Calculus-Based Introductory Physics: Example Homework Problems:

The following homework problems (with accompanying solutions) that I have written are for calculus-based introductory physics courses. They are examples of what I consider high quality problems that in themselves act as complete presentations of main/special topics and that act as supplements/complements to coverage of said topics in lectures. Not all introductory physics problems should be like these, of course, but I would like to have one for every main/special topic that I teach. I will add to the set I have written already when teaching introductory physics courses.

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006 qfull 00740 1 3 0 easy math: drag equation drag solution

1. The drag equation drag law is

$$F_{\rm DE}=\frac{1}{2}\rho v^2 C_{\rm D}A=bv^2$$

where v is flow speed, ρ is fluid density, $(1/2)\rho v^2$ is the relative kinetic energy density of the fluid, $C_{\rm D}$ is the drag coefficient (which depends on many factors in general, but the always depends on the shape of the object and probably is often determined empirically), and A is the reference area (for many objects just projected frontal area of the object), and b is a combined coefficient introduced for simplicity.

a) Taking downward as positive, write down Newton's 2nd law for an object of mass m falling under Earth's near-surface gravity with drag acting. Determine the formula for acceleration in the form $a=g[1-\ldots]$ with $v/v_{\rm ter}$ as one for the terms with a velocity parameter $v_{\rm ter}$ defined appropriately.

b) Now define $z=v/v_{\rm ter}$ and reformulate the formula for acceleration a as a formula with differential $dt=(v_{\rm ter}/g)\,dz/(\ldots)$.

c) Integrate the dt expression from part (b) to obtain $t=t(v)$ assuming initial time and velocity are zero. You will need the table integral

$$\int\frac{dz}{1-z^2}=\operatorname{artanh}(z)\qquad\text{for } z<1\ .$$

where artanh(z) is the inverse hyperbolic tangent function. Note, artanh is the preferred modern symbol for the inverse hyperbolic tangent function (Wikipedia: Inverse hyperbolic functions: Notation).

d) Invert the expression from part (c) to obtain the function $v(t)$. Note, the hyperbolic tangent function is defined and explicated by

$$\tanh(x)=\begin{cases}\dfrac{e^x-e^{-x}}{e^x+e^{-x}} & \text{in general.}\\ -\tanh(-x) & \text{odd function behavior.}\\ x & \text{for x small enough that the function is}\\ & \quad\text{in the linear regime.}\\ x-\dfrac{1}{3}x^3+\dfrac{2}{15}x^5-\dfrac{17}{315}x^7+\ldots & \text{expansion in small x.}\\ 0.761594\ldots & \text{for $x=1$, the fiducial scale value}\\ & \quad\text{between small and large x regimes.}\\ \pm(1-2e^{-2|x|}) & \text{extreme $\pm x$ asymptotic behavior.}\\ \pm 1 & \text{for $x=\pm\infty$.}\end{cases}$$

What is the terminal velocity (i.e., the velocity as $t\to\infty$)? What is the scale time $t_{\rm scale}$ (i.e, the time when hyperbolic tangent argument is 1)? What is v at the scale time? What is the meaning of the scale time?

SUGGESTED ANSWER:

a) Behold:

$$1)\quad ma=mg-bv^2\qquad 2)\quad a=g\left[1-\left(\frac{v}{\sqrt{mg/b}}\right)^2\right]=g\left[1-\left(\frac{v}{v_{\rm ter}}\right)^2\right]\ ,$$

where we define velocity parameter $v_{\rm ter} = \sqrt{mg/b}$.

b) Behold:

$$dt = \frac{v_{\rm ter}}{g}\frac{dz}{1-z^2} \ .$$

c) Behold:

$$t = \frac{v_{\rm ter}}{g}\,{\rm artanh}(z)$$

d) Behold:

$$v = v_{\rm ter}\tanh\left(\frac{t}{t_{\rm scale}}\right) \ ,$$

where we define $t_{\rm scale} = v_{\rm ter}/g = \sqrt{m/(bg)}$. Clearly, the terminal velocity is $v_{\rm ter}$ and the scale time is $t_{\rm scale}$. At the scale time,

$$v(t = t_{\rm scale}) = v_{\rm ter}\tanh(1) = (0.761594\ldots) \times v_{\rm ter} \ .$$

Th scale time is a fiducial time for the change from the linear regime where $\tanh(x) \approx x$ to the regime where $\tanh(x)$ is asymptotically approaching 1, its maximum at $t = \infty$.

Fortran-95 Code

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020 qfull 01650 1 3 0 easy math: adiabatic process and adiabatic index for an ideal gas

2. In this problem, we determine the adiabatic process curve (i.e., the adiabat) and the abiabatic index for an ideal gas.

a) An ideal gas (assuming fixed degrees of freedom and energy equipartition) has internal energy given by

$$E = \frac{d}{2}n_{\rm mol}RT \ ,$$

where d is number of degrees of freedom, (d=3: monatomic gas; d=5: diatomic gas; d=7: diatomic gas with vibration; d=6: polyatomic gas with no vibration), $n_{\rm mol}$ is the number of moles of the gas, $R = 8.31446261815324\,{\rm J\cdot K^{-1}\cdot mol^{-1}}$ (exact) is the gas constant, and T is temperature on the Kelvin scale. Remarkably, the ideal gas internal energy is independent of volume and only depends on $n_{\rm mol}$ and T. Derive $(\partial E/\partial T)_{n_{\rm mol}}/n_{\rm mol}$: the rate of change of internal with temperature for constant $n_{\rm mol}$.

b) The molar heat capacity C_V at constant volume (and implicitly a constant number of moles) is the amount of heat energy needed to raise the temperature of 1 mole of a species by 1 K at constant volume. Given the rather general version of the 1st law of thermodynamics

$$dE = \delta Q - \delta W = \delta Q - p\,dV \ ,$$

derive C_V. What is relationship of C_V and $(\partial E/\partial T)_{n_{\rm mol}}/n_{\rm mol}$? Note, the symbol δ is used in this context to mean inexact differential: i.e., a differential whose integral is path dependent (Wikipedia: Inexact differential).

c) Because, in fact, the ideal gas internal energy E is independent of volume any change in E due to a change in temperature can be written in terms of C_V whether volume is constant or not. Given this fact, determine the formula for dT in terms of C_V and $n_{\rm mol}$ moles in the case where $\delta Q = 0$ (i.e., the adiabatic case) and $p\,dV \neq 0$.

Note, the formula can be understood as a relationship between $dT/d\ell$ and $dp/d\ell$ where ℓ is a path parameter parameterizing an adiabatic path $(p(\ell), V(\ell), T(\ell))$ through pVT space. The formula must be obeyed along any adiabatic path (i.e., along any adiabat). By a simplifying convention of thermodynamics, one usually does not explicitly talk of paths and path parameters. They are taken as understood.

d) Given the ideal gas law:

$$PV = n_{\rm mol}RT \ ,$$

where P is pressure and V is volume, take the differentials of both sides and replace dT by the part (c) result which has the effect of constraining the resulting expression to adiabatic changes since the right-hand side forces the left-hand side to be consistent only with changes in temperature that result from $p\,dV$ work only. Collect all differentials of the same kind and simplify the expression recalling the molar heat capacity at constant volume $C_v = (d/2)R$ and the molar heat capacity at constant pressure for an idea gas is $C_p = C_V + R$.

Note, C_V and C_p turn up in the derivation just as parameters since neither V nor p are constant along an adiabat for an ideal gas.

e) The ideal gas adiabatic index is defined by

$$\gamma = \frac{C_p}{C_V} = \frac{d+2}{d} \ .$$

Substitute γ into the result from part (d). The resulting expression is, in fact, a diffential equation for an adiabatic curve (i.e., an adiabat). Solve the differential equation by integration to get the formula for an adiabat and simplify the formula in such way that left-hand side equals the constant $p_{\rm fid}V_{\rm fid}^\gamma$, where $p_{\rm fid}$ and $V_{\rm fid}^\gamma$ fiducial or reference values along the adiabat.

SUGGESTED ANSWER:

a) Behold:

$$\frac{1}{n_{\rm mol}}\left(\frac{\partial E}{\partial T}\right)_{n_{\rm mol}} = \frac{d}{2}R \ .$$

b) Given constant volume, the $p\,dV$ work (AKA pressure-volume work) is zero, and we find

$$1)\quad \delta Q = dQ = dE \qquad 2)\quad \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{d}{2}n_{\rm mol}R \qquad 3)\quad C_V = \frac{1}{n_{\rm mol}}\left(\frac{\partial Q}{\partial T}\right)_V = \frac{d}{2}R$$

$$4)\quad C_V = \frac{d}{2}R \ .$$

We find that C_V and $(\partial E/\partial T)_{n_{\rm mol}}/n_{\rm mol}$ are the same quantity which is a special feature of the ideal gas.

c) Behold:

$$1)\quad C_V n_{\rm mol}\,dT = dE = -p\,dV \qquad 2)\quad dT = -\frac{p\,dV}{C_V n_{\rm mol}} \ .$$

d) Behold:

$$1)\quad PV = n_{\rm mol}RT \qquad 2)\quad p\,dV + V\,dp = n_{\rm mol}R\,dT = -\frac{R}{C_V}p\,dV \qquad 3)\quad V\,dp = -\left(1+\frac{R}{C_V}\right)p\,dV$$

$$4)\quad V\,dp = -\left(\frac{C_V+R}{C_V}\right)p\,dV \qquad 5)\quad V\,dp = -\left(\frac{C_p}{C_V}\right)p\,dV \qquad 6)\quad V\,dp = -\left(\frac{d+2}{d}\right)p\,dV \ .$$

To go beyond the required answer, we can recapitulate the derivation of the molar heat capacity at constant pressure C_p for an ideal gas:

$$1)\quad dE = \delta Q - p\,dV \qquad 2)\quad C_V n\,dT = dQ - nR\,dT \qquad 3)\quad C_p = \frac{1}{n_{\rm mol}}\left(\frac{\partial Q}{\partial T}\right)_p = C_V + R \ .$$

e) Behold:

$$1)\quad V\,dp = -\left(\frac{d+2}{d}\right)p\,dV \qquad 2)\quad \frac{dp}{p} = -\gamma\frac{dV}{V}$$

$$3)\quad \ln(p) = -\gamma\ln(V) + C \qquad \text{where } C \text{ is a generic integration constant}$$

$$4)\quad \ln(pV^\gamma) = C \qquad 5)\quad pV^\gamma = p_{\rm fid}V_{\rm fid}^\gamma \ .$$

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