Cosmology Problems

David J. Jeffery

Department of Physics & Astronomy, Nevada Center for Astrophysics

University of Nevada, Las Vegas

Las Vegas, Nevada

♠

♠ ♠ ♠ ♠

♠ ♠

♠

♠ ♠

2016 January 1

Introduction

Cosmology Problems (CP) is a source book for instructors of cosmology at the low-cal end of the graduate level. The book is available in electronic form to instructors by request to the author. It is free courseware and can be freely used and distributed, but not used for commercial purposes.

The problems are grouped by topics in chapters: see Contents below. For each chapter there are two classes of problems: in order of appearance in a chapter they are: (1) multiple-choice problems and (2) full-answer problems. Almost all the problems have complete suggested answers. The answers may be the greatest benefit of CP. The problems and answers can be posted on the web in pdf format.

The problems have been suggested by many sources, but have all been written by me. Given that the ideas for problems are the common coin of the realm, I prefer to call them redactions. Instructors, however, might well wish to find solutions to particular problems from well known texts. Therefore, I give the suggesting source (when there is one or when I recall what it was) by a reference code on the extra keyword line. Caveat: my redaction and the suggesting source problem will not in general correspond perfectly or even closely in some cases. The references for the source texts and other references follow the contents. A general citation is usually, e.g., Ar-400 for Arfken, p. 400.

At the end of the book are two appendices. The first is an equation sheet suitable to give to students as a test aid and a review sheet. The second is a set of answer tables for multiple choice questions.

Cosmology Problems is a book in progress. There are gaps in the coverage and the ordering of the problems by chapters is not yet final. User instructors can, of course, add and modify as they list.

Everything is written in plain TEX in my own idiosyncratic style. The questions are all have codes and keywords for easy selection electronically or by hand. A fortran program for selecting the problems and outputting them in quiz, assignment, and test formats is also available. Note the quiz, etc. creation procedure is a bit clonky, but it works. User instructors could easily construct their own programs for problem selection.

I would like to thank the Physics & Astronomy Department of University of Nevada, Las Vegas for its support for this work. Thanks also to the students who helped flight-test the problems.

Contents

Appendices

References

Arfken, G. 1970, *Mathematical Methods for Physicists,* 2nd edition (New York: Academic Press) (Ar1970)

Arfken, G. 1985, 3rd edition *Mathematical Methods for Physicists* (New York: Academic Press) (Ar)

Baym, G. 1969, Lectures on Quantum Mechanics (Reading, Massachusetts: Benjamin-Cummings Publishing Co., Inc.) (Ba)

Bevington, P. R. 1969, Data Reduction and Error Analysis for the Physical Sciences (New York: McGraw-Hill Book Company) (Be)

Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton, New Jersey: Princeton Universitty Press) (Bi)

Boyer, C. B. 1985, A History of Mathematics (Princeton, New Jersey: Princeton Universitty Press) (Boyer)

Cimatti, A., Fraternali, F., & Nipoti, C. 2020, Introduction to Galaxy Formation and Evolution: From Primordial Gas to Present-Day Galaxies (Cambridge: Cambridge University Press) (Ci)

Coles, P., & Lucchin, F. 2002, Cosmology: The Origin and Evolution of Cosmic Structure (Chichester, United Kingdom: John Wiley & Sons Ltd.) (CL)

Enge, H. A. 1966, Introduction to Nuclear Physics (Reading, Massachusetts: Addison-Wesley Publishing Company, Inc.) (En)

Goldstein, H. 1980, Classical Mechanics, 2nd Edition (Reading, Massachusetts: Addison-Wesley Publishing Company) (Go1980)

Goldstein, H., Poole, C. P., Jr., & Safko, J. L. 2002, Classical Mechanics, 3rd Edition (San Francisco: Addison-Wesley Publishing Company) (Go)

Hecht, E., & Zajac, A. 1976, Optics (Menlo Park, California: Addison-Wesley Publishing Company) (HZ)

Liddle, A. 2015, Modern Cosmology (Chichester, United Kingdom: John Wiley & Sons Ltd.) (Li)

Mihalas, D. 1978, Stellar Atmospheres (San Francisco: W.H. Freeman & Company) (Mi)

Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1992, Numerical Recipes: The Art of Scientific Computing (Cambridge: Cambridge University Press) (Pr)

Turchin, P. 2003, Historical Dynamics: Why States Rise and Fall, (Princeton, New Jersey: Princeton University Press) (Tu2003)

Chapt. 0 General Questions

Multiple-Choice Problems

---

000 qmult 00100 1 4 1 easy deducto-memory: reading and done

1. Did you complete reading for this cosmology lecture before it was lectured/bypassed on in class and the corresponding homework by the day after?

   a) YYYessss! b) Jawohl! c) Da! d) Sí, sí. e) OMG no!

Full Answer Problems

Chapt. 2 History of Cosmology

Multiple-Choice Problems

Full Answer Problems

Chapt. 3 Miscellaneous Problems

Multiple-Choice Problems

Full-Answer Problems

002 qfull 00110 1 3 3 hard math: ancient Egyptians and unit fractions

2. The ancient Egyptian mathematicians thought there was something unfundamental about non-unit fractions (those not of the form $1/n$) though they made a bit of an exception for $2/3$ (Boyer-13–14). So they thought it a good idea to expand non-unit fractions as sums of unit fractions (those of the form $1/n$).

   There are parts a,b,c.

   a) Show the general rational number m/n can be expanded into infinitely many possible unit fraction expansions. **HINT:** This is trivial.

   b) The ancient Egyptians apparently thought some kinds of unit-fraction expansions were good, but have not left us any definite rules (Boyer-14). Probably they never formulated any. However, we can formulate a rule/algorithm. Specify an rule/algorithm for expanding general m/n in unit fractions

   $$m/n = \sum_{i=1}^{I} \frac{k_i}{n_i}$$

   where the denominators n_i are all divisors of n in increasing order of size, there are I divisors in total, and k_i are all zero or 1, except that k_I can be greater than 1. **HINT:** The proof just requires some subtraction using a recurrence relation.

   c) Use your rule/algorithm from the part (b) answer to expand $601/360$ in unit fractions. You could do this by hand or write a small computer progam do to it. Note that 360 has 24 divisors which is probably one of the main reasons why the ancient Babylonian astronomers chose it for the division of the circle—they wanted easy division. The other main reason was probably to get angle unit nearly equal to the distance the Sun moved every day on the celestial sphere. **HINT:** If you write a computer code, make it find the divisors with the mod function for you then it will be general for any denominator n. Try your code out on $1170/360$.

   d) Consider m/n and an expansion in the harmonic series with omissions:

   $$\frac{m}{n} = \sum_{\ell=2}^{K} \frac{k_\ell}{\ell} \ ,$$

   where $k_i = 1$ or zero and K is in general ∞. Why is it always possible to make this expansion? Can the series truncate with K finite? I will give one buck to the first person who finds out by themselves or from some source whether or not the expansion truncates to finite K always.

---

002 qfull 00120 1 3 3 hard math: frustum volume AKA frustrum volume

3. A general cone is 3-dimensional shape formed from a planar base and continuum of line segments from the base's perimeter to a vertex (or apex) not in the plane of the base. The height of the cone is the length of the perpendicular from the base plane to the vertex. A general frustum—or, tripping off the tongue erroneously, frustrum—is a general cone with the top sliced off parallel to the base.

   The ancient Egyptian mathematicians were very interested in frustums because of the topless pyramid kind—they are were always designing and building things like that. They even knew the rule for the volume of square pyramidal frustum which in modern formula form is

$$V = \frac{\Delta h}{3}(a^2 + ab + b^2) ,$$

   where Δh is the height of the frustum (not the height of the pyramid it's cut from), a is the base square side length, and b is the top square side length. The ancient Egyptians probably deduced this rule by constructing a square pyramidal frustum from simpler parts (Boyer-21).

   There are parts a,b.

   a) By the power of pure guess, generalize the volume formula to that of a general frustum with base area A and top area B.

   b) Prove your generalization from the part (a) answer. **HINT:** Note the following factoids. Factoid 1: You can approximately replace any cone/frustum by a **SET** of equal-base-area square cones/frustums with their base-parallel slices slid appropriately: just picture it. Factoid 2: If you slide parallel slices of 3-dimensional shape, you don't change the volume of the shape (e.g., for parallelopiped obviously).

   c) Now derive the volume of a general cone with base area A and height h without using the equation in the preamble or the formula found in the parts (a) and (b) answers. **HINT:** The area of any base-parallel slice A_z is proportional to the square of the distance from the vertex to the slice z along the perpendicular from the base plane to the vertex. This is obvious if you envisage the slice as covered by a grid: each grid line obviously scales as z.

   d) Now what is the volume of a frustum with base of area A and height to the invisible vertex h, and top with area B and height to the invisible vertex h_B?

   e) Given $\Delta h = h - h_B$, derive the formula found in the part (a) answer from the formula found in the part (d) answer. **HINT:** You will have to express h and h_B in terms of Δh, A, and B making use of the integrand used in the part (d) answer, and do some mildly tricky algebra which is accelerated by using the sum/difference of cubes formula:

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2) .$$

   f) Who is responsible for

      ...
      Come, every frustum longs to be a cone,
      And every vector dreams of matrices.
      Hark to the gentle gradient of the breeze:
      It whispers of a more ergodic zone.
      ...
      I see the eigenvalue in thine eye,
      I hear the tender tensor in thy sigh.
      Bernoulli would have been content to die,
      Had he but known such $a^2 \cos(2\phi)$.

---

002 qfull 00210 1 3 0 easy math: Pythagorean theorem I

4. The Pythagorean theorem was known to the ancient Babylonians, but not as far as known the ancient Egyptians, long before Pythagoras (c. 570–c. 495 BCE) (Boyer-42). But it is likely the ancient Babylonians never gave a general proof: they just did not think in terms of general proofs. The ancient Greek mathematicians may or may not have learnt of the Pythagorean theorem from the ancient Babylonians. However, they probably gave the first general proof. Late reports say Pythagoras himself proved it and hence its name. This may just be legend (Boyer-54; Wikipedia: Pythagorean theorem: History). Euclid (fl. 300 BCE) gives the first proof on the historical record. We will not attempt his proof, but something simpler. By the way, no one wrote equations like we do before circa 1600—they used other klutzy ways of expressing formulae (see Wikipedia: History of mathematical notation: Symbolic stage).

   Assume a Euclidean 2-dimensional space. Since the space is Euclidean or flat, a square (a 4-sided polygon with sides of equal length and right-angle vertices) has area $A = d^2$ where d is the length of a side. Prove the Pythagorean theorem for this Euclidean space. **HINT:** Draw a square with side length $a + b$ and an inscribed square of side length c where the vertices of the inscribed square touch the first square sides at the points that divide the sides into parts of length a an b.

---

002 qfull 00220 1 3 0 easy math: Pythagorean theorem II with area rule

5. In 2-dimensional Euclidean space (i.e., 2-dimensional flat space), we have a simple area principle. If you draw a general closed contour, you can tile it without overlap with squares of equal size with side length a. We define a^2 as the area of the squares. The sum of areas a^2 for closed contour in the limit that $a \to 0$ and number of squares goes to infinity is the area A of the closed contour. An identical closed contour anywhere in the space has the same area A and if you scale any the linear dimension of the contour by f, the area scales by f^2. Somewhat obviously, the area of two general closed contours (joined or separated) must equal the sum of areas of the two general closed contours since the tiled areas just equal the count of squares of area a^2 times aread a^2 before you take the limit.

   The area principle implies the Pythagorean theorem and consequently the metric of 2-dimensional flat space: $ds^2 = dx^2 + dy^2$, where x and y are general perpendicular coordinates and ds is the distance or interval between two points with coordinates that differ by dx and dy.

   There are parts a,b,c,d. The parts can be done independently, and so do not stop if you cannot do a part.

   a) Use the area principle to prove the area of a right triangle with sides of length a and b forming the right angle is $ab/2$. **HINT:** Imagine little squares of side length e and tile a rectangle with them, count the squares, find the area of the rectangle as $e \to 0$ and the number of squares goes to infinity, and then use symmetry.

   b) Draw a diagram of a square with sides of length $a + b$ and an inscribed square with side of length c with corners touching the sides of the first square (which is the circumscribed square) at points a from each corner of the first square.

   c) Use answers from the parts (a) and (b) to prove the Pythagorean theorem: i.e., $c^2 = a^2 + b^2$.

   d) Prove the metric $ds^2 = dx^2 + dy^2$ holds for a 2-dimensional flat space. **HINT:** This is easy.

---

002 qfull 00230 1 3 0 easy math: Pythagorean theorem III with area rule with Euclid's 5th postulate

6. Can we prove the Pythagorean theorem semi-rigorously? Yes.

   There are parts a,b,c,d,e,f,g,h,i. The parts can be done independently, and so do not stop if you cannot do a part.

   a) Assume an homogeneous, isotropic (homist) 2-dimensional space. Assume there is a geodesic rule: i.e., there is a rule for measuring distance and for measuring the stationary distance between two points. Draw intersecting equal length geodesics that intersect at their midpoints and that have 4-fold rotational symmetry about their intersection point. A full rotation about the intersection point is measured as 360°. How would you describe size of the angles subtended at the intersection point separating the crossed geodesic arms and why would you say this? Note draw the geodesics vertical and horizontal, so that the descriptions in the following parts are consistent with the diagram.

b) Now draw geodesics between the endpoints of your crossed geodesics, but note we are not assuming Euclidean (i.e., flat space) so that these geodesics could bend outward/inward from intersection point in some projection or another. You now have a square (but not necessarily a Euclidean square). Call it square 1. Now copy square 1 to square 2 and translate square 2 to the upper right so that the lower left corner endpoints of square 2 lie on the upper right corner endpoints of square 1. Is there a space between geodesics of the two squares joining common endpoints? Why or why not?

c) Now copy square 2 to square 3 and translate square 3 to the lower right, but otherwise with the same intructions as in part (b). Now copy square 3 to square 4 and translate square 4 to the lower left, but otherwise with the same intructions as in part (b). Does square 4 necessarily share a common geodesic with the original square 1? Why or why not?

d) The answer to part (c) was no. However, if there is a common geodesic then the space is a Euclidean plane and, at the common corner of the 4 squares, the angles between the geodesics that meet there are all 90°. Postulating that they are 90° is equivalent to Euclid's 5th postulate. For long ages mathematicians wondered if 5th postulate was derivable from Euclid's first 4 postulates. The answer is no. Even somewhat obviously no since, among other things, geodesics that are parallel on a sphere at the equator (i.e., separated by a mutually perpendicular geodesic there) meet at the poles.

Assuming a Euclidean plane, prove that ines (as we now call geodesics) parallel at one location (i.e., separated by a mutually perpendicular line) stay the same perpendicular distance apart no matter how extended. There are probably many ways of proving this, but one path is to start by noting that equal squares of any size can tile the whole Euclidean plane without overlap which actually is an immediate consequence of our considerations above.

e) The fact that one can tile the Euclidean plane completely with squares without overlap suggests an area concept. Consider differential rectangles of side lengths dx and dy. Define their area to be $dx\,dy$. We define area to be countable in the sense that the area of N rectangles is $N\,dx\,dy$. You can tile completely any region surrounded by a closed curve with equal differential rectangles with no rectangles wholly out of the region. We define the area of the region by

$$A = \lim_{N\to\infty,\, dx\,dy\to 0} N\,dx\,dy\ .$$

That such limit exists in general requires a rigorous proof that we will not do here. However, one can prove the limit exists in special cases easily and those special cases they also show why defining the area of the differential rectangles in terms of the lengths of their sides is reasonable since finite regions of sufficient symmetry also have areas specified by their defining lengths. An important point is that area is independent of the ordering of the adding up the differentials areas. As a nonce expression, we call this independence the area principle.

Determine the area of a large rectangle of sides a and b in terms of differential rectangles and take the limit so that the properties of the differential rectangles vanish.

f) Prove that the area of a right triangle with sides forming the right angle being of length a and b is $ab/2$. **HINT:** You do need to use the area principle.

g) Draw a diagram of a square with sides of length $a + b$ and an inscribed square with side of length c with corners touching the sides of the first square (which is the circumscribed square) at points a from each corner of the first square.

h) Use the area principle to prove the Pythagorean theorem: i.e., $c^2 = a^2 + b^2$.

i) Prove the metric $ds^2 = dx^2 + dy^2$ holds for a Euclidean plane. **HINT:** This is easy.

---

002 qfull 00310 1 3 0 easy math: Golden Ratio and Fibonacci sequence, golden ratio

7. The golden ratio ϕ is a special number known since Greco-Roman antiquity. But there's nothing especially special about it. There are many special numbers: all small natural numbers $(0, 1, 2, \ldots)$, all small prime numbers $(2, 3, 5, 7, 11, 13, \ldots)$, $e = 2.71828\ldots$, pi $= \pi = 3.14159\ldots$, the Euler-Mascheroni constant $\gamma = 0.57721566\ldots$, etc. Here we will investigate the golden ratio just a bit.

There are parts a,b.

a) Draw a line segment of length a and divide into two parts of of lengths b and c: thus $a = b + c$. The golden ratio is just the ratio when

$$\frac{a}{b} = \frac{b}{c} \ .$$

b) Let's do a general investigation of ratios of the form

$$\frac{a}{b} = g\frac{b}{c} \ ,$$

where $a = b + c$. Solve for the positive case of the ratio a/b as a function of g only. Find the cases for $g = \infty, 1, 0$. The case $g = 1$ gives the golden ratio itself.

c) Prove that

$$\frac{1}{\phi} = \phi - 1 \ .$$

d) In 1202, Fibonacci, perhaps independantly of Indian mathematics, discovered the Fibonnaci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ which has an interesting connection to the golden ratio.

The discovery was from the problem of reproducing pairs of rabbits. A pair takes 1 month to mature from birth and reproduces a new pair after maturity every one month: so the first reproduction happens 2 months after birth. Consider times t_i separated by 1 month periods. Say you at time t_{i-1} you had n_{i-1} adult pairs. However, only the adult pairs n_{i-2} existing at time t_{i-2} can reproduce at t_{i-1} since the new baby pairs at time t_{i-2} have only just matured at t_{i-1}. So at t_{i-1}, the old adult pairs n_{i-2} produce n_{i-2} babies who mature to be adult pairs at time t_i. So the number of adult pairs at time t_i is

$$n_i = n_{i-1} + n_{i-2}$$

which is, of course, a recurrence relation valid for $i \geq 2$.

Starting with 1 baby pair and no adult pairs at time zero, compute by inspection the Fibonnaci sequence until you get bored.

e) In the limit $i \to \infty$, the ratio of adjacent numbers following from Fibonacci recurrence relation

$$n_i = n_{i-1} + n_{i-2}$$

for $i \geq 2$, $n_0 \geq 0$, and $n_1 > 0$ obeys

$$R_i = \frac{n_i}{n_{i-1}} \to \phi \ .$$

Note we are allowing more general initial n_i values than for Fibonacci sequence. In fact, the R_i's alternate with every increment of i between being too high and too low compared to ϕ as $i \to \infty$ and they go to ϕ exactly for finite i in only one special case. Prove the above statements. **HINT:** Start from the Fibonacci recurrence relation, use the definition $\epsilon_i = (R_i - \phi)/\phi$, and remember the part (c) result.

f) Find a reasonable approximate asymptotic formula for the n_i from part (e) as $i \to \infty$. It should be exactly correct in one special case.

g) The recurrence relation

$$n_i = n_{i-1} + n_{i-2}$$

can be turned into a differential equation by changing i into continuous variable t expanding n_t and $n(t-2)$ about $t-1$ to 1st order. Make the transformation and solve the differential equation. How does the solution compare to the approximate asymptotic formula of part (f)?

---

002 qfull 00410 1 3 0 easy math: quadratic formula made numerically robust

8. The quadratic formula (which is the solution of the quadratic equation) is an infamous example of case where the standard analytic form (which is what everyone remembers) is numerically rotten. The equation and formula in standard form are, respectively,

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$

The numerical rottenness occurs if $|4ac| << b^2$: in this case, one of the roots can become affected by severe round-off error. We'll see how to fix the problem in this problem.

NOTE: There are parts a,b,c,d,e,f. The parts cannot be done independently, but parts (a) and (b) are not so hard and the later parts are just intricate.

a) Solve the quadratic equation for the standard quadratic formula using completing the square. Note we assume that a, b, and c are pure real numbers.

b) The crucial insight is that root cause of the numerical problem is the sign of b. If $|4ac| << b^2$, then the standard formula gives numerically good solution for one sign of b and numerically bad one for the other. Note if $b = 0$, there is no problem at all:

$$x = \pm\sqrt{\frac{-c}{a}} \ .$$

So the trick to getting a numerically robust quadratic formula is to isolate sign of b: i.e., to factorize b into its sign and absolute value. Rewrite the standard formula in the form

$$x_\pm = \frac{-\text{sgn}(b)|b| \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{1}{2}\text{sgn}(b)\left(\frac{|b| \pm \sqrt{b^2 - 4ac}}{a}\right) \ ,$$

where we note that the $\text{sgn}(b)(\pm 1) = (\pm 1)$ if the (± 1) is uncorrelated with the $\text{sgn}(b)$, using a bit of clairvoyance for a nice formula we put the factor of 1/2 where it's been put, and the sign function is given by

$$\text{sgn}(b) = \begin{cases} 1 & \text{for } b > 0. \\ 1 & \text{for } b = 0 \text{ which is unlike the usual definition of 0.} \\ -1 & \text{for } b < 0. \end{cases}$$

As now written, we can see that solution x_+ is numerically robust, but solution x_- is not. But you can make solution x_- robust by using the a difference of squares factor. Write the numerically robust quadratic formula for solution x_- in terms

$$q = -\frac{1}{2}\text{sgn}(b)\left(|b| + \sqrt{b^2 - 4ac}\right)$$

when the moment is right. **HINT:** Recall the difference of squares formula:

$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2 \ .$$

c) What can you say about the robust solutions when the discriminant $(b^2 - 4ac) < 0$ and what can you say about q, a, b, and c in this case?

d) What can you say about the robust solutions when $a = 0$ and $q \neq 0$, and what can you say about q, b, and c in this case?

e) What can you say about the robust solutions when $a \neq 0$ and $q = 0$, and what can you say about a, b, and c in this case?

f) What can you say about the robust solutions when $a = 0$ and $q = 0$, and what can you say about b and c in this case?

---

002 qfull 00510 1 3 0 easy math: simple 1st order DE solution

9. Consider the following linear 1st order differential equation (DE):

$$x' = A - kx \ ,$$

where t is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant. There are parts a,b,c,d. Parts (a) and (b) can be done independently at least.

a) Solve for the constant solution x_A. **HINT:** This is easy.

b) We can now write the DE as

$$x' = k\,(x_A - x) \ .$$

Without solving for non-constant solution describe what it must look like as a function of t for arbitrary initial value $x_0 = x(t = 0)$. In particular, where are its stationary points if any? **HINT:** Consider the continuity of all orders of derivative of x.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$, $x'(t)$, and the 1st order in small t solution $x_{\rm 1st}(t)$. **HINT:** You can use an integrating factor, but there is a more straightforward way.

d) What is the e-folding time t_e of your solution and what does it signify? What is the $x(t_e)$? What is the $x_{\rm 1st}(t_e)$? What is remarkable about $x_{\rm 1st}(t_e)$?

---

002 qfull 00520 1 3 0 easy math: simple 1st order DE solution variant: conflate with 00510?

Extra keywords: Has part (b) of the original 01010. Is it worth anything?

10. Consider the following linear 1st order differential equation:

$$x' = A - kx \ ,$$

where t is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant. There are parts a,b,c,d. Parts (a) and (b) can be done independently at least.

a) Solve for the constant solution x_A. **HINT:** This is easy.

b) Where is it possible for a non-constant solution of a 1st order differential equation to have a stationary point? Will there be stationary points at those t locations for the particular differenital equation of the preamble? **HINT:** Consider the differential equation written in the form

$$x' = k\left(\frac{A}{k} - x\right)$$

and consider what happens to the solution as $t \to \infty$ and remember that if the solution becomes constant, it stays constant. It helps to think graphically.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$ and the 1st order in small t solution $x_{1\text{st}}(t)$. **HINT:** You can use an integrating factor, but there is a more straightforward way.

d) What is the e-folding constant t_e and what does it signify? What is the $x(t_e)$? What is the $x_{1\text{st}}t_e)$? What is remarkable about $x_{1\text{st}}(t_e)$?

---

002 qfull 00550 1 5 0 tough math: Solving a 1st-order polynomial DE

Extra keywords: (Tu2003-12) Not very relevant to cosmology and needs reworking

11. Consider the 1st order nonlinear differential equation

$$x' = a\prod_{i=1}^{n}(x - x_i) \ ,$$

where t (which may or may not be time) is the independent variable, a is constant, and the x_i are the roots of the polynomial on the right-hand side: the roots increase monotonically with index i: i.e., they obey $x_1 \leq x_2 \leq ... \leq x_n$.

a) Solve the equation for the general solution for $n = 0$: i.e., when $x' = a$.

b) Solve the equation for the general solution for $n = 1$: i.e., when $x' = a(x - x_1)$. Since this is a warm-up question, a solution by inspection is not adequate.

c) Qualitatively and compactly describe the solutions of the differential equation in all regions for $n \geq 2$. **HINT:** The equation is a 1st order differential equation and the right-hand side is infinitely differentiable everywhere. There are 4 cases to consider. Don't forget to describe the stability of the constant solutions: i.e., does a sufficiently small perturbation lead to a restoration to the constant solution or a permanent departure from it.

d) Consider distinct roots x_{j-1} and x_j for the case with $n \geq 2$. Find an approximate interpolation solution which has the correct values at $t = \pm\infty$. The approximate solution should contain the function element ge^{-ht} where h can be positive or negative, but not zero and $g > 0$ always. The values of h and g are determined in part (e) just below. **HINT:** This is pretty easy.

e) Continuing with the problem from part (d), determine h by requiring that the approximate solution satisfy the differential equation at the midpoint $x = (x_j + x_{j-1})/2$ and g by requiring that it pass through the point (t_0, x_0), where $x_0 \in (x_{j-1}, x_j)$. **HINT:** This is a lot easier than it seems at first.

f) Continuing with the problem from part (d), show that the approximate formula is, in fact, the exact solution for the case of $n = 2$. This solution is called the logistic function. **HINT:** Simplify the formula for h and then differentiate the solution for $n = 2$ and keep substituting the solution for $n = 2$ to eliminate the h and ge^{-ht} function elements.

g) Now solve the equation for the general solution for general $n \geq 2$ and all roots the same x_r: i.e., for $x_i = x_r$ for all i. **HINT:**

---

002 qfull 00560 1 3 0 easy math: perturbation solutions for 1st order DEs

12. Consider the 1st order (ordinary, autonomous) differential equation

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. The 1st order DE rule (as yours truly calls it) applies to this DE. We have $f(x_i) = 0$ and therefore x_i yields a constant solution and a stationary point at either of $\pm\infty$.

NOTE: There are parts a,b.

a) Assuming $(df/dx)(x_i) \neq 0$, solve without words for the 1st order perturbation solution in small $\Delta x = x - x_i$. Let Δx_0 be the initial perturbation, time zero is 0, and $R_1 = (df/dx)(x_i)$ for compactness. What is the condition for convergence/divergence in the future to the constant solution? What is the condition for convergence/divergence in the past to the constant solution? **HINT:** Recall the antiderivative of $1/y$ is always $\ln(|y|)$.

b) Now assume the lowest order nonzero coefficient in the expansion of $f(x)$ in small δx is $(d^k f/dx^k)(x_i)$ where $k \geq 2$. The write the solution only in terms of $|\Delta x|$ and $|\Delta x_0|$ since that seems most clear and start from the differential form

$$\frac{d|\Delta x|}{|\Delta x|^k} = hR_k\,dt \ ,$$

where for k even $h = \pm 1$ with upper case for $\Delta x > 0$ and lower case for $\Delta x < 0$ and for k odd $h = 1$, and $R_k = (d^k f/dx^k)(x_i)$ for compactness. Show why this differential form is correct before you use it.

c) What happens as $hR_k t$ **INCREASES/DECREASES** from 0? At what time t is there an infinity?

---

002 qfull 00590 1 3 0 easy math: logistic function

13. The logistic function (called that for a darn good reason) turns up in many contexts looking like:

$$f(x) = \begin{cases} \dfrac{f_{\rm M}}{1+e^{-r(x-x_0)}} = \dfrac{f_{\rm M}}{1+(f_{\rm M}/f_0 - 1)e^{-rx}} & \text{in general form;} \\ \dfrac{1}{1+e^{-x}} = \dfrac{e^x}{e^x+1} = \dfrac{1}{2}\left[\tanh(x/2)+1\right] & \text{in natural or reduced form.} \end{cases}$$

In this question, we only use the natural form for simplicity and elegance.

There are parts a,b,c,d.

a) Determine f' (which is, in fact, called the logistic distribution), f'' (also write it as an explicitly even function which it is), the antiderivative of f (easy if you write f in terms of e^x), and the integral of f' from $-x$ to x. Use the natural form of the function.

b) Determine stationary points of f and f' and the values of f and f' at those points. Use the natural form of the function.

c) The logistic function can be used as a smooth replacement for the Heaviside step function:

$$H(x) = \begin{cases} 0 & x < 0; \\ 1/2 & x = 0; \\ 1 & x > 0. \end{cases}$$

Show that logistic function becomes the that Heaviside step function with the appropriate limiting procedure. **HINT:** This is really easy.

d) The logistic function is actually the solution of a 1st order nonlinear differential equation. This equation shows up, for example, in population dynamics. Say you have population N that grows at rate (per population) r with unlimited resources. However, the rate with resources limited by carry capacity (or maximum population) K is modeled as $r(1-N/K)$ which is zero when $N \to K$. The growth differential equation for N, sometimes called the Verhulst-Pearl equation, is

$$\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)N \ ,$$

Reduce this equation to natural form and find the solution. Then write the solution out in population-dynamics form for general initial population N_0 at $t = 0$ and show the small N/K and $t \to \infty$ asymptotic limiting cases explicitly. **HINT:** You'll need a table integral.

---

002 qfull 00610 1 3 0 hard math: 1st order DE rule I (version II better, conflate?)
Extra keywords: This version may be completely obsolete due to the 640 version

14. A 1st order homogeneous differential equation, linear or nonlinear, of the form

$$f' = g(f) \ ,$$

(with independent variable t which g has **NO** explicit dependence on) at points where it is infinitely differentiable only has solutions that are strictly in/decreasing or that are constant. Note that differentiable at a point means there is a finite derivative of the same value taken from above or below the point and there is no singularity at the point (which is usually implied by the first condition). Also note that strictly in/decreasing means there are no have stationary points and constant means constant for a finite region. The constant solutions are often stable/unstable in the sense that small perturbations from them lead to convergent/divergent behavior with increasing independent variable.

The rule actually requires the extra condition that higher derivatives of the differential equation $f^{\langle n\rangle}$ (where we use angle brackets to indicate differentiation order when primes will not do) do **NOT** generate zero-over-zero cases: i.e., cases where a f' on the right-hand side of the equation is multiplied by a factor that cancels the zero at stationary point making the higher order derivative on the left-hand side of the equation non-zero. Such a non-zero $f^{\langle n\rangle}$ means that a Taylor expansion around the stationary point will show curvature. That zero-over-zero cases occur will be proven showing important examples.

There are parts a,b,c,d,e,f,g. The parts can all be done independently, and so do not stop if you cannot do a part.

a) Prove the rule given in the preamble for a $g(f)$ that does **NOT** generate zero-over-zero cases. **HINT:** Use proof by induction using the general Leibniz rule (which is the generalization of the product rule):

$$(rs)^{\langle n\rangle} = \sum_{k=0}^{n} \binom{n}{k} r^{\langle n-k\rangle} s^{\langle k\rangle} \ ,$$

where r and s are general functions (Ar-667; Wikipedia: General Leibniz rule). Note $s^{\langle 0\rangle} = s$ not 1.

b) For this part, the preamble is long, the answer is short—have patience.

The zero-over-zero case can (but not necessarily will) occur when we have

$$(f')^p = g(f) \qquad \text{or, equivalently,} \qquad f' = e^{i\phi} g(f)^{1/p}$$

where $e^{i\phi}$ is a phase factor (and we only consider its pure real values) and where $g(f)$ does not itself lead to the zero-over-zero case. The zero-over-zero case will when

$$g^{1/p-(n-1)}(f')^(n-1) = Q \neq 0 \ ,$$

where Q is a constant and $n > 2$ and $[1/p-(n-1)]$ and $(n-1)$ are powers, **NOT** derivative orders. Note that when $n = 1$, we have

$$f' = e^{i\phi} g^{1/p} = e^{i\phi} Q$$

which means $f = at + b$ which has no stationary points and is not zero-over-zero case.

To prove the exception, we differentiate the differential equation $In - 1$ times to get

$$f^{\langle In\rangle} = Ag^{1/p-(n-1)}(f')^(n-1)f^{\langle I-1\rangle} + Bg^{1/p-(n-2)}(f')^(n-2)f^{\langle (I-1)n+1\rangle} + \ldots \ ,$$

where A and B are constants whose values are of no interest and $\{(I-1)n+1\}$ is a derivative order. Note that every term must have the sum of derivative orders equal to $In-1$: e.g., $(n-1)+(I-1)n = In-1$ and $(n-2)+(I-1)n+1 = In-1$. an **INHOMOGENEOUS** 1st order differential equation does not have to obey the rule stated in the preamble. **HINT:** Find a trivial counterexample. Think trigonometry.

b) Prove that a homogenous 1st order differential equation can have a stationary point at $\pm\infty$. **HINT:** Find a trivial example.

c) Prove the rule given in the preamble and discuss why exceptions can occur. **HINT:** Use proof by induction to show that if $x(t)$ has a stationary point where $x' = f(x)$ are infinitely differentiable that the function is constant at that point: i.e., all orders of derivatives of x are zero a that point.

d) Prove that a solution can be nonmonotonic if there is point t where $x' = f(x))$ is not infinitely differentiable. **HINT:** Find a simple example of a 1st order differential equation such a solution. Yours truly suggests differential equation with solution $x = 1/t$.

e) Prove that a solution can have a stationary point at a point t where $x' = f(x))$ is not infinitely differentiable. **HINT:** Find a simple example of a 1st order differential equation such a solution. Yours truly suggests differential equation with solution $x = |t|^3$.

---

002 qfull 00620 1 3 0 easy math: 1st order DE rule II (better version?)

Extra keywords: This version may be completely obsolete due to the 640 version

15. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P \ ,$$

where $P = (1-1/n)$ with $n \in [2,\infty)$ and we assume $g(x)$ is infinitely differentiable with respect to x. Note $g(x)$ may go negative as a function of x, but we assume it does not negative as function of t at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm[g(x)]^{1/2} \ ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

NOTE: There are parts a,b,c,d.

a) Given x_i (or in the time variable t_i) is a stationary point of $x' = f(x)$ (i.e., $x'(x_i) = f(x_i) = f[x(t_i)] = 0$), prove without words that $x''(x_i) = 0$.

b) The part (a) answer gives the base case (or 1st step) for a proof by induction that all orders of derivative of x with respect to t at x_i (or in the time variable t_i) are zero. The proof follows by inspection if your math intuition is good enough. However, do a formal proof by induction. **HINT:** For the proof, you do **NOT**, in fact, need the full general Leibniz rule for the derivative of a product (Ar-558)

$$\frac{d^m(fg)}{dx^m} = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dx^k} \frac{d^{m-k} g}{dx^{m-k}} \ .$$

Using it actually makes the proof a bit more tricky to follow. But you do need to know that the nth order derivative of x (i.e., $x^{(n)}$) is obtained by applying the general Leibniz rule for $m = n - 2$ to the result of the part (a) answer and that highest derivative of x on the right-hand side of that application is $x^{(n-1)}$. Note that $f(x)$ is general to the degree specified in the preamble, and so the proof is unchanged if any order of derivative $f(x)$ with respect to x is zero at x_i.

c) Given the part (b) result, give an argument for why the stationary point t_i must be all points (i.e., is actually a static solution) or at time equals infinity.

d) A 1st order DE system given a small perturbation from a static solution either asymptotically goes back to it (i.e., is asymptotic to it at positive infinity, and so is called stable) or grows away from it (i.e., is asymptotic to it at negative infinity, and so is called unstable). Assuming the df/dx is nonzero at x_i, prove without words that a 1st order DE system given a small perturbation (i.e., a perturbation Δx_0 which requires only 1st order expansion of $f(x)$ in small $\Delta x = x - x_i$) varies exponentially and determine the condition for stability.

---

002 qfull 00630 1 3 0 easy math: main exception to the 1st order DE rule

Extra keywords: This version may be completely obsolete due o 640 version

16. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P \ ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to x. Note $g(x)$ may go negative as a function of x, but we assume it does not negative as function of t at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm\sqrt{g(x)} \ ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

NOTE: There are parts a,b,c,d,e.

a) Given x_i (or in the time variable t_i) is a stationary point of $x' = \pm\sqrt{g(x)}$ (i.e., $x'(x_i) = \pm\sqrt{g(x_i)} = \pm\sqrt{g[x(t_i)]} = 0$), prove without words that $x''(x_i) \neq 0$ for $g(x_i) \neq 0$.

b) What does the part (a) answer imply about x_i? What does the part (a) answer imply about x_i given the sign of $dg/dx(x_i)$?

c) Given $(dg/dx)(x_i) = 0$, prove by induction that for general $n \in [1\infty]$ that $x^{(n)}(x_i) = 0$. **HINT:** Consider $x^{(4)}(x_i) = 0$ as step 1 (i.e., the base case) of the proof. Note that the right-hand side of the expressions in the proof will always have a derivative of x two orders lower than the left-hand side.

d) Given $(dg/dx)(x_i) = 0$, what does the part (c) answer imply about x_i?

e) Given $(dg/dx)(x_i) = 0$, and therefore there is a static solution $x = x_i$ for all time t, we can consider what the lowest order solution is for a small perturbation from the static solution. The expansion of the differential equation in small $\Delta x = x - x_i$ is

$$\frac{d\Delta x}{dt} = \pm\sqrt{\sum_{k=\ell}^{\infty} \Delta x^k \left[\frac{d^k g}{dx^k}(x_i)\right]} \ ,$$

where ℓ is the lowest power for which there is a nonzero coefficient $(d^\ell g/dx^\ell)(x_i)$. What possible signs can Δx when ℓ is even and $(d^\ell g/dx^\ell)(x_i) > 0$? What possible signs can Δx when ℓ is even and $(d^\ell g/dx^\ell)(x_i) < 0$? What possible signs can Δx when ℓ is odd?

---

002 qfull 00640 1 3 0 easy math: 1st order autonomous DE and stationary points, definitive version

Extra keywords: This is the definitive version as of 2025mar09

17. First order autonomous ordinary differential equations (FAODEs), linear or nonlinear, only have solutions with stationary points at infinity (SPIs), (except for special cases which are not all that rare) and constant solutions. Actually, each SPI corresponds to a constant solution which could also be viewed as a continuum of stationary points. Note an autononous differential equation depends only on functions of the dependent variable, and so has no explicit dependence on the independent variable.

To investigate the SPI behavior of FAODEs consider the (somewhat general) FAODE

$$x^{(1)} = [f(x)]^{1/k} \ ,$$

where t (not necessarily time) is the independent variable, the superscript (1) means 1st derivative with respect to t, $f(x)$ is an infinitely differentiable function with zeros at set of values $\{x_i\}$, and $k > 0$. We limit k to being greater than zero to avoid uninteresting generality. Since $f(x)$ is infinitely differentiable at (general) x_i, we can expand $f(x)$ about x_i with some radius of convergence: i.e.,

$$f(\Delta x) = \sum_{j=\ell}^{\infty} \Delta x^j f_j = \Delta x^\ell f_\ell + \ldots \ ,$$

where $\Delta x = x - x_i$, the f_j are expansion constants, and $\ell > 0$ is the lowest (nonzero) order in the expansion. Note $\ell \neq 0$ since we have assumed x_i is a zero of $f(x)$: i.e., $f(x_i) = 0$.

We will primarily be examining the lowest order solutions in Δx, and so we will be dealing with $\Delta x^{\ell/k} f_\ell^{1/k}$ and related expressions. Mathematically, if ℓ/k is not an integer, complex numbers can arise in these expressions. However, we are only interested FAODEs and their solutions corresponding to physical systems involving real numbers. In these systems, the solutions just never evolve into the complex number realm. So we are not going to concern ourselves with question what happens mathematically if some our expressions can give rise to complex numbers. They never give rise to complex numbers physically.

NOTE: There are parts a,b,c,d,e,f,g,h,i,j,k. On exams, do **ONLY** parts i,j.

a) What is the behavior of x as a function of t between the points in the set $\{x_i\}$.

b) In this question we are only interested in the SPI behavior and constant solution behavoir, and so we are only interested in the behavior of $x(t)$ when it is arbitrarily close to x_i where

SPI and constant solutions occur. Therefore expand the FAODE about x_i with dependent variable Δx to lowest order in the exponent.

c) Determine the formula $p(n)$ for the exponent of Δx in the n derivative of Δx (for the lowest order of the FAODE) with respect to t. **HINT:** Drop all constants that turn up in the differentiations.

d) What is behavior of the t derivatives of Δx when $x = x_i$ for $\ell/k \geq 1$? What solutions $x(t)$ are implied by $\ell/k \geq 1$?

e) What is behavior of the t derivatives of Δx for $f(x_i)$ for $\ell/k < 1$ assuming the formula $p(n)$ never equals zero? What solution $x(t)$ behavior is implied by $\ell/k < 1$ in this case? Only a short answer is expected to the last question.

f) If $\ell/k < 1$ and the formula $p(n)$ goes to zero for a stopping n_{st}, what is the formula for ℓ/k as a function of n_{st} and what are the values of ℓ/k for the set $n_{\text{st}} = 1, 2, 3, \ldots, \infty$ and what do the $n_{\text{st}} = 1$ and $n_{\text{st}} = \infty$ cases mean? What is the formula n_{st} as a function of ℓ/k? What is this formula good for?

g) What is implied by a stopping $n_{\text{st}} \in [2, \infty)$ (i.e., an actual integer n_{st} in this range)? Give the solution for small $\Delta x(t)$ with with initial condition $\Delta x(t = 0) = 0$. Describe the function behavior at $\Delta x(t = 0) = 0$: i.e., maximum or minimum stationary point or rising or falling inflection point.

h) What would you expect the two likeliest values for ℓ to be for physically relevant FAODEs? What would you expect the two likeliest value for $k \neq 1$ to be for physically relevant FAODEs?

i) Now we intuited for the case of $\ell/k \geq 1$ that the stationary point would be a stationary point at infinity (i.e., an SPI), but we did not prove this directly. To prove directly, we need to show that the small Δx (meaning small in absolute value) solutions of

$$\Delta x^{(1)} = \Delta x^{\ell/k} f_\ell^{1/k}$$

that go to zero only do so as $t \to \infty$. Solutions that go to zero are convergent solutions. This means that the constant solutions they correspond to are stable solutions: small perturbations from the constant solutions damp out. Those that do not go to zero are divergent solutions. This means that the constant solutions they correspond to are unstable solutions: small perturbations from the constant solutions cause non-stopping divergence from the constant solutions.

Here consider the $\ell/k = 1$ case and the solutions for $\Delta x(t)$ starting from $t = t_0$ and $\Delta x = \Delta x_0$ as initial conditions. Determine the solutions and under what conditions they are convergent/divergent. Does the convergent solution, in fact, have a SPI? **HINT:** Let $y = \pm \Delta x$ where the upper/lower case is for positive/negative Δx_0.

j) Repeat part (i) for the case of $\ell/k > 1$.

k) An optional continuation of the discussion of the part (h) answer.

---

002 qfull 00650 1 3 0 easy math: a FAODE with a stationary point that is not a SPI

18. In this problem, we will get some more insight into first order autonomous ordinary differential equations (FAODEs) with stationary points that are not stationary points at infinity (SPIs) by examining a solution beyond solution to lowest (nonzero) order around the stationary points. Consider the FAODE

$$x^{(1)} = f(x) ,$$

where $f(x_i) = 0$ (i.e., $x = x_i$ gives a stationary point of some kind) and the independent variable is t (not necessarily time). However,

$$x^{(2)} = \frac{df}{dx} x^{(1)} = \frac{df}{dx} f(x) \neq 0$$

for $x = x_i$. This means the stationary point is not a SPI.

NOTE: There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c.

a) Let

$$g(x) = \frac{df}{dx} f(x)$$

and determine a formal solution for $f(x)$.

b) Assume $x(t)$ has maximum and minimum at, respectively, x_i and $-x_i$. Now invent the simplest $f(x)$ you can starting from the part (a) answer, except it has a general constant coefficient so as to give a general scale to the derivative $x^{(1)}$.

c) Now solve for $x(t)$ given the part (b) answer. **HINT:** You could do this by integrating $x(t)$, but differentiating $x(t)$ lead to solution by inspection.

d) Say a FAODE is given by

$$x^{(1)} = [f(x)]^{1/k} \ ,$$

where t is the independent variable (not necessarily time), $k > 0$, $f(x)$ is infinitely differentiable, and $f(x) = \Delta x^\ell f_\ell + \ldots$ is the expansion of $f(x)$ around the stationary point x_i with $\Delta x = x - x_i$ starting with the lowest nonzero order. Then the lowest order FAODE is

$$\Delta x^{(1)} = x^{\ell/k} f_\ell^{1/k} \ ,$$

In order for a solution of the FAODE to have stationary point that is not a SPI, there must be a stopping (derivative order) $n_{\rm st}$ given the formula

$$n_{\rm st} = \frac{1}{1 - \ell/k}$$

where an actual stopping $n_{\rm st}$ must be an integer. If the formula gives a non-integer value, then there is a singularity in the behavior of some order of derivative of $x(t)$ at $x = x_i$ and that behavior takes some analysis to determine. An actual stopping $n_{\rm st}$ gives the only nonzero derivative order of $x(t)$ at $x = x_i$. What are the ℓ and k values for the FAODE used in the part (c) and are they consistent with a nonzero derivative order $n = 2$ which is what we imposed in the preamble?

---

002 qfull 00850 1 3 0 easy math: iteration equation solution convergence: On exams, only do parts f,g,h.

19. Say you need to find a root to equation

$$g(y) = 0$$

and no analytic solution is available. The equation my be transcendental: i.e., no finite number operations results in a solution. There are many sophisticated of doing this (e.g., Pr-340ff), but a simple one is by an iteration function suitable if you can constrain the root you are looking for to some interval $y \in [a, b]$. First reaarange the equation as iteration equation

$$y = f(y)$$

and then iterate by feeding the output of function $f(y)$ back into function $f(y)$ as an argument or input. The iteration starts with an initial estimate solution y_0 and proceeds via iterates y_1, y_2, ..., y_{i-1}, y_i, etc. using equation

$$y_i = f(y_{i-1}) \ .$$

But how do you know you will get convergence and not divergence or just wandering. We will investigate convergence in this question.

Note the iteration equation approach (assuming it converges) may be very slow both in computer time and iterations especially if you are trying to converge to high machine precision and, of course, for transcendental equations you will never find exact numerical solution. Faster methods are available (e.g., the Brent method (Pr-352) and Newton-Raphson method (Pr-355)), but if you are just solving a simple one-off problem, the iteration equation method may be fine. In vast multiple variable problems like astrophysical atmosphere problems, a multivariable iteration "equation" may be all you have.

HINT: Drawing diagrams as needed helps.

NOTE: There are parts a,b,c,d,e,f,g,h,i. On exams, do **ONLY** parts f,g,h.

a) First, without loss of generality adjust the variables such that root is zero. Of course, course you cannot do this in an actual problem unless you already know the answer, but for the proof you can assume you do know the answer. Define two functions

$$y = \pm x \qquad \text{and} \qquad y = f(x) \ .$$

The first function divides the Cartesian plane into 4 quadrants. Show that if $f(x)$ is confined to the side quadrants and never tounches the lines defined by $y = \pm x$ in interval $[-a, a]$ (except at the origin itself which is in the interval $[-a, a]$) that convergence is guaranteed for zeroth iterate $y_0 \in [-a, a]$.

b) In a real problem the interval surrounding the root may not be symmetric about the root. This can lead to divergence with some easily imagined bad behavior in the side-quadrant-confined iteration function $f(x)$. How is divergence easily prevented?

c) In terms of sufficient and necessary conditions for convergence how would you describe the side-quadrant-confined iteration function $f(x)$ condition?

d) What is a simple sufficient, but not necessary, condition side-quadrant-confined iteration function $f(x)$ to give convergence?

e) How would iterates behave if side-quadrant-confined iteration function $f(x)$ were monotonically increasing/decreasing?

f) What makes an iteration function to solve for a root (AKA a zero) better thinking in the simplest sense? Think of the ideal limit.

g) Consider the transcendental equation

$$\frac{1}{2} = (x+1)e^{-x} \ .$$

Find an iteration function to solve for a that is probably divergent at a first guess. Note this is a real problem, and so the solution is not the origin.

h) For the transcendental equation of part (g)

$$\frac{1}{2} = (x+1)e^{-x}$$

find an iteration function guaranteed to converge for some interval about the solution. Find the interval of convergence and prove convergence in the interval.

i) Try to solve your convergent iteration equation from part (h) by series expansion in small x. You may have to consult Wikipedia (Wikipedia: Natural logarithm) to see where the series expansion is covergent and where a truncated version is a valid approximation. Then just use the Wikipedia plot to estimate the solution: i.e., the point where $y = x$ and the y value from the iteration function.

j) If you know how to code, iterate to function you found in part (h) to convergence to within machine precision and give the number iteration needed and the result.

---

002 qfull 00900 1 3 0 easy math: Monte Carlo sampling

20. In a Monte Carlo simulation, you want want to sample a random variable x drawn from a probability density function (pdf) $\rho(x)$. The trick is to set another random variable

$$y = P(x) = \int_0^x \rho(x')\,dx'$$

where P(x) is the cumulative probability distribution function (cdf). You then generate y values from a computer random number generator that gives them with uniform probability over the range $(0,1)$. You then obtain the sample random variables x from

$$x = P^{-1}(y)$$

where P^{-1} is the inverse function of P. The probability of y values in general range Δy is exactly the probability of x values in the corresponding range Δx since

$$\Delta y = \Delta P = \int_{\Delta x} \rho(x')\,dx' \ .$$

An odd point is that random number generators generate y values completely deterministically. So the y values are deterministic relative to source, but, for a good random number generators such as those discussed by Pr-191ff, the y values are random to all useful statistical tests relative to receiver. This fact invites the philosophical question: Is there any fundamental difference between a deterministic universe that mimics some amount of intrinsic randomness to all detection and one that has some intrinsic randomness as quantum mechanics as ordinarily discussed posits?
In any case, let's investigate how to do Monte Carlo sampling for photons for a couple of interesting cases.

There are parts a,b.

a) A stream of photons in a certain direction is scattered out that direction obeying

$$dN = -N\,d\tau$$

where N is the number of photons traveling in the direction and τ is the optical depth. What is the cdf for photon being scattered by general τ if it started at $\tau = 0$? What is the pdf?

b)

---

002 qfull 01010 1 3 0 easy math: variational calculus and Euler's equation

21. To determine geodesics (stationary paths through spaces) one needs to apply variational calculus in general which in the end amounts to solving a differential equation. The most famous variational calculus differential equation is Euler's equation (or Euler's equations if the plural is needed). Euler's equation can be used to find geodesics and it can be specialized to the Euler-Lagrange equations of classical mechanics whose use is justified by Hamilton's principle. We will derive Euler's equation now.

You have integral

$$I = \int_a^b f(x_i, \dot{x}_i, t)\,dt$$

where the set of coordinate functions $x_i = x_i(t)$ constitute a path through space with path parameter t and f is general function for its arguments. We want to determine the path $x_i(t)$

that makes the integral stationary for fixed endpoints $x(a)$ and $x(b)$. Note that following a general relativity convention, the subscript i means that x_i is one of set of coordinates and that it stands for all of them if that is what the context means.

We define

$$x_i(t,\alpha) = x_i(t) + \alpha\eta_i(t) \ ,$$

where $x_i(t)$ is the stationary path, $x_i(t,\alpha)$ is the varied path, α is a variational parameter, and η_i is a general function of t except that it vanishes at the endpoints of the integral. It is helpful to think of η_i as any little blip deviation from the stationary path you care to think of. Since η_i is general it and is derivative $\dot\eta_i$ can be varied independently, and thus x_i and $\dot x_i$ can be treated as independent in the variation. We now determine the condition on the stationary path as follows:

$$\begin{aligned}
0 = \frac{dI}{d\alpha} &= \int_a^b \left(\frac{\partial f}{\partial x_i}\eta_i + \frac{\partial f}{\partial \dot x_i}\dot\eta_i \right) dt \\
&= \int_a^b \left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot x_i} \right) \right] \eta_i \, dt + \frac{\partial f}{\partial \dot x_i}\eta_i \bigg|_a^b \\
&= \int_a^b \left[\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot x_i} \right) \right] \eta_i \, dt \\
0 &= \frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot x_i} \right)
\end{aligned}$$

where repeated indices in a product means summed over all index values (which is Einstein's summation rule), where we have used integration by parts, and the last line follows since the only way the integral (including all the Einstein summed terms) can be zero in general for general η_i is if the bracketed expression in the second to last line vanishes everywhere. Euler's equations (regarding subscript i as indicating a set of equations) are, in fact,

$$\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot x_i} \right) = 0 \ .$$

There are certain special cases. First is the case when f has no dependence on a particular x_k (which does not stand for the set of coordinate functions x_i). In this case, Euler's equation for x_k reduce to

$$\frac{\partial f}{\partial \dot x_k} = C_k \ ,$$

where C_k is a constant of integration. Second is the case when f has no dependence on a particular $\dot x_k$. In this case, Euler's equations reduce to

$$\frac{\partial f}{\partial x_k} = 0$$

which implies that f is independent of the particular x_k. This result may have a profound significance that altogether escapes yours truly.

Third is the case when f has no intrinsic dependence on t: i.e., f is just $f(x_i, \dot x_i)$, and so $\partial f/\partial t = 0$. To progress, we invoke the Einstein-when-off-track-contract rule and contract Euler's equation with the clairvoyantly chosen $\dot x_i$ (i.e., multiply by $\dot x_i$ and Einstein sum on i):

$$\begin{aligned}
0 &= \dot x_i \frac{\partial f}{\partial x_i} - \dot x_i \frac{d}{dt}\left(\frac{\partial f}{\partial \dot x_i} \right) \\
0 &= \frac{df}{dt} - \ddot x_i \frac{\partial f}{\partial \dot x_i} - \frac{\partial f}{\partial t} - \left[\frac{d}{dt}\left(\dot x_i \frac{\partial f}{\partial \dot x_i} \right) - \ddot x_i \frac{\partial f}{\partial \dot x_i} \right] \\
0 &= -\frac{\partial f}{\partial t} + \frac{d}{dt}\left(f - \dot x_i \frac{\partial f}{\partial \dot x_i} \right) \ .
\end{aligned}$$

The last equation is the single-alternative Euler's equation. Because of the sum on i it can only replace one of the set of Euler's equations for x_i. But if there is only one coordinate function x_i, then the single-alternative Euler's equation can be useful. The single-alternative Euler's equation is mostly likely to be useful (no matter how many function coordinates x_i there are) when f has no intrinsic dependence on t (i.e., when $\partial f/\partial t = 0$) which is the case we have been working toward in this paragraph. So when $\partial f/\partial t = 0$, we obtain

$$f - \dot{x}_i \frac{\partial f}{\partial \dot{x}_i} = C ,$$

where C is a constant of integration. Now if, in fact, there is only one coordinate function x_i, the last equation is likely to be very useful.

There are parts a,b.

a) The metric for a Euclidean space is

$$ds^2 = \sum_j dx_j^2 ,$$

where we have not used Einstein summation—we turn it on and off as convenient. Using Euler's equation, prove that the stationary path between any two points is a straight line. **HINT:** First, find what the function f is in this case.

b) What kind of a stationary path is the answer from part (a): global minimum, local minimum, global maximum, local maximum, inflection? Explain your answer.

c) The metric for the surface of sphere of radius R is

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) .$$

Using Euler's equations, prove that the stationary path between any two points is a great circle (i.e., a circle that cuts the sphere in half). **HINT:** First, find what the function f is in this case. Second, without loss of generality you can choose one endpoint to be the pole (i.e., the place where $\theta = 0$). Third, find the Euler equation result for ϕ first and check its behavior at pole.

d) What kind of a stationary path is the answer from part (c)? Note there are two cases. Explain your answer.

---

002 qfull 01110 1 3 0 easy math: law of reflection/refraction from Fermat's principle I

22. The laws of reflection and refraction can be proven from the modern version Fermat's principle (HZ-69; Wikipedia: Fermat's principle)—which yours truly for some reason keeps thinking of as Fermat's last principle. Fermat's principle states that a light ray traveling between two points follows a path that is stationary in optical path length which is defined by the differential ds/λ, where ds is differential physical length and λ is local wavelength. In the wave theory of light, Fermat's principle follows from the idea that along stationary paths multiple coherent wave fronts are in phase to 1st order, and so an add constructively: along other paths the multiple coherent wave fronts cancel out by destructive interference virtually totally.

There are parts a,b.

a) Write down the laws of reflection and refraction.

b) Give an argument why the stationary optical path must be in a perpendicular plane to the interface of reflection/transmission for the two laws. This plane is called the plane of incidence (AKA incidence plane) in optics jargon.

c) Draw a diagram in of incidence plane with a reflection/transmission interface. Mark point 1, a source, at (x_1, y_1), and point 2, a receiver, at (x_2, y_2). For niceness, x_1 is measured to the left from the origin at the point of reflection/transmission, x_2 is measured to the right from the origin at the point of reflection/transmission, and y_2 can be on either side of the interface and is positive either way.

d) Continuing with the part (c) setup, consider the source and receiver points as fixed, but the origin as free to vary along the interface in the incidence plane. Now write down the formula for h which is the optical path length between source and receiver for reflection and transmission plus a Lagrange multiplier term.

e) Solve for the stationary point of the formula of part (d) and show that it is a minimum.

f) Now complete the proof of the laws of reflection and refraction.

---

002 qfull 01120 1 3 0 easy math: law of reflection/refraction from Fermat's principle II

23. The first variational principle in physics was discovered by Hero of Alexandria (10?–70? CE) (Wikipedia: Hero of Alexandria: Inventions). He noted that the law of reflection followed from the idea that a light ray traveled the shortest path of light from source to receiver during a reflection of a planar surface. In equation form the law of reflection is

$$\theta_1 = \theta_2 \ ,$$

where θ_1 is the angle of incidence and θ_2 is the angle of reflection both measured in the plane of incidence (i.e., the plane defined by the source and the normal to the surface). Pierre de Fermat (1607–1665) generalized the Hero's idea by saying a light ray traveled the shortest time between source to receiver and from this idea was able to prove the law of refraction as well as the law of reflection. In modern form, the law of refraction is

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \ ,$$

where θ_1 is the angle of incidence and θ_2 is the angle of transmission both measured from the normal to the surface between the media, the $n_i = c/v_i$ are the refractive indices of the media v_i is the light speed in the media, and angles are both in the plane of incidence.

Fermat's idea in modern form is called Fermat's principle and states that a light ray moves along a stationary path in optical path length (i.e., length divided by wavelength). Fermat's principle and the earlier notions of Hero and Fermat himself are variational principles in that variations from the stationary path are used to find it. In fact, a key law of classical mechanics is a variational principle: the principle of least action—more accurately, the principle of stationary action. The classical principal of least action is actually derivable from quantum mechanics. Particles propagate as waves and phase variation tend to cause destructive interference, except for the stationary path for action which is the wave phase itself (Ba-69ff). In the macroscopic limit, the destructive interference causes virtually complete cancellation of propagation, except along the stationary path. Actually, Fermat's principle is, we can now see, the special case for light of the principle of least action.

There are parts a,b.

a) Draw a diagram with a source P_1 a distance y away from a planar surface and a general receiver P_2 that is y above the surface for reflection and y below for refraction. The separation along the direction parallel the planar surface is ℓ. A light ray from the source hits the surface at the origin 0. Draw a normal to the surface at origin 0. The incident angle is θ_1 and the reflection/refraction angle is θ_2. The incident wavelength is λ_1 and the reflection/refraction is λ_2.

b) What is the ray optical path length s from P_1 to P_2 expressed in terms of y, θ_1, θ_2 λ_1, and λ_2?

c) The elegant way to prove the laws of reflection and refraction is to use Lagrange multipliers. The general form is

$$L = f + \alpha g \ ,$$

where L is called the Lagrangian function, f is the function whose contrained stationary point you want find, α is the Lagrange multiplier, and g is the contraint function: i.e., $g =$ constant when the constraint is imposed. Write down the Lagrangian function for the optical path length case. Find the formula for θ_i that makes s stationary consistent with the constraint.

d) From the results of part (c), prove the laws of reflection and refraction.

e) Why can't the stationary path be outside of the planet of incidence?

---

002 qfull 01130 1 3 0 easy math: Euler-Lagrange equations

24. The Euler equations (Ar-928) (AKA the Euler-Lagrange equations: Go45) are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where t is a general independent variable (but we are already thinking of specializing it to time), i the representative index for a set of indices j, q_j is set of unknown functions that one solves Euler equations for (but we are already thinking of them as being generalized coordinates in classical mechanics), $\dot{q}_j$ are the t partial derivatives of the q_j, and $L = L(q_j, \dot{q}_j, t)$ is a known function.

Now whence the Euler equations and what for their solutions. The solutions of Euler equations, are the functions that make the functional (i.e., function of functions)

$$S(q_j) = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)$$

stationary with respect to general small varitions in q_i: i.e., unchanging to 1st order in a variational parameter that actually never needs to be specified. The Euler equations themselves are obtained by variational calculus on S. Note we are already thinking of specializing S to the action in physics jargon in which case L is the Lagrangian for a system (which is a function of the generalized coordinates q_j) and the Euler-Lagrange equations become the Lagrange equations of motion for the system (Go-45). That stationarized S yields the equations of motions is a variational principle called the principle of least action (though more precisely of stationary action). The specific version of the principle of least action that yields the Lagrange equations is formally called Hamilton's principle (Go-34), but I think most people refer to it just by generic name principle of least action.

There are parts a,b.

a)

b)

Chapt. 4 The Friedmann Equations

Multiple-Choice Problems

003 qmult 00100 1 4 5 easy deducto-memory: Friedmann equation derivation

25. "Let's play *Jeopardy*! For $100, the answer is: It was derived from general relativity in 1922 with the assumptions of a homogeneous and isotropic universe and that all mass-energy in the universe could be modeled by a perfect fluid. A Newtonian derivation (which required extra natural hypotheses) was given in 1934.

    What is the ____________, Alex?

    a) Einstein equation b) Milne-McCrea equation c) Synge equation
    d) Bondi equation e) Friedmann equation

003 qmult 00110 1 4 3 easy deducto-memory: Why Newtonian derivation of FE not found
Extra keywords: in the 19th century.

26. A Newtonian derivation of the Friedmann equation (with extra natural hypotheses) could easily have been done in the 19th century, but it wasn't. There were probably 3 reasons why 19th century astronomers did not think of such a derivation. First, many were still thinking of a universe that was static on average even though dynamic equilibrium seemed hard to arrange, even though the universe was obviously not in thermodynamic equilibrium (and so why should be in dynamic equalibrium), and even though idea existed that the Milky was held up by rotation around its center of mass located somewhere. Second, they did not know that other galaxies existed though some believed this and they had not observed the general redshifts of the objects they thought might be other galaxies. Third, they thought in terms of Newton's absolute space (i.e., a single fundamental inertial frame) and did not think of the alternative idea completely compatible with their data that all ____________ unrotating with respect to the observable universe were elementary inertial frames (i.e., frames with respect to which Newtonian physics and all other known physics could be referenced to). The elementary inertia frames could be incorporated into more general inertial frames (e.g., center-of-mass inertial frames) and the whole observable universe could organized into the more general inertial frames. Going beyond what 19th century astronomers probably could have thought of, there is whole hierarchy of general inertial frames that tops out with the comoving frames of the expanding universe.

    What is the ____________, Alex?

    a) star frames b) planet frames c) free-fall frames d) thermodynamics frames
    e) gravity frames

003 qmult 00250 1 1 2 easy memory: shell theorem to point masses interaction

27. "Let's play *Jeopardy*! For $100, the answer is: This theorem (originally proven by Newton by primitive means) allows one to show by means of a **COROLLARY** that spherically symmetric masses should interact gravitationally as though they are point masses as long as they are do not interpenetrate.

    What is the ____________, Alex?

    a) Newton theorem b) shell theorem c) point-mass theorem d) sphere theorem
    e) waste book theorem

---

003 qmult 00410 1 4 5 easy deducto-memory: Bertrand's theorem, the inverse-square and linear forces

Extra keywords: (Go3-92)

28. "Let's play *Jeopardy*! For \$100, the answer is: The theorem that states that the only attractive central forces that give closed orbits for all bound orbits are the inverse-square law force and the attractive linear force (AKA Hooke's law force or the radial harmonic oscillator force). All attractive central forces give closed **CIRCULAR** orbits, of course."

    What is ____________, Alex?

    a) the virial theorem b) Euler's theogonic proof c) the brachistochrone problem
    d) Schubert's unfinished symphony e) Bertrand's theorem

---

003 qmult 00420 1 4 4 easy deducto-memory: linear force Gauss's law shell theorem

Extra keywords: (Go3-92)

29. "Let's play *Jeopardy*! Bertrand's theorem implies a symmetry between the inverse-square law force and the linear force. Another symmetry is that the linear force gives a Gauss' law analogue and the analogue of this important theorem that Newton needed to derive in order to show that spherically symmetric bodies interact through gravity like point masses.

    What is ____________, Alex?

    a) the virial theorem b) Euler's theorem c) Turchin's theorem
    d) the shell theorem e) the clam theorem

---

003 qmult 00750 1 4 5 easy deducto-memory: Hubble parameter

30. "Let's play *Jeopardy*! For \$100, the answer is: Characteristic time time and length scales can be derived from this parameter of the Friedmann equation models for the universe."

    What is the ____________ parameter, Alex?

    a) Lemaître b) de Sitter c) Einstein d) Eddington e) Hubble

---

003 qmult 00950 1 1 3 easy memory: age of universe for single density component solutions

31. The single density component (power-law) solutions to the Friedmann equation are

$$a = a_0 \left[\sqrt{\Omega_{p,0}} \left(\frac{p}{2} \right) H_0 t \right]^{2/p} = a_0 \left[\left(\frac{p}{2} \right) H_0 t \right]^{2/p} = a_0 \left[\frac{t}{(2/p) H_0^{-1}} \right]^{2/p} ,$$

    where 0 indicates cosmic present, a_0 is the cosmic present scale factor (conventionally set to 1), H_0 is the Hubble constant, $\Omega_{p,0} = 1$ for a single density component solution, and $-p$ is the single power of the a that occurs on the right-hand side of the Friedmann equation which in a scaled form is

$$\left(\frac{\dot{a}}{a} \right)^2 = \sum_p \Omega_{p,0} \left(\frac{a_0}{a} \right)^p .$$

    What is the age of the universe for a single density component solution?

    a) $(p/2)H_0$. b) $(2/p)H_0$. c) $(2/p)H_0^{-1}$. d) $(p/2)H_0^{-1}$ e) H_0/p.

---

003 qmult 01100 1 1 1 easy memory: cosmological and Hubble quantities

32. The solutions of the Friedmann equation have characteristic cosmological quantities some of which are called Hubble quantities since the Hubble constant is one of their ingredients. The table below displays some the cosmological quantities. Since the currently determined values of the quantities always fluctuate a bit depending on whose analysis is used, we have written the quantities as fiducial values with correction factors that are 1 to within a few percent: h_{70} is the Hubble constant divided by 70 (km/s)/Mpc (i.e., $H_0/(70\,\text{(km/s)/Mpc})$), $\omega_{\text{m},0} = \Omega_{\text{m},0}/0.3$, and

$\omega_\Lambda = \Omega_\Lambda/0.7$. The asymptotic Hubble quantities are those that will be the Hubble quantities as cosmic time goes to infinity if the Λ-CDM model is correct.

Table: Cosmological Quantities

Cosmic scale factor for the present cosmic time $a_0 = 1$ by convention
Hubble constant $H_0 = 70h_{70}$ (km/s)/Mpc
Hubble time $t_H = 1/H_0 = (13.968\ldots)/h_{70}$ Gyr
Hubble length $\ell_H = c/H_0 = (13.968\ldots)/h_{70}$ Gly $= (4.2827\ldots)/h_{70}$ Gpc
Critical density $\rho_{\text{critical}} = [3H_0^2/(8\pi G)] = (9.2039 \times 10^{-27})h_{70}^2$ kg/m^3
$= (1.3599 \times 10^{11})h_{70}^2$ $\text{M}_\odot/\text{Mpc}^3$
AKA Hubble density (i.e., the density implied by the Hubble constant at cosmic present)
Cosmological constant matter density parameter $\Omega_{\text{m},0} = 0.3\omega_{\text{m},0}$
Cosmological constant Λ density parameter $\Omega_\Lambda = 0.7\omega_\Lambda$
Asymptotic Λ Hubble parameter $H_\Lambda = H_0\sqrt{\Omega_\Lambda} = \sqrt{\Lambda/3} = (58.566\ldots)h_{70}\sqrt{\omega_\Lambda}$ (km/s)/Mpc
Asymptotic Λ Hubble time $t_{H_\Lambda} = (16.6955\ldots)/(h_{70}\sqrt{\omega_\Lambda})$ Gyr

Given that the Λ-CDM model is correct, to 1st order, the observable universe is already expanding like a cosmological-constant universe with $a = a_0 \exp(\Delta t/t_{H_\Lambda})$ (where $\Delta t = t - t_0$) and this formula becomes more correct as time advances. On what time scale Δt will the matter mass-energy density of the observable universe fall to of order 2 % of the total mass-energy? Note you have to solve for a/a_0 from

$$\Omega_{\text{m}} = \Omega_{\text{m},0}\left(\frac{a_0}{a}\right)^3 \approx 0.02\Omega_\Lambda$$

and then solve for Δt.

a) t_{H_Λ}. b) $2t_{H_\Lambda}$ c) $3t_{H_\Lambda}$. d) $4t_{H_\Lambda}$. e) $5t_{H_\Lambda}$.

Full-Answer Problems

003 qfull 00210 1 3 0 easy math: Gauss' law derivation

33. In this problem, we will derive the generic Gauss' law in its integral form and then specialize to the gravity and Coulomb force cases.

NOTE: There are parts a,b,c,d. Some of the parts can be done independently, and so do not stop if you cannot do a part.

a) Consider the generic inverse-square law central force

$$\vec{f} = \frac{q}{r^2}\hat{r} ,$$

where q is a generic charge for the force located at the origin (which is the center of force), r is the distance to a point where the force is evaluated, and $\hat{r}$ is the direction to that point. Now consider a (**SINGLE**) differential surface area vector $d\vec{A} = dA\,\hat{n}$ for a **CLOSED** surface. The unit vector $\hat{n}$ is normal to the differential surface and points in outward direction. The differential solid angle subtended by the differential surface area is $d\Omega$. Prove

$$\vec{f} \cdot d\vec{A} = q(\pm\, d\Omega) ,$$

where the upper/lower cases are for the solid angle cone going outward/inward through the differential surface area. Note the charge could be inside or outside the closed surface. **HINT:** This is easy, but a few words of explanation and a diagram are needed.

b) Consider a differentally small cone extending from the origin. It intersects the closed surface n times. Note that closed surface is finite, and so the cone must exit the closed surface for good at some point. We form the sum

$$\sum_{i=1}^{n} \vec{f} \cdot d\vec{A}_i \ ,$$

where sum is over all intersections. What is the sum equal to in terms of solid angle for all cases? **HINT:** A few words of explanation and a diagram are needed.

c) Say you had multiple charges q_i with total charge Q and total charge Q_{enclosed} inside a closed surface. Evaluate

$$\oint \vec{f} \cdot d\vec{A} \ .$$

The result is the generic Gauss' law in its integral form. Specialize the result for the cases of gravity and the Coulomb force.

d) What is the necessary condition for a force to obey Gauss' law?

---

003 qfull 00250 1 3 0 easy math: linear-force Gauss's law and shell theorem

34. Remarkably the linear force obeys analogues to Gauss's law and shell theorem for the inverse-square law force. Let the linear-force field (force per unit charge) for a point charge be

$$\vec{f} = kqr\hat{r} \ ,$$

where k is a constant which could be positive or negative, q is the charge (of some unspecified kind), and r is the distance from the point charge. We assume Newtonian physics, and so to maintain Newton's 3rd law, we require

$$\vec{F}_{1,2} = kq_1q_2r_{1,2}\hat{r}_{1,2} \ ,$$

where $\vec{F}_{1,2}$ is the force of point charge 1 on point charge 2.

There are parts a,b,c,d,f. Some of the parts can be done independently, and so do not stop if you cannot do a part. Omit part (f) during exams.

a) Without words, for a close surface derive the linear-force Gauss' law

$$\oint \vec{f} \cdot d\vec{A} = kQ \ ,$$

where $\vec{f}$ is the field due to the entire charge distribution, the integral is over the whole close surface, and Q is the total charge of the charge distribution wherever it is in space. **HINT:** Recall the divergence theorem (AKA Gauss' theorem)

$$\oint \vec{Y} \cdot d\vec{A} = \int \nabla \cdot \vec{Y} \, dV \ ,$$

where Y is a general vector field and the volume integral is over all volume V inclosed by the closed surface (Wikipedia: Divergence theorem). Recall also the divergence operator for spherically symmetric system in spherical coordinates obeys

$$\nabla \cdot \vec{Z} = \frac{1}{r^2}\frac{\partial(r^2 Z_r)}{\partial r} \ ,$$

where Z is spherically symmetric, but otherwise general, and Z_r is the radial component of $\vec{Z}$ (Arfken-104).

b) For what symmetries can the linear-force field be easily solved for directly from the linear-force Gauss' law?

c) Without words, solve for the linear-force field for a spherically symmetric charge distribution. What simple charge distribution would give an equivalent linear-force field for all radius r? What can this result be called? How is this equivalent linear-force field different from the analogue result with the inverse-square-law force?

d) Without words, show for a general charge distribution 1 and a spherical symmetric charge distribution 2 that the force of distribution 1 on distribution 2 is exactly the same as when distribution 2 is replaced point-charge 2. If charge distribution 1 were also spherically symmetric, what be the force between them be equal to and what would it be if their centers coincided exaclty?

e) Say you had a charge distribution that maintained spherically symmetry no matter what, that had its center of mass at its center, and the only external forces that acted on it were external linear forces. How would described its motion? Recall Newton's 2nd law:

$$\vec{F}_{\text{net external}} = m\vec{a}_{\text{cm}} ,$$

where $\vec{F}_{\text{net external}}$ is the net external force on a body of mass m and a_{cm} is the center of mass of the body. Given the result of part (d) Without words, show for two spherically symmetric distribution charges that the force of distribution 1 on distribution 2 is exactly the same **HINT:** Recall the part (d) answer.

f) Is the linear force for spherically symmetric mass distribution with mass as its charge consistent with linear force that occurs in the Newtonian derivation of the Friedmann equation:

$$\vec{F} = \frac{\Lambda}{3} mr\hat{r} ,$$

where m is a test particle mass. There is no right answer. This is a discussion question.

---

003 qfull 00260 1 3 0 easy math: the linear force or cosmological force in cosmology

35. The (Newtonian) cosmological constant force **PER UNIT MASS** is given by

$$\vec{f} = \frac{\Lambda}{3}\vec{r} ,$$

where Λ is the cosmological constant, the 1/3 factor is for consistency with cosmological constant as it appears in the Einstein field equations, and $\vec{r}$ is the displacement vector from any point in space. In an extra Newtonian hypothesis, one can hypothesize that Λ is set somehow by a universal force charge density that is constant in space and time and the Newtonian-3rd-law equal-and-opposite force caused by the cosmological constant force on a particular mass is exterted by the particular mass on on this charge throughout the universe. But this may be a useless hypothesis.

Consider a system of point masses m_i at displacements $\vec{r}_i$ relative to an external origin. The total mass of the system is $m = \sum_i m_i$. The center of mass of the system is $\vec{r}_{\text{cm}}$ and the relative displacements are $\Delta\vec{r}_i = \vec{r}_i - \vec{r}_{\text{cm}}$.

NOTE: There are parts a,b,c.

a) Write down the cosmological constant force $\vec{F}_i$ on point mass m_i relative to the **ORIGIN** both in terms of $\vec{r}_i$ and $\Delta\vec{r}_i$. **HINT:** This is easy.

b) Determine the net cosmological constant force $\vec{F} = \sum_i \vec{F}_i$ on the system and simplify as much as possible. **HINT:** Recall the definition of center of mass.

c) What simplifying conclusion can you draw from the part (b) answer?

---

003 qfull 00700 1 3 0 easy math: Friedmann equation and Hubble law derivations

36. The Friedmann equation of general relativity (GR) cosmology in its most standard form (e.g., Wikipedia: Friedmann equations: Equations) is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} ,$$

where H is the Hubble parameter (which at current cosmic time is the Hubble constant H_0 and has fiducial value $70\,(\mathrm{km/s})/\mathrm{Mpc}$), a is the cosmic scale factor, $\dot{a}$ is the time derivative of the cosmic scale factor with respect to cosmic time t, $G = 6.67430(15) \times 10^{-11}\,\mathrm{J\,m/kg^2}$ is the gravitational constant, ρ is the density of a uniform perfect fluid (in old-fashioned jargon AKA the cosmological substratum: Bo-75–76) which is used to model the universal mass distribution, k is called the curvature (Li-24,28) $k/(c^2a^2)$ is called Gaussian curvature (CL-12,29), $c = 2.99792458 \times 10^8\,\mathrm{m/s}$ is the vacuum light speed as usual. and Λ is the cosmological constant which is the simplest form of the dark energy even though is only a from of energy in one interpretation. Note k is often defined with an unabsorbed c^2: i.e., the shown k is replaced by kc^2.

There are parts a,b,c. Some of the parts can be done independently, and so do not stop if you cannot do a part. During exams do **ONLY** parts a,b,c,d.

a) Without words prove the Friedmann equation starting from the work-energy theorem

$$E_{\mathrm{mechanical}} = \frac{1}{2}mv^2 - \frac{GMm}{r} - \frac{1}{2}\frac{\Lambda}{3}mr^2 ,$$

where m is the mass of a test particle.

b) Without words prove the general Hubble law $v = Hr$, where v is recession velocity (i.e., the velocity between comoving frames) and r is proper distance (i.e., the distance measurable in with a ruler at one instant in cosmic time).

c) What is the asymptotic Hubble law (i.e., Hubble law valid in the limit $z \to 0$)?

---

003 qfull 00820 1 3 0 easy math: 1-component solutions to the scaled Friedmann equation

37. The scaled Friedmann equation for multi-component (power-law) density components is

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_p \Omega_{p,0} x^{-p} ,$$

where 0 indicates the fiducial time which may be cosmic present, $h = H/H_0$ is the scaled Hubble parameter with H_0 being the Hubble constant, $x = a/a_0$ is the scaled cosmic scale factor, $x_0 = 1$, $\dot{x} = dx/d\tau$ is the rate of change of the scaled cosmic scale factor, $\tau = H_0 t = t/t_{H_0}$ is the scaled time with t_{H_0} being the Hubble time, the $\Omega_{p,0}$ are the density parameters for the density components at the fiducial time with their sum being 1, and p are the powers of the power-law density components.

NOTE: There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c,d.

a) Without words, derive the general asymptotic solution $\tau(x)$ and its inverse $x(\tau)$ for the leading density component as $\tau \to 0$ (i.e., the density component with highest p). As a shorthand, this solution can be called the early universe solution. Assume $p > 0$. To avoid pointless generality, assume $x(\tau = 0) = 0$ (i.e., there is a point origin at time zero).

b) Without words, derive early universe formula for $\Omega_p(\tau) = \Omega_p[x(\tau)]$ for $p > 0$.

c) Without words, derive the special case early universe solutions for $p = 1, 2, 3, 4$.

d) Without words, derive the Hubble parameter $h = \dot{x}/x$ and the deceleration parameter $q = -\ddot{x}x/(\dot{x})^2 = -\ddot{x}/(xh^2)$ for the general early universe with $p > 0$. Simplify the latter as much as possible. For what p values is the universe in positive/zero/negative acceleration?

e) We now assume the universe has only one density component with power $p > 0$. Without words, derive the generic age of the universe formula (which we assume to the fiducial time where $x = 1$) for τ and t and give the fiducial value version for t with the Hubble time $t_{H_0} = (13.968\ldots\,\text{Gyr})/h_{70}$, where $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$. special case solutions for $p = 1, 2, 3, 4$. Note the fi

f) We assume the universe has only one density component with power $p = 0$. Without words, derive $x(\tau)$ and $x(t)$ assuming $x(0) = 1$. Note this universe is the de Sitter universe and the Hubble constant $H_0 = \sqrt{\Lambda/3}$.

g) Students are now welcome to view a table in the answer to this part that presents the single density component solutions plus relevant features for powers $p = 4, 3, 2, 1, 0$. Note that if we assume that the dependence of the density components on the scale factor is due to a perfect fluid pressure obeying the equation of state $p_{\text{pressure}} = w\rho c^2$ where w is a constant parameter (with no special name), then power

$$p = 3(1 + w) \ .$$

The w values are included in the table.

---

003 qfull 01130 1 3 0 easy math: perfect fluid solutions

38. The differential equation (DE) for the perfect fluid of Friedmann equation cosmology is

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \ ,$$

where ρ is mass-energy in the comoving frames of Friedmann equation cosmology and p is isotropic pressure in those frames (in some sense) (Liddle 26). The perfect fluid DE can be derived rigorously from general relativity (Carroll 333–334) and, perhaps somewhat fudgily, from classical thermodynamics and special relativity. Remarkably, this equation does not guarantee conservation of energy in the ordinary sense of classical physics: it does embody the general relativity feature that the covariant derivative of the energy-momentum tensor is zero (Carroll 117,120): i.e., the energy-momentum conservation equation. General relativity may or may not in some sense conserve energy for cosmology, but certainly gravitating mass-energy is allowed to appear and disappear by the perfect fluid DE.

Multiple perfect fluids can exist and if they are assumed to act independently (which is the usual cosmological assumption), then they all obey there own perfect fluid DE: i.e., for perfect fluid i,

$$\dot{\rho}_i = -3\frac{\dot{a}}{a}\left(\rho_i + \frac{p_i}{c^2}\right) \ .$$

In current standard cosmology (i.e., the ΛCDM model or simple variations thereof), it is assumed that the perfect fluid equation of state (EOS) is of the form

$$p = w\rho c^2 \ ,$$

where w is a constant parameter that seems to have no special name. Most standard/interesing values of w are given by

$$w = \begin{cases} 0 & \text{for nonrelativistic (NR) mass-energy (AKA “matter”} \\ & \qquad \text{or “dust”: Liddle-40);} \\ 1/3 & \text{for extreme relativistic (ER) mass-energy: most obviously photons,} \\ & \qquad \text{but also the ER neutrinos of the Big Bang era} \\ & \qquad \text{and to some later time not perfectly known cosmic time;} \\ -1 & \text{for cosmological constant or equivalently (constant) dark energy;} \\ -1/3 & \text{for zero-acceleration (or constant } \dot{a}\text{) universes such as} \\ & \qquad \text{Fulvio Melia's } R_{\text{h}} = ct \text{ universe or a universe with cosmic scale} \\ & \qquad \text{determined only by negative curvature } k. \end{cases}$$

Solve for the formula for $\rho(a)$ for general w and the 4 special cases of w listed above. Assume a_0 and ρ_0 for cosmic present values.

---

003 qfull 01150 1 3 0 easy math: Friedmann equation solutions for general EdS universes: redundant to 00820

Extra keywords: Look over and see if anything to salvage in this question

39. The Friedmann equation and acceleration equation are, respectively,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} = H_0^2\frac{\rho}{\rho_{\rm C}} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p_{\rm pressure}}{c^2}\right) + \frac{\Lambda}{3} = -\frac{1}{2}H_0^2\frac{\rho}{\rho_{\rm C}}(1+3w) + \frac{\Lambda}{3} ,$$

where following a usual convention c^2 has been absorbed into k and Λ (Li-55) and we have assumed a single equation of state for the second version of the acceleration equation. A standard set of solutions follows for perfect fluids with equation of state $p_{\rm pressure} = w\rho c^2$ (with w constant) for the cases with $k = 0$ and $\Lambda = 0$ and density ρ obeying an inverse-power law of a. Following CL-36, we will call these solutions Einstein-de-Sitter universes (EdS universes) although originally only the $w = 0$ case was called an EdS universe. Note EdS universes do not the include the Einstein universe (which is a static, positive curvature universe), but do include the flat de Sitter universe with $k = 0$. The original de Sitter universe had positive curvature (O'Raifearty et al., 2017, p. 38). Explicitly, density as an inverse-power law of a is

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^p \qquad \text{with} \qquad p = 3(1+w) \qquad \text{and} \qquad \gamma = 2/p ,$$

where p is a power and not pressure $p_{\rm pressure}$.

There are parts a,b.

a) For the EdS universes, determine the general solutions for $a(t)$ (assuming a(0)=0, except for $w = -1$), t_0, q_0, and $\rho(t)$ in terms of a_0, t, H_0, w (or any convenient combination of w, p, and γ), and ρ_0 which equals

$$\rho_{\rm C} = \frac{3H_0^2}{8\pi G}$$

for Euclidean universes (i.e., flat universes). Note the subscript 0 means present cosmic time and the $w = -1$ cases require special treatment. Recall the deceleration parameter formula

$$q = -\frac{\ddot{a}}{a}\frac{1}{H^2}$$

(Li-53).

b) Specialize the results of part (a) for w values 0 ("matter"), 1/3 ("radiation"), -1 (de Sitter universe: cosmological constant, constant dark energy, or steady-state universe), and $-1/3$ (zero acceleration universe). Organize the results in a table for easy understanding.

---

003 qfull 01250 2 5 0 moderate thinking: the Friedmann equation: long derivation: rework?

Extra keywords: This problem needs reworking and probably cannot be used for students ever

40. The Friedmann equation of general relativity (GR) cosmology in standard form (e.g., Wikipedia: Friedmann equations: Equations) is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} ,$$

where H is the Hubble parameter (which at current cosmic time is the Hubble constant H_0 and has fiducial value 70 (km/s)/Mpc), a is the cosmic scale factor, $\dot{a}$ is the time derivative of the cosmic scale factor with respect to cosmic time t, $G = 6.67430(15) \times 10^{-11}\,\mathrm{J\,m/kg^2}$ is the gravitational constant, ρ is the density of a uniform perfect fluid (in old-fashioned jargon AKA the cosmological substratum: Bo-75–76) which is used to model the universal mass distribution, k is called the curvature (Li-24,28) $k/(c^2a^2)$ is called Gaussian curvature (CL-12,29), and $c = 2.99792458 \times 10^8\,\mathrm{m/s}$ is the vacuum light speed as usual. Note k is often defined with an unabsorbed c^2: i.e., the shown k is replaced by kc^2.

The Friedmann equation is, as one can see, a 1st order nonlinear ordinary differential equation. The fact that is nonlinear means that linear combinations of solutions are not in general solutions though they may be in special cases or approximately. The Friedmann equation is also a homogeneous differential equation at least in the sense that it can be written $\dot{a} = g(a)$. The form $\dot{a} = g(a)$ implies that a must be strictly increasing or decreasing except possibly at $\pm\infty$ and possibly at points where the some order of derivative of g have infinities. Both exceptions do occur for some solutions of the Friedmann equation. For example, the latter exception occurs for the closed universe model (with only matter). The closed universe model solution is closely related to throwing a ball into the air: the maximum size of the closed universe model corresponds to the maximum height of the ball.

The Friedmann equation actually has an interesting nature in that its independent variable is cosmic time t, but the solution the cosmic scale factor $a(t)$ is the factor by which all distances scale with time in expanding universe models.

Let's derive the Friedmann equation from Newtonian physics with extra natural hypotheses as needed. A priori, it not clear that the Newtonian derivation must yield the Friedmann equation with the extra natural hypotheses. But it can be shown that it should (C.G. Wells 2014, ArXiv:1405.1656). Note that the Newtonian derivation can say nothing about the curvature of space and assumes any curvature does not affect the derivation. We will do a long preamble wherein, with any luck, the extra hyptheses are shown to be natural.

First, just as in the GR derivation, we assume for our universe model the cosmological principle which states that the universe has a homogeneous, isotropic mass-energy distribution when averaged on a sufficiently large scale. The cosmological principle is what allows us to approximate the observable universe in our model with a perfect fluid. Observationally, the cosmological principle has been verified to a degree, but some tension remains. The observational scale for the validity of the cosmological principle is 100 Mpc or maybe a factor of a few times that larger (Wikipedia: Cosmological principle: Observations). Note that well beyond the observable universe, the cosmological principle may well fail, but, just as in the GR derivation, we assume this has negligible effect for the observable universe.

As to the perfect fluid of our model, it has uniform rest-frame mass-energy density ρ (uniform in space, not in time). The mass-energy gravitating mass-energy, of course. The perfect fluid has no viscosity and has an isotropic pressure p in its own rest frame (Ca-34). The perfect fluid equation of state (EOS) is $p = p(\rho)$. Actually, the perfect fluid can have internal energy (i.e., thermal energy), but that is counted as part of ρ as follows from $E = mc^2$. Also note that we said "rest-frame mass-energy" which can be the energy of massless particles. In fact, a photon gas is a good realization of the perfect fluid. The actual cosmic background radiation since the recombination era approximates a perfect fluid to high accuracy. Its photons do pass through gravitational wells, scatter off free electrons, and sometime hit planets, etc., but to good approximation the photons act as if they never interacted with anything except gravitationally.

Next, we note a corollory of Birkhoff's theorem (a theorem in GR): a spherical cavity at the center of spherical symmetric mass-energy distribution (static or not, finite or infinite) is a flat Minkowski spacetime (CL-24; We-337–338, 474). The spherical symmetric mass distribution can be, in fact, an unbounded homogeneous, isotropic mass-energy distribution: it can be infinite or finite. Note that if the spherical symmetric mass distribution is finite, it must have positive curvature and be a closed universe model. We assume, just as in the GR derivation, that

Birkhoff's theorem applies to good approximation even if the cosmological principle fails well beyond the observable universe. Inside the cavity, we can put mass-energy and it should behave exactly as superimposed on a universe of flat Minkowski spacetime (CL-24; We-337–338, 474) as long as it does not break spherical symmetry significantly, which would cause a significant perturbation of the spherical symmetry of the surroundings. The mass-energy we put in the cavity used for our derivation does not break spherical symmetry.

The situation for the Birkhoff-theorem cavity is analogous to a cavity in spherically symmetric mass distribution in Newtonian physics. Inside the Newtonian cavity, the gravitational field is zero: this is a corollory of the shell theorem first proven by Newton himself. However, what happens if the mass distribution is infinite is not defined by pure Newtonian physics. Analogous to the GR case, inside the cavity, we can put mass-energy and it should behave exactly as superimposed a region where there is no external gravitational field as long as it does not break spherical symmetry significantly which would cause a significant perturbation of the spherical symmetry of the surroundings.

Now consider general relativistic space infinite or finite and unbounded (which would be positive curvature space: Li-33). The space is filled with the aforementioned uniform perfect fluid. The fluid density ρ is a function of cosmic time t in general. The fluid's motions are determined only by gravity (i.e., the geometry of spacetime) and initial conditions, and so each element of the fluid moves along a geodesic in a GR interpretation and in free fall in the Newtonian physics interpretation. Since we demand homogeneity and isotropy, we can only have uniform expansion/contraction of the whole model. Note the fluid can have pressure (positive or negative), but uniformity means the pressure force cancels out everywhere locally. The fluid can also have a formal pressure that does not have to push/pull on anything. However, formal pressure does have a global effect as we will show below.

Now consider a Birkhoff-theorem cavity of radius r for our model which is also filled with the perfect fluid with density ρ. Everything inside the cavity behaves just as everything outside, and so the cosmological principle is maintained. The cavity fluid has total mass M. We assume that gravitational field due to the cavity fluid is asymmptotically Newtonian. This requires

$$\frac{R_{\rm Sch}}{r} = \frac{2GM/c^2}{r} = \frac{8\pi}{3}\frac{G\rho}{c^2}r^2 << 1 ,$$

where $R_{\rm Sch} = 2GM/c^2$ is the Schwarzschild radius (Wikipedia: Schwarzschild radius). So we just assume r is small enough. Note that Newtonian gravitational field is actually the classical limit of the left-hand side of the Einstein field equations (i.e., the spacetime geometry structure side: We-152), and so it does not itself contribute mass-energy (which comes from the right-hand side of the Einstein field equations and is described by the energy-momentum tensor). So we do not have to worry about the mass-energy contribution of the gravitational field to gravitating mass-energy since it does not contribute.

We also have to assume that r is small enough that the gravitational effects propagate with negligible time delay. Really, they propagate at the vacuum light speed relative to their local inertial frame.

We also have to assume that all relative velocities v of the fluid elements inside the cavity satisfy $v/c << 1$ so that we can employ Newtonian physics. This assumption is also asymptotically valid for small enough cavity radius r since the relative velocities between fluid elements are proportional to their separation distances as shown by Hubble's law which we derive nonrigorously below.

Recall all fluid elements in the perfect fluid are in free fall as aforesaid. This raises an interesting point. Special relativity gives the vacuum light speed c as the highest speed relative to inertial frames, but not between inertial frames. And the strong equivalence principle of GR shows that free-fall frames with uniform external gravity are exact inertial frames. The strong equivalence principle has been verified to very high accuracy (Archibald et al. 2018, arXiv:1807.02059). So the free-fall frames (which we will call comoving frames) of our model can grow apart at faster than c. In fact, Hubble's law shows that they must for large enough separation distances. Note that a light signal between comoving frames can only propagate at the vacuum speed light relative to the comoving frames it propagates through. So the fact that space can grow faster than the vacuum light speed does not imply there is faster-than-light signaling.

To summarize our assumptions for the Newtonian derivation, we require Birkhoff's theorem and that r be sufficiently small so that all relativistic and time-delay effects are small. If the aforesiad effects vanish in the differential limit as $r \rightarrow$, then the Newtonian derivation should be valid. Recall the Friedmann equation holds at every point in the universe model according to the GR derivation. Perhaps, there is some way that the Newtonian proof is still invalid, but it would have to be a very odd way.

Now we are ready to tear into the derivation of the Friedmann equation. We put a test particle of mass m at the surface of our cavity (i.e., at radius r). Given our setup, we have conservation of mechanical energy E:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv^2 - \frac{4\pi G}{3}\rho r^2 m \ ,$$

where the first term to the right of the equal signs is the kinetic energy of our test particle and the second is its gravitational potential energy which is also its gravitational field energy in Newtonian physics which as discussed above does not itself contribute to gravitating mass-energy. We now write

$$r = ar_0 \ ,$$

where a is the dimensionless cosmic scale factor and r_0 is a time-independent covoming distance. By usual convention the scale factor for the current cosmic time t_0 is defined to be 1: i.e., $a_0 = a(t_0) = 1$. This means that the r_0 are the proper distances for the current cosmic time: i.e., distances that you could measure with a ruler at current instant in cosmic time. Note $v = \dot{a}r_0$. Now defining the Hubble parameter $H = \dot{a}/a$, we get

$$v = Hr$$

which is the general-time Hubble's law. The current cosmic time Hubble's law (with the current Hubble parameter being Hubble's constant) is

$$v_0 = H_0 r_0 \ ,$$

The validity of this derivation of Hubble's law follows from the Friedmann equation itself, and so is valid insofar as our Newtonian derivation of the Friedmann equation is valid. A rigorous GR derivation is given by CL-13–14.

Re Hubble's law: it is an exact law for recession velocities (which are velocities between comoving frames: i.e., free-fall frames that are exact inertial frames) and proper distances (which are true physical distances that can be measured at one instant in cosmic time with a ruler). In fact, neither recession velocities nor proper distances are observables, except asymptotically as $r \rightarrow 0$. The exception allows Hubble's constant to be measured from cosmologically nearby galaxies.

We divide the conservation of mechanical energy equation by $-mr_0^2/2$ to get

$$-\frac{2E}{mr_0^2} = -\dot{a}^2 + \frac{8\pi G}{3}\rho a^2 \ .$$

The right-hand side of the second to last equation is independent of E, m, and r_0 and depends only on universal quantities of the universe model, and therefore the constant on the left-hand side must be a universal constant independent of the peculiarities of the test particle: i.e., E, m, and r_0. We use the symbol k for this universal constant: thus,

$$k = -\frac{2E}{mr_0^2} .$$

The constant k is called the curvature since GR tells us it describes the curvature of space which we cannot know from Newtonian physics (Li-24, CL-12–13). Note $k > 0$ gives positive curvature (hyperspherical geometry), $k < 0$ gives negative curvature (hyperbolical geometry), and $k = 0$ gives zero curvature (flat or Euclidean geometry): see Wikipedia: Shape of the universe. (As noted above, k is often defined with an unabsorbed c^2: i.e., $kc^2 = -2E/mr_0^2$.) Rearranging the second to last equation gives us the Friedmann equation itself:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} = H_0^2\left[\Omega + \Omega_k\left(\frac{a_0}{a}\right)^2\right] ,$$

(Li-24), where we have defined

$$\Omega = \frac{\rho}{\rho_{\rm c}} , \qquad \rho_{\rm c} = \frac{3H_0^2}{8\pi G} , \qquad \Omega_k = -\frac{k}{a_0^2 H_0^2} .$$

(Li-51,56). Note Ω is the density parameter (Li-51), $\rho_{\rm c} = 3H_0^2/(8\pi G)$ is the critical density (Li-51), and Ω_k is the curvature density parameter (Li-56). If $\Omega = 1$ at the current cosmic time (or any other cosmic time defined as current cosmic time), one has

$$H_0^2 = H_0^2(1 + \Omega_k)$$

implying $\Omega_k = 0$. So a universe model that is exactly flat at any cosmic time is exactly flat at all times.

There are several interesting points to be made about the Friedmann equation. First, we demanded r be small enough so that we could neglect relativistic and time travel effects. But we would derive the same Friedmann equation no matter what r we choose. So actually, all the effects we have neglected must cancel out for any r due to the conditions we imposed on the universe model: the cosmological principle and the perfect fluid.

A second interesting point is that Friedmann equation allows for mass-energy to appear or disappear as function of a. To explicate, mass-energy that is conserved (which called matter in cosmology jargon) has $\rho_{\rm m} \propto 1/a^3$. We show this below, but is in fact it is somewhat obvious: if the volume of a fluid element scales of up as a^3 and mass-energy is conserved, then density must decrease as $1/a^3$. But we allow other kinds of mass-energy dependence on a. For one example of mass-energy appearance/disappearance is that the cosmic background radiation and cosmic neutrino background (which in cosmology jargon is collectively called radiation) has $\rho_{\rm r} \propto 1/a^4$. The extra power of a is due to the cosmological redshift of extreme relativistic mass-energy which just causes radiation mass-energy to vanish from universe—it's just gone as gravitating mass-energy. Note general relativity cosmology does not have ordinary conservation of mass-energy: it just has the energy-momentum conservation equation $\nabla^\mu T_{\mu\mu} = 0$ (Carroll-120). Another point is that Noether's theorem that gives energy conservation when time invariance applies does not apply in an evolving universe model that does not have time invariance (Carroll-120). Another example of mass-energy appearance/disappearance is that constant dark energy (which is equivalent to the cosmological constant Λ in effect in the Friedmann equation if not otherwise) has ρ_Λ constant. The appearing/disappearing mass-energy contributes both gravitational field energy and, by the conservation of mechanical energy, the kinetic energy of the comoving frames which is sort of energy of expansion. (The disappearance of radiation also removes the kinetic energy of the comoving frames). To make more obvious the way mass-energy appearance/disappearance balances the gravitational field energy and the kinetic energy of the comoving frames , consider the Friedmann equation version

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k \ .$$

Holding a and k fixed, and increasing ρ (mass-energy) proportionally increases $\dot{a}^2$ (kinetic energy of comoving frames). This balanced contribution of gravitational field energy and kinetic energy for appearing/disappearing mass-energy arises only from starting our derivation from the conservation of mechanical energy equation. If we had started from Newton's 2nd law, we would have had no obvious path to include appearing/disappearing mass-energy.

You might ask what if k is a function of time or appearing/disappearing mass-energy is an explicit function of time not merely a function of a which is a function time. We have no guiding theory for these cases, and so far no observational or theoretical need for them.

We will now derive the fluid equation as it is called in cosmology jargon: i.e., the equation for $\dot{\rho}$. We assume that the perfect fluid obeys the 1st law of thermodynamics (which is actually implicit in the energy-momentum tensor for a perfect fluid: C.G. Wells 2014, ArXiv:1405.1656, p. 4). The 1st law is

$$dE = T\,dS - p\,dV + \mu\,dN \ ,$$

where here E is total mass-energy and not mechanical energy as above, T is temperature, S is entropy, p is pressure, V is volume, μ is chemical potential, and N is number of particles. The perfect fluid is adiabatic (i.e., $dS = 0$) and so the 1st law reduces to

$$dE = -p\,dV + \mu\,dN \ ,$$

For simplicity, we allow change in number of particles only to a species that is spontaneously created in such a way that N stays proportional to volume V. This means that $N = nV$ where n is the constant density of the spontaneously created particles. The spontaneously created particles are created at rest in the comoving frames, and so their chemical potential is just their rest-mass mass-energy. Given a volume $V \propto a^3$ for an amount of perfect fluid, we have

$$\begin{aligned}
&E = \rho c^2 V \\
&\dot{E} = (\dot{\rho}V + \rho\dot{V})c^2 = -p\dot{V} + \mu n\dot{V} \\
&\dot{\rho} = -\frac{\dot{V}}{V}\left(\rho + \frac{p}{c^2} - \frac{\mu n}{c^2}\right) \qquad \text{and using} \qquad \frac{\dot{V}}{V} = \frac{3a^2\dot{a}}{a^3} = 3\frac{\dot{a}}{a} \\
&\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2} - \frac{\mu n}{c^2}\right)
\end{aligned}$$

(Li-26). At the expense of clutter, we can explicitly allow for different species in the fluid equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}\sum_i \left(\rho_i + \frac{p_i}{c^2} - \frac{\mu_i n_i}{c^2}\right) \ ,$$

where $\mu_i = 0$ for those species which are not the spontaneously created particles we allowed for.

We note that in cosmology the equation of state is often parameterized thusly

$$p = \begin{cases} w\rho c^2 & \text{where } w \text{ is constant parameter just called w;} \\ 0 & \text{for matter where } w = 0; \\ \frac{1}{3}\rho c^2 & \text{for radiation where } w = 1/3; \\ -\rho c^2 & \text{for constant dark energy where } w = -1; \\ -\frac{1}{3}\rho c^2 & \text{for a non-accelerating universe where } w = -1/3. \end{cases}$$

One might well ask what the heck is the negative pressure of constant dark energy. Well for a hypothetical laboratory gas, its something with suction. So expanding it, requires adding internal energy. But the constant dark energy negative pressure may be just formal. There is no reason to require it to couple to anything except maybe itself, and so maybe nothing feels negative pressure, except maybe dark energy itself. In any case, the dark energy is uniform, and so there are no pressure gradients. Where does the mass-energy come from to keep dark energy constant as the universe expands? Well in simplest theory, it just appears as a fundamental fact. However, there are quantum field theory reasons for believing there could be dark energy, but quantum field theory in its simplest prediction gets the size of constant dark energy too big by more than 100 orders of magnitude. So maybe quantum field theory does not know what its talking about.

Why do we allow for constant dark energy? The universal expansion is positively accelerating and constant dark energy supplies a cause. Of course, constant dark energy insofar as it affects Friedmann equation (but perhaps not otherwise) can be replaced by Einstein's cosmological constant Λ with the appropriate positive value. The cosmological constant (if it exists) is a fundamental aspect of gravity and not mass-energy form at all.

The negative pressure for the non-accelerating universe is just a fix to get a non-accelerating universe which has been argued for by some (e.g., Melia 2015, arXiv:1411.5771). So it's just a formal pressure.

Why did we allow for spontaneously created particles? They represent an alternative idea to constant dark energy and the cosmological constant. In the Friedmann equation, they have the same effect as constant dark energy and the cosmological constant Λ with the appropriate positive value. What could such particles be? Very speculatively, dark matter particles, nonrelativistic neutrinos (which can exist even if we have never detected them), and/or baryonic matter (pairs of protons and electrons). All of these would have other effects than just giving a positively accelerating universe. They could clump eventually and affect large-scale structure evolution, and in the case of baryonic matter lead to new star formation. The particles, by the way, certainly have only positive pressure, but to first approximation that is negligible compared to their mass-energy contribution. The case of spontaneous creation of baryonic matter leads to the unlikely hypothesis that the observable universe started with a Big Bang, but is now evolving to the steady-state universe as hypothesized by Bondi, Gold, and Hoyle in 1948. Actually, Einstein anticipated the steady-state universe in unpublished work in 1931.

Now for some problems.

a) Derive the acceleration equation (AKA the 2nd Friedmann equation)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2} - \frac{3\mu n}{c^2}\right) \ .$$

HINT: Start by mulitpying the Friedmann equation through by a^2.

b) The deceleration parameter q is a dimensionless measure of the acceleration of the universal expansion. It is defined

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}$$

(Li-53), where the negative sign was included to get a positive value when people expected the acceleration to be negative. Some simple analytic solutions for $a(t)$ have only two unknown parameters and the observational determination of H_0 and q_0 determine those. This is why Allan Sandage (1926–2010) once, with admitted vast simplification, called the cosmology the search for two numbers: i.e., H_0 and q_0. Write q for general time in terms of general-time Ω, $\rho_{\rm c}$, p, and μn.

c) As discussed in the preamble, the cosmological constant is the alternative to constant dark energy insofar as the Friedmann equation alone is considered. One can derive it from the given standard form of the Friedmann equation by replacing ρ by $\rho + \rho_\Lambda$, where $\rho_\Lambda \equiv \Lambda/(8\pi G)$ (Li-56). Make this replacement in the Friedmann equation and then reverse engineer the derivation of the Friedmann equation to find the Newtonian potential energy U_Λ and the Newtonian force F_Λ implied by the cosmological constant.

d) What is peculiar about the Newtonian force F_Λ? **HINT:** The short answer is expected.

e) Write down the Friedmann equation and the acceleration equation with the explicit cosmological constant term. Set $\mu n = 0$ for simplicity. **HINT:** This is easy given $\rho_\Lambda \equiv \Lambda/(8\pi G)$, but you have to remember ρ_Λ has a formal pressure if it is attributed to constant dark energy as follows from the fluid equation for $\dot{\rho}_\Lambda = 0$.

f) The de Sitter solution of the Friedmann equation—which is grandly called the de Sitter universe—is obtained for the case where $\rho = 0$, $k = 0$, and $\Lambda > 0$. Find this solution in terms of current cosmic time t_0 and find the expressions for the Hubble parameter, the Hubble constant, and the deceleration parameter in general and for the current cosmic time. By the by, the de Sitter solution with the cosmological constant interpreted as constant density of ordinary matter is the steady-state universe.

---

003 qfull 01260 1 3 0 easy math: quick derivation Friedmann, fluid, and acceleration equations

41. Here we do the quick derivations of the Friedmann equation, the fluid equation, the Friedmann acceleration equation, and some other results.

NOTE: There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c,d,e. Some of the parts can be done independently, and so do not stop if you cannot do a part.

a) Without words, derive the Friedmann equation in standard form (with the cosmological constant force $F_\Lambda = (\Lambda/3)mr$ included) from classical physics with the hypotheses that all free-fall frames are elementary inertial frames (as told to us by general relativity) and that the shell theorem for a spherically symmetric mass distribution can be extended to infinite distance (which is validated by Birkhoff's theorem from general relativity). The derivation makes use of the classical conservation of mechanical energy. You should end up with a $-k/a^2$ term among other things. You can a draw diagram if you like. **HINT:** Start with the conservation of mechanical energy of a test particle of mass m:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} - \left(\frac{1}{2}\right)\frac{\Lambda}{3}mr^2 \; .$$

b) Without words and starting from the 1st law of thermodynamics

$$dE = T\,dS - p\,dV + \mu\,dN \; ,$$

derive the cosmological fluid equation in standard form (which means with $dS = 0$ and $dN = 0$) and in a form with $\dot{\rho}a/\dot{a}$ equal to something for use in part (d). Recall the rest-frame energy is $E = \rho c^2 V$.

c) Specialize the fluid equation to the special case where the equation of state is $p = w\rho c^2$ where w is the equation-of-state constant (which seems to have no special name). Determine the explicit solution $\rho(a)$ for the special case where $\rho_0 = \rho(a_0)$. **HINT:** You will have to eliminate the time derivative.

d) Without words, derive the acceleration equation (or Friedmann acceleration equation) in standard form using parts (a) and (b). A subtle point is that you have to assume that the gravitational potential energy formula continues to be valid (though perhaps with a different meaning) for cases where mass is not conserved. There is an argument why it should, but that is beyond the scope of this question.

e) Without words, derive from the Friedmann equation the de Sitter universe solution which has $\rho = 0$ and $k = 0$, but $\Lambda \neq 0$.

f) Without words, derive the scaled Friedmann equation

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \Omega_{\text{non-}k,\Lambda} + \Omega_k + \Omega_\Lambda$$

with the scalings $x = a/a_0$, $\tau = H_0 t$, $h = H/H_0$, $k_a = k/a_0^2$, and $\rho_c = 3H_0^2/(8\pi G)$. Note the subscript 0 indicates fiducial time t_0 which is often cosmic present and is not in general the Hubble time. Implicitly show expressions for ρ_k, and ρ_Λ and the density parameters in the derivation. What is the curvature equation at the fiducial time: i.e., the formula for $\Omega_{k,0}$. What does it mean if $\Omega_{k,0} = 0$ exactly.

g) Without words, derive the scaled accelertion equation using the same scalings and expressions as in part (f) and $p = w\rho c^2$.

Chapt. 5 The Geometry of the Universe

Multiple-Choice Problems

004 qmult 00100 1 4 1 easy deducto-memory: geometry of the universe curvature k

42. "Let's play *Jeopardy*! For $100, the answer is: This quantity according to general relativity and the Robertson-Walker metric determines the geometry of the universe: hyperspherical, flat, or hyperbolical."

    What is curvature ____________, Alex?

    a) k b) n c) a d) v e) e

004 qmult 00120 1 4 1 easy deducto-memory: factoring the curvature term

43. The Friedmann equation written in terms of density parameter components with some specializations is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\Omega + \Omega_k + \Omega_\Lambda\right)$$

where H is the Hubble parameter, H_0 is the Hubble constant, Ω is the sum of all density parameter components (excluding the curvature and Λ components),

$$\Omega_k = -\frac{kc^2}{H_0^2 a^2}$$

is the curvature density parameter component, and

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2} = \frac{\Lambda/(8\pi G)}{3H_0^2/(8\pi G)} = \frac{\rho_\Lambda}{\rho_{\text{crit},0}}$$

is the Λ density parameter component (i.e., the cosmological constant component). At the fiducial cosmic present,

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2 a_0^2}$$

and we are free to factorize k/a_0^2 as we like. In fact, the Robertson-Walker metric choice is to make $k = 0$ for flat space (i.e., Euclidean space), $k = 1$ for positive curvature space (i.e., hyperspherical space with $\Omega_{k,0} < 0$), and $k = -1$ for negative curvature space (i.e., hyperbolical space with $\Omega_{k,0} > 0$). For non-flat space, this implies a definite physical scale for a_0:

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\text{Gpc})/h_{70}}{\sqrt{|\Omega_k|}} = \frac{(13.968\ldots\,\text{Gly})/h_{70}}{\sqrt{|\Omega_k|}}$$

(where $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$) which can be called the curvature radius of the universe. Note formally the Gaussian curvature radius is defined

$$R_G = \frac{a_0}{\sqrt{k}}$$

which is imaginary for $k = -1$ (CL-12).

The Friedmann equation as written has 3 free parameters for cosmic present which we can choose to be H_0, Ω_0, and Ω_Λ. This means we have the constraint $\Omega_0 + \Omega_{k,0} + \Omega_\Lambda = 1$, and so $\Omega_{k,0} = 1 - \Omega_0 - \Omega_\Lambda$, and so $\Omega_{k,0}$ follows if all other density parameters are known by assumption or a fit to data. Tristram et al. (2023) give $\Omega_{k,0} = -0.012(10)$ consistent with 0, and so consistent with flat space.

Assuming $\Omega_k = -0.01$, what is the approximate curvature radius and how does that compare with the radius of the observable universe according to the Λ-CDM model 14.25 Gpc which must be approximately true whatever the correct universe model is (Wikipedia: Observable universe).

a) 43 Gpc; large. b) 430 Gpc; large. c) 43 Gpc; small. d) 430 Gpc; small.
e) 0.043 Gpc; small.

---

004 qmult 00150 1 1 2 easy memory: proper distance to the antipodal point

44. For a positive curvature space (i.e., $k = 1$ space), the proper distance to the antipodal point according to the Robertson-Walker metric formulation at cosmic present is

a) a_0. b) πa_0. c) $2\pi a_0$. d) $a_0/2$. e) $a_0/4$.

---

004 qmult 00180 1 1 4 easy memory: geodesic is a stationary path

45. A geodesic is a ____________ between two points in a general geometry. It is not in general a global minimum path nor a global maximum ____________. However, a sufficiently small segment is always the shortest distance between points in that segment.

a) non-stationary path b) straight line c) great circle d) stationary path
e) small circle

---

004 qmult 00182 1 1 3 easy memory: great circle geodesic

46. A geodesic on a sphere (i.e., an ordinary 2-sphere) is:

a) longitude. b) small circle. c) great circle. d) semicircle. e) meridian.

---

004 qmult 00200 1 1 3 easy memory: general metric

47. The spacetime interval (which in relativity is also called the metric) in general is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\mu$$

where $g_{\mu\nu}$ is the ____________ or sometimes just the metric in another meaning of the term. Note Einstein summation on repeated indices is used.

a) Minkowski tensor b) geodesic c) metric tensor d) gravity tensor
e) stress-energy tensor

---

004 qmult 00210 1 1 3 easy memory: Minkowski metric tensor tests

48. The ____________ is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(CL-24). Some authors define ____________ with an overall negative sign compared to the definition above.

a) Robertson Walker metric tensor b) geodesic tensor c) Minkowski metric tensor
d) gravity tensor e) stress-energy tensor

---

004 qmult 00220 1 4 5 easy deducto-memory: Robertson-Walker metric identified

49. "Let's play *Jeopardy*! For \$100, the answer is:

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] .$$

What is the ____________ metric, Alex?

a) Einstein-Hilbert b) de-Sitter-Schwarzschild c) Eddington-Lemaître
d) Milne-McCrea e) Robertson-Walker

---

001 qmult 00240 1 1 3 easy memory: radial and transverse proper distances

50. The Robertson-Walker metric is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] ,$$

where $ds^2 = d\tau^2$ is the spacetime interval (CL-10) and also the squared proper time differential in the convention adopted here (CL-10). The $a(t)$ is the physical curvature radius and r is the conventional dimensionless comoving coordinate and t is cosmic time. The r coordinate is proportional to tangential proper distance at any time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. The χ is proportional to the radial proper distance at any time. Note

$$r = \begin{cases} \sin(\chi) & \text{for } k=1 \text{ (positive curvature);} \\ \chi & \text{for } k=0 \text{ (flat space);} \\ \sinh(\chi) & \text{for } k=-1 \text{ (negative curvature)} \end{cases}$$

and

$$dr = \begin{cases} \cos(\chi)\,d\chi = \sqrt{1-\sin^2(\chi)}\,d\chi = \sqrt{1-r^2}\,d\chi & \text{for } k=1 \text{ (positive curvature);} \\ d\chi & \text{for } k=0 \text{ (flat space);} \\ \cosh(\chi)\,d\chi = \sqrt{1+\sinh^2(\chi)}\,d\chi = \sqrt{1+r^2}\,d\chi & \text{for } k=-1 \text{ (negative curvature),} \end{cases}$$

where we have used the hyperbolic identity $\cosh^2 - \sinh^2 = 1$ (Wikipedia: Hyperbolic functions: Useful relations). We now find

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} .$$

The differential radial proper distance is

$$dD_{\text{proper,radial}} = a(t)\left(\frac{dr}{\sqrt{1-kr^2}}\right) = a(t)\,d\chi .$$

The differential transverse proper distance $dD_{\text{proper,transverse}}$ is:

a) $4\pi[a(t)r]^2$. b) $a(t)r$. c) $a(t)r\sqrt{d\theta^2 + \sin^2\theta\,d\phi^2}$. d) $\pi a(t)$. e) $2\pi a(t)$.

Full-Answer Problems

---

004 qfull 00350 1 3 0 easy math: some of the geometry of Robertson-Walker metric

51. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] ,$$

where ds is the (differential) spacetime interval (also equal to $d\tau$ the proper time in the present convention: CL-10), dt is the differential cosmic time interval, the coordinates are for an arbitrary origin in the homogeneous and isotropic spacetime of the Robertson-Walker metric, θ and ϕ are the ordinary polar coordinates, r a dimensionless (i.e., unitless) comoving coordinate for the tangential direction, t is cosmic time, $a(t)$ is the cosmic scale factor with dimensions of length, and $k = 0$ for Euclidean space (i.e., flat space), $k = 1$ for hyperspherical space (i.e., positive curvature space with the geometry of the surface of a 3-sphere which is sphere in 4-dimensional Euclidean space: see Wikipedia: n-sphere) and $k = -1$ for hyperbolical space (i.e., negative curvature space). Note an ordinary sphere is a 2-sphere in math jargon. For $ds^2 > 0$ / $ds^2 = 0$ / $ds^2 < 0$, the interval is timelike / lightlike (or null) / spacelike (CL-10; Carroll-9).

For non-flat space, the Robertson implies a definite physical scale for a_0:

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\text{Gpc})/h_{70}}{\sqrt{|\Omega_k|}} = \frac{(13.968\ldots\,\text{Gly})/h_{70}}{\sqrt{|\Omega_k|}}$$

(where $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$) which can be called the curvature radius of the universe. Note formally the Gaussian curvature radius is defined

$$R_G = \frac{a_0}{\sqrt{k}}$$

which is imaginary for $k = -1$ (CL-12).

The Friedmann equation as written has 3 free parameters for cosmic present which we can choose to be H_0, Ω_0, and Ω_Λ. This means we have the constraint $\Omega_0 + \Omega_{k,0} + \Omega_\Lambda = 1$, and so $\Omega_{k,0} = 1 - \Omega_0 - \Omega_\Lambda$, and so $\Omega_{k,0}$ follows if all other density parameters are known by assumption or a fit to data. Tristram et al. (2023) give $\Omega_{k,0} = -0.012(10)$ consistent with 0, and so consistent with flat space. For $k = 0$, there is no physically determined a_0 value and one can set it for convenience: e.g., $a_0 = 1\,\text{Gpc}$ or $a_0 = c/H_0 = [4.2827\ldots)/h_{70}]\,\text{Gpc}$ which is the Hubble length. However, for flat universe models, one usually makes $a(t)$ dimensionless and sets $a_0 = 1$. In these models, the comoving coordinates are dimensioned and given units (e.g., Gpc).

The r coordinate is the tangential comoving coordinate since it is proportional to tangential proper distance at any time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. The χ is proportional to the radial proper distance at any time.

The radial proper distance $D_{\text{P,radial}}$ is given by

$$D_{\text{P,radial}} = a(t)\begin{cases}\chi & \text{for } k = 1 \text{ with } \chi \in [0,\pi];\\ \chi & \text{for } k = 0 \text{ with } \chi \in [0,\infty];\\ \chi & \text{for } k = -1 \text{ with } \chi \in [0,\infty],\end{cases}$$

The r coordinate is related to the χ by

$$r = \begin{cases}\sin(\chi) & \text{for } k = 1 \text{ (positive curvature)};\\ \chi & \text{for } k = 0 \text{ (flat space)};\\ \sinh(\chi) & \text{for } k = -1 \text{ (negative curvature)}\end{cases}$$

and

$$dr = \begin{cases}\cos(\chi)\,d\chi = \sqrt{1-\sin^2(\chi)}\,d\chi = \sqrt{1-r^2}\,d\chi & \text{for } k = 1 \text{ (positive curvature)};\\ d\chi & \text{for } k = 0 \text{ (flat space)};\\ \cosh(\chi)\,d\chi = \sqrt{1+\sinh^2(\chi)}\,d\chi = \sqrt{1+r^2}\,d\chi & \text{for } k = -1 \text{ (negative curvature)},\end{cases}$$

where we have used the hyperbolic identity $\cosh^2 - \sinh^2 = 1$ (Wikipedia: Hyperbolic functions: Useful relations). We now find

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} \ .$$

The transverse proper distance $D_{\rm P,transverse}$ is given by

$$D_{\rm P,transverse} = a(t)r\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2} \ .$$

The general differential the proper distance $D_{\rm P}$ formula is

$$\begin{aligned} dD_{\rm P}^2 &= a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \\ &= a(t)^2\left[d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \ . \end{aligned}$$

NOTE: There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c. The parts can be done independently, and so do not stop if you cannot do a part.

a) For the $k = 1$ case, what directions from the origin do radial geodesics lead to the antipodal point (i.e., the antipode)? How far in proper distance is it from the origin to the antipodal point along a radial geodesic? How far in proper distance to make the geodesic round trip from origin to origin?

b) What is the general formula for circumference C in proper distance for a circle at r in terms of r and χ? Sketch a plot of C as a function of χ for all cases of k.

c) Integrate over all solid angle to find the proper surface area A of the curved-space 2-sphere surrounding the origin at comoving coordinate r. This area is analogous to the circumference of a small circle on a ordinary sphere at polar angle θ. Sketch a plot of A as a function of χ for all cases of k. **HINT:** The integration is really easy and $d\theta^2 + \sin^2\theta\, d\phi^2$ is a differential path distance created using the differential Pythagorean theorem and not a differential piece of solid angle.

d) The differential volume for the sphere is $dV = A(\chi)a\, d\chi$. For all k, determine explicit formulae for $V(\chi)$ small χ and then for general χ. What is the maximum value of $V(\chi)$ for $k = 1$? **HINT:** You will need the identities $\sin^2(x) = (1/2)[1 - \cos(2x)]$ and $\sinh^2(x) = (1/2)[\cosh(2x) - 1]$.

---

004 qfull 00400 1 3 0 easy math: prove Hubble's law from the RW metric

52. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\, dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \ .$$

Note that r is the radial comoving coordinate chosen so that r is proportional to proper distance in the transverse direction (i.e., perpendicular to the radial direction).

Prove Hubble's law in general form from the Robertson-Walker metric: i.e., prove

$$v_{\rm R} = HD_{\rm P} \ ,$$

where $v_{\rm R} = \dot{D}_{\rm P}$ is the recession velocity, $H = \dot{a}/a$ is the Hubble parameter, and $D_{\rm P}$ is proper (radial) distance. Note proper distance is distance that can be measured at one instant in cosmic time using a ruler: i.e., with $dt = 0$, it is

$$D_{\rm P} = \int \sqrt{-ds^2} \ .$$

The general form of Hubble's law is an exact result, but alas containing two quantities that are not direct observables, $v_{\rm R}$ and $D_{\rm P}$, except asymptotically as $z \to 0$ or, in other words, in the limit where the 1st-order-in-small-z formulae can be treated as exact. The observational Hubble's law is

$$v_{\rm red} = H_0 D_{\rm P,1st} ,$$

where $v_{\rm red} = zc$ is redshift velocity (a direct observable) and $D_{\rm P,1st}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance (which are direct observables). The observational Hubble's law is very plausible a priori, but a formal proof is left to a later problem.

---

004 qfull 00500 1 3 0 easy math: cosmological time dilation and cosmological redshift

53. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] .$$

Note that r is the radial comoving coordinate chosen so that r is proportional to proper distance in the transverse direction (i.e., perpendicular to the radial direction).

NOTE: There are parts a,b,c,d. The parts can be done independently, so don't stop if you can't do one.

a) For a lightlike interval $ds^2 = 0$ for a light source at comoving coordinate r distant from an observer, prove that

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = f(r) = F(t_0) - F(t) = \int_t^{t_0} \frac{c\,dt'}{a(t')} ,$$

where $f(r)$ is just r integral and $F(t)$ is the antiderivative (or indefinite integral) of $c/a(t)$. The right-hand side integral is the conformal time for light to travel from the light source at comoving coordinate r to the observer. What does the proven result imply about the conformal time in this case?

b) For light signals coming from comoving coordinate r to the observer, prove with few words the cosmological time-dilation effect (CL-16,19):

$$\frac{dt_0}{a_0} = \frac{dt}{a(t)} \qquad \text{or} \qquad \frac{dt_0}{dt} = \frac{a_0}{a(t)} ,$$

where t is the cosmic time of emission, t_0 is the cosmic time of observation (i.e., the cosmic present), $a_0 = a(t_0)$, dt is the differential time between two emitted light signals, and dt_0 is the differential time between the corresponding two observed signals.

c) Prove without words the cosmological redshift formula $1 + z = a_0/a(t)$. **HINT:** You will have to use the part (b) answer to relate frequency/wavelength of emission to frequency/wavelength of reception.

d) The cosmological redshift formula is a very useful connecting the direct observable cosmological redshift z and the scaling up of the universe since a light signal was emitted $a_0/a(t)$. Why can't it be used to directly determining the function $a(t)$?

---

004 qfull 00610 1 3 0 easy math: Robertson-Walker metric and observables

54. The basic Λ-CDM model has its cosmic scale factor $a(t)$ fully specified via the Friedmann equation (FE) by the Hubble constant H_0 and three density parameters: i.e., $\Omega_{\rm R,0}$ ("radiation"), $\Omega_{\rm m,0}$ ("matter"), and Ω_Λ (cosmological constant or constant dark energy). Obtaining the parameters is a major observational goal. In principle, only 3 are independent, but observational uncertainties make obtaining all 4 somewhat independently useful goal.

If the FE model is not flat, the Friedmann equation (in its derivation from general relativity) plus Robertson-Walker metric tells us that the physical scale of the FE models at cosmic present t_0 is given by

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_0 - 1|}} = \frac{c/H_0}{\sqrt{|\Omega_{k,0}|}} = \frac{(4.2827\ldots\,\text{Gpc})/h_{70}}{\sqrt{|\Omega_{k,0}|}} = \frac{(13.968\ldots\,\text{Gly})/h_{70}}{\sqrt{|\Omega_{k,0}|}} ,$$

where Ω_0 is the sum of all density parameters, except $\Omega_{k,0}$, and $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$ is the reduced Hubble constant which must be 1 to within a few percent. If the FE model is flat, there is no physical scale for the model and a_0 can be chosen arbitrarily or set to dimensionless 1 in which case the comoving distances r have length units and are equal to the proper distances of the cosmic present. In all cases with a_0 set to a dimensioned physical scale, the proper distance to an object at comoving distance r is

$$D_{\text{P}} = a_0 \int_0^r \frac{dr}{\sqrt{1 - kr^2}} = a_0 f(r) ,$$

where r is comoving coordinate independent of time and $k = 1$ for hyperspherical space, $k = 0$ for Euclidean space (i.e., flat space in which case $f(r) = r$), and $k = -1$ for hyperbolical space. The variable k is called the curvature (Li-24).

One way to test a FE model or fit it to observations is to plot some observable cosmic distance measures D_{C} for objects versus their cosmological redshifts z (which are the only easily obtained direct observables) and then compare to the theoretical cosmic distance measure D_{C} plotted as a function of z. The two best known observable cosmic distance measures (other than cosmological redshift z itself) are the luminosity distance D_{L} and the angular diameter distance D_{A} both of which have explicit dependence on z, but also depend on z via the comoving coordinate $r(z)$ whose z dependence is an observational constraint, not an intrinsic dependence.

NOTE: There are parts a,b,c,d. On exams, omit part d. Use minimal words. Some of the parts can be done independently, and so not stop if you cannot do one.

a) Recall the Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\,d\phi^2 \right) \right] .$$

For a light signal traveling from a source at comoving coordinate r, time t, and cosmological redshift z to the origin (i.e., us) at time t_0 along a radial path, derive an equation from the Robertson-Walker metric relating spatial integral $f(r)$ to time integral $\chi(t)$ (which is actually an alternative comoving coordinate though the symbol χ used by CL-11 may not a standard for it). The left-hand side should depend only on parameters r and k and the right-hand side only on t and t_0. Do **NOT** use any words: just the expressions. **HINT:** The interval is lightlike for a light signal: i.e., $ds = 0$.

b) Formal expressions for r, t, and lookback time t_{LB} for a light signal are, respectively,

$$r = f^{-1}[\chi(t)] = f^{-1}[\chi(a)] = f^{-1}\left[\chi\left(\frac{a_0}{1+z}\right)\right] = f^{-1}[\chi(z)] , \qquad t = t(a) = t\left(\frac{a_0}{1+z}\right) = t(z) ,$$

and

$$t_{\text{LB}} = -\Delta t = -[t(a) - t_0] ,$$

where we have used the cosmological redshift formula

$$1 + z = \frac{a_0}{a(t)} .$$

Note that $f(r) = r$ and $f^{-1}(r) = r$ if curvature $k = 0$.

In order to obtain the proper distance $D_{\rm P} = a_0 f(r) = a_0\chi(z)$ explicitly, from the foregoing formulae, we need to specify an FE model. In general, only numerical results can be obtained. However, the de-Sitter universe (with k general) allows explicit simple formulae for some cosmological distance measures. For the de-Sitter universe,

$$a(t) = a_0 e^{H_0 \Delta t} ,$$

where in this case the Hubble constant $H_0 = \sqrt{\Lambda/3}$ is time-independent and Δt is the time relative to cosmic present.

For the de-Sitter universe, determine in order the explicit formulae for $\Delta t(z)$, $t_{\rm LB}(z)$, $\chi(z)$, radial proper distance $D_{\rm P}(z)$, and recession velocity $v_{\rm R}(z)$.

What is odd about lookback time $t_{\rm LB}$ as $z \to \infty$ relative to the case of a cosmological model with a point origin (AKA Big Bang singularity)?

c) What is the explicit expression for the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$ for the de Sitter universe?

d) The formal expressions for the standard cosmological distance measures (expressed in observational form if it exists and is distinct from theoretical forms and then in the theoretical forms) are as follows:

Cosmological redshift: $$z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{a_0}{a(t)} - 1$$

Lookback time: $$t_{\rm LB} = t_0 - t(a) = -\Delta t$$

Comoving coordinate r: $$r = f^{-1}\left[\chi(z)\right] = f^{-1}\left[\chi(t)\right]$$

Comoving coordinate χ: $$\chi(z) = \chi(t) = \int_t^{t_0} \frac{c\,dt'}{a(t')}$$

Radial proper distance: $$D_{\rm P} = a_0\chi(z) = a_0\chi(t) = a_0 f(r)$$

Recessional velocity: $$v_{\rm R} = H_0 D_{\rm P}$$

Redshift velocity: $$v_{\rm red} = zc$$

Luminosity distance: $$D_{\rm L} = \sqrt{\frac{L}{4\pi f}} = a_0 r(1+z)$$

Angular diameter distance: $$D_{\rm A} = \frac{D_{\rm ruler}}{\theta} = \frac{a_0 r}{(1+z)}$$

Distance-duality relation: $$\frac{D_{\rm L}}{D_{\rm A}} = (1+z)^2 ,$$

where the distance-duality relation is also called the Etherington reciprocity relation.

Determine special case expressions (if they exist) for the cosmological distance measures above as a functions of z for the de Sitter universe. Note that some were already determined in part (b) and some already functions of z. What is odd about $D_{\rm A}$ as z goes to infinity in the case of $k = 0$?

---

004 qfull 00650 1 3 0 easy math: conformal time and cosmological redshift

55. The alternative comoving coordinate

$$\chi = \int_t^{t_0} \frac{c\,dt}{a(t)}$$

is also what is called conformal time.

NOTE: There are parts a,b,c,d,f.

a) Starting from the scaled Friedmann equation form

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\sum_p \Omega_{p,0} x^{-p}\right)$$

(where $x = a/a_0$) derive without words an integral formula for $\chi(x)$.

b) Now change the integral formula so that we have $\chi(z)$.

c) In what limit would $\chi(z)$ have an analytic formula?

d) Assuming there is only a single density component with $p > 0$, derive the exact solution for $\chi(z)$.

e) Assuming there is only a single density component with $p = 0$, derive the exact solution for $\chi(z)$.

f) Give the formula for radial proper distance $D_{\rm P}$ with $\chi(z)$ expanded into the integral form. Does $D_{\rm P}$ depend on a_0? Give the formula for $a_0 r$ for all cases of k with $\chi(z)$ unexpanded. Does $a_0 r$ depend on a_0?

---

004 qfull 00700 1 3 0 easy math: deceleration parameter

56. The theoretical cosmological distance measures to 2nd order in small cosmological redshift z are conventionally written in terms of the Hubble constant $H_0 = \dot{a}_0/a_0$ and the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$ (which is unitless or rather has natural units). In fact in the 1970s, cosmology was sometimes comically oversimplified as a search for two numbers: H_0 and q_0 (see A.R. Sandage, 1970, Physics Today, 23, 34, *Cosmology: A search for two numbers*). Nowadays, q_0 has lost some of its glamor. It is now not regarded as a basic parameter of cosmological models, but just one of the derived parameters and its peculiar definition just a historical convention. The fact that the universal expansion is accelerating makes the deceleration parameter negative which is an incongruity.

There are parts a,b.

a) Taylor expand $a(t)$ in small $\Delta t = t - t_0$ to 2nd order and rewrite the coefficients in terms of H_0 and q_0. The rewritten expansion should begin $a(t) = a_0[1 + \ldots$.

b) Recalling the cosmological redshift formaula $1 + z = a_0/a$, rewrite the formula from the part (a) answer as an expansion for z to 2nd order small Δt. **HINT:** You will need the geometric series:

$$\frac{1}{1-x} = \sum_{\ell=0}^{\infty} x^\ell ,$$

which converges for $|x| < 0$ (Ar-279).

c) Now we need to invert the power series for z to find lookback time $t_{\rm LB} = t_0 - t = -\Delta t$ to 2nd order in small z. We will need the power series inversion cofficients. Given

$$\Delta y = \sum_{\ell=1}^{\infty} a_\ell \Delta x^\ell \qquad \text{and} \qquad \Delta x = \sum_{\ell=1}^{\infty} b_\ell \Delta y^\ell ,$$

where the inversion coefficients b_i run $b_1 = 1/a_1$, $b_2 = -a_2/a_1^3$, ... (Ar-316–317).

d) The Friedmann acceleration equation can be used to get a useful expression for the deceleration parameter q_0. Behold:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda}{3}$$

$$
\begin{aligned}
\frac{\ddot{a}a}{\dot{a}^2}H^2 &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2} + \rho_\Lambda + 3\frac{p_\Lambda}{c^2}\right) \\
-qH^2 &= -\frac{4\pi G}{3}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{4\pi G}{3H^2}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{1}{2}\frac{1}{\rho_{\rm critical}}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{1}{2}\left[\Omega_{\rm M}(1+3w) + \Omega_\Lambda(1+3w_\Lambda)\right] \\
q &= \frac{1}{2}\left[\Omega_{\rm M} - 2\Omega_\Lambda\right] = \frac{\Omega_{\rm M}}{2} - \Omega_\Lambda \qquad \text{with } w = 0 \text{ and } w_\Lambda = -1 \text{ as per usual} \\
q &= \frac{1}{2}\left[0.3\alpha_{\rm M} - 2\times(0.7\alpha_\Lambda)\right] = \frac{1}{2}\left[0.3\alpha_{\rm M} - 1.4\alpha_\Lambda\right] = 0.15\alpha_{\rm M} - 0.7\alpha_\Lambda \ ,
\end{aligned}
$$

where $\alpha_{\rm M} = \Omega_{\rm M}/0.3$ (0.3 being a modern fiducial value) and $\alpha_\Lambda = \Omega_\Lambda/0.7$ (0.7 being a modern fiducial value). Wit the modern fiducial values, one obtains a fidicial modern value $q_0 = -0.55$. Before 1998, people mostly thought $\Omega_\Lambda = 0$ which with $\Omega_{\rm M} = 0.3$ (which was what it seemed then as well as now) gives $q_0 = 0.15$. However, some people then hoped that $\Omega_{\rm M} = 1$ which would give $q_0 = 1/2$ which many thought was the great good value. Why?

---

004 qfull 00710 1 3 0 easy math: small z expressions for the cosmological distance measures

57. To get the small cosmological redshift z formulae for cosmological distance measures one expands $a(t)$ around current time t_0 to 2nd order in $\Delta t = t - t_0$, parameterizes the first expansion coefficients with the Hubble constant $H_0 = \dot{a}_0/a_0$ and the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$, substitutes for $a(t)$ with z (and thereby assuming t is the start time for a light signal coming from z), and inverts the power series to get lookback time $t_{\rm LB}$ to 2nd order in small z:

$$ t_{\rm LB} = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2}q_0\right)z + \ldots\right] \ . $$

One then uses the $t_{\rm LB}$ formula with the Robertson-Walker metric applied to the light signal to get the comoving coordinate r to 2nd order in z:

$$ r = \frac{zc}{a_0 H_0}\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right] \ . $$

There are parts a,b,c,d. The parts can be done be at least semi-independently, so don't stop necessarily if you can't do a part.

a) Use the 2nd-order-in-z formulae given in the preamble to get the **2nd-order-in-z** formulae (simplified so that there is only one second order term appearing) and **1st-order-in-z** formulae (expressed just one term appearing) for the following standard cosmological distance measures (expressed in observational form if it exists and then theoretical form), except for expression for z itself included for completeness:

Cosmological redshift: $\quad z = \dfrac{\lambda_0 - \lambda_{\rm e}}{\lambda_{\rm e}} = \dfrac{a_0}{a(t)} - 1 \qquad 1 + z = \dfrac{a_0}{a(t)}$

Lookback time: $\quad t_{\rm LB} = t_0 - t(a)$

Comoving coordinate r: $\quad r = f^{-1}\left\{A\left[t_0, t\left(\dfrac{a_0}{1+z}\right)\right]\right\}$

Proper distance: $D_{\rm P} = a_0 f(r)$

Recessional velocity: $v_{\rm R} = H_0 D_{\rm P}$

Redshift velocity: $v_{\rm red} = zc$

Luminosity distance: $D_{\rm L} = \sqrt{\dfrac{L}{4\pi f}} = a_0 r(1+z)$

Angular diameter distance: $D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1+z)}$.

b) Under what conditions are the cosmological distances measures direct observables to 1st and 2nd order given that one can measure z?

c) Prove that all the standard cosmological distance measures are the same to 1st order in small z aside from constants. Show what they are in terms of quantity zc/H_0, where $c/H_0 = (13.968\ldots\,\text{Gly})/h_{70} = (4.2827\ldots\,\text{Gpc})/h_{70}$ is the Hubble length with $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$.

d) Prove the observational Hubble's law:

$$v_{\rm red} = H_0 D_{\text{P-1st}} ,$$

where $D_{\text{P-1st}}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance.

e) Given that $|q_0| \lesssim 1$, at what z values would one expect the standard cosmological distance measures (with constants applied as needed to make them all all equal to 1st order in z) to diverge by of order or less than 1 %, 10 %, 30 %, 50 %, and 100 %.

Chapt. 6 Advanced Cosmological Models

Multiple-Choice Problems

Full-Answer Problems

005 qfull 00110 1 3 0 easy math: radiation-matter universe somewhat completely

58. The Friedmann equation for the radiation-matter universe (which applies to the observable universe from cosmic time zero to of order cosmic time 10 Gyr) in general scaled form is

$$\left(\frac{\dot{x}}{x}\right)^2 = \Omega_{4,0}x^{-4} + \Omega_{3,0}x^{-3}$$

where x is the cosmic scale factor with $x_0 = 1$ for cosmic present, $\tau = H_0 t$ is the scaled cosmic time with t being cosmic time in standard time units and H_0 being the Hubble constant, $\Omega_{4,0}$ is the radiation density parameter for cosmic present, and $\Omega_{3,0}$ is the matter density parameter for cosmic present.

NOTE: There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c. The parts a,b,c can be done independently, and so don't stop if you can't do one.

a) Determine the radiation-matter equality scale factor $x_{\rm eq}$: i.e., the x value that makes the radiation and matter mass-energy equal.

b) Defining $y = x/x_{\rm eq}$, rewrite the Friedmann equation into a nice integrable form $dw = f(y)\,dy$ (i.e., a special case scaled form), where $w = \tau/\tau_{\rm sc}$ is rescaled time and the form has no constants. What is $\tau_{\rm sc}$ in terms of the density parameters?

c) Solve the Friedmann equation form found in part (b) for $w(y)$ with $w(y=0) = 0$. You will need the table integral

$$\int \frac{y\,dy}{\sqrt{1+y}} = \frac{2}{3}(y-2)\sqrt{1+y}\ .$$

d) For $w(y)$, write out the special cases $w(y=0)$ $w(y)$ to 2nd order in small y, $w(y=1)$ (at the radiation-matter equality) $w(y=2)$ (at 2 times the radition-matter equality) $w(y=3)$ (at 3 times the radition-matter equality which is where the exact y(w) formula changes form), and $w(y >> 1)$ (the large y asymptotic limit).

d) Solve for the asymptotic limiting small w and large w forms of $y(w)$.

f) Transform the limiting forms found in part (d) into the general scaled forms: i.e., into $x(\tau)$ forms.

g) This a challenging part if you have some time. Yours truly has probably spent more time than it is worth trying to find good analytic approximate for solutions $x(\tau)$ for cases where no exact solution exists or the exact solution exists, but is too complex for easy understanding. In fact, the V model solutions (Jeffery 2025) provide understandable exact solutions which are analogues to the standard traditional, but non-exact, solutions for the

Friedmann equation found by Alexander Alexandrovich Friedmann (1888–1925), Georges Lemaitre (1894–1966), Willem de Sitter (1872–1934), and others long ago. There may be no better way in general to understand those standard traditional, but non-exact, solutions than using those V model solution analogues. However, in special cases, there may be. One special case, is the radiaton-matter universe. In fact, an exact solution for $y(w)$ exists with two mathematically equivalent formulae that look rather different (Jeffery 2026). But both formulae are too complex for easy understanding. However, a fairly accurate, easy-to-understand interpolation formula does exist that agrees asymptotically with the symptotic limiting small w and large w forms of $y(w)$ found in part (d). See if you can find it. **HINT:** The formula uses $\arctan[y_{\text{small}}(w)]$ and it takes some playing around to find it.

---

005 qfull 00310 1 3 0 easy math: two-power-law models, two-inverse-power-law models

Extra keywords: This question needs reworking a bit

59. There are, in fact, some cases where there exist simple exact analytic solutions $t(a)$ for the Friedmann equation for a combination of two inverse-power-law dependences on the cosmic scale factor a. In this problem, we investigate these solutions.

NOTE: There are parts a,b,c,d,e,f,g. Part groups d,e and f,g are independent, and so do not stop if you cannot to one of those groups.

a) The Friedmann equation for a combination of two inverse-power-law dependences is

$$\frac{1}{x}\frac{dx}{d\tau} = (bx^{-p} + x^{-q})^{1/2}$$

where $x = a/a_0$, $d\tau = \sqrt{\Omega_{q,0}}H_0\,dt$, and $b = \Omega_{p,0}/\Omega_{q,0}$. The Friedmann equation can be rewritten in the form

$$\frac{x^r\,dx}{(b+x)^{1/2}} = d\tau$$

and solved simply by integration by parts if r is integer greater than or equal to zero, and p and q have allowd values. What must q equal as a function of r for this rewrite? What must p equal as a function of r for this rewrite? What case of the above Friedmann equation is important in actual cosmology and what r is needed in that case?

b) Solve the equation in part (a) for τ (with initial condition $x = 0$ when $\tau = 0$) for one integration-by-parts step. Note r stays general: i.e., r stays an integer greater than or equal to zero. After the one integration-by-parts step, there is an integrated part and an integral to do, and the if you knew what r was, you could in further steps get an explicit solution without integrals.

c) Specialize the solution in part (b) for the case of early universe where radiation and matter are the only significant mass-energy forms. Do the integration to get the explicit exact solution $\tau(x)$.

d) The solution obtained in part (c) is the analytically exact solution. However, it has a complex appearance and also it becomes numerically inaccurate as $x \to 0$. Expand the solution in small $y = x/b$ to the first two nonzero terms.

e) The solution obtained in part (d) is the small x asymptotic solution to the radiation-matter solution. From that obtain the pure radiation solution $\tau(a)$ that is asymptotic to the radiation-matter solution. Also obtain the radiation solution $a(\tau)$. **HINT:** This is easy.

f) Now expand the exact analytic solution obtained in part (c) in small b/x for the first four leading terms in b/x. This is the asymptotic solution for large x.

g) From the part (f) solution obtain the pure matter solution $\tau(a)$ that is asymptotic to the radiation-matter solution. with the optimum start time for $x = 0$ which is not $\tau = 0$. Also obtain the matter solution $a(\tau)$. **HINT:** This is easy.

005 qfull 00480 1 3 0 easy math: exact age of the universe formula for the Lambda-CDM model

60. The exact solution $t(a)$ in scaled parameters for matter-Λ universe (which is the Λ-CDM universe not counting the comparatively brief radiation era) is

$$w = \ln\left(z + \sqrt{z^2+1}\right) ,$$

where the scalings are

$$w = \frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t \qquad \text{and} \qquad z = \left[\frac{a/a_0}{(\Omega_{\mathrm{m},0}/\Omega_{\Lambda,0})^{1/3}}\right]^{3/2} ,$$

where 0 indicates cosmic present, a_0 is the cosmic present scale factor (conventionally set to 1), $\Omega_{\mathrm{m},0}$ is the cosmic present matter density parameter (fiducial value 0.3), $\Omega_{\Lambda,0}$ is the cosmic present Λ density parameter or constant dark energy density parameter (fiducial value 0.7), and H_0 is the Hubble constant (fiducial value 70 (km/s)/Mpc).

NOTE: There are parts a,b,c,d,e,f. The parts (c) and (f) can be done independently of part (a), but the other parts cannot.

a) Undo the scalings, replace $\Omega_{\mathrm{m},0}$ by $(1-x)$, $\Omega_{\Lambda,0}$ by x, set $a = a_0$, and scale time to τ using $\tau = H_0 t$ for a simplified age of the universe formula. Simplify the formula as much as you reasonably can.

b) Starting from the part (a) result, derive the Taylor expansion formula for τ to all orders small x **HINT:** You will need the Taylor expansion

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} .$$

The Taylor expansion formula for τ is remarkably simple.

c) Why might you want a small-x Taylor expansion even if you have the exact formula?

d) Write a pseudocode fragment to sum the Taylor expansion of part (b) to the Kth term. Make it numerically accurate (by adding from smallest terms up) and efficient.

e) Derive the 2-term asymptotic formula for τ as $x \to 1$.

f) The exact formula for τ can be replaced by an interpolation formula accurate to within 3 % for all $x \leq 0.99$ and also at $x = 1$:

$$\tau_{\mathrm{interp}} = -\frac{1}{3}\left[\ln(1-x) + \sum_{k=1}^{2}\frac{x^k}{k}\right] + \frac{2}{3}\left[\sum_{k=0}^{2}\frac{x^k}{2k+1}\right] .$$

Why in general might one want a simple interpolation formula to complement a complex exact formula or procedure of evaluation?

005 qfull 00490 1 3 0 easy math: exact age of the universe formula for the Lambda-CDM model 2

61. The exact cosmic scale factor for the matter-Λ universe (which is the Λ-CDM universe not counting the comparatively brief radiation era) is

$$a = a_0\left(\frac{\Omega_{\mathrm{m},0}}{\Omega_{\Lambda,0}}\right)^{1/3}\sinh^{2/3}\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t\right) ,$$

where 0 indicates cosmic present, a_0 is the cosmic present scale factor (conventionally set to 1), $\Omega_{\mathrm{m},0}$ is the cosmic present matter density parameter (fiducial value 0.3), $\Omega_{\Lambda,0}$ is the cosmic present Λ or constant dark energy density parameter (fiducial value 0.7), and H_0 is the Hubble constant (fiducial value 70 (km/s)/Mpc).

There are parts a,b,c,d,e,f. The part groups (a,b,c), (d), and (e,f) can be done independently. So don't stop if you can't start one of these part groups.

a) Derive the inverse formula $w(z)$ in terms of

$$w = \frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t \qquad \text{and} \qquad z = \left[\frac{a/a_0}{(\Omega_{\mathrm{m},0}/\Omega_{\Lambda,0})^{1/3}}\right]^{3/2}$$

given

$$\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right) .$$

b) Derive 1st-order-in-small-z formula for $w(z)$. **HINT:** You will need to remember how to Taylor expand.

c) Derive large-z-asymptotic formula for $w(z)$.

d) Directly from the exact formula for $a(t)$ (or by other means) derive the 1st-order-in-small-t formula for $a(t)$. What universe model does the resulting formula give?

e) Directly from the exact formula for $a(t)$ (or by other means) derive the large-t-asymptotic formula for $a(t)$. What universe model does the resulting formula give?

f) From the part (e) result, determine the fiducial value of the asymptotic or Λ Hubble parameter.

---

005 qfull 00630 1 3 0 easy math: Einstein universe, einstein universe

62. The Einstein universe (proposed by Einstein in 1917) was the first cosmological model derived consistently from a physical theory (i.e., general relativity) and was the beginning of modern cosmology. Einstein assumed the cosmological principle (i.e., a homogeneous, isotropic universe) and represented the mass-energy by a pressureless perfect fluid where the density scaled as a^{-3}. In modern cosmology jargon, this kind of perfect fluid is called "matter" and approximates ordinary baryonic matter and dark matter For cosmological purposes, matter has approximately zero kinetic energy relative its local comoving frame.

Einstein believing in 1917 that the universe was one of stars (which seemed on average at rest) and not galaxies wanted a static model, but found that impossible with his field equations as originally formulated (O'Raifeartaigh et al. 2017). So he added the cosmological constant term Λ to the field equations which was the simplest possible modification and had no significant effect on smaller-than-cosmological-scale phenomena. The Einstein universe he obtained is a finite, boundless, positively curved universe or hyperspherical universe. It is geometrically the 3-dimensional surface of the a 3-sphere (which is actually a 4-dimensional sphere in Euclidean or flat space). The distance to return to the same point along a geodesic is $2\pi a_0$, where a_0 is the Gaussian curvature radius a hyperspherical universe. (CL-11–12). For considering the Einstein universe, a_0 is not the conventional dimensionless quantity but a physical proper distance with units of length.

Einstein in 1931 abandoned the Einstein universe since observations showed an expanding universe and because the Einstein universe had been shown to be unstable by Eddington in 1930 (O'Raifeartaigh et al. 2017 p. 36, 41).

Note that Einstein did not have the Friedmann equation and acceleration equation when he derived the Einstein universe. He used a general relativity directly and followed a "rough and winding road" (O'Raifeartaigh et al. 2017, p. 18).

In this problem, we investigate the Einstein universe. There are parts a,b,c,d,e,f,g,h. In exam environments, do **ONLY** parts a,b,c,d.

a) The Friedmann equation and acceleration equation in forms appropriate for solving for the Einstein universe and investigating its stability are

$$H^2 = \left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a_0^2 x^2} + \frac{\Lambda}{3} = \frac{8\pi G\rho_0}{3}\left[\Omega_{\rm M} x^{-3} + \Omega_k x^{-2} + \Omega_\Lambda\right]$$

and

$$\frac{\ddot{x}}{x} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3} = -\frac{4\pi G\rho_0}{3}\left[\Omega_{\rm M} x^{-3} - 2\Omega_\Lambda\right]$$

(Li-55 *mutatis mutandis*), where $x = a/a_0$, a_0 is the Gaussian curvature radius of the Einstein universe (as aforesaid), ρ_0 is the density of Einstein universe, $k = 1$ for a positive curvature universe,

$$\Omega_{\rm M} = 1 \ , \qquad \Omega_k = -\frac{kc^2}{a_0^2(8\pi G\rho_0/3)} \ , \qquad \text{and} \qquad \Omega_\Lambda = \frac{\Lambda}{8\pi G\rho_0} \ .$$

Note we cannot use the Hubble parameter H in defining the density parameter Ω_i quantities since $H = 0$ for the Einstein universe.

The Einstein universe has $x = 1$, $\dot{x} = 0$, $\ddot{x} = 0$ and $\rho = \rho_0$. Given the Einstein-universe values, determine formula for Λ from the first form of the acceleration equation and the numerical value of Ω_Λ from the second form.

b) Given the Einstein-universe values, determine the formula for a_0 as function of ρ_0 and then the formula for a_0 as a function of Λ. **HINT:** Start from the second form of the Friedmann equation and recall the given formula for Ω_k.

c) Given $G = 6.67430(15) \times 10^{-11}$ MKS, vacuum light speed $c = 2.99792458 \times 10^8$ m/s, $\rho_0 = 0.85 \times 10^{-26}$ kg/m^3 (which is suggest value of the critical density circa 2021), and 1 Gpc = $(3.085677581\ldots) \times 10^{25}$ m, calculate the Gaussian curvature radius a_0 in units of gigaparsecs (Gpc). You can use your phone for the calculations—but only for those.

d) Now write the Friedmann equation in the dimensionless form

$$\frac{dx}{d\tau} = \pm\sqrt{f(x)} \ ,$$

where the dimensionless time τ is given by

$$\tau = t\sqrt{\frac{8\pi G\rho_0}{3}} \ .$$

Sketch a plot the radicand $f(x)$ for $x \geq 0$ going left from $x = 1$ to $x = 0$ and right from $x = 1$ to $x = \infty$. Using the first two derivatives of $f(x)$ as a function of x (not τ) prove that the Einstein universe (i.e., the $x = 1$ case) is a unique static universe for $x \geq 0$.

e) For the initial condition x_1 greater/less than 1 at τ_1 and the positive/negative case for $x' = \pm\sqrt{f(x)}$, describe the evolution of x with τ increasing and in particular what happens if $x \to 0$. Explain the evolutions and describe the stability of the Einstein universe to perturbations in these cases. **HINT:** It might help to draw a figure of the evolutions.

f) For the initial condition x_1 greater/less than 1 at τ_1 and the negative/positive case for $x' = \mp\sqrt{f(x)}$, describe the probable evolution of x with $\tau \to \infty$. Prove these evolutions and describe the stability of the Einstein universe to perturbations in these cases. **HINT:** The proof requires that you show that all orders of derivative of x are zero when x is stationary. You will need to determine the x'', x''', and $x^{(4)}$, notice some things about these orders of derivative, and add some explanatory words. Also, it might help to draw a figure of the evolutions.

g) From parts (d) and (e), what is the stability of the Einstein universe to general perturbations of a? Note a solution is unstable to general perturbations if it is unstable to any kind of perturbations.

h) Given all the answers to the other parts, discuss how an Einstein universe filled with real gas (including dark matter gas) and/or stars might evolve.

---

005 qfull 00710 1 3 0 easy math: crude asymptotic fit to the Lemaitre-Eddington universe

Extra keywords: Needs to be completed.

63. Lemaître (1925) gave a solution for the Friedmann equation which started from a small positive perturbation Δx_0 in scaled cosmic scale factor from the Einstein universe (with scaled cosmic scale factor $x = 1$ that initially and asymptotically grew exponentially, but with different e-folding constants. The initial growth could be made as small as you like Lemaître himself lost interest in this model, but Eddington favored it and it had a vogue up to 1935 (e.g., Bondi 1960, p. 84–85, 117–121, 159, 175, 180). It is now called the Lemaître-Eddington universe (which is not the Lemaître universe). Apparently, Eddington favored the Lemaître-Eddington universe since was an expanding universe model (which agreed with observational expansion of the universe discovered 1929), it could have a slow an initial growth phase as you liked (which thus allowed one to avoid the age problem of circa 1930), and it avoided dealing with the primeval universe that set the initial condition. Lemaître, however, wanted to deal with the primeval universe and so came to favor his Lemaître universe with his primeval atom which can be called a cold big bang theory.

Recall the scale Friedmann equation for the Einstein universe:

$$\dot{x} = \pm\sqrt{x^{-1} - \frac{3}{2} + \frac{1}{2}x^2} ,$$

where the first term is the matter component, the second the positive curvature component, and the third is the cosmological constant (i.e., Λ) component. Note the scaled time is τ.

NOTE: There are parts a,b.

a) Keeping only the leading term in large x of the Friedmann equation, solve for the the positive case asymptotic solution for $x = 1 + \Delta x$ given the asymptotic initial condition $x_{\rm a} = 1 + \Delta x_{\rm a}$ Note the asymptotic initial condition $x_{\rm a} = 1 + \Delta x_{\rm a}$ is not the true initial condition $x_0 = 1 + \Delta x_0$, but is the initial condition that allows the asymptotic solution to track into the exact solution as $\tau \to \infty$. Given only what you now know, what is the best estimate of $x_{\rm a}$?

b) Keeping only the two leading term in large x of the Friedmann equation for the positive case, gives

$$\dot{x} = \sqrt{\frac{1}{2}x^2 - \frac{3}{2}} = \sqrt{\frac{1}{2}}\,x\sqrt{1 - \frac{3}{x^2}} .$$

Substitute for the $1/x^2$ from the solution obtained in part (a) to create a correction term, expand the square root factor to 1st order for small correction term, and solve the Friedmann equation for an improved asymptotic solution.

---

005 qfull 00950 1 3 0 easy math: The matter-positive-curvature universe

Extra keywords: Need to rewrite in scaled form throughout, but no time 2023nov26.

64. The Friedmann equation is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

(Li-55). Let's consider the matter-positive-curvature universe (i.e., a universe with $\rho \propto 1/a^3$, $k > 0$, $\Lambda = 0$). The geometry of this universe is the surface of hypersphere (specifically a

3-sphere) which is finite, but unbounded. Here, however, we are only interested in the solution for cosmic scale factor a, not in the geometry.

There are parts a,b,c,d,e.

a) Rewrite the Friedmann the form $\dot{a} = f(a)$ with $\Lambda = 0$, $\rho = \rho_{\rm M}(a_{\rm M}/a)^3$. We define $a_{\rm M}$ to be the a value for the minimum density $\rho_{\rm M}$ that is allowed by the differential equation. Determine the value for k in terms of the minimum density $\rho_{\rm M}$. What is $a_{\rm M}$ in the solution $a(t)$?

b) Given that the Friedmann equation is of the form $f' = \pm\sqrt{g(f)}$ and that for small a we must have the Einstein-de-Sitter universe behavior ($a \propto t^{2/3}$ assuming $a(t=0) = 0$), describe what the solution must look like qualitatively.

c) Rewrite the Friedmann equation in natural units: $\sqrt{k}\,t \to t$ and $a/a_{\rm M} \to a$.

d) An approximate simple analytic solution for the Friedmann equation (in natural units) suggested by part (b) is

$$a = \sin^{2/3}\left(\frac{\pi}{2}\frac{t}{t_{\rm M}}\right) ,$$

where $t_{\rm M}$ is the location of the maximum. This approximate solution is an interpolation formula since it gives the right behavior at the endpoints and the maximum. But $t_{\rm M}$ has to be determined. What are natural guesses for $t_{\rm M}$? Now use a 1-step Euler method to obtain a reasonable estimate of a good value for the approximate solution.

e) Actually, an exact analytic solution can be obtained to the differential equation in terms of a new independent variable η. One needs a trick:

$$\dot{a} = \frac{da}{d\eta}\dot{\eta} = \frac{da}{d\eta}\frac{1}{a} \qquad \text{with requirement} \qquad \dot{\eta} = \frac{1}{a} .$$

The trick gets rid of an a in a denominator, but in the way that clairvoyance says is the Tao. Using the trick solve for $a(\eta)$ using a table integral and with the constant of integration chosen so that $a(\eta = 0) = 0$. Then find $t(\eta)$. What the limits of η? Why can we write an analytic formula for $a(t)$? but it has no analytic form

Chapt. 7 Cosmic Background Radiation, Cosmic Temperature, Recombination, and Reio

Multiple-Choice Problems

006 qmult 00110 1 1 3 easy memory: logarithmic representation of specific intensity

65. The logarithmic representation of specific intensity satisfies equation:

   a) $I_E = I_\nu = I_\lambda$.
   b) $I_E/E = I_\nu/\nu = I_\lambda/\lambda$.
   c) $EI_E = \nu I_\nu = \lambda I_\lambda$.
   d) $I_E = I_\nu = 1/I_\lambda$.
   e) $I_E = 1/I_\nu = 1/I_\lambda$.

Full-Answer Problems

006 qfull 00110 1 3 0 easy math: nu,lambda,log represenions: On exams, omit part d,e

66. Specific intensity and related quantities (e.g., energy density per unit wavelength) are conventionally given in three representations: photon energy representation I_E, frequency representation I_ν, and wavelength representation I_λ. These representations are related by differential expression

$$I_E\,dE = I_\nu\,d\nu = I_\lambda\,(-d\lambda) \ ,$$

where the minus sign is occasionally omitted if one knows what one means—which is that a differential increase in photon energy/frequency corresponds to a differential decrease in wavelength.

There are parts a,b,c,d,e. On exams, omit parts d,e and use mimimal words. Parts a,b,c can done independently, and so do not stop if you can't do a part.

   a) As well as the three conventional representations, there is a logarithmic representation

$$EI_E = \nu I_\nu = \lambda I_\lambda$$

   which has the same value whichever of E, ν, or λ is used as the independent variable. Prove the logarithmic representation equality. **HINT:** You will have to use differentials of the logarithm of the independent variables (e.g., $d[\ln(E)]$) and make use of the de Broglie relations $E = h\nu = hc/\lambda$.

   b) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1} \ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} \ .$$

   Derive the explicit energy representation B_E, wavelength representation B_λ, and logarithmic representation $EB_E = \nu B_\nu = \lambda B_\lambda$ in all three of the E, ν and λ forms.

c) Write the Planck's law in the dimensionless frequency representation expression $B_x\,dx$ and derive for $B_x\,dx$ the Rayleigh-Jeans law form (small x) and the Wien approximation form (large x).

d) Suggest one or two reasons why people might want to use the logarithmic representation for plots.

e) Derive the Rayleigh-Jeans law (small x, small E, small ν, large λ approximation) and the Wien approximation (large x, large E, large ν, small λ approximation) for B_E, B_ν, and B_λ **HINT:** This pretty easy albeit tedious.

---

006 qfull 00220 1 3 0 easy math: Debye function and blackbody radiation results

67. The total Debye function (i.e., the sum of the first and second Debye functions) is

$$D_z = \int_0^\infty \frac{x^z}{e^x - 1}\,dx = z!\zeta(z+1)\ ,$$

(e.g., Wolfram Mathworld: Debye functions; Wikipedia: Debye function) where the factorial function

$$z! = \begin{cases} \int_0^\infty x^z e^{-x}\,dx = z(z-1)! & \text{for } z \text{ not a negative integer and also} \\ & \quad z \neq 0 \text{ for the second form (Ar-543);} \\ n! & \text{for integer } n \geq 0; \\ \sqrt{\pi} & \text{for } z = -1/2 \text{ (Ar-543,544);} \\ \dfrac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi} & \text{for half-integer } z \geq -1/2 \text{ with } (-1)!! = 1; \end{cases}$$

and Riemann zeta function (without anayltic continuation considered)

$$\zeta(s) = \begin{cases} \displaystyle\sum_{\ell=1}^{\infty} \frac{1}{\ell^s} & \text{in general;} \\ \displaystyle\zeta(1) = \sum_{\ell=1}^{\infty} \frac{1}{\ell} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots & \text{the divergent harmonic series (Ar-279);} \\ \displaystyle\zeta(2) = \frac{\pi^2}{6} = \frac{\pi^2}{2\cdot 3} = 1.644934066848226436472415166646\ldots & \\ \zeta(3) = 1.2020569031595942853997381615114\ldots & \\ \displaystyle\zeta(4) = \frac{\pi^4}{90} = \frac{\pi^4}{2\cdot 3^2\cdot 5} = 1.082323233711138191516003696541\ldots & \\ \zeta(5) = 1.036927755143369926331365486457\ldots & \\ \displaystyle\zeta(6) = \frac{\pi^6}{945} = \frac{\pi^6}{3^3\cdot 5\cdot 7} = 1.0173430619844491397145179297909\ldots & \\ \zeta(7) = 1.008349277381922826839797549849\ldots & \\ \displaystyle\zeta(8) = \frac{\pi^8}{9450} = \frac{\pi^8}{2\cdot 3^3\cdot 5^2\cdot 7} = 1.004077356197944339378685238508\ldots & \\ \zeta(9) = 1.002008392826082214417852769232\ldots & \\ \displaystyle\approx \sum_{\ell=1}^{k-1} \frac{1}{\ell^s} + \int_{k-1/2}^{\infty} \frac{1}{x^s}\,dx = \sum_{\ell=1}^{k-1} \frac{1}{\ell^s} + \frac{1/(k-1/2)^{s-1}}{s-1} & \text{integral approximation for } s>1; \\ \displaystyle 1 + \frac{1}{2^s} & \text{2nd simplest asymptotic form as } s\to\infty; \\ 1 & \text{asymptotic form as } s\to\infty \end{cases}$$

(e.g., Wikipedia: Riemann zeta function; OEIS: Riemann zeta function).

There are parts a,b,c,d,e,f. On exams, do **ONLY** parts a,b,c. Parts a,b,c can be done independently, so don't stop if you can't do one.

a) Prove $D_z = z!\zeta(z+1)$.

b) Determine the general moment formula M_n (where n is the moment power) for the distribution $f(x) = Ax^z/(e^x - 1)$, where A is the normalization constant which you must determine too. Specialize for $n = 0$ (the normalization), $n = 1$ (the mean), and $n = 2$. Determine the general formula for the variance σ^2.

c) From the Planck's law specific intensity,

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^x - 1}\ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda}\ ,$$

show the total energy density of a blackbody radiation field is

$$\epsilon = a_{\mathrm{R}}T^4\ ,$$

where the radiation constant

$$a_{\mathrm{R}} = \frac{8\pi^5 k^4}{15h^3c^3} = (7.56573325028000\ldots)\times 10^{-16}\,\mathrm{J/m^3/K^4} = 1\,\mathrm{J/m^3}\times\left(\frac{1}{6029.61649612301\ldots\quad\mathrm{K}}\right)^4$$

and $T = 6029.61649612301$ is the temperature that gives $1\,\mathrm{J/m^3}$. The numerical values are **NOT** required for the answer. **HINT:** Remember to change an isotropic specific intensity into a density you must multiply by $4\pi/c$.

d) Show that the mean photon energy of blackbody radiation field is

$$E = \frac{\zeta(4)}{\zeta(3)}(3kT) = (2.70117803291906\ldots) \times kT$$
$$= 2.327695131004933 \times 10^{-4}\,\mathrm{eV} \times T = 1\,\mathrm{eV} \times \left(\frac{T}{4296.09525182222\ldots\,\mathrm{K}}\right) ,$$

where $k = (0.8617333262\ldots) \times 10^{4}\,\mathrm{eV/K}$. The numerical values are **NOT** required for the answer.

e) Prove by induction that

$$z! = \frac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi}$$

for half-integer $z \geq -1/2$ with $(-1)!! = 1$.

f) For $s > 1$ and $k \geq 2$,

$$\zeta(s) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^s} \approx \sum_{\ell=1}^{k-1} \frac{1}{\ell^s} + \int_{k-1/2}^{\infty} \frac{1}{x^s}\,dx = \sum_{\ell=1}^{k-1} \frac{1}{\ell^s} + \frac{1/(k-1/2)^{s-1}}{s-1} ,$$

where the summation-to-integral approximation is just the reverse of the Riemann integral-to-midpoint-summation rule which remarkably is more accurate than the trapezoid rule (Wikipedia: Riemann sum: Midpoint rule). The series truncated at term k is always a lower limit on the Riemann zeta function since all the terms are positive. Prove that the integral approximation is always larger (except in the limit that $s \to \infty$) than the term k which means the integral approximation never underestimates the Riemann zeta function. **HINT:** You will need to use L'Hôpital's rule.

---

006 qfull 00240 1 3 0 easy math: radiation distribution function: complete for forget

68. In blackbody radiation distributions, the function

$$f(x) = \frac{x^z}{e^x - 1} \qquad \text{where} \qquad x = h\nu/(kT) .$$

The power z is 2 for photon number, 3 for photon energy analyzed by frequency, 5 for photon energy analyzed by wavelength, and 4 for photon energy analyzed by $\ln(\nu)$ or $\ln(\lambda)$. In this question, we will analyze this function generally, but leave its integration to a later question.

There are parts a,b.

a) Analyze $f(x)$ in the limit of $x \to 0$ for the qualitatively possibilities for z: $z < 1$, $z = 1$, $z \in (1,2)$, $z = 2$, and $z > 2$. Find the limiting, non-constant form of the $f(x)$ and determine whether it is a stationary point, a minimum/maximum for the x range $[0,\infty]$, or a singularity.

b)

---

006 qfull 00320 1 3 0 easy math: The cosmic evolution of the primordial photon gas/CMB.

69. The primordial photon gas (which is conventionly called the cosmic microwave background (CMB) even before it redshifts into the microwave band) after recombination does not signficantly interact with itself, matter, or anything again and photon number in any box scaling with the expansion of the universe is conserved to excellent approximation.

There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c. Parts a,b,c can be done independently, so don't stop if you can't do one.

a) Prove that the energy density of the CMB obeys

$$\epsilon = \epsilon_0 \left(\frac{a_0}{a}\right)^4 ,$$

where 0 refers to a fiducial cosmic time which could be cosmic present and a is the cosmic scale factor. Note we are not assuming the specific intensity has any particular distribution.

b) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1} , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} .$$

Show that the CMB obeys this law at any general time t provided it obeys it at the fiducial time t_0 where $a = a_0$ and temperature is T_0. **HINT:** The photons in a frequency bin $d\nu = (a_0/a)\,d\nu_0$ stay in that frequency bin as the universe evolves, and so obey the same energy scaling as the overall CMB. Thus at general time t, we have

$$I_\nu\,d\nu = \left(\frac{a_0}{a}\right)^4 B_{\nu_0}\,d\nu_0 ,$$

where we have assumed the specific intensity at the fiducial time obeys Planck's law. The proof requires showing that $I_\nu\,d\nu = B_\nu\,d\nu$ using a temperature parameter T that obeys a simple formula depending on the cosmic scale factor a. Why is this temperature parameter T the actual temperature at general time t?

c) Given that the CMB specific intensity obeys Planck's law, its energy density is

$$\epsilon = a_{\rm R} T^4 ,$$

where $a_{\rm R}$ is the radiation density constant (usually symbolized by a) and T is the temperature. Using the part (b) answer find the energy density at general time t in terms of the energy density ϵ_0 at fiducial time t_0. Is the result consistent with the part (a) answer?

d) It is quite possible to have a radiation field with a Planck's law shape, but not size. Say for example, say you have blackbody radiator sphere of radius R and you are a distance $r \geq R$ from the sphere center. The emitted specific intensity beams all have B_ν, and so the shape of the spectrum at r obeys Planck's law, but its size is smaller. The effect is called geometrical dilution. Determine the geometrical dilution factor $W(\mu)$ (where radial cosine $\mu = \cos(\theta)$) from the integral for mean specfic intensity J_ν at r

$$J_\nu = \frac{1}{4\pi}\int_0^\theta \int_0^{2\pi} B_\nu \sin(\theta')\,d\theta'\,d\phi = W B_\nu .$$

HINT: Transform the θ integral to a μ integral and draw a diagram.

---

006 qfull 00410 1 3 0 easy math: recombination studied

70. Let's consider the recombination of the cosmic radiation field: i.e., recombination.

There are parts a,b.

a) Consider the differential equation

$$\frac{dN_e}{dt} = -CN_e^2 + CN_{\rm I}(N_{\rm H} - N_e) .$$

This is very simplified equation for recombination assuming a pure hydrogen gas with number density $N_{\rm H}$ and ionizing photon density $N_{\rm I}$: both we assume to be constant over the short time scales. The N_e is the electron density which is also the hydrogen ion density by charge conservation. The two C's are rate coefficients which are equal by a detailed balancing argument that yours truly is none too certain of. The products of the densities arise since the reactions are fluxes of one kind of particle on density of another. Find the steady-state solution in terms of $X = N_e/N_{\rm H}$ and $R = N_{\rm I}/N_{\rm H}$ and argue why it must be asymptotically approached as time goes to infinity.

Actually, the idea is that the steady-state solution is really a quasistatic process: "a thermodynamic process that happens slowly enough for the system to remain in internal equilibrium." We are crudely/vaguely attempting to understand recombination in this question. But we don't get too far.

b) Find the limiting forms of solution X for $R \to 0$ (to 1st order in small R), $R = 1$, and $R \to \infty$ to first order in small $1/R$). What is special about $X(R = 1)$ from a number point of view?

c) For the nonce, let's define the recombination temperature of the cosmic radiation field by $R(T) = 1$. Let N be the photon density, we have

$$1 = R = \frac{N_{\rm I}}{N_{\rm H}} = \frac{N_{\rm I}/N}{N_{\rm H}/N} = \frac{1}{\eta} f_{\rm I} = \frac{1}{\eta}\frac{D_2^{(2)}(x)}{D_n} \approx \frac{1}{\eta}\frac{e^{-x}x^2}{2\zeta(3)} ,$$

where we have approximated the second Debye function by leading term which is valid for $x >> 1$ and where $x = E_{\rm R}/(kT)$ where $E = 13.605693009(84)\,{\rm eV}$ is the Rydberg energy (i.e., the ionization energy of hydrogen) and T is the recombination temperature that we are solving for. The baryon-to-photon ratio $\eta = 6 \times 10^{-10}$ for a fiducial value, $\zeta(3) = 1.2020569031595942853997381615114\ldots$, and $k = 0.86173303 \times 10^{-4}\,{\rm eV}$.

Solve for x by iteration and then determine T. Remember a iteration formula tends to converge/diverge when its slope is low/high relative to 1. You could write a small computer program to do the solution. **HINT:** In a test *mise en scène*, just do the zeroth order solution: i.e., no iteration.

Chapt. 8 The Classical Mechanics of Special Relativity

Multiple-Choice Problems

007 qmult 00210 1 1 3 easy memory: marginalization

71. The Bayesian analysis iteration formula for iteration ℓ is

$$P(T_i|K_\ell) = \frac{P(D_\ell|T_iK_{\ell-1})P(T_i|K_{\ell-1})}{\sum_j P(D_\ell|T_jK_{\ell-1})P(T_j|K_{\ell-1})} \ ,$$

where $\{T_i\}$ is an exhaustive set of possible theories about some aspect of reality, K_ℓ is background knowledge after iteration ℓ, D_ℓ is data acquired in iteration ℓ, $P(T_i|K_\ell)$ is the posterior probabiltity of theory T_i to your knowledge for iteration ℓ, $P(D_\ell|T_iK_{\ell-1})$ is the probability of D_ℓ given theory T_i and background knowledge $K_{\ell-1}$, and $P(T_i|K_{\ell-1})$ is the prior probabiltity of theory T_i to your knowledge for iteration ℓ. That the iteration formula exists in principle is vital since it proves that the ideal Bayesian analysis leads to true theories. That the ideal Bayesian analysis can be approached in practice is also vital since that means it is a useful path to true theories. In toy cases, one can actually do ideal Bayesian analysis. But in toy cases, you know the true theory is included in the set of the set of possible theories which is exhaustive by definition.

However, in practice, you usually only do iteraion 1 formally. Initial background knowledge K_0 implicitly contains vague Bayesian analysis iterations going back to vaguely negative infinity. Also, you usually do not have and are not interested in having an exhaustive set of theories $\{T_i\}$. You usually just have interest in a set of interesting theories $\{T_i\}$: i.e., a set of theories that seem likely a priori. You usually just assign the theories equal priors following the principle of indifference, unless you has some other guidance. Evaluating the denominator of the iteration formula is useless in this practical Bayesian analysis, and so is seldom done explicitly. What you do do is evaluate the Bayesian odds ratio for any two of theories to compare them. The Bayesian odds ratio for theories T_i and T_j is

$$\frac{P(T_i|K_\ell)}{P(T_j|K_\ell)} = \frac{P(D_\ell|T_iK_{\ell-1})}{P(D_\ell|T_jK_{\ell-1})}\frac{P(T_i|K_{\ell-1})}{P(T_j|K_{\ell-1})} = k_{\rm B}\frac{P(T_i|K_{\ell-1})}{P(T_j|K_{\ell-1})} \ ,$$

where

$$k_{\rm B} = \frac{P(D_\ell|T_iK_{\ell-1})}{P(D_\ell|T_jK_{\ell-1})}$$

is the Bayesian k factor or Bayesian evidence. If you have made used the principle of indifference, all you have is the Bayesian evidence to compare the theories by. But most theories have free parameters. How are they accounted for? You expand $P(D_\ell|T_iK_{\ell-1})$ in the terms of the free parameters: i.e.,

$$P(D_\ell|T_iK_{\ell-1}) = \int P(D_\ell|T_i(\theta)K_{\ell-1})\rho(\theta)\,d\theta \ ,$$

where θ stands symbollically for all free parameters, $\rho(\theta)$ is the probability density for all free parameters, the integration is over all free parameter space, and $P(D_\ell|T_i(\theta)K_{\ell-1})$ is, in fact, the likelihood or likelihood function. The hard part of Bayesian analysis is usually choosing $\rho(\theta)$ which is really the hard prior to evaluate. Usually, you just assign a flat prior $\rho(\theta)$: i.e.,

$$\rho(\theta) = \begin{cases} \dfrac{1}{\Delta\theta_{\rm range}} & \text{for } \theta \text{ in the range } \Delta\theta_{\rm range}; \\ 0 & \text{for } \theta \text{ not in the range } \Delta\theta_{\rm range}. \end{cases}$$

The hard part is thus reduced to determining $\Delta\theta_{\rm range}$. Independent Bayesian analyses can find very different Bayesian evidences depending on how the researchers choose $\Delta\theta_{\rm range}$. This is why Bayesian evidence is usually not considered decisive if $k_{\rm B}$ is of order a few or even of order 10. If $k_{\rm B}$ is order 100 or 1000, then that may be decisive depending on who is judging.

Note maximizing the likelihood gives you the best set of free parameters assuming a theory is true. Using a theory with maximum likelihood parameters biases in favor of the theory in Bayesian analysis since the theory is not assumed to be true or usually even more likely than other interesting theories. In fact, eliminating the free parameters by the integration above implements Occam's razor: *"Numquam ponenda est pluralitas sine necessitate"* ("Plurality must never be posited without necessity"). You eliminate unnecessary and misleading hypotheses about the free parameters. This elimination process is called:

a) Occamization. b) dithering. c) marginalization. d) buffering.
e) obscuration.

Full Answer Problems

---

007 qfull 00100 1 3 0 easy math: dice problem

72. You are in Las Vegas, right? So you know dice (singular die). Let's see if we can predict the odds for a throw of two dice.

    There are parts a,b.

    a) Let's start being general, but not too general. You have two identical dice. They each have I faces with dot count running $i = 1, 2, \ldots, I$. The probability of any face (i.e., any face landing facing up) is P_i. What is the probability for a dice throw yielding faces i and i? What is the probability for a throw yielding first face i and then face j where $i \neq j$. What is the probability for a dice throw yielding faces i and j where $i \neq j$ and you do not distinguish the order or, in other words, you sum over the probabilities for the different orders.

    b) Let the sum of the face dots yielded by a throw be $k = i + j$. What is the run of possible k values (i.e., the ordered sequence of possible k values) and how many values are there? Is there always a middle value? Why? What is the middle value and how many values are above and below it?

    c) Now what we really want to know is what is the probability P_k of the summation of face dots being $k = i + j$: i.e., the probability distribution for a throw of two dice which is our random variable. Determine the two summation formulae needed and the number of terms in each summation. **HINT** The two formulae can be adjusted to look the same, except for their limits. The real hard part is determining limits. Draw an outcome square for the throw results with row index i and column index j. The squares to include in the summation are on the diagonals with $i + j = k$ with k constant.

    d) Specialize the P_k formulae to the case of equal face probability: i.e., all $P_i = 1/I$. Conflate the two formulae into one with transformation $k = k' + (I + 1)$, and show that $P_{k'}$ is an even function of k', find the limits on k', and find the maximum P'_k value.

e) Specialize the P_k and $P_{k'}$ formulae to the case of ordinary dice with $I = 6$ and $P_i = 1/6$. Tabulate the probability distributions P_k and $P_{k'}$ for the random variables k and k'.

---

007 qfull 00120 1 3 0 easy math: multinomial probability distribution

73. The multinomial theorem (from which the multinomial probability distribution is derived) is generated by the generating function (using that expression loosely)

$$F_N = F_1^N = \left(\sum_{i=1}^{I} P_i\right)^N = \sum_{i,j,\ldots} P_i P_j \ldots \ ,$$

where N is the number of factors in a sequence of factors, F_1 is the multinomial theorem for sequences of length 1, I is the number of variables and the order of the multinomial theorem (e.g., $I = 2$ for the binomial theorem), P_i is factor i (which for the multinomial probability distribution becomes probability of event i), the sequence of factors $P_i P_j \ldots$ are the terms in the multinomial expansion resulting from a straightforward branching multiplication before collecting terms into multinomial terms, $\sum_{i,j,\ldots}$ is the sum over the sequences (i.e., uncollected terms), and the total number of sequences is I^N. Note that all possible sequences of factors P_i must occur uniquely in the $\sum_{i,j,\ldots}$ since the branching pattern of all possibilities is exhaustive and there can be no duplications since obviously the first factor in each sequence is different.

There are parts a,b.

a) There are, as aforesaid, I^N sequences of factors. But what is the count of sequences for each combination: i.e., for each set of sequences have the same sets of factors P_i without distinguishing order. Such a count of sequences is called a multinomial coefficient. Note that sequences differing by undistinguishable factors are the same sequence in the branching multiplication that creates the whole set of sequences.

Let the multinomial coefficient for each combination be $C(N, \{n_i\})$, where $\{n_i\}$ stands for the set of factors P_i in the sequences. To be explicit, every distinct combination has a unique set $\{n_i\}$ otherwise it would not be a distinct combination. Note $\sum_{i=1}^{I} = N$, of course. Derive the formula for $C(N, \{n_i\})$ in terms of N and $\{n_i\}$. **HINT:** You will need factorials. Also, note the odd fact that you have consider permutations of the same factor P_i in a sequence even though these permutations just give the same sequence as it would occur in actually creating the sequences by the branching multiplication.

b) The individual distinct sequences are usually not of interest. What one usually wants is collect all the sequences corresponding to each unique combination since they all have the same numerical value, and so in the probability distribution all have the same probability. The collections are the multinomial terms for the multinomial theorem. Using the result of part (a), derive the formula for a multinomial term

$$\tilde{P}(N, \{n_i\}) \ ,$$

and the formula for multinomial theorem itself in terms of multinomial terms. Just use $\sum_{\{n_i\}}$ for the summation of the multinomial terms since there is no simple way in general to explicitly order them in a summation.

c) If the factors P_i are identified as probabilities of events i, then we require

$$\sum_{i=1}^{I} P_i = 1 \ .$$

What is value of F_N in this case and what does this value mean? What is the probability of obtaining the combination of events $\{n_i\}$?

d) The multinomial term $\tilde{P}(N, \{n_i\})$ is the multinomial probability distribution itself. We can easily obtain some ancillary formulae about the multinomial probability distribution. For example, the mean number of events j for the multinomial probability distribution is

$$\mu_j = \langle n_j \rangle = \sum_{\{n_i\}} n_j \tilde{P}(N, \{n_i, P_i\}) \ ,$$

where j is just a representative index. Derive the explicit formula for μ_j for the multinomial probability distribution. **HINT:** The trick is treat the P_i as variables in the multinomial theorem in both the forms

$$F_N = \sum_{\{n_i\}} \tilde{P}(N, \{n_i, P_i\})$$

and

$$F_N = F_1^N = \left(\sum_{i=1}^{I} P_i \right)^N \ .$$

You then apply operator $P_j(\partial/\partial P_j)$ to both of forms and afterward impose the constraint that the constraint $F_1 = \sum_i P_i = 1$.

e) The variance/covariance of a multinomial probability distribution is given by

$$\sigma_{jk}^2 = \langle (n_j - \mu_j)(n_k - \mu_k) \rangle = \langle n_j n_k \rangle - \mu_j \mu_k \ .$$

Derive the explicit formula for σ_{jk}^2 for the multinomial probability distribution. Explain the striking feature of covariance case (i.e., the case when $j \neq k$). **HINT:** The trick is used in part (d) still works *mutatis mutandis*.

e) Specialize the results of parts (a), (b), and (c) of the binomial theorem: i.e., the case where $I = 2$. For best understanding, let $n_1 = k$ and $n_2 = N - k$, where $k \in [0, N]$ is a usual parameter for specify all the sets $\{n_i\}$.

---

007 qfull 00150 1 3 0 easy math: Poisson distribution

74. The Poisson (probability) distribution is

$$P = \frac{\mu^x}{x!} e^{-\mu} \ ,$$

where μ is the mean of integer random variable x, $\sigma = \sqrt{\mu}$, and there is no upper limit on x.

The Poisson distribution is appropriate for analyzing two kinds counting observations which not completely distinct. The first kind of counting observation is where the events occur randomly in time (or some similar variable), but there is a mean number of events per unit time μ and the time of each event is zero or approximately that. In this case, the Poisson distribution is exact if the time of an event is zero. An obvious example of this kind of observation is counting the radioactive decays from a long-lived radioactive sample.

The second kind of counting observation is where the random variable x (the count of events) obeys an extreme binomial distribution where p the probability of x on an individual trial is very small (i.e., $p << 1$) and consequently $\mu << n$, where n is the number of trials. If you actually do know n and p, you could just use the binomial distribution itself, but the Poisson distribution may be an adequate approximation. Note for the binomial distribution $\mu = np$ and $\sigma = \sqrt{np(1-p)} \approx \sqrt{np} = sqrt\mu$.

In both cases, you may often just have one count x, and not know μ nor σ. However, you can estimate $\mu \approx x$, and thus $\sigma \approx \sqrt{x}$ and this is often done.

There are parts a,b.

NOTE: This question has **MULTIPLE PAGES** on an exam.

a) Derive a cute formal general formula for the moments

$$\langle x^\ell \rangle = e^{-\mu} \sum_{x=0}^{\infty} x^\ell \frac{\mu^x}{x!}$$

of the Poisson distribution, where ℓ runs $0, 1, 2, \ldots$. Use the formula to solve moments for $\ell = 0, 1, 2$ and for the formulae for μ and σ. **HINT:** Operating with the operator $[\mu(\partial/\partial\mu)]^\ell$ is the trick.

b) The derivation of the Poisson distribution for the first kind of counting observation mention in the preamble is straightforward. Say τ is the average rate of random events. The probability of observing no events in time t (starting from time $t = 0$) obeys the differential equation

$$dP(x = 0, t) = -P(x = 0, t)\frac{dt}{\tau} ,$$

where $P(x = 0, t)$ is the probability of having no events to t and dt/τ is the differential probability of an event in dt. The solution for $P(x = 0, t)$ is clearly

$$P(x = 0, t) = e^{-t/\tau} .$$

The differential formula for x events in t is

$$dP(x, t) = e^{-t/\tau} \prod_{i=1}^{x} \frac{dt_i}{\tau} ,$$

where we assume the events are instantaneous. Simple integration of all dt_i gives the Poisson distribution plus accounting for overcounting with events pass each other on the time line. Complete the proof of the Poisson distribution. Give the explicit μ and σ formulae for this case.

c) Prove the Poisson distribution by taking the limit of the binomial distribution

$$P(x, n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

where $n \to \infty$, $p \to 0$, $np \to \mu$ (which is a finite nonzero value), and x is fixed. **HINT:** You will need to expand $(1-p)^n = (1-\mu/n)^n$ in a binomial theorem expression.

---

007 qfull 00200 1 3 0 easy math: Bayes' Theorem and Bayesian analysis

75. Bayes' theorem in symmetric form is

$$P(AB) = P(A|B)P(B) = P(B|A)P(A) ,$$

where P is probability, A and B are events, $P(A)$ is the probability of A, $P(B)$ is the probability of B, $P(AB)$ is the probability of A and B, $P(A|B)$ is the conditional probability of A given B, and $P(B|A)$ is the conditional probability of B given A. In unsymmetric form,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \qquad \text{or equivalently} \qquad P(B|A) = \frac{P(A|B)P(B)}{P(A)} .$$

Note that in the notation we are using, AB is not the product of A and B, but the union of A and B: i.e., AB is A and B.

There are parts a,b.

a) Prove the expansion rule

$$P(AB) = P(A|B)P(B)$$

and Bayes' theorem from frequentist definition of probability. Frequentist definition states given population of events N, the probability of sampling events with property A is $P(A) = N_A/N$ where N_A is the number of events in the population with property A.

Yours truly believes that probability only has meaning from the frequentist definition. You can do a lot of probability formalim without the definition, but it seems to have no meaning without the definition. Maybe yours truly is just ignorant. However, the limitation to the frequentist definition isn't really a limitation in yours truly view since frequentist definition always applies even if you can't can't calculate the probabilities with high accuracy from it. Thus, Bayesian analysis can be applied generally.

b) Yours truly is not going to give a general description of Bayesian analysis procedure here, but just a description of an ideal procedure that concerns itself with the theories in order to find the true one. Say we have a system, the exhaustive finite set of all theories of nonzero probability $\{T_i\}$ that apply to the system $\{T_i\}$ and inital knowledge K_0 about the system (which includes the set of theories, of course). Given that the set of theories is exhaustive, their probabilities to our knowledge (i.e., K_0) is normalizable: i.e., we have $\sum_i P(T_i|K_0) = 1$.

Now how is it possible to assign a probability to a theory T_i? Well if we know the theory is true, $P(T_i|K_0) = 1$ and if we know it is false, $P(T_i|K_0) = 0$. What if you don't know whether T_i is true or false? Well there are procedures of assigning numerical probabilities to theories based background knowledge. After all people are always assessing theories as probable, very probable, improbable, or very improbable based on their background knowledge. This assessment must be based on some fuzzy frequentist analysis of the features that make up a theory. Now the procedure of assigning (numerical) probabilities doesn't have to be perfect—and probably rarely is in practice—but the better it is, the faster in all probability the Bayesian analysis will converge to the true theory. One procedure is the principle of indifference: just assign equal probabilies to the theories. By the principle of indifference, if there are I theories, $P(T_i|K_0) = 1/I$ for all i.

In fact, the completely fuzzy assignments of probability only happens prior to the first iteration of Bayesian analysis when our background knowledge is K_0. The zeroth probabilities $P(T_i|K_0)$ are our zeroth prior probabilities (AKA zeroth priors). After completing Bayesian analysis iteration $(\ell - 1)$ we have posterior probabilities (AKA posteriors) $P(T_i|K_{\ell-1})$ relative to the $(\ell-1)$th iteration; they are the priors for the iteration ℓ.

In iteration ℓ, we acquire new data D_ℓ which gives us updated knowledge $K_\ell = D_\ell K_{\ell-1}$, where $D_\ell K_{\ell-1}$ recall is a union, not a product. To get the posteriors for the ℓth iteration, we apply Bayes' theorem:

$$\begin{aligned} P(T_i|K_\ell) = P(T_i|D_\ell K_{\ell-1}) &= \frac{P(D_\ell|T_i K_{\ell-1})P(T_i K_{\ell-1})}{P(D_\ell K_{\ell-1})} \\ &= \frac{P(D_\ell|T_i K_{\ell-1})P(T_i|K_{\ell-1})P(K_{\ell-1})}{P(D_\ell|K_{\ell-1})P(K_{\ell-1})} = \frac{P(D_\ell|T_i K_{\ell-1})P(T_i|K_{\ell-1})}{P(D_\ell|K_{\ell-1})} \ . \end{aligned}$$

Note that $P(K_{\ell-1})$ has canceled out, and so the result is valid no matter what the value of $P(K_{\ell-1})$ though if we are doing the Bayesian analysis correctly it should be 1.

Now if we actually have data D_ℓ, then $P(D_\ell) = 1$. But $P(D_\ell)$ is not what is in the denominator of the result. We have $P(D_\ell|K_{\ell-1})$ which the probability of getting data D_ℓ given that we know $K_{\ell-1}$ which recall includes the knowledge that the set $\{T_i\}$ exists. We can, in fact, expand $P(D_\ell|K_{\ell-1})$ in the set $\{T_i\}$:

$$P(D_\ell|K_{\ell-1}) = \sum_j P(D_\ell|T_j K_{\ell-1})P(T_j|K_{\ell-1}) \ ,$$

where the summation is over all the set $\{T_i\}$. Now we have the Bayesian analysis iteration formula

$$P(T_i|K_\ell) = \frac{P(D_\ell|T_iK_{\ell-1})P(T_i|K_{\ell-1})}{\sum_j P(D_\ell|T_jK_{\ell-1})P(T_j|K_{\ell-1})} \ .$$

We note that $P(D_\ell|K_{\ell-1})$ is the weighted mean of the $P(D_\ell|T_jK_{\ell-1})$'s where the $P(T_j|K_{\ell-1})$'s are the weights.

The last equation is in fact the probability update formula. Those theories T_i whose ℓth posteriors are greater/lesser/equal relative to their $(\ell-1)$th priors gain/lose/conserve credence.

We now assume that there is enough potential knowledge K_L for a decisive determination: i.e.,

$$P(T_i|K_L) = \begin{cases} 1 & \text{if } T_i \text{ is true;} \\ 0 & \text{if } T_i \text{ is false.} \end{cases}$$

This means that the Bayesian analysis converges to truth as $\ell \to L$. Note that convergence happens no matter how imperfect our method of assigning probabilities is provided we keep iterating until we reach K_L where only one viable theory remains. However, the amount of K_L actually varies depending on which data sets D_ℓ we acquire and how accurate are our probability assignments for $P(T_i|K_0)$ and $P(D_\ell|T_iK_{\ell-1})$. Obviously, if we make really good choices for the data sets D_ℓ and for probability assignments, convergence should be fast. If we make really poor choices, we may be iterate to a very large K_L and all the probabilities calculated in the iteration may be wildly in accurate except that we can calculate the $P(T_i|K_L)$'s accurately and end the iteration. In this extreme case, the Bayesian analysis wasn't very useful, except as a tactic to keep going. We just accumulated data until we had exhausted the possibilities and arrived at truth.

There's a relevant aphorism attributed to Ernest Rutherford (1871–1937): "If you need statistics, you are doing the wrong experiment." In fact, all aphorisms are true and false (including this one). Howsoever, the point of Rutherford's aphorism is that you choose data acquisitions as decivively as possible to speed the Bayesian analysis iteration (in a formal or informal sense) to completion.

The Bayesian analysis procedure described above is an ideal one which is probably very seldom fully carried out. Much less ideal procedures are usually used—and for darn good reasons. But it is important that the ideal procedure exists: a procedure which guarantees the arrival at truth. We could not trust Bayesian analysis if there were no ideal procedure to approach. If there were no ideal procedure to approach, Bayesian analysis might fail in some cases no matter how well we did it.

Does the foregoing seem OK to you? If not, why not?

c) From the Bayesian analysis iteration formula given in part (b) prove that the $P(T_i|K_\ell)$'s are normalized even if the the $P(T_i|K_{\ell-1})$'s are not. Why does this normalization inevitably happen?

d) What does it mean if all $P(T_i|K_\ell)$ are zero in Bayesian iteration step?

e) What does it mean if $P(T_i|K_\ell) = 1$, but your set of theories $\{T_i\}$ was not actually exhaustive.

f) What does it mean if $P(T_i|K_\ell) = 1$ and your set of theories $\{T_i\}$ was actually exhaustive.

Chapt. 9 The Density of the Universe and Dark Matter

Multiple-Choice Problems

Full-Answer Problems

009 qfull 00410 1 3 0 easy math: Virial theorem derivation

76. The virial theorem was first derived by Rudolf Clausius (1822–1888) in 1870 (before going off to serve in the Franco-Prussian War) in the context of thermodynamics. Various versions of the virial theorem have been derived since including one for quantum mechanics version was later dervived. Here we are only interested in the classical mechanics version for a system of interacting particles that have arrived at a static (or if you prefer steady-state) distribution in sense of having a constant time-averaged spherical rotational inertia $\langle I \rangle$ which will usually imply a constant time-average position and momentum distribution. The system, of course, conserves total mechanical energy. Such a system is called a virialized system. The general form of the virial theorem (for classical mechanics) is

$$\langle K \rangle = -\frac{1}{2}\sum_i \langle \vec{r}_i \cdot \vec{F}_i \rangle = -\frac{1}{2}\Upsilon \ ,$$

where averaging is over time, K is the total kinetic energy

$$K = \sum_i \frac{1}{2} m_i v_i^2 \ ,$$

the sum is over all particles i, $\vec{r}_i$ is the particle position vector relative to a general origin, $\vec{v}_i = \dot{\vec{r}}_i$ is particle velocity, m_i is particle mass, $\vec{F}_i$ is the net force on a particle, and

$$\Upsilon = \sum_i \langle \vec{r}_i \cdot \vec{F}_i \rangle$$

is the standard definition of the virial or virial of Clausius (see Wikipedia: Virial theorem; Clayton 1983, p. 134). The word "virial" does not seem to have any use other than in the context of the virial theorem itself.

If the only internal forces act and, except for ideal collision forces (i.e., ones that obey Newton's 3rd law and act a point), they are derivable from a single inverse power-law potential energy with power p that just depends on inter-particle separation, the virial theorem specializes to the inverse-power-law-potential-energy form

$$\langle K \rangle = -\frac{p}{2}\langle V \rangle \ ,$$

where V is the total potential energy. One can drop the angle brackets for the virial theorem if one knows what one means. If the one has the super-important case of the inverse-square law, $p = 1$ and the virial theorem specialized to

$$\langle K \rangle = -\frac{1}{2}\langle V \rangle \ .$$

There are parts a,b,c,d,e,f,g. The parts can all be done independently. So don't stop if you can't do a part.

a) Why is the virial theorem of interest? This is a discussion question, and so there is no single right answer, just useful or interesting ones.

b) Clairvoyance tells us that the derivation of the virial theorem starts from the spherical rotational inertia formula

$$I = \sum_i m_i r_i^2 ,$$

where the position vectors are relative to the system origin and **NOT** from and perpendicular to an axis as for the ordinary scalar rotational inertia (see Clayton 1983, p. 134). If the system has relaxed to a time-average static state, then based on general trends for relaxed statistical systems (which are formally reached at time infinity), we expect

$$\left\langle \dot{I} \right\rangle = 0 \qquad \text{and} \qquad \left\langle \ddot{I} \right\rangle = 0 .$$

As a first step in the derivation, take the 2nd derivative of $I/2$ and substitute using $\vec{p_i} = m_i v_i$, Newton's 2nd law, and the kinetic energy formula. Then take the time average assuming relaxation to get the general virial theorem.

c) Assume the system has only internal forces, where F_{ji} is the force of paricle j on particle i. Use some relabeling trickery and Newton's 3rd law to obtain

$$\sum_i \vec{r}_i \cdot \vec{F}_i = \frac{1}{2} \sum_{ij} \vec{r}_{ji} \cdot \vec{F}_{ji} ,$$

where $\vec{r}_{ji} = \vec{r}_i - \vec{r}_j$. Determine the contribution of ideal collisions to the sum in the equation above. Note ideal collisions obey Newton's 3rd law and happen at a single point.

d) Assume all the internal forces can be derived from the same inverse-power-law potential energy dependent only on inter-particle separation: i.e.,

$$\vec{F}_{ji} = -\frac{\partial V_{ji}}{\partial r_{ji}} \hat{r}_{ji} \qquad \text{with} \qquad V_{ji} \propto \frac{1}{r_{ji}^p} .$$

Prove the inverse-power-law-potential-energy virial theorem. What changes to the theorem if ideal collisions are included in the system? Prove that the theorem holds if $p = 0$.

e) Derive the inverse-power-law-potential-energy virial theorem for the special case of gravity (i.e., the gravity virial theorem).

f) Prove that the gravity virial theorem holds for 2-body system where the interaction is only gravitation, one body has infinite mass M and the other finite mass m, and the orbit is circular with radius r.

g) Generalize the inverse-power-law-potential-energy virial theorem to the case of multiple inverse-power-law potential energies V_p. Can kinetic energy be zero in this case for nonzero potential energies? **HINT:** Proof by inspection is OK.

---

009 qfull 00420 1 3 0 easy math: spherical symmetrical system potential energy

77. The general differential equation for gravitational potential energy and the special case for a spherically symmetric system are, respectively,

$$dU = V\,dm \qquad \text{and} \qquad dU = -\frac{Gm(r)}{r}\,dm ,$$

where V is gravitational potential and $m(r)$ is the enclosed mass to radius r. In this problem, we investigate the potential energy of spherical symmetric systems with power-law density profiles.

There are parts a,b.

a) Given

$$\rho = \rho_R \left(\frac{r}{R}\right)^p , \qquad \text{derive} \qquad m(r) = \frac{4\pi R^3 \rho_R}{p+3} \left(\frac{r}{R}\right)^{p+3} ,$$

where R is the maximum radius and $m(R) = M$ the total mass. Write out the total mass formula $M = m(R)$ explicitly. When will it equal the constant-density total mass formula? Under what condition will $m(r)$ diverge?

b) Now derive the formula for the total potential energy

$$U = -\left(\frac{p+3}{2p+5}\right)\frac{GM^2}{R} .$$

Under what condition will U diverge? **HINT:** Write dm in terms of dr.

c) How does $f(p) = (p+3)/(2p+5)$ behave as a function of p?

d) Determine the special cases of $f(p)$ for p with values $-5/2$, $-9/4$, -2, -1, 0, 1, 2, and ∞.

---

009 qfull 00430 1 3 0 easy math: astrophysical virial theorem

78. The (most common) astrophysical virial theorem formula is

$$\sigma^2 = \frac{GM}{R} ,$$

where σ is the dispersion (i.e., standard deviation) of line-of-sight velocities of a gravitationally-bound, virialized (i.e., evolved to time-averaged steady state) astronomical system (most commonly a galaxy cluster) relative to its center of mass within the virial radius, R is the virial radius defined to be where $\sigma(R)$ is a maximum, and M is the virial mass which is just a characteristic mass for the astronomical system. Virial mass is a simple characteristic mass for comparison between astronomical systems. If one wants a better determination of an astronomical system's mass, one must do a more elaborate calculation. The astrophyical virial theorem can be written in the fiducial-value form for galaxy clusters

$$M = \frac{R\sigma^2}{G} = 2.3251 \times 10^{14} M_\odot \times \left(\frac{R}{1\,\text{Mpc}}\right)\left(\frac{\sigma}{1000\,km/s}\right)^2 ,$$

where the fiducial values have been chosen to be typical for galaxy clusters: diameters of order 2 to 10 Mpc and velocituy dispersions of order 1000 km/s.

Given the gravitational virial theorem and the power-law gravitational potential energy formula (with power p),

$$T = -\frac{1}{2}U \qquad \text{and} \qquad U = -\left(\frac{p+3}{2p+5}\right)\frac{GM^2}{R} ,$$

derive the astrophysical virial theorem formula making reasonable assumptions as needed.

Chapt. 10 The Cosmic Microwave Background

Multiple-Choice Problems

Full-Answer Problems

Chapt. 11 Classical Chaos

Multiple-Choice Problems

Full Answer Problems

Chapt. 12 Canonical Perturbation Theory

Multiple-Choice Problems

Full-Answer Problems

Chapt. 13 Lagrangian and Hamiltonian Formulations for Continuous Systems and

Multiple-Choice Problems

Full-Answer Problems

Chapt. 14 Cosmic Present Galaxies as a Benchmark for Evolutionary Studies

Multiple-Choice Problems

023 qmult 00110 1 4 5 easy deducto-memory: specific intensity and surface brightness

79. "Let's play *Jeopardy*! For \$100, the answer is: It and surface brightness are the same physical quantities though in some conventions surface brightness has an extra factor of 4π. The name used just depends on context."

    What is ____________, Alex?

    a) radiant flux b) absolute magnitude c) apparent magnitude
    d) mean intensity e) specific intensity

023 qmult 00330 1 1 4 easy memory: elliptical radius and circularized radius

80. Many galaxies seen on projection on the sky have approximately elliptical isophotes characterized by semi-major axis a and semi-minor axis b which are approximated as having the same ratio for all isophotes with ellipticity defined $\epsilon = 1 - b/a$ which is not eccentricity $e = \sqrt{1-(b/a)^2}$. One definition of projected radius used for labeling isophotes and in calculating surface brightness behavior is just elliptical radius $R = a$ and the other is the:

    a) circularized radius $R = (ab)$. b) ellipticized radius $R = (ab)$.
    c) ellipticized radius $R = (ab)^{1/2}$. d) circularized radius $R = (ab)^{1/2}$.
    e) ellipticized radius $R = (ab)^{1/4}$.

023 qmult 00420 1 4 2 easy deducto-memory: the Sérsic profile specified

81. "Let's play *Jeopardy*! For \$100, the answer is:

$$I_\lambda = I_{\lambda,0} \exp\left(bx^{1/n}\right) = I_{\lambda,\mathrm{e}} \exp\left[b(x^{1/n} - 1)\right] ,$$

where I_λ is the surface brightess as a function x, $x = R/R_\mathrm{e}$ is the radius (elliptical or circularized radius) in units of the effective radius R_e, $I_{\lambda,0} = I_\lambda(x = R/R_\mathrm{e} = 0)$, $I_{\lambda,\mathrm{e}} = I_\lambda(x = R/R_\mathrm{e} = 1)$, n is an index parameter typically in the range $[1, 2.5]$ for star forming galaxies (SFGs) and in the range $[2.5, 10]$ for early type galaxies (ETGs), and b is a function of n (i.e., $b = b(n)$).

    What is the ____________, Alex?

    a) de Vaucouleurs profile b) Sérsic profile
    c) Navarro-Frenk-White profile (NFW profile) d) Burkert profile
    e) Brownstein profile

Full-Answer Problems

Chapt. 15 Cosmic Present Star-Forming Galaxies (SFGs)

Multiple-Choice Problems

024 qmult 00180 1 1 3 easy memory: Milk Way stellar mass and virial mass

82. In solar mass units, the Milky Way stellar mass (i.e., the mass in stars M_*) is $\sim$ ____________ $M_\odot$ and its virial mass ($M_{\rm vir}$: i.e., fiducial total mass which is mostly dark matter) is $\sim$ ____________ $M_\odot$. At least these values were standard circa 2023. However, a downward revision may have become accepted just about that year.

    a) 10^{12}; 10^{10} b) 5×10^{10}; 5×10^{10} c) 5×10^{10}; 10^{12} d) 10^9; 5×10^{10}
    e) 10^9; 5×10^8

024 qmult 00230 1 4 2 easy deducto-memory: exponential profile for face-on spiral galaxies

83. "Let's play *Jeopardy*! For \$100, the answer is:

$$I_\lambda = I_{\lambda,0} e^{-(R/R_{\rm d})} ,$$

where I_λ is the surface brightness, $I_{\lambda,0}$ is the central surface brightness, R is the radius coordinate, and $R_{\rm d}$ disk scale length (and not the effective or half-light radius)."

What is the standard ____________ surface brightness profile, Alex?

a) edge-on spiral disc b) face-on spiral disc c) elliptical d) dwarf irregular
e) general Sérsic

024 qmult 00400 1 1 3 easy memory: an inclined circle is an ellipse

84. An inclined circle (i.e., not seen face-on; inclination angle $> 0°$) is in projection a/an:

    a) square b) oval c) ellipse d) line segment e) invisible

024 qmult 00600 1 1 4 easy memory: two main classes of galaxy bulges

85. There are two main classes of galaxy bulges:

    a) classical bulges and non-classical bulges b) big bulges and disc-like bulges
    c) little bulges and disc-like bulges d) classical bulges and disc-like bulges
    e) little bulges and big bulges

024 qmult 00620 1 4 2 easy deducto-memory: Schmidt-Kennicutt law

86. "Let's play *Jeopardy*! For \$100, the answer is:

$$\Sigma_{\rm SFR} = B \left(\frac{\Sigma_{\rm gas}}{1\, M_\odot/{\rm pc}^2} \right)^\alpha M_\odot/{\rm yr/kpc}^2 ,$$

where SFR means star formation rate, $\Sigma_{\rm SFR}$ is surface star formation rate in units of $M_\odot/{\rm yr/kpc}^2$, $\Sigma_{\rm gas}$ is gas surface density in units $M_\odot/{\rm pc}^2$ (the denominator below $\Sigma_{\rm gas}$ makes the overall factor dimensionless), $B \approx 10^{-4}$ is an empirical constant, and $\alpha = 1.40(15)$ is another empirical constant with some theoretical understanding.

What is the ____________, Alex?

a) Press-Kennicutt law b) Schmidt-Kennicutt law c) Press-Schechter law
d) Martin-Schmidt law e) Martin-Schmidt-Kennicutt law

Full-Answer Problems

024 qfull 00100 1 3 0 easy math: inclined circle analyzed: On exams, do all parts with minimal words.

87. An inclined circle (ideal disc galaxy is) is seen in projection as an ellipse.

NOTE: There are parts a,b. On exams, do all parts with minimal words.

a) The equation for a circle is written elaborately is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1 ,$$

where a is the radius. Find the explicit formula for y. The circle is rotated on its x-axis to inclination angle i where inclination angle is measured from the direction to the observer to a normal to the circle. What is the projected height of every y point (i.e., what is the inclined y_i)? Prove the inclined circle (i.e., projected circle) is an ellipse and find its semi-minor axis b.

b) The area of an ellipse is $A = \pi ab$ and the circularized radius an ellipse created by inclination is defined by

$$R_i = \sqrt{ab} = a\sqrt{\cos(i)} .$$

Prove that the differential area of an inclined circle is

$$dA = 2\pi R_i \, dR_i .$$

024 qfull 01030 1 3 0 easy math: galaxy potential energy and escape velocity: On exams, do **ONLY** parts a,b,c,d.

88. In this question, we consider escape velocities from galaxies. The path is long if one does not gloss over tricky points like Ci-86–87.

NOTE: There are parts a,b,c,d,e,f. On exams, do **ONLY** parts a,b,c,d and answer with minimal words.

a) From introductory physics, the change mechanical energy of particle is

$$\Delta E = \Delta KE + \Delta PE = W_{\rm noncon} ,$$

where KE is kinetic energy, PE is potential energy, and $W_{\rm noncon}$ work done by nonconservative forces. If there are no nonconservative forces, mechanical energy is conserved and

$$1) \quad \Delta E = 0 \qquad 2) \quad \Delta KE = -\Delta PE \qquad 3) \quad E = KE + PE \quad \text{is constant.}$$

The escape velocity from some point (with no nonconservative forces) can be found from some point noting that $KE = 0$ at infinity where the gravitational potential Φ (which is potential energy PE per unit mass) is defined to be zero. Find the general formula for escape velocity $v_{\rm esc}$ given that kinetic energy is initially KE and gravitational potential is initially Φ.

b) Assume a spherically symmetric mass distribution for a galaxy which seems to be often approximately true since dark matter halos are often quite spherically symmetric it seems though not always. Let the density profile be a power law

$$\rho = \rho_{\rm s}\left(\frac{r}{r_{\rm s}}\right)^{-\alpha} = \rho_{\rm s} x^{-\alpha} \ ,$$

where $\rho_{\rm s}$ is a scale density, $r_{\rm s}$ is a scale radius, $x = r/r_{\rm s}$ is a dimensionless radius, and α is the power. Determine the formula for interior mass $M(r)$ (i.e., mass interior to radius r) in terms of a scale $M_{\rm s}$ and x assuming $\alpha < 3$.

c) Why can't a galaxy have pure power law density profile from $r = 0$ to $r = \infty$, in fact? **HINT:** Consider the divergence behavior of the interior mass formula.

d) There is a tricky point in considering potential change. When integrating up the potential energy of a gravitating sphere, we use

$$PE(r) = \int_0^r \left[\frac{-GM(r)}{r}\right] 4\pi r^2 \rho \, dr \ ,$$

where $M(r)$ is the interior mass and $\Phi = -GM(r)/r$ is the gravitational potential r. This is the right thing to do, but $-GM(r)/r$ is not the potential at r in the fully assembled gravitating sphere. Why not? Show what the potential at r is (relative to infinity which is zero) for a gravitating sphere of total radius R. **HINT:** Getting the signs right for potential is tricky. You have to do the sign on every step right—or chance of being right is only 50 %.

e) Making use of the part (b) and the part (d) answers find the potential from $x \leq X$ for $\alpha < 3$. Show explicitly the cases for 1) $\alpha \neq 2$, 2) $\alpha \in (2, 3)$ and $x << X$, 3) $\alpha < 2$ and nd $x << X$, and 4) $\alpha = 2$.

f) From the part (e) answer from the escape velocity formula for the case of $\alpha \in (2, 3)$ and $x << X$ in terms of the circular velocity for scaled radius $x = 1$. What is the escape velocity if circular velocity is 200 km/s and $\alpha = 9/8$? Why are galactic outflows hard to understand if α gets very close to 2? Having α close to 2 is what is implied by the flat velocity curve ranges of observed disc galaxies.

---

024 qfull 01050 1 3 0 easy math: metallicity saturation in galaxies: On exams, do **ONLY** parts a,b,d,e.

89. The metallicity of galaxies does not generally increase with cosmic time, but reaches an (approximate) plateau due to gas inflow from the intergalactic/circumgalactic medium (which if intergalactic is of nearly primordial gas: primordial cosmic gas fiducial mass fractions $X = 0.75$ H, $Y = 0.25$ He, $Z = 0.001$ metallicity which is overwhelmingly deuterium counted as a metal: Wikipedia: Big Bang: Abundance of primordial elements) and the outflow of metal enriched gas from stellar evolution (i.e., stellar winds and supernovae) back to the intergalactic/circumgalactic medium or into compact astro-bodies (compact remnants, long-lived small mass stars, brown dwarts, planets, and smaller astro-bodies). The plateau phase will probably not last forever since cosmological constant acceleration isolates all bound systems not participating in the mean expansion of the universe from fresh primordial gas. So a slow metallicity increase should occur despite gas inflow/outflows as the overall isolated bound system gas gradually enriches. However, this enrichment seems very slow since cosmic time ~ 5 Gyr after the Big Bang (Weinberg 2016, arXiv:1604.07434) and will gradually turn off with the end of the stelliferous era (theoretically cosmic time ~ 0.15–10^5 Gyr: Wikipedia: Graphical timeline of the Stelliferous Era; Wikipedia: Future of an expanding universe: The Stelliferous Era). In this question, will do a simple modeling of the plateauing of galaxy metallicity.

NOTE: There are parts a,b,c,d,e,f. On exams, do **ONLY** parts a,b,d,e and answer using minimal words.

a) Write a (1st order ordinary autonomous) differential equation for galaxy gas density ρ (assumed to be uniform) in terms of a constant inflow rate of gas $F = (d\rho/dt)_{\rm inflow}$ (not necessarily primordial gas) and an outflow rate $-\kappa\rho = -\rho/\tau$, where κ is the rate constant and $\tau = 1/\kappa$ is the time constant. The outflow rate includes both outflow of gas back to the intergalactic/circumgalactic medium and into compact objects.

b) Using an integrating factor solve the differential equation of part (a) with ρ_0 as the initial density at time zero (i.e., $t = 0$). Give the 1st-order-in-small-t solution and the asymptotic solution as $t \to \infty$ (which is also the constant solution of the differential equation). What name can be given to the time constant τ?

c) Why do we get an asymptotic solution in part (b)?

d) Write a (1st order ordinary autonomous) differential equation for galaxy gas metal density $Z\rho$ (assumed to be uniform) in terms of a constant inflow rate of metal-only gas $Z_{\rm in}F = Z_{\rm in}(d\rho/dt)_{\rm inflow}$, where $Z_{\rm in} \in [0,1]$. Let the outflow rate be the same as in part (b): i.e., $-\kappa\rho = -\rho/\tau$, where κ is the rate constant and $\tau = 1/\kappa$ is the time constant. There is also a rate constant γ for the creation metal-only gas in the galaxy from zero-metallicity gas with density $(1 - Z)\rho$.

e) The differential equation in part (c) can be solved for Z for general time t using the solution of part (b), but it seems a bit tedious to get this solution. However, finding the asymptotic solution $Z_{\rm asy}$ as $t \to \infty$ is easy. Find it. Check that $Z_{\rm asy}$ is dimensionally correct and show that it satisfies $Z_{\rm asy} \in [0,1]$.

f) We can make a crude estimate of current cosmic $Z_{\rm asy}$. First, let

$$\kappa = \frac{(d\rho/dt)_{\rm outflow}}{\rho} = \frac{3\,{\rm M}_\odot/{\rm yr}}{\rho} \ ,$$

where $3\,{\rm M}_\odot/{\rm yr}$ is roughly the rate of star formation for a galaxy like the Milky Way (Ci-383) and we assume this is of order the overall gas loss rate due gas outflow back to the intergalactic/circumgalactic medium and locking up of gas in compact astro-bodies. Second, let

$$\gamma = \frac{[d(Z\rho)/dt]_{\rm metal\,creation}}{\rho} = \frac{[5\,{\rm SNe}/(100\,{\rm yr})] \times (1\,{\rm M}_\odot\ {\rm metals/per\,SNe})}{\rho} \ ,$$

where $5\,{\rm SNe}/(100\,{\rm yr})$ is roughly the rate of supernovae for a galaxy like the Milky Way (Wikipedia: Supernova: Milky Way candidates) and we assume that this is of order the metal creation given that each supernovae yields of order $1\,{\rm M}_\odot$ of metals. Let $Z_{\rm in} = 0.001$ the fiducial primordial cosmic metallicity. Calculate $Z_{\rm asy}$ with these values and discuss whether the result is reasonable or not.

Chapt. 16 Cosmic Present Early Type Galaxies (ETGs)

Multiple-Choice Problems

---

025 qmult 00130 1 1 3 easy memory: virial mass range for ellipticals

90. The range of virial mass (which is the fiducial total mass of galaxies determined in a tricky way) for elliptical galaxies (e.g., dwarf ellipticals (dEs), ellipticals (Es), and bright cluster ellipticals (BCEs)) is

    a) $\sim 10^5$–$10^6\,M_\odot$. b) $\sim 10^5$–$10^7\,M_\odot$. c) $\sim 10^8$–$10^{13}\,M_\odot$ or more.
    d) $\sim 10^5$–$10^{15}\,M_\odot$. e) $\sim 10^5$–$10^{20}\,M_\odot$.

---

025 qmult 00230 1 1 2 easy memory: galaxy ellipticity for ellipticals

91. More so than for disk galaxies, the shape and orientation of isophotes is dependent on projected radius R (which could be the circularized radius) and position angle ϕ (measured counterclockwise from north on the sky) and therefore there is an ellipticity profile $\epsilon(R,\phi)$ (Ci-126–127). However, since a fiducial or characteristic ellipticity is useful is the galaxy ellipticity (Ci–127) defined by:

    a) $\epsilon(R_{\rm d})$. b) $\epsilon(R_{\rm e})$. c) $\epsilon(R_{\rm f})$. d) $\epsilon(R_{\rm g})$. e) $\epsilon(R_{\rm h})$.

---

025 qmult 00330 1 1 5 easy memory: ETG Sersic indices

92. When a Sérsic profile is fitted to the **CENTRAL** surface brightness of ETGs, the Sérsic index range is ~ 2 to ~ 10. However, when a single Sérsic profile is fitted to a galaxy the dividing line between later type galaxy Sérsic indices and ETG Sérsic indices is taken to be:

    a) 1. b) 1.25. c) 1.3. d) 1.5. e) 2.5.

---

025 qmult 00430 1 1 1 easy memory: Hubble sequence E number

93. The Hubble sequence E number (E in range $[0,7]$) is nowadays determined by

$$E = 10 \times \epsilon = 10 \times \left(1 - \frac{b}{a}\right) ,$$

    where ϵ is the:

    a) ellipticity. b) eccentricity. c) effectiveness. d) e-folding. e) error.

---

025 qmult 00510 1 4 5 easy deducto-memory: LOSVD defined

94. "Let's play *Jeopardy*! For \$100, the answer is: It is the distribution of velocity measured by the Doppler shift of some line along a line of sight (LOS) through an ETG using integral field spectroscopy (whereby a spectrum is obtained at each spatial pixel in the field of view)."

    What is a LOS ____________, Alex?

    a) Doppler distribution b) velocity disperson c) Doppler dispersion
    d) integral dispersion e) velocity distribution

---

025 qmult 00630 1 1 3 easy memory: fast and slow rotators dividing line

95. The simple measure of ordered to random motions in ETGs is the ratio V/σ. The symbol V/σ seems also to be the name of the measure (Ci-132). Now V and σ have various definitions, but

V is often the maximum line-of-slight (los) velocity $v_{\rm max}$ and σ is the central velocity dispersion defined by the surface brightness weighted los velocity dispersion formula

$$\sigma_0^2 = \frac{\int_{R_{\rm ap}} \sigma_{\rm los}(R)^2 I(R)\, d^2R}{\int_{R_{\rm ap}} I(R)\, d^2R} ,$$

where $R_{\rm ap}$ stands for some aperature radius that is used for the determination (Ci-133). For low-redshift galaxies, R is usually in the range $0.1R_{\rm e}$ to $R_{\rm e}$. When $R_{\rm e}$ is used, one denotes σ_0 by $\sigma_{\rm e}$. For some darn good reason, the dividing line between fast rotators (above) and slow rotators (below) on a V/σ versus $\epsilon_{\rm e}$ plot is:

a) $\sim (1/5)\epsilon_{\rm e}$. b) $\sim (1/5)\sqrt{\epsilon_{\rm e}}$. c) $\sim (1/3)\sqrt{\epsilon_{\rm e}}$. d) $\sim (1/3)\epsilon_{\rm e}$. e) $\epsilon_{\rm e}$.

Full-Answer Problems

025 qfull 01000 1 3 0 easy math: proof of the virial theorem: On exams, do all parts.

96. The virial theorem is one most basic theorems of statistical mechanics taking the term statistical mechanics to include stellar systems formalism (which is about point-mass systems interacting by gravity) and other systems not ordinarily considered in conventional statistical mechanics. Here we consider only the classical virial theorem and not the quantum mechanical version. The general (non-quantum mechanical) virial theorem for a system of interacting particles isolated from all other forces.

$$\langle K\rangle = -\frac{1}{2}\left\langle \sum_i \vec F_i\cdot\vec r_i \right\rangle ,$$

where the average is over time and the average is constant in time (i.e., the system is stationary), K is kinetic energy, the sum is over all particles in the system, $\vec F_i$ is the net force on particle i, and $\vec r_i$ position vector to particle i from a defined origin. The right-hand side of the equation is the virial itself (Wikipedia: Virial theorem).

When all the forces in the system are interparticle forces derivable from potentials that depend only powers ℓ of interparticle of distances, the virial theorem specializes to

$$\langle K\rangle = \frac{1}{2}\sum_\ell \ell\,\langle U_\ell\rangle ,$$

where sum is over all the potential energies.

NOTE: There are parts a,b,c,d. All the parts can be done independently. So do not stop if you cannot do any part. On exams, do all parts with minimal words.

a) Prove the general virial theorem starting from the scalar moment of inertia

$$I = \sum_i m_i \vec r_i\cdot\vec r_i .$$

HINT: Take the first and second time derivatives of I and making use of the definitions of momentum and kinetic energy and Newton's 2nd law as needed.

b) Prove the special case virial theorem specified in the preamble: i.e., the important special case of the virial theorem where all the forces are derivable from potentials depending on power-law interparticle forces: i.e., the force of particle j on particle i is given by

$$\vec F_{ji} = -\sum_\ell \nabla U_{\ell,ji} r_{ji}^\ell = -\sum_\ell \ell U_{\ell,ji} r_{ji}^{\ell-1}\hat r_{ji} .$$

HINT: Just start from $\sum_i \vec{F}_i \cdot \vec{r}_i$ and march forward. You will need to do some trickery with indices.

c) Why must a stationary system have negative energy? What does this imply about a system to which the virial theorem applies: i.e., to a virialized system? What does the last implication imply about the kinds of potential energies of the special case virial theorem and what does it imply if there is only one kind of potential energy?

d) Specialize the special case virial theorem to the case where only the inverse-square force and linear force are present. This case actually the case for the large-scale structure of the universe where there is only the gravitation force and the cosmological constant force. Of course, this version of the virial theorem cannot apply to the universe as whole since one needs general relativistic physics for that.

---

025 qfull 01230 1 3 0 easy math: mass determination using the virial theorem: On exams do all parts.

97. The crudest way of determining a galaxy mass is by a simple use of the virial theorem.
NOTE: There are parts a,b,c,d. On exams, do all parts with minimal words.

a) What is called the virial velocity dispersion $\sigma_{\rm vir}$ is defined by

$$K = \frac{1}{2} M_{\rm vir} \sigma_{\rm vir}^2 \ ,$$

where K is the total kinetic energy and $M_{\rm vir}$ is the mass out to some cutoff radius $r_{\rm cutoff}$. If you actually knew everything about self-gravitating system that was virialized within the shell defined by the cutoff radius $r_{\rm cutoff}$, then you would know K and $M_{\rm vir}$. What is the formula for $\sigma_{\rm vir}$ in this case?

b) What is called the gravitational radius $r_{\rm g}$ (which is not the cutoff radius $r_{\rm cutoff}$) is defined by

$$U = -\frac{GM_{\rm vir}^2}{r_{\rm g}} \ ,$$

where U is total gravitational potential energy out to the cutoff radius. The gravitational radius is just a characteristic radius since it is not the radius of anything in general. If you actually knew everything about self-gravitating system that was virialized within the shell defined by the cutoff radius $r_{\rm cutoff}$, then you would know U and $M_{\rm vir}$. What is the formula for $r_{\rm g}$ in this case?

c) Since for actual galaxies, we do not know a priori $M_{\rm vir}$, K, or U, we do not know $\sigma_{\rm vir}$ and $r_{\rm g}$ exactly and they are actually what we want in order to estimate $M_{\rm vir}$. However, we can guess that $\sigma_{\rm vir}$ and $r_{\rm g}$ will be of order, respectively, the central velocity dispersion σ_0 (however specified exactly) and the effective radius $R_{\rm e}$, but maybe only to within a factor of 10 for each one. So we parameterize

$$\sigma_0^2 = a\sigma_{\rm vir}^2 \qquad \text{and} \qquad R_{\rm e} = br_{\rm g} \ ,$$

where a and b are fudge factors. Use the virial theorem for gravity to solve for $M_{\rm vir}$ eliminating $\sigma_{\rm vir}$ and $r_{\rm g}$ via the fudge factors and then eliminate the fudge factors via the virial coefficient $k_{\rm vir} = 1/(ab)$.

d) In fact, $k_{\rm vir}$ can only be known accurately from detailed modeling. However, the fiducial value is 5, but deviations from this can be large. Write the virial mass formula in terms of fiducial values $k_{\rm vir} = 5$, $R_{\rm e} = 1\,{\rm kpc} = (3.08567758\ldots) \times 10^{19}\,{\rm m}$, $\sigma_0 = 200\,{\rm km/s}$, and solar masses $M_\odot = 1.98847 \times 10^{30}\,{\rm kg}$. Note the gravitational constant $G = 6.67430 \times 10^{-11}\,{\rm MKS}$.

In fact, the fiducial formula with $k_{\rm vir}$ actually set to 5 is called the dynamical mass $M_{\rm dyn}$ (Ci-147). When resolved kinematic information is not available for a galaxy, the virial mass from the formula (with $k_{\rm vir}$ set to 5 or some other good value) given in the answer to this question may be the best estimate of total mass one can get from observations of stellar light.

Multiple-Choice Problems

026 qmult 00210 1 1 3 easy memory: groups and clusters differentiated

98. There is no sharp distinction, in fact, between groups and clusters and one can regard groups as just very poor clusters. However, by Ci-165,167,174's account, fiducially groups have 3 to x non-dwarf galaxies and clusters have x to a few thousand non-dwarf galaxies, and groups have virial mass $\lesssim y\,M_\odot$ and clusters have virial mass y to $10^{15}\,M_\odot$. Now x and y are, respectively:

    a) 10 and 10^{10}. b) 20 and 10^{11}. c) 50 and 10^{14}. d) 500 and 3×10^{14}. e) 1000 and 3×10^{14}.

026 qmult 00320 1 1 5 easy memory: local group non-dwarf galaxies

99. The Local Group has only 3 non-dwarf galaxies (all spiral galaxies): the Milky Way, the Andromeda Galaxy (M31), and the:

    a) Aries Galaxy (M33). b) Boötes Galaxy (M33). c) Monoceros Galaxy (M33). d) Pegasus Galaxy (M33). e) Triangulum Galaxy (M33).

026 qmult 00620 1 1 4 easy memory: cluster mass components and the cosmic baryonic mass fraction

100. Galaxy clusters have $(1-x)$ to 90 % of their mass as dark matter. Baryonic matter mostly in the form of intracluster gas is the rest of the mass. Stars make up only 1 to 5 % of the mass. The value x is, in fact, the cosmic baryonic mass fraction set by Big Bang nucleosynthesis and other information. Among dark matter halo structures in the observable universe, only the largest clusters have baryonic mass fraction as large as the cosmic mass fraction x whose value is:

    a) 50 %. b) 40 %. c) 33 %. d) 16 %. e) 12 %.

026 qmult 00810 1 1 1 easy memory: intracluster medium (ICM) temperature

101. The intracluster medium (ICM) temperature is

    a) $(2\text{–}10)\times10^{7}$ K. b) $(5\text{–}10)\times10^{7}$ K. c) $(2\text{–}10)\times10^{8}$ K. d) $(5\text{–}10)\times10^{8}$ K. e) $(2\text{–}10)\times10^{9}$ K.

Full-Answer Problems

026 qfull 01020 1 3 0 easy math: classical pressure force deriviation: On exams, do all parts

102. In this problem, we derive the general classical pressure for formula and some special cases. A remarkable fact is that the same formula follows from a quantum mechanical derivation with box quantization (Wikipedia: Particle in a box). This suggests that the formula is really very general.

    NOTE: There are parts a,b,c,d. On exams, do all parts with minimal words.

    a) Draw a diagram with a horizontal differential surface area vector $d\vec{A}$ with the vector pointing up. Now consider a flow of particles in a general direction through $d\vec{A}$. Write down the formula for the differential $dP\,dt\,dA$ (where capital P is pressure) for the flow of particles of momentum p through dA with velocity v in differential time dt, in differential particle momentum range dp, in differential angle $d\Omega = d\mu\,d\phi$ (where $\mu = \cos(\theta)$ and $d\mu = d\cos(\theta) = -\sin(\theta)\,d\theta$), and given the isotropic directional distribution of particles per volume per momentum $N(p)/(4\pi)$ (where the angle-integrated distribution is $N(p)$). **HINT:** You will need two factors of $\cos(\theta)$: one to account for the fact that it is only the component of mometum in the direction of $d\vec{A}$ that contributes to pressure and the other to account for the fact that there is reduced area for the beam of particles going through dA obliquely.

b) Now write down the momentum integral for pressure after having integrated over all angle.

c) Let ε be kinetic energy density. Write for formula for pressure as a function of ε in two limits: the non-relativistic (NR) limit where $p = mv$ and the extreme relativistic (ER) limit where $v = c$ and $p = K/c$, where K is kinetic energy per particle.

d) We note that the pressure formulae of parts (b) and (c) are independent of the nature of the distribution $N(p)$. It could be a thermodynamic equilibrium distribution, but also anything else. For the (thermodynamic equilibrium) Maxwell-Boltzmann distribution for NR classical particles $N(p)\,dp = nf(v)\,dv$ (where n is particle density),

$$\langle v^2 \rangle = \frac{3kT}{m} \qquad \text{and} \qquad E_{\text{energy per particle}} = \frac{3}{2}kT$$

(Wikipedia: Maxwell-Boltzmann distribution; Wikipedia: Ideal gas law: Energy associated with a gas). On the other hand for a photon gas (which is made of ER particles),

$$\varepsilon = aT^4 \ ,$$

where a is the radiation density constant (Wikipedia: Photon gas; Wikipedia: Stefan-Boltzmann law). Write down the pressure formulae for the cases of the Maxwell-Boltzmann distribution and the photon gas.

---

026 qfull 01040 1 3 0 easy math: mean atomic weight with electrons: On exams, do all parts

103. The mean atomic mass is defined

$$\frac{1}{\mu} = \sum_i \frac{X_i}{A_i} \ ,$$

where the sum is over all species present, X_i is the mass fraction of species i and A_i is the atomic mass (i.e., the mass in a standard microscopic unit). Cimatti (2020) uses the proton mass m_{p} as the standard microscopic mass probably since the universe is made of hydrogen and not of daltons (i.e., 1/12 of an unperturbed carbon-12 atom). Written it as we have, the quantity $1/\mu = n/(\rho/m_{\text{p}})$ (where n is the number of free particles) is the mean number of free particles per proton mass in the substance and $\mu = (\rho/m_{\text{p}})/n$ is the mean mass in units of the proton mass of the free particles.

NOTE: There are parts a,b,c. On exams, do all parts with minimal words.

a) What is the formula for the number density of a substance with mass density ρ?

b) Say you have a gas of completely ionized hydrogen. What is the exact formula for $1/\mu$ and what is the approximate value of $1/\mu$. Assume m_{p} is the exactly the proton mass and not just the hydrogen atom mass.

c) In this part, assume that the sum is only over nuclides and does not include free electrons. Say you have a completely ionized gas with the hydrogen mass fraction X_1 and everything else collective mass fraction $(1 - X_1)$. Assume the atomic mass of hydrogen is $A_1 = 1$ and for everything not hydrogen approximate $(Z_i + 1)/A_i = 1/2$. What is the formula for the approximate mean atomic mass in terms of X_1? Give the special cases where X_1 equals 1, 3/4, 1/2, 1/3, and 0.

---

026 qfull 01050 1 3 0 easy math: the beta-model: On exams, do only parts a,b,c

104. In this problem we investigate the β-model of (galaxy cluster) gas particle density. The β-model is probably only order of magnitude accurate, but it is a standard fiducial model for the gas particle density.

NOTE: There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c with minimal words.

a) The hydrostatic equililibrium equation for a spherically symmetric mass distribution is

$$\frac{dp}{dr} = -\frac{Gm(r)}{r^2}\rho ,$$

where r is radius coordinate, p is pressure, ρ is density, and $m(r)$ is interior mass (i.e., the mass interior to a shell of radius r). In fact, the equation can be written separately for each species in the distribution if they are decoupled: i.e., the pressure of one species is felt only by that species. In galaxy clusters, there are 3 decoupled species:

1) Gas with particle density $n = \rho/(\mu m_{\rm p})$ and $p = nkT$ (with T approximated as constant for the β-model).
2) Galaxies with galaxy number density $n_{\rm gal}$, galaxy mass density $n_{\rm gal}m_{\rm gal}$ (with $m_{\rm gal}$ approximated as constant for the β-model), and pressure approximated $n_{\rm gal}m_{\rm gal}\sigma_{\rm los}^2$ (where $\sigma_{\rm los}$ the mean line-of-sight dispersion for the galaxies for the β-model). Note $d\ln(n_{\rm gal}m_{\rm gal}\sigma_{\rm los}^2) = d\ln(n_{\rm gal})$ since $m_{\rm gal}$ and $\sigma_{\rm los}^2$ are constants.
3) Dark matter with density $\rho_{\rm DM}$ and effective pressure $p_{\rm DM}$ (whatever that may be).

Write the hydrostatic equililibrium equation in terms of the $(p/\rho)d\ln(p)/d\ln(r)$ specialized for each species compactly on one line.

b) Using the results of part (a), solve for the proportionality between n and $n_{\rm gal}$ in terms of the constant

$$\beta = \frac{\sigma_{\rm los}^2}{kT/(\mu m_{\rm p})} = \frac{(\mu m_{\rm p})\sigma_{\rm los}^2}{kT} .$$

c) Given fiducial (but probably only order a magnitude accurate) King profile

$$n_{\rm gal}(r) = n_{\rm gal,0}\left[1+\left(\frac{r}{r_{\rm c}}\right)^2\right]^{-3/2} = \frac{n_{\rm gal,0}}{\left[1+(r/r_{\rm c})^2\right]^{3/2}}$$

(where $n_{\rm gal,0}$ is central galaxy density and $r_{\rm c}$ a core radius), determine a β-model density profile. Given that fiducial range for β is $[1/2, 1]$, what can one say about the gas density profile relative to the galaxy density profile.

d) The X-ray emissivity from galaxy clusters is approximately given $j_{\rm X} \propto n^2$ and the line-of-sight surface brightness at project radius R is for optically thin gas

$$I_{\rm X}(R) = 2\int_R^\infty \frac{j_{\rm X} r\,dr}{\sqrt{r^2-R^2}} ,$$

where r radial coordinate to the line-of-sight coordinate and spherical symmetry is assumed. Solve the integral approximately to within an unspecified factor for part (c) gas density profile. You will have to make an order of magnitude approximation whose chief virtue is that it makes the integral analytically tractable.

---

026 qfull 01550 1 3 0 easy math: 2-point correlation function. On exams, do all parts.

105. There are many statistical measures for the distribution of galaxies. All of them are trying to capture aspects of large-scale structure. The ideal statistical measure would capture all aspects and would exactly specify large-scale strurcture completely. But the ideal has not been reached, and so multiple statistical measures are used. Comparing a statistical measure's values for large-scale structure simulations and those for observed large-scale structure is a test of the simulations.

Probably the simplest statistical measure is the 2-point correlation function $\xi(r)$ which appears in the following equation

$$dN = n[1 + \xi(r)]\,dV \ ,$$

where n is the mean number of galaxies per unit volume in the observable universe and dN is the mean number of galaxies in volume dV located at a distance r from a reference galaxy at $r = 0$ (Ci-188–190). There must be some probability distribution from which this mean is derived, but yours truly cannot located it at the moment. However, if the $\xi(r) = 0$ everywhere, the distribution is the Poisson distribution and the mean number of galaxies in dV is just $n\,dV$. Note if $\xi(r) > 0$ for small r, galaxies tend to cluster and if $\xi(r) < 0$ for small r, galaxies tend to avoid each other.

NOTE: There are parts a,b,c. On exams, do all parts with minimal words.

a) Prove

$$\lim_{V\to\infty} \int_V \xi(r)\,dV = 0 \ .$$

b) For $r \leq 10\,\text{Mpc}$, the fiducial version of the 2-point correlation function is

$$\xi(r) = \left(\frac{r_{\text{s}}}{r}\right)^{\alpha} = x^{-\alpha} \ ,$$

where scale radius $r_{\text{s}} = 5\,\text{Mpc}$, $\alpha = 1.8$, and $x = r/r_{\text{s}}$ (Ci-189). For $r \gtrsim 10\,\text{Mpc}$, $\xi(r)$ oscillates around zero, but there is a positive feature at the baryon acoustic oscillation (BAO) scale $\sim (140/h_{70})\,\text{Mpc}$ (Ci-189). Determine the function $N(x)$ for $x \leq 2$ and give expressions for $N(0)$, $N(1)$, and $N(2)$: numerical evaluation is not required.

c) As mentioned above, the probability distribution from which the mean given by the 2-point correlation function is derived has not been located at the moment by yours truly. However, the Poisson distribution is

$$P(k) = e^{-\mu}\frac{\mu^k}{k!} \ ,$$

where k is the number of events and μ is the mean of the distribution (i.e., the mean number of events) (Be-36–43,53). The Poisson distribution can be viewed as the extreme limit of the binomial distribution where total possible number of events is infinity, and so the number of events observed is always small and the mean number of events is nonzero. The ℓth moment of the Poisson distribution is given by

$$\langle k^\ell \rangle = e^{-\mu}\sum_{k=0}^{\infty}\frac{k^\ell \mu^k}{k!} = e^{-\mu}\left(\mu\frac{d}{d\mu}\right)^{\ell} e^{\mu} \ ,$$

where the last formula is a trick where you treat μ as variable. Evaluate the moments for $\ell \in [0, 2]$ and the standard deviation for the Poisson distribution.

Chapt. 18 Formation, Evolution, and Properties of Dark Matter Halos

Multiple-Choice Problems

027 qmult 01110 1 4 5 easy deducto-memory: MOND defined

106. "Let's play *Jeopardy*! For \$100, the answer is: A paradigm (i.e., a broad theory with many versions) that posits that there is no dark matter and that dark matter effects are really caused by some modification of dynamics and gravity (both classically and relativistically) for very low accelerations of order 10^{-13} km/s."

What is ____________, Alex?

a) dark energy b) baryonic dark matter c) WHIM d) WHINE e) MOND

Full-Answer Problems

027 qfull 01020 1 3 0 easy math: standard dark matter halo profiles: On exams, omit part d.

107. There several standard dark matter parameterized density profiles (i.e., profiles of density as a function of radius from the center of dark matter halos) that can be fitted to observed galaxy rotation curves with varying goodness. Here we study the behavior of some of them.

NOTE: There are parts a,b,c,d. On exams, omit part d.

a) The NFW profile (i.e., Navarro-Frenck-White profile, 1996) is

$$\rho(r) = \frac{4\rho_{\rm s}}{(r/r_{\rm s})(1+r/r_{\rm s})^2}$$

where the parameters are $r_{\rm s}$ the scale radius and $\rho_{\rm s}$ the density at the scale radius (e.g., Lin & Li 2019, p. 4). The NFW profile was suggeted by N-body simulations with dark matter particles, and so is a true theoretical dark matter halo density profile. It is a cusp profile in that $\rho(r \to 0)$ diverges. Show the limiting behaviors of $\rho(r)$ for $r/r_{\rm s} << 1$, $r/r_{\rm s} = 1$, and $r/r_{\rm s} >> 1$. Find the outer shell mass $M(r)$ from radius $r_{\rm outer} >> r_{\rm s}$ to general r. Discuss the converge/divergence properties of $M(r)$.

b) The Burkert profile (1995) is

$$\rho(r) = \frac{4\rho_{\rm s}}{(1+r/r_{\rm s})[1+(r/r_{\rm s})^2]}$$

where the parameters are $r_{\rm s}$ the scale radius and $\rho_{\rm s}$ the density at the scale radius (e.g., Lin & Li 2019, p. 4). The Burkert profile is a phenomenological profile chosen to fit galaxy rotation curves. If dark matter exists, $\rho_{\rm s}$ is true density parameter. If dark matter does not exist and MOND is true, then $\rho_{\rm s}$ is parameter with dimensions of density, but whose meaning is vague. The Burkert profile is a core profile in that $\rho(r \to 0)$ does not diverge. Show the limiting behaviors of $\rho(r)$ for $r/r_{\rm s} = 0$, $r/r_{\rm s} << 1$, $r/r_{\rm s} = 1$, and $r/r_{\rm s} >> 1$. Find the outer shell mass $M(r)$ from radius $r_{\rm outer} >> r_{\rm s}$ to general r. Discuss the converge/divergence properties of $M(r)$.

c) The Einasto profile (in the version of Wang 2020 September, Nature, p. 40) is

$$\rho(r) = \rho_{-2} \exp\left\{-\left(\frac{2}{\alpha}\right)\left[\left(\frac{r}{r_{-2}}\right)^{\alpha} - 1\right]\right\} ,$$

where the parameters are r_{-2} the scale radius where the logarithmic slope is -2, ρ_{-2} the density at the scale radius, and $\alpha = 0.16 \approx 1/6$. The Einasto profile (in this version) is a fit to a huge number of high accuracy N-body simulation that span 20 orders of magnitude in dark matter halo mass. Almost everywhere the fit is accurate to within a few percent. The NFW profile is accurate to within 10 % almost everywhere, but has distinct shape relative to the Einasto profile. The Einasto profile is a core profile in that $\rho(r \to 0)$ does not diverge. Show the limiting behaviors of $\rho(r)$ for $r/r_{-2} = 0$, $r/r_{-2} << 1$, $r/r_{-2} = 1$, and $r/r_{-2} >> 1$.

d) For the Einasto profile of part (c), find the interior $M(r)$ from radius $r = 0$ to general r in terms of the incomplete factorial function

$$z(y')! = \int_0^{y'} e^{-y} y^z \, dy$$

(e.g., Ar-543). making the approximation $\alpha = 1/6$. You will find it convenient to make two transformations of the variable of integration. Determine the total mass $M(r = \infty)$ for general r_{-2} and ρ_{-2} by evaluating the factorial function (i.e., $z(y' = \infty)!$) making the approximation $\alpha = 1/6$.

---

027 qfull 01030 1 3 0 easy math: the NFW profile explored: On exams, do only parts b,c.

108. The Navarro-Frenck-White (NFW) profile for the density profile of a quasi-equilibrium spherically symmetric dark matter halo derived from N-body simulations with scale radius $r_{\rm s}$, scale density, $\rho_{\rm s}$, and $x = r/r_{\rm s}$ is

$$\rho(r) = \begin{cases} \dfrac{4\rho_{\rm s}}{x(1+x)^2} = \dfrac{4\rho_{\rm s}}{x + 2x^2 + x^3} & \text{in general;} \\ \dfrac{4\rho_{\rm s}}{x} & \text{for } x << 1; \\ \rho_{\rm s} & \text{for } x = 1; \\ \dfrac{4\rho_{\rm s}}{x^3} & \text{for } x >> 1. \end{cases}$$

(Wikipedia: Navarro-Frenk-White profile). The logarithmic slope is

$$\frac{d\ln(\rho)}{d\ln(r)} = \frac{d\ln(\rho)}{d\ln(x)} = \frac{x}{\rho}\frac{d\rho}{dx} = -\frac{x}{\rho}(4\rho_{\rm s})\left[\frac{1+4x+3x^2}{(x+2x^2+x^3)^2}\right]$$
$$= \begin{cases} -\dfrac{1+4x+3x^2}{1+2x+x^2} & \text{in general;} \\ -2 & \text{for } x = 1. \end{cases}$$

The scale radius $r_{\rm s}$ and scale density, $\rho_{\rm s}$ were chosen to yield logarithmic slope -2 when $x = 1$ and density is $\rho_{\rm s}$.

The logarithmic slope -2 gives a flat (circular) velocity profile everywhere if it applies everywhere and gives an asymptotically flat velocity profile as radius $r \to \infty$ if it applies in the outer region of a mass distribution. However, the NFW profile actually only has logarithmic slope -2 at one point and does not yield an exactly flat density profile anywhere as we shall see.

Note an approximately flat velocity profile over some extended range of radius is characteristic of galaxy rotation curves for disk galaxies. However, the approximate flatness is a combination of dark matter and baryonic matter in actual galaxies and not of dark matter alone.

NOTE: There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts b,c.

a) In fact, there is a semi-analytic argument for the NFW profile. Given that a dark matter halo density profile is approximately $\propto 1/r^2$ in its most characteristic region (which we center on $x = 1$), one might be tempted to Taylor expand around the point where the logarithmic slope is exactly -2: i.e., where $x = 1$. Argue that it is better to expand the specific volume $V_{\rm sp}$ (i.e., $1/\rho$) around $x = 1$? Do the expansion for $V_{\rm sp}$ to 3rd order, collect like terms, and take the inverse using general symbols for the coefficients: i.e., ρ_0, $\rho_1 = c$, $\rho_2 = b$, and $\rho_3 = a$, where c, b, and a are chosen to conform to the conventions of tables of integrals. Why set the zeroth coefficent to zero? Why choose the 1st, 2nd, and 3rd order coefficients to be, respectively 1, 2, and 1 (given overall coefficient is set to be $\rho_{\rm s}$ times the sum of the coefficients in order to yield $\rho_{\rm s}$) other than the fact that that choice turns out to be good fitting parameters? **HINT:** To answer the last question, look at a table of integrals for the integrals needed to integrate density to get mass interior to radius x?

b) Determine the formula for $M(x)$ as a function of $r_{\rm s}$ and $\rho_{\rm s}$. You will have to use the table integrals:

$$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln(ax^2+bx+c) - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}\,,$$
$$\int \frac{dx}{ax^2+bx+c} = -\frac{2}{2ax+b} \qquad \text{for } b^2-4ac=0.$$

c) Rewrite the formula using the coefficient $M_{\rm s} = M(x = 1)$ parameterized by $r_{\rm s}v_{\rm s}^2$ where $v_{\rm s}$ is the circular velocity Do not forget to normalize the function of x (i.e., the dimensionless mass function $f(x)$) that is required in the rewritten formula) to 1 at $x = 1$ using a normalization constant A. In fact, a vast set of N-body simulations purely for dark matter particles shows that the NFW profile can be expected to hold usually to within 10 % for $x \in [0, 30]$, but with some systematic deviations (Jie Wang et al. 2020, Nature, Sep02). For $x > 30$, large deviations from the NFW profile can be expected.

d) Compute $f(x)$ for x values 0, 0.1, 0.3, 1, 3, 10, and 30. What is $f(x)$ for $x \to \infty$ and what does this mean? **HINT:** Write a small computer program for the calculation.

e) Write the dimensionless circular velocity formula $g(x)$ normalized to 1 at $x = 1$.

f) Compute a list of $g(x)$ values from $x = 0$ to $x = 30$. Describe the behavior. **HINT:** Extend your write small computer program to do the calculation.

g) The machine precision maximum characteristics of $g(x)$ can be determined by numerical methods. Setting the derivative of $g(x)$ to zero gives you a non-anaytically solvable equation for the maximizing x. An iteration formula that always converges can be obtained by isolating x on the left-hand side on on the right-hand side having a function where the expression under the square-root sign is never negative for $x > 0$. Convergence to machine precision however is slow. Convergence to machine precision is faster using the Newton-Raphson method (Wikipedia: Newton's method). If you feel ambitious, use one or other some combination of both approaches to solve for $x_{\rm max}$ and $g(x_{\rm max})$. **HINT:** Extend your write small computer program to do the calculation.

Chapt. 19 Galaxy Formation Theory

Multiple-Choice Problems

Full-Answer Problems

028 qfull 00350 1 3 0 easy math: free-fall time and collapse to star time: On exams, only do parts a,b,c.

109. The free-fall time for a straight line fall of a particle of mass m starting from rest to a point source or spherically symmetric source of mass M (always interior to the infalling particle) is

$$t_{\rm ff} = \frac{t_{\rm orbit}}{2} = \frac{\pi}{\sqrt{G(M+m)}} \left(\frac{r}{2}\right)^{3/2} \quad ,$$

where $t_{\rm orbit}$ is the orbital period predicted by the Newtonian physics version of Kepler's 3rd law and r is the initial distance from the particle to the source center and is twice the relative semi-major axis of an elliptical orbit of the particle to the source (Wikipedia: Free-fall time; Wikipedia: Kepler's laws of planetary motion Third law; Ci-246). The Kepler's 3rd law orbital period is independent of eccentricity $e \leq 1$, and so half of it is the free-fall time.

NOTE: There are parts a,b,c,d,e,f. On exams, do **ONLY** parts a,b,c.

a) What is the free-fall time for test particle (i.e., one of negligible mass)?

b) What is the free-fall time as a function of r for a spherical mass distribution with initially constant density ρ and outer radius $r \leq R$. The matter is initiall all at rest and there is zero pressure at all times. Assume the (infinitely thin) shells of matter in the distribution at all the r values never cross during free fall which is true and plausible, but seems tricky to prove. Describe the order of arrival of the shells at the center?

HINT: Remember the shell theorem

$$\vec{g} = -\frac{GM(r)}{r^2}\hat{r}$$

where the mass distribution is sperically symmetric and $M(r)$ is the interior mass to radius r. Note $M(r)$ must increase monotonically since there is no negative mass, but it can be zero out some radius r.

NOTE: For all subsequent parts, we assume a spherically symmetric mass distribution at all times with initial outer radius R and there is zero pressure at all times.

c) Say that the interior mass $M(r)$ to radius r obeys a power law $M(r) = M_0(r/r_0)^\alpha$ where $\alpha \leq 3$. When does the mass all collapse to the center assuming that it magically all stops there on arrival and the shells of matter at all the r values never cross during free fall which is true and plausible, but seems tricky to prove.

d) For star formation, we want to relate density ρ to the particle density n which can be measured more directly. The relating formula is

$$n = \rho \left(\sum_i \frac{X_i}{A_i m_{\rm p}} \right) \quad ,$$

where X_i is the mass fraction of species i (which could be any atom or a molecule including those that are distinct due to their isotopic nature), A_i is the atomic mass number (which could be a molecular mass number), and $m_{\rm p} = 1.67262192369(51) \times 10^{-24}$ g is the proton mass. Note this special case atomic mass number is in units of proton masses, not daltons (symbol u or Da and AKA atomic mass units). The fact is most of the universe is made of hydrogen (which made of protons) and not made of daltonium (which is made of daltons). Worrying about corrections due to electron masses, binding energies, and isotopes abundances (which aside from hydrogen and helium are rather uncertain) is below the level of accuracy of this problem. The mean atomic mass is defined by

$$\mu^{-1} = \sum_i \frac{X_i}{A_i}$$

which gives

$$n = \frac{\rho}{\mu m_{\rm p}} \qquad \text{or} \qquad \rho = n\mu m_{\rm p} \ .$$

Fiducial cosmic values for X_i are: $X = 0.73$ for H, $Y = 0.25$ for He, and $Z = 0.02$ for metals. Two fiducial mean atomic masses are given by

$$\mu_{\rm H_1,dominated} = \left(\frac{X}{1} + \frac{Y}{2} + \frac{Z}{30}\right)^{-1} \qquad \text{and} \qquad \mu_{\rm H_2,dominated} = \left(\frac{X}{2} + \frac{Y}{2} + \frac{Z}{30}\right)^{-1} \ ,$$

where the atomic mass for Z is a rough fiducial average based on the fiducial atomic masses of very abundant metals: i.e., $A_{\rm C,6} = 12$, $A_{\rm O,8} = 16$, $A_{\rm Si,14} = 28$, and $A_{\rm Fe,28} = 56$. Compute the $\mu_{\rm H_1,dominated}$ and $\mu_{\rm H_2,dominated}$ values to 3-digit precision which probably 1 more digit than is significant, but it is useful to know insignificant digits sometimes to check for consistency between different calculations.

HINT: Write a small computer program to do the calculation.

f) The part (b) answer gives a fiducial lower limit for the formation time for a star. It is just a fiducial lower limit since real initial clouds of molecular gas do not have uniform density, are not spherically symmetric, and do not have zero pressure and zero initial kinetic energy. It is just a lower limit since the pressure force and kinetic energy in the molecular cloud resist collapse during the collapse process and delay collapse to a star sized object. However, it is useful to rewrite the part (b) answer in terms fiducial values: particle density $10^3\,{\rm cm}^{-3}$, $\mu_{\rm H_2,dominated}$ from part (e), and Julian years (i.e., 365.25 days). Do the rewrite.

HINT: Write a small computer program to do the calculation.

---

028 qfull 00360 1 3 0 easy math: Free-fall time and shells crossing: On exams, do all parts.

110. Consider free-falling spherical shells of matter that only interact gravitationally.

NOTE: There are parts a,b,c. On exams, do all parts, but answer with minimal words.

a) First we consider a single infinitely thin spherical shell of radius $r_{\rm s}$ and mass m. What is the gravitational field $\vec{g}$ at $r < r_{\rm s}$? What is the gravitational field $\vec{g}$ at $r > r_{\rm s}$? Justify your answers.

b) What of the gravitational field $\vec{g}$ at $r_{\rm s}$? In one sense, the field is indeterminate since there is a discontinuity in the field r and which value you get depends on the direction you take the limit in. However, a limiting value often depends on the limiting process and some limiting processes are physically realistic and others are not. A physically realistic limit gravitational field at r does exist. The trick is consider tiny cylinder Gaussian surface (see Wikipedia: Gaussian surface) placed on the shell of radius r that extends inward and outward from r and whose top and bottom are parallel to the shell surface. In the small limit, the cylindar straddles an infinite infinitely thin plane of surface mass density $\sigma = m/(4\pi r_{\rm s}^2)$. Determine the gravitational field on the top and bottom of the cylinder just due to the enclosed mass. Then find the gravitational field due the rest of the shell on enclosed mass in the cylinder for all r including $r = r_{\rm s}$. That gravitational field is the gravitational field that can accelerate the enclosed mass treating it as test particle.

c) Do infalling spherical shells ever cross for any possible mass distribution? Prove your answer. **HINT:** Recall, the free-fall time for a straight line fall of a particle of mass m starting from rest to a point source or spherically symmetric source of mass M (always interior to the infalling particle) is

$$t_{\rm ff} = \frac{t_{\rm orbit}}{2} = \frac{\pi}{\sqrt{G(M+m)}}\left(\frac{r}{2}\right)^{3/2} ,$$

where $t_{\rm orbit}$ is the orbital period predicted by the Newtonian physics version of Kepler's 3rd law and r is the initial distance from the particle to the source center and is twice the relative semi-major axis of an elliptical orbit of the particle to the source (Wikipedia: Free-fall time; Wikipedia: Kepler's laws of planetary motion Third law; Ci-246). The Kepler's 3rd law orbital period is independent of eccentricity $e \leq 1$, and so half of it is the free-fall time.

Appendix 1 Classical Mechanics Equation Sheet

Note: This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things.

---

111 Some Operator Expressions

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

---

112 Binomial Theorem and Biderivative Formula

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \qquad \text{binomial theorem}$$

$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k f}{dx^k}\frac{d^{n-k}g}{dx^{n-k}} \qquad \text{biderivative formula}$$

---

113 Trigonometric Identities

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin\theta = 2\sin(\theta/2)\cos(\theta/2)$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

---

114 Kronecker Delta and Levi-Civita Symbol

$$\delta_{ij} = \begin{cases} 1, & i=j; \\ 0, & \text{otherwise} \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} 1, & ijk \text{ cyclic;} \\ -1, & ijk \text{ anticyclic;} \\ 0, & \text{if two indices the same.} \end{cases}$$

$$\varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell} \qquad \text{(Einstein summation on } i\text{)}$$

---

115 Lagrangians and Lagrange Equation Versions

$$L = T - V \qquad L = T - U \qquad L_F = L + \frac{dF(q_j,t)}{dt} \qquad L_{\text{ext}} = L + \sum_k \lambda_k f_k(q_j,\dot{q}_j,t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

---

116 **Forces and Potentials**

$$f(x) = -kx \qquad V(x) = \frac{1}{2}kx^2$$ linear force

$$\mathbf{F}(\mathbf{r}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad U = q(\phi - \mathbf{A} \cdot \mathbf{v})$$ Lorentz force

$$F = -\sum_i k_i v_i \qquad \mathcal{F} = \frac{1}{2}\sum_i k_i v_i^2$$ Rayleigh's dissipation function

$$\mathbf{F}_{12} = -\frac{k}{r_{12}^2}\hat{\mathbf{r}}_{12} \qquad V_{12} = -\frac{k}{r_{12}}$$ inverse-square law

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{\mathbf{r}}_{12} \qquad V_{12} = -\frac{Gm_1m_2}{r_{12}} \qquad \oint \mathbf{f} \cdot d\mathbf{A} = -4\pi G M_{\text{enc}}$$ gravitation

---

117 **Central Force Problem**

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \qquad \mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{M}\mathbf{r} \qquad \mathbf{r}_1 = \mathbf{R} - \frac{m_2}{M}\mathbf{r}$$

$$M = m_1 + m_2 \qquad m = \frac{m_1 m_2}{m_1 + m_2}$$

$$\ell = mr^2\dot{\theta} \qquad \frac{dA}{dt} = \frac{1}{2}\frac{\ell}{m}$$

$$m\ddot{r} - mr\dot{\theta}^2 = f(r) \qquad m\ddot{r} - \frac{\ell^2}{mr^3} = f(r)$$

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V = E \qquad \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} + V = E$$

$$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} \qquad \overline{T} = -\frac{(\ell - 1)}{2}\overline{V} \qquad \overline{T} = -\frac{1}{2}\overline{V}$$

$$\left(\frac{x}{a}\right)^2 \pm \left(\frac{y}{b}\right)^2 = 1 \qquad r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} = \frac{p(1+\epsilon)}{1+\epsilon\cos\theta} \qquad r_{\text{focus 1}} + r_{\text{focus 2}} = 2a$$

$$\frac{\ell}{mk} = a(1-\epsilon^2) = p(1+\epsilon) \qquad E = -\frac{k}{2a} \qquad \epsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}} = \sqrt{1 - \frac{\ell^2}{mka}}$$

$$\tau = 2\pi\sqrt{\frac{ma^3}{k}} \qquad \theta_{\text{mean}} = \omega t \qquad \omega = \sqrt{\frac{k}{ma^3}} \qquad \theta_{\text{mean}} = \omega t$$

$$r = a(1 - \epsilon\cos\psi) \qquad \omega t = \psi - \epsilon\sin(\psi)$$

$$\cos\theta = \frac{\cos\psi - \epsilon}{1 - \cos\psi} \qquad \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\left(\frac{\psi}{2}\right)$$

118 Hamiltonian Formulation

$$H = p_i\dot{q}_i - L \qquad \dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{q} = \frac{\partial H}{\partial p} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\text{if} \qquad L = L_0(q,t) + L_{1,j}(q,t)\dot{q}_j + L_{2,jk}(q,t)\dot{q}_j\dot{q}_k \ ,$$
$$\text{then} \qquad p_i = L_{1,i} + L_{2,ik}(q,t)\dot{q}_k + L_{2,ji}(q,t)\dot{q}_j$$
$$\text{and} \qquad H = p_i\dot{q}_i - L = L_{2,j,k}(q,t)\dot{q}_j\dot{q}_k - L_0$$

$$H = T + V$$

$$\text{if} \qquad L(q,\dot{q},t) = L_0(q,t) + \dot{\mathbf{q}}^T\mathbf{a}(q,t) + \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{T}(q,t)\dot{\mathbf{q}} \ , \qquad \text{then} \qquad \mathbf{p} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{a}$$

$$\text{and} \qquad H = \frac{1}{2}(\mathbf{p}^T - \mathbf{a}^T)\mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}) - L_0(q,t)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad I = \int_{t_1}^{t_2}\left[p_i\dot{q}_i - H(q,p,t) + \frac{dF(q,p,t)}{dt}\right]dt$$

119 Canonical Transformations

$$\lambda\left[p_i\dot{q}_i - H(q,p,t)\right] = P_i\dot{Q}_i - K(Q,P,t) \qquad \lambda = uv \qquad K = \lambda H \qquad Q_i = uq_i \qquad P_i = vp_i$$

$$p_i\dot{q}_i - H(q,p,t) = P_i\dot{Q}_i - K(Q,P,t) + \frac{dF}{dt} \qquad K = H + \frac{\partial F_i}{\partial t}$$

$$F = F_1(q,Q,t) \qquad p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$F = F_2(q,P,t) - Q_iP_i \qquad p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$F = F_3(p,Q,t) + q_ip_i \qquad q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F = F_4(p,P,t) + q_ip_i - Q_iP_i \qquad q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i}$$

$$F_2 = f_i(q,t)P_i + g(q,t) \qquad Q_j = f_j(q,t) \qquad p_j = \frac{\partial f_i}{\partial q_j}P_i + \frac{\partial g}{\partial q_j}$$

$$\mathbf{p} = \mathbf{P}\mathbf{f_q} + \mathbf{g_q} \qquad \mathbf{P} = (\mathbf{p} - \mathbf{g_q})\mathbf{f_q}^{-1}$$

120 Hamilton-Jacobi Theory

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \qquad S = S(q, \alpha, t)$$

$$P_i = \alpha_i \qquad p_i = \frac{\partial S}{\partial q_i} \qquad \beta_i = \frac{\partial S}{\partial \alpha_i} \qquad q_i = q(\alpha, \beta, t) \qquad p_i = p(\alpha, \beta, t)$$

$$S(q, \alpha, t) = W(q, \alpha) - \alpha_t t \qquad p_i = \frac{\partial W}{\partial \alpha_i} \qquad H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_t$$

$$P_i = \alpha_i \qquad \dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = \begin{cases} 1\ , & i = t \\ 0\ , & i \neq t \end{cases} \qquad Q_i = \frac{\partial W}{\partial \alpha_i} = \begin{cases} t + \beta_t\ , & i = t \\ \beta_i\ , & i \neq t \end{cases}$$

$$H\left[q', \frac{\partial W'}{q'}, f\left(q_j, \frac{\partial W_j}{q_j}\right)\right] = \alpha_t \qquad W(q, \alpha) = W'(q', \alpha) + W_j(q_j, \alpha)$$

$$f\left(q_j, \frac{\partial W_j}{q_j}\right) = \alpha_j$$

Appendix 2 Multiple-Choice Problem Answer Tables

Note: For those who find scantrons frequently inaccurate and prefer to have their own table and marking template, the following are provided. I got the template trick from Neil Huffacker at University of Oklahoma. One just punches out the right answer places on an answer table and overlays it on student answer tables and quickly identifies and marks the wrong answers

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
121.	O	O	O	O	O	6.	O	O	O	O	O
122.	O	O	O	O	O	7.	O	O	O	O	O
123.	O	O	O	O	O	8.	O	O	O	O	O
124.	O	O	O	O	O	9.	O	O	O	O	O
125.	O	O	O	O	O	10.	O	O	O	O	O

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e
126.	O	O	O	O	O
127.	O	O	O	O	O
128.	O	O	O	O	O
129.	O	O	O	O	O
130.	O	O	O	O	O
131.	O	O	O	O	O
132.	O	O	O	O	O
133.	O	O	O	O	O
134.	O	O	O	O	O
135.	O	O	O	O	O

	a	b	c	d	e
11.	O	O	O	O	O
12.	O	O	O	O	O
13.	O	O	O	O	O
14.	O	O	O	O	O
15.	O	O	O	O	O
16.	O	O	O	O	O
17.	O	O	O	O	O
18.	O	O	O	O	O
19.	O	O	O	O	O
20.	O	O	O	O	O