \frac{d}{dx}p\frac{d}{dx} + q$$

(with p and q being functions in general) is a Hermitian operator for a Hilbert space of normalizable function vectors defined on interval $[a,b]$ for its set of eigenfunctions $\{u_i\}$ that satisfy the boundary conditions

$$pu_j^* \frac{du_k}{dx}\bigg|_{x=a}^{x=b} = 0 ,$$

where u_j and u_k are general eigenfunctions of the set $\{u_i\}$ (Ar-430). The eigenfunctions are solutions of the eigen equation

$$Lu = \lambda wu ,$$

where *lambda* is an eigenvalue and w is a weight function. We also require that all Hilbert-space functions including those eigenfunctions in the set $\{u_i\}$ be normalizable over the interval where the normalization rule will in general include a weight function w: i.e., we require for general Hilbert space function f that

$$\int_a^b |f|^2 w\,dx$$

exist (i.e., be non-divergent). Show that only the polynomial solutions of Laguerre equation (and also the self-adjoint Laguerre equation) satisfy the boundary conditions for the interval $[0,\infty]$.

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009 qfull 01012 3 3 0 tough math: associated Laguerre equation

Extra keywords: WA-493-9.1.1

82. The associated Laguerre equation

$$xy'' + (\nu + 1 + -x)y' + \alpha y = 0$$

(where α and ν are constants) is immensely important in the solution of the radial part of the wave function of the hydrogen atom (and other hydrogenic atoms too) in quantum mechanics. The associated Laguerre equation is, in fact, a transformed version of the radial part of the Schrödinger equation with x being dimensionless scaled radial coordinate. The hydrogen atom solution is a solution for the behavior of an electron in the spherical symmetric potential well provided by the proton nucleus. Spherical polar coordinates are the natural coordinates for the solution. Since x is a radial coordinate the interval of interest for it is $[0, \infty]$,

a) Find the asymptotic solution of the associated Laguerre equation equation for large x. **HINT:** Approximate the equation for very large x, solve, and then show that asymptotic solution validates the approximations made.

b) The asymptotic solution of the associated Laguerre equation for large x has the nasty property that it grow exponentially with x in magnitude. The quantum mechanical wave function of which the associated Laguerre equation solution must be a factor must go to zero as x goes to infinity as a necessary, but not sufficient, condition to be normalizable. It turns out that the asymptotic solution grows too strongly with x in magnitude to be allowed (Gr-152). But there are solutions that don't grow exponentially in magnitude for certain values of α. These solutions are polynomial solutions (i.e., finite power series solutions).

   Find the α values that yield polynomial solutions by substituting the general power series

$$y = \sum_{j=0}^{\infty} c_j x^j$$

   into the assciatedLaguerre equation and finding the recurrence relation for the coefficients c_ℓ. Is our asymptotic solution wrong for these values of α?

Chapt. 8 Legendre Polynomials and Spherical Harmonics

Multiple-Choice Problems

011 qmult 05531 1 4 2 easy deducto-memory: generating function
Extra keywords: mathematical physics

83. "Let's play *Jeopardy*! For \$100, the answer is: It is a function that in many important cases can be used for relatively easily determining general properties of special sets of functions (which are often complete sets of solutions of Sturm-Liouville Hermitian operator eigenvalue problems). There seems to be a bit of black magic though in finding the function for particular cases."

    What is a ____________, Alex?

    a) general function b) generating function c) generic function
    d) genitive function e) genuine function

011 qmult 05551 1 1 3 easy memory: uniqueness theorem and Leg. poly.

84. It using the generating function to determine properties of the Legendre polynomials, one frequently makes use of the uniqueness theorem of:

    a) the harmonic series. b) the geometric series. c) power series.
    d) the Legendre series. e) the world series.

011 qmult 05571 1 4 5 easy deducto-memory: [n/2] function
Extra keywords: Really one could just int(n/2) or floor(n).

85. "Let's play *Jeopardy*! For \$100, the answer is: It is an integer function used among other things in the general power series formula for the Legendre polynomials."

    What is ____________, Alex?

    a) (n) which equals $(n-1)/2$. b) (n) which equals $n/2$. c) ceiling$(n/2)$.
    d) $[n/2]$ which equals $n/2$ for n even and $(n+1)$ for n odd. e) $[n/2]$ which equals $n/2$ for n even and $(n-1)/2$ for n odd.

011 qmult 05631 1 3 1 easy math: Leg. poly. recurrence relation

86. Given the Legendre polynomial recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

and $P_0 = 1$, find $P_1(x)$ and $P_2(x)$.

    a) $P_1 = x$ and $P_2(x) = (1/2)(3x^2 - 1)$. b) $P_1 = 2x$ and $P_2(x) = 4x^2 - 2$.
    c) $P_1 = -x + 1$ and $P_2(x) = (1/2!)(x^2 - 4x + 2)$. d) $P_1 = x$ and $P_2(x) = x^2$.
    e) $P_1 = 1/x$ and $P_2(x) = 1/x^2$.

011 qmult 05651 1 1 5 easy memory: Leg. poly. properties

87. As the non-degenerate solutions of the Sturm-Liouville Hermitian operator eigenvalue problem for the interval $[-1, 1]$, the Legendre polynomials:

    a) form a complete set for $[-1, 1]$. b) are orthogonal for $[-1, 1]$. c) have real associated eigenvalues. d) are normalized as obtained from the generating function. e) all of the above, except (d).

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011 qmult 05791 1 4 4 easy deducto-memory: Rodriques's formula
Extra keywords: mathematical physics

88. "Let's play *Jeopardy*! For \$100, the answer is: It is an altnerative definition of the Legendre polynomials."

    What is ____________, Alex?

    a) Borracho's formula b) Gomez's formula c) Morales's formula
    d) Rodrigues's formula e) Ruiz's formula

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011 qmult 05841 1 4 2 easy deducto-memory: spherical harmonics
Extra keywords: mathematical physics

89. "Let's play *Jeopardy*! For \$100, the answer is: They form the standard complete orthonormal set for the 2-dimensional spherical surface subspace of the 3-dimensional Euclidean space."

    What are the ____________, Alex?

    a) harmonies of the spheres b) spherical harmonics c) Harmonices Mundi
    d) associated Legendre functions e) Legendre polynomials

---

011 qmult 05851 1 1 1 easy memory: memorable spherical harmonic

90. The one spherical harmonic that everyone can remember is

    a) $Y_{0,0} = 1/\sqrt{4\pi}$. b) $Y_{0,0} = 4\pi$. c) $Y_{1,0} = \cos\theta$. d) $Y_{1,0} = \sin\theta$.
    e) $Y_{0,m} = e^{im\phi}$.

Full-Answer Problems

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011 qfull 00110 2 3 0 moderate math: Legendre generating function

91. The Legendre polynomial generating function is

$$g(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n \ ,$$

where the Legendre polynomials $P_n(x)$ come from arranging the infinite sum as a power series in t (WA-553). The series is absolutely convergent for $|t| < 1$ provided the $|P_n(x)|$ have a finite upper bound which they do. This follows because the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is absolutely convergent for $|x| < 1$ (WA-258). First, we note that if the terms u_n of a series obey $|u_n| \leq |a_n|$, then the u_n series absolutely converges if the a_n series does (WA-262,271). If U is the upper bound of the $|P_n(x)|$, then $|P_n t^n| \leq U t^n$ and the generating function absolutely converges for $|t| < 1$ since the Ut^n is just the geometric series times a constant U. (Actually $U = 1$ [WA-566].) The absolute convergence of the generating function for $|t| < 1$ is useful because it means the series converges for any ordering of the terms (WA-271) which is a property often needed in making use of the generating function.

We will make use of the generating function to prove a number of general results about the Legendre polynomials.

a) Use the generating function to prove

$$P_n(\pm 1) = (\pm 1)^n$$

or

$$P_n(1) = 1 \qquad \text{and} \qquad P_n(-1) = (-1)^n \ .$$

HINT: Use the uniqueness of power series (WA-292).

b) Prove

$$P_n(-x) = (-1)^n P_n(x) \ :$$

i.e., prove that the even order P_n are even functions and the odd order P_n are odd functions. **HINT:** Consider $g(-x,-t)$ and use the uniqueness of power series (WA-292).

c) Show that

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} x^n \ .$$

HINT: It helps to make use of the double factorials:

$$(2n)!! \equiv \begin{cases} 2n \cdot (2n-2) \cdot (2n-4) \cdot \ldots \cdot 4 \cdot 2 & \text{for } n \geq 1; \\ 1 & \text{for } n = 0 \text{ by convention if one wishes;} \\ 2^n n! & \text{in general,} \end{cases}$$

$$(2n-1)!! \equiv \begin{cases} (2n-1) \cdot (2n-3) \cdot (2n-5) \cdot \ldots \cdot 3 \cdot 1 & \text{for } n \geq 1; \\ 1 & \text{for } n = 0 \text{ by convention if one wishes} \end{cases}$$

(Ar-457).

d) Find the expressions for $P_n(0)$. **HINT:** Use $g(0,t)$, the part (c) identity, and the uniqueness of power series (WA-292).

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011 qfull 00240 1 3 0 easy math: satisfying the Legendre equation

92. Legendre's differential equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \ .$$

a) Prove that the Legendre operator

$$L = (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$$

is a Sturm-Liouville self-adjoint operator for the interval $[-1, 1]$.

b)

---

011 qfull 00250 2 3 0 moderate math: Legendre orthogonality

Extra keywords: Legendre inner product

93. Because the Legendre polynomials are the non-degenerate eigensolutions of a Sturm-Liouville Hermitian operator eigenvalue problem for interval $[-1, 1]$, they are guaranteed to have real eigenvalues, be orthogonal for $[-1, 1]$, and form a complete set for $[-1, 1]$ (WA-496). But Legendre polynomials as obtained from the generating function

$$g(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

are not normalized for $[-1, 1]$. In expanding functions in the Legendre polynomials, normalized versions are needed. Here we show how to use the generating function to find the general normalization constant.

a) Integrate the square of the generating function

$$g(x,t) = (1 - 2xt + t^2)^{-1/2}$$

over $[-1,1]$.

b) Expand

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

in a Taylor's series about $t = 0$. For what values of t is the series absolutely convergent? **HINT:** It is easier to expand $\ln(1+t)$ and then obtain the required series.

c) Integrate the square of the generating function series

$$g(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$$

over $[-1,1]$ to obtain a series with coefficients that are the inner product $\langle P_n|P_n\rangle$.

d) Using the uniqueness of power series theorem (WA-292), determine the values of $\langle P_n|P_n\rangle$ and thus the general normalization constant of the Legendre polynomials.

---

011 qfull 00650 1 3 0 easy math: sph. har. expansion
Extra keywords: WA-588-11.5.10

94. Recall that the spherical harmonics form a complete set for 2-dimensional angular subspace of the Euclidean 3-dimensional space. This means that any piecewise continuous non-divergent angular function can be expanded in the spherical harmonics. Consider following function expanded in spherical harmonics:

$$f(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m} r^{\ell} Y_{\ell m} \ .$$

Recall the definition of $Y_{\ell m}$:

$$Y_{\ell m} = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta)e^{im\phi} \ ,$$

where $P_\ell^m(\cos\theta)$ is an associated Legendre function and $e^{im\phi}$ is an azimuthal eigenfunction (WA-584).

a) Find the azimuthal-angle averaged value of $f(r,\theta,\phi)$: i.e., $\langle f(r,\theta,\phi)\rangle_\phi$. **HINT:** Make use of orthogonality.

b) Now find the full angle averaged value of $f(r,\theta,\phi)$ i.e., $\langle f(r,\theta,\phi)\rangle_{\theta,\phi}$. **HINT:** Make use of orthogonality. Recall the inner product relation for pairs of Legendre polynomials:

$$\langle P_m|P_n\rangle = \int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2\delta_{mn}}{2n+1} \ .$$

c) What is $f(0,\theta,\phi)$?

Chapt. 9 Calculus of Variations

Multiple-Choice Problems

Full-Answer Problems

017 qfull 00680 1 3 0 easy math: Fermat's principle and law of reflection/refraction
Extra keywords: Lagrange multipliers

95. Fermat's principle (in modern form) states that light going from point 1 to point 2 traverses a stationary optical path length (HZ-68–69). Optical path length is the path integral of the index of refraction:

$$\ell = \int n(s)\, ds \ ,$$

where ℓ is the optical path length, $n(s)$ is the index of refraction, and s is the path length variable. From the wave point of view, Fermat's principle is a consequence of the fact that light traveling along or nearly along the stationary optical path adds coherently to give a finite signal and elsewhere adds incoherently to give virtually zero. Fermat's principle can be used to prove the laws of reflection and refraction.

a) Draw a diagram with a horizontal interface between region/medium 1 and region/medium 2, and then draw a normal to the interface where the intersection of the two lines is the origin. Put the light source at point 1 above the interface to the left of the normal and point 2 above/below the interface to the right of the normal. The incident angle 1 and reflection/refraction angle 2 are measured from the normal.

For mental convenience, we treat the coordinates for the points and the angles as all being positive numbers.

b) From point 1 with coordinates (x_1, y_1) to point 2 with coordinates (x_2, y_2), the optical path length is

$$\ell = n_1\sqrt{x_1^2 + y_1^2} + n_2\sqrt{x_2^2 + y_2^2} \ .$$

The angles 1 and 2 can be varied freely by varying x_1 and x_2 to find the stationary optical path length subject to the constraint

$$x = x_1 + x_2$$

is constant. Use Lagrange multipliers to find the stationary path and prove the laws of reflection and refraction.

Appendix 1 Introductory Physics Equation Sheet

Note: This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

Equation Sheet for Physical Sciences Courses

The equations are mnemonic. Students are expected to understand how to interpret and use them. Usually, non-vector forms have been presented: i.e., forms suitable for one-dimensional calculations.

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96 Geometry

$$C_{\rm cir} = 2\pi r \qquad A_{\rm cir} = \pi r^2 \qquad A_{\rm sph} = 4\pi r^2 \qquad V_{\rm sph} = \frac{4}{3}\pi r^3$$

$$c^2 = a^2 + b^2 \quad \text{Pyth.Thm.}$$

---

97 Kinematics

$$d = vt \qquad v_{\rm ave} = \frac{d_{\rm final} - d_{\rm initial}}{t} \qquad v = at \qquad a_{\rm ave} = \frac{v_{\rm final} - v_{\rm initial}}{t}$$

$$\text{Amount} = \text{Constant Rate} \times \text{time} \qquad \text{time} = \frac{\text{Amount}}{\text{Constant Rate}} \qquad a_{\rm centripetal} = \frac{v^2}{r}$$

---

98 Dynamics

$$F_{\rm net} = ma \qquad 1\,\text{N} \approx 0.225\,\text{lb} \qquad F_{\rm centripetal} = \frac{mv^2}{r} \qquad p = mv$$

---

99 Gravity

$$F = \frac{Gm_1m_2}{r^2} \qquad F_g = mg \qquad v_{\rm circular} = \sqrt{\frac{GM}{r}} \qquad v_{\rm escape} = \sqrt{\frac{2GM}{r}}$$

$$G = 6.6742 \times 10^{-11}\,\text{MKS units (circa 2002)}$$

$$g = 9.80\,\text{m/s}^2 \quad (\text{latitude range } \sim 9.78030\text{–}9.8322\,\text{m/s}^2\ [\text{CAC-72}])$$

---

100 Energy and Work

$$W = Fd \qquad 1\,\text{J} = 1\,\text{N}\cdot\text{m} \qquad P = \frac{W}{t} \qquad KE = \frac{1}{2}mv^2 \qquad PE_{\rm gravity} = mgy$$

$$c = 2.99792458 \times 10^8\,\text{m/s} \approx 2.998 \times 10^8\,\text{m/s} \approx 3 \times 10^8\,\text{m/s}$$

$$E = mc^2 \qquad E_{\rm rest} = m_{\rm rest}c^2 \qquad \Delta t_{\rm proper} = \Delta t\sqrt{1 - (v/c)^2}$$

---

101 Thermodynamics and Buoyancy

$$T_{\rm absolute} = T_{\rm Celsius} + 273.15 \qquad T_{\rm Fahrenheit} = \frac{9}{5}T_{\rm Celsius} + 32 \qquad \Delta Q = C_{\rm specific}m\Delta T$$

$$\rho = \frac{m}{V} \qquad n = \frac{N}{V} \qquad p = \frac{F}{A} \qquad p = p_{\rm surface} + \rho g y \qquad F_{\rm buoyant} = m_{\rm dis}g = \rho_{\rm fluid}V_{\rm dis}g$$

$$\rho_{\rm fluid}V_{\rm dis} = m_{\rm floating} \qquad PV = NkT \qquad k = 1.3806505 \times 10^{-23}\,\text{J/K}$$

$$\varepsilon = \frac{W_{\rm done}}{Q_{\rm absorbed}} \qquad \varepsilon_{\rm upperlimit} = 1 - \frac{T_{\rm cold}}{T_{\rm hot}}$$

---

102 Electricity and Magnetism

$$F = \frac{kQ_1Q_2}{r_{12}} \qquad k = 8.99 \times 10^9\,\mathrm{N\,m^2/C^2} \qquad e = 1.60217733 \times 10^{-19}\,\mathrm{C}$$

$$1 \text{ ampere (A)} = 1\,\frac{\text{coulomb (C)}}{\text{second (s)}}$$

$$1 \text{ volt (V)} = 1\,\frac{\text{joule (J)}}{\text{coulomb (C)}} \qquad 1 \text{ ohm } (\Omega) = 1\,\frac{\text{volt (V)}}{\text{ampere (A)}}$$

$$\sum \Delta V_{\text{rise}} = \sum \Delta V_{\text{drop}} \qquad V = IR \qquad P = VI$$

---

103 Waves

$$v = f\lambda \qquad p = 1/f \qquad n\frac{\lambda_n}{2} = L \qquad \lambda_n = \frac{2L}{n} \qquad f_n = \frac{v}{2L}n$$

$$v_{\text{sound }20^\circ\text{C }1\,\text{atm}} = 343\,\mathrm{m/s} \qquad v_{\text{sound }0^\circ\text{C }1\,\text{atm}} = 331\,\mathrm{m/s}$$

---

104 Nuclear Physics

$${}^{A}_{Z}X \qquad n(t) = \frac{N_0}{2^{t/t_{1/2}}} \qquad 1\,\text{amu} = 931.494043\,\text{MeV}$$

---

105 Quantum Mechanics

$$h = 6.6260693 \times 10^{-34}\,\mathrm{J\,s} \qquad m_e = 9.1093826 \times 10^{-31}\,\mathrm{kg} \qquad E = hf \qquad \lambda = \frac{h}{p} = \frac{h}{mv}$$

$$KE_{\text{photoelectron}} = hf - w \qquad H\Psi = \frac{ih}{2\pi}\frac{\partial \Psi}{\partial t}$$

---

106 Astronomy

$$v = Hd \qquad H = 71^{+4}_{-3}\,\frac{\text{km/s}}{\text{Mpc}} \quad \text{(circa 2004)}$$

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107 Geometrical Formulae

$$C_{\text{cir}} = 2\pi r \qquad A_{\text{cir}} = \pi r^2 \qquad A_{\text{sph}} = 4\pi r^2 \qquad V_{\text{sph}} = \frac{4}{3}\pi r^3$$

---

108 Trigonometry Formulae

$$x = h\cos\theta \qquad y = h\sin\theta \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)]$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

---

109 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \;:\; (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

110 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\ , \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

111 Vector Formulae

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z) \qquad \vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

112 Differentiation and Integration Formulae

$$\frac{d(x^n)}{dx} = nx^{n-1} \quad \text{except for } n = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d[\ln(x)]}{dx} = \frac{1}{x}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln(x)$$

113 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = v_0 + at \qquad x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$x = x_0 + \frac{1}{2}(v_0 + v)t \qquad v^2 = v_0^2 + 2a(x - x_0) \qquad g = 9.8\,\text{m/s}^2$$

$$x' = x - v_{\text{frame}}t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

114 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta\vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = \frac{v^2}{r}(-\hat{r}) \qquad a_{\text{centripetal}} = \frac{v^2}{r}$$

115 Projectile Motion

$$x = v_{x,0}t \qquad y = y_0 + v_{y,0}t - \frac{1}{2}gt^2 \qquad v_{x,0} = v_0\cos\theta \qquad v_{y,0} = v_0\sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0\cos\theta} \qquad y = y_0 + x\tan\theta - \frac{x^2 g}{2v_0^2\cos^2\theta}$$

$$x_{\text{for max}} = \frac{v_0^2\sin\theta\cos\theta}{g} \qquad y_{\text{max}} = y_0 + \frac{v_0^2\sin^2\theta}{2g}$$

$$x_{\text{gen}} = \frac{\tan\theta \pm \sqrt{\tan^2\theta - 2g(y-y_0)/(v_0^2\cos^2\theta)}}{g/(v_0^2\cos^2\theta)} = \frac{v_0^2\sin\theta\cos\theta}{g}\left[1 \pm \sqrt{1 - \frac{2g(y-y_0)}{v_0^2\sin^2\theta}}\right]$$

$$x(y=y_0) = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g} \qquad \theta_{\text{for max/min}} = \pm\frac{\pi}{4} \qquad x_{\text{max}}(y=y_0) = \frac{v_0^2}{g}$$

$$x(\theta=0) = \pm v_0\sqrt{\frac{2(y_0-y)}{g}} \qquad t(\theta=0) = \sqrt{\frac{2(y_0-y)}{g}}$$

116 Very Basic Classical Mechanics

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{\text{opp}} = -\vec{F} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2 \qquad F = -kx$$

$$F_{\text{f static}} = \min[F_{\text{applied}}, F_{\text{f static max}}] \qquad F_{\text{f static max}} = \mu_{\text{static}}F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}}F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = \frac{v^2}{r}(-\hat{r}) \qquad \vec{F}_{\text{centripetal}} = m\frac{v^2}{r}(-\hat{r})$$

117 Work and Energy

$$dW = \vec{F}\cdot d\vec{r} \qquad W = \int \vec{F}\cdot d\vec{r} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + U$$

$$P = \frac{dW}{dt} \qquad P_{\text{avg}} = \frac{W}{\Delta t} \qquad P = \vec{F}\cdot\vec{v}$$

$$KE_f = KE_i + W_{\text{net}} \qquad \Delta U_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \qquad E_f = E_i + W_{\text{nonconservative}}$$

$$F = -\frac{dU}{dx} \qquad \vec{F} = -\nabla U \qquad U = \frac{1}{2}kx^2 \qquad U = mgy$$

118 Systems of Particles

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}} \vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}} \qquad \vec{F}_{\text{net ext}} = m\vec{a}_{\text{cm}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = m\frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net ext}} = m\frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a} = \vec{F}_{\text{net non-transferred}} + (\vec{v}_{\text{transferred}} - \vec{v})\frac{dm}{dt} = \vec{F}_{\text{net non-transferred}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_i + v_{\text{thrust}} \ln\left(\frac{m_i}{m}\right) \qquad \text{rocket in free space}$$

119 Collisions

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\,dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{J}}{\Delta t} \qquad \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total } f} = KE_{\text{total } i} \qquad v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \qquad v_{\text{rel } f} = -v_{\text{rel } i}$$

1-d Elastic Expressions

$$P_{\text{perfect gas pressure}} = \frac{1}{3}\int_0^\infty pvn(v)\,dp \qquad P_{\text{ideal gas pressure}} = nkT$$

120 Rotation

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = f^{-1} = \frac{2\pi}{\omega}$$

$$\omega = \omega_0 + \alpha t \qquad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \qquad \theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \qquad \vec{L} = \vec{r} \times \vec{p}$$

$$\tau = rF\sin\theta \qquad \tau_{\text{net}} = I\alpha \qquad L = I\omega$$

$$I = \sum_i m_i r_{xy,i}^2 \qquad I = \int \rho r_{xy}^2 \, dV \qquad I_{\text{par-axis}} = m r_{xy,\text{cm}}^2 + I_{\text{cm}}$$

$$a = \frac{g \sin\theta}{1 + I/(mr^2)}$$

$$KE_{\text{rot}} = \frac{1}{2} I\omega^2 \qquad KE_{\text{total}} = KE_{\text{trans}} + KE_{\text{rot}} = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 \qquad dW = \tau \, d\theta \qquad P = \tau\omega$$

$$\Delta KE_{\text{rot}} = W_{\text{net}} = \int \tau_{\text{net}} \, d\theta \qquad \Delta U = -W = -\int \tau \, d\theta \qquad E_f = E_i + W_{\text{nonconservative}}$$

121 Static Equilibrium in Two Dimensions

$$0 = F_{\text{net } x} = \sum F_x \qquad 0 = F_{\text{net } y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$

122 Gravity

$$G = 6.67407 \times 10^{-11} \, \text{m}^3 \, \text{kg}^{-1} \, \text{s}^{-2} \quad \text{(2002 result)} \qquad \vec{F}_{1 \text{ on } 2} = \frac{Gm_1 m_2}{r_{12}^2} (-\hat{r}_{12})$$

$$\vec{f}_g = \frac{GM}{r^2} (-\hat{r}) \qquad \oint \vec{f}_g \cdot d\vec{A} = -4\pi GM$$

$$U = -\frac{Gm_1 m_2}{r_{12}} \qquad V = -\frac{GM}{r} \qquad v_{\text{escape}} = \sqrt{\frac{2GM}{r}} \qquad v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

$$P^2 = \left(\frac{4\pi^2}{GM} \right) r^3 \qquad P = \left(\frac{2\pi}{\sqrt{GM}} \right) r^{3/2} \qquad \frac{dA}{dt} = \frac{1}{2} r^2 \omega = \frac{L}{2m} = \text{Constant}$$

123 Fluids

$$\rho = \frac{\Delta m}{\Delta V} \qquad p = \frac{F}{A} \qquad p = p_0 + \rho g d_{\text{depth}}$$

Pascal's principle	$p = p_{\text{ext}} - \rho g (y - y_{\text{ext}}) \qquad \Delta p = \Delta p_{\text{ext}}$
Archimedes principle	$F_{\text{buoy}} = m_{\text{fluid dis}} g = V_{\text{fluid dis}} \rho_{\text{fluid}} g$
equation of continuity for ideal fluid	$R_V = Av = \text{Constant}$
Bernoulli's equation	$p + \frac{1}{2} \rho v^2 + \rho g y = \text{Constant}$

124 Simple Harmonic Oscillator

$$P = f^{-1} \qquad \omega = 2\pi f \qquad F = -kx \qquad U = \frac{1}{2}kx^2 \qquad a(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad P = 2\pi\sqrt{\frac{m}{k}} \qquad x(t) = A\cos(\omega t) + \sin(\omega t)$$

$$E_{\text{mec total}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kx_{\text{max}}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$P = 2\pi\sqrt{\frac{I}{mg\ell}} \qquad P = 2\pi\sqrt{\frac{\ell}{g}}$$

125 **Waves**

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \qquad v = \sqrt{\frac{F_{\text{T}}}{\mu}} \qquad y = f(x \mp vt) \qquad y = y_{\text{max}}\sin[k(x \mp vt)] = y_{\text{max}}\sin(kx \mp \omega t)$$

$$\text{Period} = \frac{1}{f} \qquad k = \frac{2\pi}{\lambda} \qquad v = f\lambda = \frac{\omega}{k} \qquad P \propto y_{\text{max}}^2$$

$$y = 2y_{\text{max}}\sin(kx)\cos(\omega t) \qquad n = \frac{L}{\lambda/2} \qquad L = n\frac{\lambda}{2} \qquad \lambda = \frac{2L}{n} \qquad f = n\frac{v}{2L}$$

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \qquad n\lambda = d\sin(\theta) \qquad \left(n + \frac{1}{2}\right)\lambda = d\sin(\theta)$$

$$I = \frac{P}{4\pi r^2} \qquad \beta = (10\,\text{dB}) \times \log\left(\frac{I}{I_0}\right)$$

$$f = n\frac{v}{4L} \;:\; n = 1, 3, 5, \ldots \qquad f_{\text{medium}} = \frac{f_0}{1 - v_0/v_{\text{medium}}}$$

126 **Thermodynamics**

$$T_{\text{K}} = T_{\text{C}} + 273.15\,\text{K} \qquad T_{\text{F}} = 1.8T_{\text{C}} + 32^\circ \qquad \Delta L = L\alpha\Delta T \qquad \Delta V = V\beta\Delta T \qquad \beta = 3\alpha$$

$$Q = mc\Delta T = mc(T_{\text{f}} - T_{\text{i}}) \qquad Q = \sum_k m_{\text{k}}c_{\text{k}}(T_{\text{f}} - T_{\text{i},k})$$

$$W = \int_{V_{\text{i}}}^{V_{\text{f}}} p\,dV \qquad dE = dQ - dW \qquad dE = T\,dS - p\,dV$$

$$F_{\text{cond}} = -k\frac{dT}{dx} \qquad F_{\text{surface}} = \varepsilon_1 \sigma T_{\text{surface}}^4 \qquad F_{\text{env}} = \varepsilon_2 \sigma T_{\text{env}}^4 \qquad \sigma = 5.67 \times 10^{-8}\,\text{W}\,\text{m}^{-2}\,\text{K}^{-4}$$

$$pV = nRT = NkT \qquad R = 8.31\,\text{J}\,\text{mol}^{-1}\,\text{K}^{-1} \qquad k = 1.38 \times 10^{-23}\,\text{J/K}$$

$$W_{\text{isothermal}} = nRT\ln(V_{\text{f}}/V_{\text{i}}) \qquad E = \frac{f}{2}nRT = nC_V T \qquad C_V = \frac{f}{2}R \qquad C_p = C_V + R$$

$$PV^\gamma = \text{Constant} \qquad \gamma = \frac{C_p}{C_V} = \frac{C_V + R}{C_V} = 1 + \frac{2}{f}$$

$$0 = \Delta E = (Q_{\text{h}} - Q_{\text{c}}) - W \qquad \varepsilon = \frac{W}{Q_{\text{h}}} = 1 - \frac{Q_{\text{c}}}{Q_{\text{h}}} \qquad \eta = \frac{Q_{\text{c}}}{Q_{\text{h}}} = 1 - \frac{W}{Q_{\text{h}}}$$

$$\varepsilon_{\text{Carnot}} = 1 - \frac{T_{\text{c}}}{T_{\text{h}}} \qquad \eta_{\text{Carnot}} = \frac{T_{\text{c}}}{T_{\text{h}}} = 1 - \varepsilon_{\text{Carnot}} \qquad \varepsilon_{\text{Carnot}} + \eta_{\text{Carnot}} = 1$$

$$\Delta S = \int_{\text{i}}^{\text{f}} dS = \int_{\text{i}}^{\text{f}} \frac{dQ}{T} \qquad \Delta S = nR\ln\left[\left(\frac{T_{\text{f}}}{T_{\text{i}}}\right)^{f/2}\left(\frac{V_{\text{f}}}{V_{\text{i}}}\right)\right]$$

---

127 **Electrostatics**

$$e = 1.602 \times 10^{-19}\,\text{C} \qquad m_e = 9.109 \times 10^{-31}\,\text{kg}$$

$$\varepsilon_0 = 8.854 \times 10^{-12}\,\text{C}^2/(\text{N-m}^2) \approx 10^{-11} \qquad k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9\,\text{N-m}^2/\text{C}^2 \approx 10^{10}$$

$$\vec{F}_{1\text{ on }2} = \frac{kq_1q_2}{r_{1,2}^2}\hat{r}_{1,2} \qquad \vec{F} = q\vec{E} \qquad \vec{E}(\vec{r}) = \frac{kq}{r^2}\hat{r} = \frac{q}{4\pi\varepsilon r^2}\hat{r} \qquad \oint \vec{E}\cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \vec{\tau} = \vec{p} \times \vec{E} \qquad PE = -\vec{p}\cdot\vec{E}$$

$$\Delta U = q\Delta V \qquad \Delta V = -\int_{\text{i}}^{\text{f}} \vec{E}\cdot d\vec{s} \qquad V = \frac{kq}{r} \qquad V = k\sum_i \frac{q_i}{r_i} \qquad V = k\int\frac{dq}{r} \qquad \vec{E} = -\nabla V$$

$$C = \frac{q}{V} \qquad C = \frac{\varepsilon_0 A}{d} \qquad C_{\text{parallel}} = \sum_i C_i \qquad \frac{1}{C_{\text{series}}} = \sum_i \frac{1}{C_i}$$

$$U_C = \frac{q^2}{2C} = \frac{1}{2}CV^2 \qquad C_{\text{dielectric}} = \kappa C_{\text{vacuum}} \qquad \vec{E}_{\text{macroscopic charge}} = \kappa\vec{E}_{\text{net}} \qquad \oint \kappa\vec{E}\cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

128 **Current and Circuits**

$$I = \frac{dq}{dt} \qquad I = \int \vec{J} \cdot d\vec{A} \qquad J = \frac{I}{A} \qquad \vec{J} = nq\vec{v}_{\text{drift}} \qquad \vec{J} = \sigma\vec{E} \qquad V = IR$$

$$\sigma = \frac{nq^2\tau}{m} \qquad \rho = \frac{1}{\sigma} \qquad V = IR \qquad R = \rho\frac{L}{A} \qquad P = IV \qquad P = I^2R = \frac{V^2}{R}$$

$$0 = \sum_i^{\text{node}} I_i \qquad 0 = \sum_i^{\text{loop}} \Delta V_i \qquad R_{\text{series}} = \sum_i R_i \qquad \frac{1}{R_{\text{parallel}}} = \sum_i \frac{1}{R_i}$$

$$\tau = RC \qquad I = \frac{\Delta V_{\text{initial}}}{R} e^{-t/\tau} \qquad \Delta V = \Delta V_{\text{final}} \left(1 - e^{-t/\tau}\right)$$

129 **Magnetic Fields and Forces**

$$\mu_0 = 4\pi \times 10^{-7}\,\text{T-m/A} \qquad 1\,\text{tesla (T)} = 10^4\,\text{gauss (G)}$$

$$\vec{F} = q\vec{v} \times \vec{B} \qquad \vec{dF} = I\,d\vec{\ell} \times \vec{B}$$

$$\vec{r}_{\text{cyclotron}} = \frac{mv}{qB} \qquad \omega_{\text{cyclotron}} = \frac{qB}{m} \qquad f_{\text{cyclotron}} = \frac{qB}{2\pi m} \qquad P = \frac{2\pi m}{qB}$$

$$\vec{\mu} = NIA\hat{A} \qquad \tau = \vec{\mu} \times \vec{B} \qquad U = -\vec{\mu} \cdot \vec{B}$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\,d\vec{s} \times \hat{r}}{r^2} \qquad B_{\text{wire}} = \frac{\mu_0 I}{2\pi r} \qquad B_{\text{arc}} = \frac{\mu_0 I\theta}{4\pi r} \qquad \vec{F}_{1\text{ on }2} = \frac{\mu_0 \ell I_1 I_2}{2\pi r}\hat{r}_{2\text{ to }1}$$

$$B_{\text{solenoid}} = \mu_0 nI \qquad B_{\text{toroid}} = \frac{\mu_0 NI}{2\pi r} \qquad \vec{B}_{\text{circ. loop}} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}\hat{z} \qquad \vec{B}_{\text{dipole}} = \frac{\mu_0\vec{\mu}}{2\pi z^3}$$

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I \qquad \Phi_B = \int_{\text{linked area}} \vec{B}\cdot d\vec{A} \qquad V_{\text{emf}} = \oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_{\text{linked area}} \vec{B}\cdot d\vec{A}$$

$$L = \frac{\Phi_B}{I} \qquad L_{\text{solenoid}} = \mu_0 n^2 V_{\text{vol}} \qquad V_{\text{emf drop}} = L\frac{dI}{dt} \qquad U_L = \frac{1}{2}LI^2 \qquad u_B = \frac{B^2}{2\mu_0}$$

130 **LR, LC, LRC Circuits and Alternating Current**

$$\tau_L = \frac{L}{R} \qquad I = \frac{V_{\text{emf}}}{R}\left(1 - e^{-t/\tau_L}\right) \qquad I = I_0 e^{-t/\tau_L}$$

$$\omega = \frac{1}{\sqrt{LC}} \qquad I_{\rm max} = \frac{V_{\rm max}}{\sqrt{R^2 + \left[w_{\rm d}L - 1/(w_{\rm d}C)\right]^2}}$$

131 Electromagnetic Waves and Radiation

$$c = 2.99792458 \times 10^8 \approx 3.00 \times 10^8\,{\rm m/s} \qquad {\rm Period} = \frac{1}{f} \qquad c = f\lambda = \frac{\omega}{k} \qquad k = \frac{2\pi}{\lambda} \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\vec{E} = \vec{E}_0 \sin(kx - wt) \qquad \vec{B} = \vec{B}_0 \sin(kx - wt) \qquad \frac{B}{E} = \frac{1}{c}$$

$$I_{\rm monochr} = \frac{1}{2}\varepsilon E_0^2 = \varepsilon E_{\rm rms}^2 \qquad I_{\rm point} = \frac{L}{4\pi r^2}$$

$$\Omega_{\rm sphere} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

$$I_{\rm trans.\ polarized} = I_{\rm inc}\cos^2\theta \qquad I_{\rm trans.\ non\text{-}polarized} = \frac{1}{2}I_{\rm inc}$$

132 Geometrical Optics

$$\theta_{\rm inc} = \theta_{\rm refl} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad \theta_{1,\rm critical} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

$$n = \frac{c}{v_{\rm n}} \qquad \lambda_n = \frac{\lambda_{\rm vacuum}}{n} \qquad f = \frac{v_{\rm n}}{\lambda_n} = \frac{c}{\lambda_{\rm vacuum}}$$

133 Interference and Diffraction

2-slit $\quad n\lambda = d\sin\theta_{\rm max} \qquad I = \dfrac{4I_0}{r^2}\cos^2(\alpha) \qquad \alpha = \pi\dfrac{d}{\lambda}\sin\theta$

single-slit $\quad n\lambda = a\sin\theta_{\rm zero} \qquad I = \dfrac{I_0}{r^2}\dfrac{\sin^2(\alpha)}{\alpha^2} \qquad \alpha = \pi\dfrac{a}{\lambda}\sin\theta$

grating $\quad n\lambda = d\sin\theta_{\rm max} \qquad$ X-ray $\quad n\lambda = 2d\sin\theta_{\rm max}$

134 Special Relativity

$$c = 2.99792458 \times 10^8\,{\rm m/s} \approx 2.998 \times 10^8\,{\rm m/s} \approx 3 \times 10^8\,{\rm m/s} \approx 1\,{\rm lyr/yr} \approx 1\,{\rm ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$

$$y' = y \qquad\qquad y' = y$$
$$z' = z \qquad\qquad z' = z$$
$$\tau' = \tau \qquad\qquad \tau' = \gamma(\tau - \beta x)$$

$$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta \qquad\qquad \beta'_{\text{obj}} = \frac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$$

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \theta_{\text{Mink}} = \tan^{-1}(\beta)$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2} \qquad KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating} \qquad f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

---

135 **Quantum Mechanics**

$$\hbar = \frac{h}{2\pi} \qquad E = hf \qquad p = \frac{h}{\lambda} = \hbar k \qquad \Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t} \qquad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi \qquad \Psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

$$\Delta x\Delta p \geq \frac{\hbar}{2} \qquad \int_{-\infty}^{\infty}|\Psi|^2\,dx = 1 \qquad E_n = -\frac{1}{2}m_ec^2\alpha^2\frac{Z^2}{n^2} = -E_{\text{ryd}}\frac{Z^2}{n^2}$$

$$E_{\text{ryd}} = 13.6056981\,\text{eV} \qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.0359895}$$

$$a_{\text{Bohr}} = 0.529177249 \times 10^{-10} = 0.529177249\,\text{Å}$$

$$m_{\text{amu}} = 1.6605402 \times 10^{-27}\,\text{kg} = 931.49432\,\text{MeV} \qquad m_e = 9.1093897 \times 10^{-31}\,\text{kg} = 0.510999906\,\text{MeV}$$

$$m_p = 1.6726231 \times 10^{-27}\,\text{kg} = 938.27231\,\text{MeV} \qquad m_n = 1.6749286 \times 10^{-27}\,\text{kg} = 939.56563\,\text{MeV}$$

Appendix 2 Table of Integrals, Etc.

Note: There are no guarantees of accuracy with these integrals, etc. I include some derivatives at the start, but they are such a minor component that they do not merit a change in title.

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136. Derivatives

(1)
$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \text{for } \sin^{-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(2)
$$\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \text{for } \cos^{-1} \in [0, \pi]$$

---

137. Functions containing $ax^2 + b$

(1)
$$\int \frac{dx}{ax^2+b} = \begin{cases} \frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{a}{b}}\right) & a > 0 \text{ and } b > 0; \\ \frac{1}{2\sqrt{-ab}}\ln\left(\frac{x\sqrt{a}-\sqrt{-b}}{x\sqrt{a}+\sqrt{-b}}\right) & a > 0 \text{ and } b < 0; \\ \frac{1}{2\sqrt{-ab}}\ln\left(\frac{\sqrt{b}+x\sqrt{-a}}{\sqrt{b}-x\sqrt{-a}}\right) & a < 0 \text{ and } b > 0; \\ -\frac{1}{ax} & b = 0; \\ \frac{x}{b} & a = 0 \end{cases}$$

(2)
$$\int \frac{dx}{(ax^2+b)^2} = \frac{1}{2b}\frac{x}{(ax^2+b)} + \frac{1}{2b}\frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{a}{b}}\right) \qquad a > 0 \text{ and } b > 0$$

(3)
$$\int \frac{dx}{\sqrt{ax^2+b}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(x\sqrt{-\frac{a}{b}}\right) = -\frac{1}{\sqrt{-a}}\cos^{-1}\left(x\sqrt{-\frac{a}{b}}\right) \qquad a < 0$$

---

138. Functions containing $ax^2 + bx + c$

(1)
$$\int \frac{dx}{(ax^2+bx+c)^{3/2}} = -\frac{2(2ax+b)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

---

139. Functions containing $\sin(ax)$

(1) $$\int \sin^3(ax)\,dx = -\frac{1}{a}\cos(ax) + \frac{1}{3a}\cos^3(ax)$$

(2) $$\int \sin^5(ax)\,dx = -\frac{1}{5a}\sin^4(ax)\cos(ax) + \frac{4}{15a}\cos^3(ax) - \frac{4}{5a}\cos(ax)$$

(3) $$\int \sin^n(ax)\,dx = -\frac{1}{na}\sin^{n-1}(ax)\cos(ax)\frac{n-1}{n}\int \sin^{n-2}(ax)\,dx$$

---

140. Inverse Trigonometric Functions

(1) $$\int \sin^{-1}(ax)\,dx = x\sin^{-1}(ax) + \frac{1}{a}\sqrt{1-a^2x^2}$$

(2) $$\int \cos^{-1}(ax)\,dx = x\cos^{-1}(ax) - \frac{1}{a}\sqrt{1-a^2x^2}$$

---

141. Algebraic and Trigonometric Functions

(1) $$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

(2) $$\int x^2\sin(ax)\,dx = -\frac{x^2}{a}\cos(ax) + \frac{2x}{a^2}\sin(ax) + \frac{2}{a^3}\cos(ax)$$

(3) $$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) - \frac{x}{a}\sin(ax)$$

(4) $$\int x^2\sin(ax)\,dx = \frac{x^2}{a}\sin(ax) + \frac{2x}{a^2}\cos(ax) - \frac{2}{a^3}\sin(ax)$$

---

142. Gaussian Function and Factorial Integral

(1) $$G(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \qquad \text{standard form}$$

(2) $$\int_{-\infty}^{\infty} e^{-(a+ib)x^2}\,dx = \sqrt{\frac{\pi}{a+ib}}$$

(3) $$\int_{-\infty}^{\infty} x^2 e^{-(a+ib)x^2}\, dx = \frac{\sqrt{\pi}}{2}\frac{1}{(a+ib)^{3/2}}$$

(4) $$z! = \int_0^{\infty} e^{-t} t^z\, dt$$

(5) $$0! = 1 \qquad 1! = 1 \qquad \left(-\frac{1}{2}\right)! = \sqrt{\pi} \qquad (z-1)! = \frac{z!}{z}$$

(6) $$g(n,x) = \int_0^x e^{-t} t^n\, dt = n!\left(1 - e^{-x}\sum_{\ell=0}^{n} \frac{x^\ell}{\ell!}\right)$$

(7) $$g(0,x) = 1 - e^{-x}$$

(8) $$g(1,x) = 1 - e^{-x}(1+x)$$

(9) $$g(2,x) = 2\left[1 - e^{-x}\left(1 + x + \frac{1}{2}x^2\right)\right]$$

Appendix 3 Multiple-Choice Problem Answer Tables

Note: For those who find scantrons frequently inaccurate and prefer to have their own table and marking template, the following are provided. I got the template trick from Neil Huffacker at University of Oklahoma. One just punches out the right answer places on an answer table and overlays it on student answer tables and quickly identifies and marks the wrong answers

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
143.	O	O	O	O	O	6.	O	O	O	O	O
144.	O	O	O	O	O	7.	O	O	O	O	O
145.	O	O	O	O	O	8.	O	O	O	O	O
146.	O	O	O	O	O	9.	O	O	O	O	O
147.	O	O	O	O	O	10.	O	O	O	O	O

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
148.	O	O	O	O	O	11.	O	O	O	O	O
149.	O	O	O	O	O	12.	O	O	O	O	O
150.	O	O	O	O	O	13.	O	O	O	O	O
151.	O	O	O	O	O	14.	O	O	O	O	O
152.	O	O	O	O	O	15.	O	O	O	O	O
153.	O	O	O	O	O	16.	O	O	O	O	O
154.	O	O	O	O	O	17.	O	O	O	O	O
155.	O	O	O	O	O	18.	O	O	O	O	O
156.	O	O	O	O	O	19.	O	O	O	O	O
157.	O	O	O	O	O	20.	O	O	O	O	O

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
158.	O	O	O	O	O	16.	O	O	O	O	O
159.	O	O	O	O	O	17.	O	O	O	O	O
160.	O	O	O	O	O	18.	O	O	O	O	O
161.	O	O	O	O	O	19.	O	O	O	O	O
162.	O	O	O	O	O	20.	O	O	O	O	O
163.	O	O	O	O	O	21.	O	O	O	O	O
164.	O	O	O	O	O	22.	O	O	O	O	O
165.	O	O	O	O	O	23.	O	O	O	O	O
166.	O	O	O	O	O	24.	O	O	O	O	O
167.	O	O	O	O	O	25.	O	O	O	O	O
168.	O	O	O	O	O	26.	O	O	O	O	O
169.	O	O	O	O	O	27.	O	O	O	O	O
170.	O	O	O	O	O	28.	O	O	O	O	O
171.	O	O	O	O	O	29.	O	O	O	O	O
172.	O	O	O	O	O	30.	O	O	O	O	O

NAME:

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
173.	O	O	O	O	O	26.	O	O	O	O	O
174.	O	O	O	O	O	27.	O	O	O	O	O
175.	O	O	O	O	O	28.	O	O	O	O	O
176.	O	O	O	O	O	29.	O	O	O	O	O
177.	O	O	O	O	O	30.	O	O	O	O	O
178.	O	O	O	O	O	31.	O	O	O	O	O
179.	O	O	O	O	O	32.	O	O	O	O	O
180.	O	O	O	O	O	33.	O	O	O	O	O
181.	O	O	O	O	O	34.	O	O	O	O	O
182.	O	O	O	O	O	35.	O	O	O	O	O
183.	O	O	O	O	O	36.	O	O	O	O	O
184.	O	O	O	O	O	37.	O	O	O	O	O
185.	O	O	O	O	O	38.	O	O	O	O	O
186.	O	O	O	O	O	39.	O	O	O	O	O
187.	O	O	O	O	O	40.	O	O	O	O	O
188.	O	O	O	O	O	41.	O	O	O	O	O
189.	O	O	O	O	O	42.	O	O	O	O	O
190.	O	O	O	O	O	43.	O	O	O	O	O
191.	O	O	O	O	O	44.	O	O	O	O	O
192.	O	O	O	O	O	45.	O	O	O	O	O
193.	O	O	O	O	O	46.	O	O	O	O	O
194.	O	O	O	O	O	47.	O	O	O	O	O
195.	O	O	O	O	O	48.	O	O	O	O	O
196.	O	O	O	O	O	49.	O	O	O	O	O
197.	O	O	O	O	O	50.	O	O	O	O	O

Answer Table

Name:

	a	b	c	d	e
198.	O	O	O	O	O
199.	O	O	O	O	O
200.	O	O	O	O	O
201.	O	O	O	O	O
202.	O	O	O	O	O
203.	O	O	O	O	O
204.	O	O	O	O	O
205.	O	O	O	O	O
206.	O	O	O	O	O
207.	O	O	O	O	O
208.	O	O	O	O	O
209.	O	O	O	O	O
210.	O	O	O	O	O
211.	O	O	O	O	O
212.	O	O	O	O	O
213.	O	O	O	O	O
214.	O	O	O	O	O
215.	O	O	O	O	O
216.	O	O	O	O	O
217.	O	O	O	O	O
218.	O	O	O	O	O
219.	O	O	O	O	O
220.	O	O	O	O	O
221.	O	O	O	O	O
222.	O	O	O	O	O
223.	O	O	O	O	O
224.	O	O	O	O	O
225.	O	O	O	O	O
226.	O	O	O	O	O
227.	O	O	O	O	O

	a	b	c	d	e
31.	O	O	O	O	O
32.	O	O	O	O	O
33.	O	O	O	O	O
34.	O	O	O	O	O
35.	O	O	O	O	O
36.	O	O	O	O	O
37.	O	O	O	O	O
38.	O	O	O	O	O
39.	O	O	O	O	O
40.	O	O	O	O	O
41.	O	O	O	O	O
42.	O	O	O	O	O
43.	O	O	O	O	O
44.	O	O	O	O	O
45.	O	O	O	O	O
46.	O	O	O	O	O
47.	O	O	O	O	O
48.	O	O	O	O	O
49.	O	O	O	O	O
50.	O	O	O	O	O
51.	O	O	O	O	O
52.	O	O	O	O	O
53.	O	O	O	O	O
54.	O	O	O	O	O
55.	O	O	O	O	O
56.	O	O	O	O	O
57.	O	O	O	O	O
58.	O	O	O	O	O
59.	O	O	O	O	O
60.	O	O	O	O	O