A Heuristic Two-Dark-Energy-Components Model for Cosmic Scale Factor Evolution

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ABSTRACT

We present a heuristic two-dark-energy-components model for cosmic scale factor evolution. The first dark energy component is just the standard cosmological constant equivalent with pressure $P_\Lambda = w_\Lambda \rho_\Lambda c^2$, where Λ stands for cosmological constant and the equation of state parameter is constant $w_\Lambda = -1$. We will call the first dark energy the Λ dark energy and, for brevity, the model itself the $\Lambda\Gamma$ model where Γ is the symbol adopted for the second dark energy. The Γ dark energy component has pressure $P_\Gamma = w_\Gamma \rho_\Gamma c^2$ with $w_\Gamma = -1/2$. The motivation for the $\Lambda\Gamma$ model is that the DESI DR2 results (Lodha et al. 2025; Abdul Karim et al. 2025) suggest that the dark energy density has been decreasing from cosmological redshift $z = 0.45$ (i.e., lookback $\lesssim 4.5\,\mathrm{Gyr}$ or cosmic time $\gtrsim 9\,\mathrm{Gyr}$ assuming Λ-CDM evolution: see, e.g., Lodha et al. 2025, Fig. 2) and the $\Lambda\Gamma$ model can give that effect. The $\Lambda\Gamma$ model also gives (exact) analytic solutions for cosmic scale factor $a(t)$ and its inverse $t(a)$ which solutions give the $\Lambda\Gamma$ model physical elegance and makes it easy to test and use as a standard of comparison: these factors constitute a secondary motivation for introducing the $\Lambda\Gamma$ model. However, we have no physical motivation for the Γ dark energy with $w_\Gamma = -1/2$. (Incidentally, the $\Lambda\Gamma$ model solutions are special cases of what we call the V models solutions which are analytic solutions that include analogues to the non-analytic standard solutions of the Friedmann equation reviewed by Bondi (1961, esp. p. 80–86). We present the V models solutions in catalogue form. Noteworthily, there is a V model solution with negative Λ dark energy that permits a sinusoidal cosmic scale factor evolution that never goes to zero: a kind of evolution not noted by Bondi (1961)). Using universe age (time from a Big Bang to cosmic present) as a metric, we study the overall behavior of the $\Lambda\Gamma$ model with the variation of its parameters. A crude test of the $\Lambda\Gamma$ model with the $Om(z)$ diagnostic shows that $\Lambda\Gamma$ model may be crudely adequate to the fit the new observations with a cosmic present density parameters of order 0.53 for Λ

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dark energy, 0.18 for Γ dark energy, and 0.29 for matter. Given that ΛΓ model is just a heuristic model with no physical reason for its equation of state parameter $w_\Gamma = -1/2$, any expectation that even a good fit of it to observations will be meaningful is very modest.

Unified Astronomy Thesaurus concepts: Cosmology (343); Accelerating universe (12); Cosmological constant (334); Cosmological evolution (336); Cosmological models (337); Cosmological parameters (339); Dark energy (351); Density parameter (372); Einstein universe (452); Expanding universe (502); Friedmann universe (551); Lambda density (898); Lemaître universe (914); Matter density (1014);

1. Introduction

Section 2 introduces ΛΓ model and § 3 presents the analytic solutions for it and the more general V model. In § 4, we use the universe age (time from a Big Bang to cosmic present τ_0 in scaled time (see § 2)) as a metric to test the overall behavior of the ΛΓ model solutions as their parameters are varied. We make a preliminary test of the ΛΓ model using the $Om(z)$ diagnostic and data from (Lodha et al. 2025, Fig. 9) in § 5 and also present some ancillary formulae there. For future reference, we specialize the deceleration parameter diagnostic for the ΛΓ model and present some ancillary formulae in § 6. A discussion is given in § 7.

2. The ΛΓ Model

First, to be general, we assume $\rho_p \propto x^{-p}$ (where power $p \geq 0$) and then obtain

$$\frac{\dot{\rho}_p}{\rho_p} = -p\frac{\dot{x}}{x} \ , \tag{1}$$

where x is cosmic scale factor and the time derivative is with respect to scaled cosmic time $\tau = H_0 t$ with H_0 being the Hubble constant and t being unscaled cosmic time. We equate $\dot{\rho}_p/\rho_p$ to the usual fluid equation of cosmology (e.g., Liddle 2015, p. 26) with pressure P_p parameterized by equation of state $P_p = w_p \rho_p c^2$ (with w_p being the equation of state parameter for power p) to obtain

$$-p\frac{\dot{x}}{x} = \frac{\dot{\rho}_p}{\rho_p} = -3\frac{\dot{x}}{x}\left(1 + \frac{P_p}{\rho_p c^2}\right) = -3\frac{\dot{x}}{x}\left(1 + w_p\right) \tag{2}$$

which we solve to obtain

$$p = 3\,(1+w_p) \ , \quad w_p = \left(\frac{1}{3}\right) p - 1 \ , \quad \text{and} \quad p - 2 = 1 + 3w_p \ , \tag{3}$$

where the last expression is used in § 6. For Γ dark energy component density and as aforesaid in the abstract,

$$w_\Gamma = -\frac{1}{2} \ , \quad \text{and thus} \quad p_\Gamma = \frac{3}{2} \quad \text{and} \quad p_\Gamma - 2 = 1 + 3w_\Gamma = -\frac{1}{2} \ . \tag{4}$$

As well as the Γ dark energy, as aforesaid in the abstract, there is the also the Λ dark energy (i.e., the ordinary constant dark energy or cosmological constant) and matter. The brief radiation-dominated era (i.e., before cosmic time $\sim 50\,\text{kyr}$ (e.g., Hergt & Scott 2024, p. 6). of the observable universe is not being considered. Thus, the ΛΓ model has three density components: Λ dark energy, Γ dark energy, and matter.

The motivation for the heuristic ΛΓ model is that the DESI DR2 results (Lodha et al. 2025; Abdul Karim et al. 2025) suggest that the dark energy density has been decreasing from cosmological redshift $z = 0.45$ (i.e., lookback $\lesssim 4.5\,\text{Gyr}$ or cosmic time $\gtrsim 9\,\text{Gyr}$ assuming Λ-CDM evolution: see, e.g., Lodha et al. 2025, Fig. 2) and the ΛΓ model can give that effect. The ΛΓ model also gives (exact) analytic solutions for cosmic scale factor $a(t)$ and its inverse $t(a)$ which solutions give the ΛΓ model physical elegance and makes it easy to test and use as a standard of comparison: these factors constitute a secondary motivation for introducing the ΛΓ model. However, we have no physical motivation for Γ dark energy with equation of state parameter $w_\Gamma = -1/2$ (or equivalently $p_\Gamma = 3/2$). Nevertheless, it is possible that ΛΓ model could fit the cosmic scale factor evolution to some degree after the brief radiation-dominated era of the observable universe: i.e., after cosmic time $\sim 50\,\text{kyr}$ (e.g., Hergt & Scott 2024, p. 6). In which case, the ΛΓ model might become physically interesting.

3. The Friedmann Equation Solutions for ΛΓ Model and the More General V Model

Analytic three density component solutions to the Friedmann equation are available if the components have (inverse) power dependencies on cosmic scale factor with powers p, q and r, where $q = p/2$, $r = 0$, and ancillary constant $V = 1/(p-q) = 2/p$ (which implies $pV = 2$). We will call the model with these dependences the V model to give it a name. We derive the general V model solutions below. For further discussion of the V model, see Jeffery (2026, App. B).

The $\Lambda\Gamma$ model is special case of the V model with $p = 3$ for matter, $q = 3/2$ for Γ dark energy component, $r = 0$ for Λ dark energy component, and $V = 2/3$. Another special case of the V model of interest has $p = 4$, $q = 2$, $V = 1/2$. This is the radiation-curvature-Λ universe which is an analogue (when the q density parameter component is negative) to the Lemaître universe (a matter-positive curvature-Λ universe) which has no analytic solution (e.g., Bondi 1961, p. 82,84–85,120–122,165,168–170,175–176).

Now the density parameters for the Friedmann equation are usually symbolized using Greek capital letter Ω. However, multiple Ω's are difficult to distinguish and they are not consistent with the symbols used in standard tables of integral (which we have made use of). Therefore, here we use c for the cosmic present matter density parameter, b for the cosmic present Γ dark energy density parameter, and a for the Λ dark energy density parameter (which is constant), and so a is mostly not used for the cosmic scale factor in this paper. (We use x for cosmic scale factor as already noted in § 2.) We will call a, b, and c collectively the model parameters and, respectively, just the a, b, and c parameters though adding the descriptive terms Λ parameter, Γ parameter, and matter parameter when needed for greater clarity. The values of model parameters are model parameter weights. Note we consider cases where a and b are positive, negative, or zero, but $c \geq 0$ always since matter always has positive mass.

The Friedmann equation for the V model in terms of cosmic scale factor x and scaled cosmic time τ (see § 2) is

$$\frac{\dot{x}}{x} = \pm\sqrt{a + bx^{-q} + cx^{-p}} \ . \tag{5}$$

Equation (5) actually has a fair number of special case solutions. However, for the $\Lambda\Gamma$ model, we restrict solutions to those that start from a Big Bang (i.e., $x(\tau = 0) = 0$), that strictly increase with time thereafter, and have $b > 0$ (except we let what we call below the Γ_1 solution have $b \geq 0$). Thus, we restrict what we call $\Lambda\Gamma$ model solutions to those that at least minimally match the observable universe. There are 4 of these solutions which we call the Γ_i solutions (more explicitly $\Gamma_i(\tau)$ solutions) with index i running 1 to 4. We also find 5 solutions not conforming to our requirements for the $\Lambda\Gamma$ model solutions: we call these the Γ_{i-} solutions (more explicitly the $\Gamma_{i-}(\tau)$ solutions). The first 4 of the Γ_{i-} solutions are the same functions as like-numbered Γ_i solutions, but with different choices of model parameters and/or initial conditions. The inverse solutions (which are obtained first) are called the Γ_i^{-1} and Γ_{i-}^{-1} solutions (more explicitly the $\Gamma_i^{-1}(x)$ and $\Gamma_{i-}^{-1}(x)$ solutions).

For the Γ_i solutions, we choose $x = 1$ at cosmic present time τ_0 (which is the universe age defined in § 1) to yield the scaled Hubble constant 1 (and unscaled Hubble constant H_0).

By this choice, the model parameters obey the constraint

$$a + b + c = 1 \ . \tag{6}$$

The fiducial round-number Λ-CDM model parameter weights are $c = 0.3$ and $a = 0.7$ (e.g., Wikipedia: Lambda-CDM model: Parameters), and so $b = 0$. If we moved some model parameter weight from a to b, we would clearly increase the overall effect of dark energy and cause more rapid growth from the the Big Bang, and so decrease the universe age τ_0. On the other hand, if we moved some model weight from c to b, we would weaken the initial growth in the matter dominated era, but would strengthen the later dark-energy dominated era, and so a priori it is not certain how τ_0 would change. In either case, if $b > 0$, the total dark energy would be decreasing as the universe age τ_0 is approached which is the effect found in the DESI DR2 results (Lodha et al. 2025; Abdul Karim et al. 2025) and, as aforesaid, matching this effect is main the motivation for introducing the $\Lambda\Gamma$ model.

We do not impose the $a + b + c = 1$ constraint on the Γ_{i-} solutions, unless otherwise noted, and so the model parameters a, b, and c are independent for those solutions, unless otherwise noted.

We now define $y = x^{1/V}$ implying $x = y^V$ and $dx/x = V\,dy/y$. Using these expression, the Friedmann equation transformed for direct solution for τ as a function of x and also as function of y is

$$d\tau = \pm \frac{dx}{x\sqrt{a + bx^{-q} + cx^{-p}}} = \pm \frac{V\,dy}{y\sqrt{a + by^{-pV/2} + cy^{-pV}}} = \pm \frac{V\,dy}{\sqrt{ay^2 + by + c}} \ , \tag{7}$$

where recall $pV = 2$ and we omit the negative case for the Γ_i solutions, but not for all the Γ_{i-} solutions. In order for the $\tau(x)$ and $x(\tau)$ solutions (which we derive just below) to apply to the V model as well the $\Lambda\Gamma$ model, we leave V general in all our formulae for these solutions.

For $\tau(x)$ solutions using a standard table of integrals (Wikipedia: List of integrals of irrational functions: Integrals involving $R = \sqrt{ax^2 + bx + c}$) and aided by Google AI, we obtain

$$
\tau = \begin{cases}
\dfrac{V}{\sqrt{a}} \ln \left| \dfrac{2ay + b + 2\sqrt{a}\sqrt{ay^2 + by + c}}{b + 2\sqrt{ac}} \right| & \text{Conditions: } a > 0, \\
& \quad \tau(y = x^{1/V} = 0) = 0. \\
& \text{Solution } \Gamma_{123}^{-1}(x)\text{: } b \geq 0, \\
& \quad \text{equivalent to all of } \Gamma_1^{-1}(x),\ \Gamma_2^{-1}(x), \\
& \quad \text{and } \Gamma_3^{-1}(x). \\
& \quad \text{Valid for solutions that increase} \\
& \quad \text{monotonically from } y = x^{1/V} = 0. \\
\dfrac{V}{\sqrt{a}} \ln \left| \dfrac{2a + b + 2\sqrt{a}}{b + 2\sqrt{ac}} \right| = \tau_0 & \text{Extra Conditions: } y = x^{1/V} = 0 \text{ and} \\
& \quad a + b + c = 1 \text{ is explicitly applied.} \\
\dfrac{V}{\sqrt{a}} \ln \left| \dfrac{1 - c + a + 2\sqrt{a}}{1 - c - a + 2\sqrt{ac}} \right| = \tau_0 & \text{Extra Condition: } b = 1 - (a + c) \\
& \quad \text{is explicitly applied.}
\end{cases} \tag{8}
$$

$$
\tau = \begin{cases}
\dfrac{V}{\sqrt{a}} \operatorname{arsinh}\left(\dfrac{2ay + b}{\sqrt{4ac - b^2}} \right) - \tau_{\mathrm{zero}\,x} & \text{Conditions: } a > 0,\ b^2 - 4ac < 0, \\
& \quad \text{zero time } \tau_{\mathrm{zero}\,x} \text{ chosen} \\
& \quad \text{to give } \tau(y = x^{1/V} = 0) = 0 \\
& \quad \text{so that } x(\tau = 0) = 0. \\
& \text{Solution } \Gamma_1^{-1}(x)\text{: } b \geq 0. \\
& \text{Solution } \Gamma_{1-}^{-1}(x)\text{: } b < 0. \\
\tau_{\mathrm{zero}\,x} = \dfrac{V}{\sqrt{a}} \operatorname{arsinh}\left(\dfrac{b}{\sqrt{4ac - b^2}} \right) & \text{Zero time } \tau_{\mathrm{zero}\,x}.
\end{cases} \tag{9}
$$

$$
\tau = \begin{cases}
\pm \dfrac{V}{\sqrt{a}} \ln |2ay + b| + C & \text{Conditions: } a > 0,\ b^2 - 4ac = 0. \\
& \text{Solution } \Gamma_2^{-1}(x)\text{: upper case only, } b > 0, \\
& \quad C \text{ chosen to give } \Gamma_2(\tau = 0) = 0. \\
& \text{Solution } \Gamma_{2-}^{-1}(x)\text{: No constraint on } b \\
& \quad \text{and C chosen to make } \Gamma_{2-}(\tau) \\
& \quad \text{heuristically interesting.}
\end{cases} \tag{10}
$$

$$
\tau = \begin{cases}
\pm\dfrac{V}{\sqrt{a}}\operatorname{arcosh}\left(\left|\dfrac{2ay+b}{\sqrt{b^2-4ac}}\right|\right) & \text{Conditions: } a>0,\ b^2-4ac>0, \\
& \quad \text{upper/lower case time} \\
& \quad \text{increasing/decreasing with } y. \\
& \text{Solution } \Gamma_3^{-1}(x)\text{: } b>0, \text{ upper case,} \\
& \quad \text{zero time } \tau_{\text{zero}\,x} \text{ exists,} \\
& \quad \text{and } \tau \geq \tau_{\text{zero}\,x}. \\
& \text{Solution } \Gamma_{3-}^{-1}(x) \text{ is for all other} \\
& \quad \text{physical cases.} \\
\tau_{\text{zero}\,x} = \dfrac{V}{\sqrt{a}}\operatorname{arcosh}\left(\left|\dfrac{b}{\sqrt{b^2-4ac}}\right| \geq 1\right) & \text{For } y = x^{1/V} = 0. \\
& \text{Condition: The zero time } \tau_{\text{zero}\,x} \\
& \quad \text{is chosen to give } x(\pm\tau_{\text{zero}\,x}) = 0. \\
& \text{Note } \tau_{\text{zero}\,x} \text{ does not exist} \\
& \quad \text{if the arcosh function} \\
& \quad \text{argument} < 1 \\
& \quad \text{(e.g., when } b=0 \text{ and } c<0\text{).} \\
& \quad \text{In this case, there is no } x=0 \text{ line} \\
& \quad \text{intersection and the } y \text{ and} \\
& \quad x \text{ solutions that open upward} \\
& \quad \text{are physical and those that} \\
& \quad \text{open downward are not.} \\
& \text{Note for } \tau_{\text{zero}\,x} = 0, \text{ one needs } c=0. \\
& \text{Note if } \tau_{\text{zero}\,x} \text{ exists and } b>0, \\
& \quad \text{then the } y \text{ solution needs to} \\
& \quad \text{go negative (or at least to zero)} \\
& \quad \text{to reach the point where} \\
& \quad \text{the arcosh function has} \\
& \quad \text{argument 1 which gives } \tau=0. \\
& \quad \text{Note the cosh function solutions} \\
& \quad \text{for } y \text{ and } x \text{ are necessarily even} \\
& \quad \text{about } \tau=0. \\
& \quad \text{The upshot is that } \tau=0 \text{ is the} \\
& \quad \text{minimum value time for the } y \\
& \quad \text{and } x \text{ solutions, and therefore} \\
& \quad \text{they have positive coefficients} \\
& \quad \text{and open upward.} \\
& \text{By a corresponding argument,} \\
& \quad \text{if } \tau_{\text{zero}\,x} \text{ exists and } b<0, \\
& \quad \text{the } y \text{ and } x \text{ solutions} \\
& \quad \text{have negative coefficients} \\
& \quad \text{and open downward.}
\end{cases}
\tag{11}
$$

$$
\tau = \begin{cases}
\pm V\left(\dfrac{2}{b}\right)\sqrt{by+c} + \text{constant} & \text{General solution.} \\
 & \text{Conditions: } a=0,\ b\neq 0. \\
V\left(\dfrac{2}{b}\right)\left(\sqrt{by+c}-\sqrt{c}\right) & \text{Solution } \Gamma_4^{-1}(x)\text{: } b>0, \\
 & \quad \tau(y=x^{1/V}=0)=0 \text{ chosen.} \\
V\left(\dfrac{y}{\sqrt{c}}\right) & \text{Solution } \Gamma_4^{-1}(x)\text{: } b=0,\ c=1. \\
\pm V\left(\dfrac{2}{b}\right)\sqrt{by+c} & \text{Solution } \Gamma_{4-}^{-1}(x)\text{: } b<0, \\
 & \quad \tau(y=c/|b|)=0, \\
 & \quad \tau_{\text{zero}\,x}=V(2/|b|)\sqrt{c} \\
 & \quad \text{gives } x(\pm\tau_{\text{zero}\,x})=0. \\
V\left(\dfrac{2}{b}\right)\left(\sqrt{b+c}-\sqrt{c}\right) & \text{Solution } \Gamma_4^{-1}(x)\text{: } b>0. \\
 & \text{Extra Conditions: } y=x^{1/V}=1. \\
V\left[\dfrac{2(1-\sqrt{c}\,)}{1-c}\right] = V\left(\dfrac{2}{1+\sqrt{c}}\right) & \text{Extra Condition:} \\
 & \quad b+c=1 \text{ applied.} \\
\quad = V\left(\dfrac{2}{1+\sqrt{1-b}}\right) & \\
\quad = V\left(\dfrac{2}{b}\right)\left(1-\sqrt{1-b}\,\right) &
\end{cases} \tag{12}
$$

$$
\tau = \begin{cases}
-\left(\pm\dfrac{V}{\sqrt{|a|}}\right)\arcsin\left(\dfrac{2ay+b}{\sqrt{b^2-4ac}}\right) & \text{Conditions: } a<0,\ b^2-4ac>0, \\
 & \quad |2ay+b|<\sqrt{b^2-4ac}\,. \\
 & \text{Solution } \Gamma_{5-}^{-1}(x). \\
 & \text{No } \Gamma_5^{-1}(x) \text{ nor } \Gamma_5(\tau) \text{ exists.}
\end{cases} \tag{13}
$$

The inverses of the cosmic time solutions $\tau(x)$ give the cosmic scale factor solutions $x(\tau)$

where only the region where $x(\tau)^{1/V} > 0$ are physical. The scale factor solutions are:

$$x = \begin{cases} \left\{ \dfrac{-b + \sqrt{4ac - b^2}\,\sinh[\sqrt{a}\,V^{-1}(\tau + \tau_{\mathrm{zero}\,x})]}{2a} \right\}^V & \begin{array}{l} \text{Conditions: } a > 0,\ b^2 - 4ac < 0, \\ \quad c > 0. \\ \text{Solution } \Gamma_1(\tau)\text{: } b \geq 0. \\ \text{Solution } \Gamma_{1-}(\tau)\text{: } b < 0. \\ \text{The } \Lambda\text{-CDM model} \\ \quad \text{for } b = 0. \\ \text{An analogue to the Lemaître} \\ \quad \text{universe for } b < 0. \end{array} \\ \tau_{\mathrm{zero}\,x} = \dfrac{V}{\sqrt{a}} \operatorname{arsinh}\left(\dfrac{b}{\sqrt{4ac - b^2}} \right) & \begin{array}{l} \text{Zero time } \tau_{\mathrm{zero}\,x} \text{ is chosen} \\ \quad \text{to give } x(\tau = 0) = 0. \end{array} \end{cases} \tag{14}$$

$$x = \begin{cases} \left(\dfrac{-b + \Delta y_0 e^{\pm\sqrt{a}\,V^{-1}\tau}}{2a} \right)^V & \begin{array}{l} \text{Conditions: } a > 0,\ b^2 - 4ac = 0. \\ \text{Extra Conditions: } \tau \geq 0,\ -b + \Delta y_0 \geq 0 \\ \quad \text{for a physical solution at } \tau = 0. \\ \text{Solution } \Gamma_2(\tau)\text{: upper case only, } b > 0 \text{ and } \Delta y_0 = b \\ \quad \text{implying } x(\tau = 0) = 0. \\ \text{Solution } \Gamma_{2-}(\tau)\text{: } b > 0,\ \Delta y_0 > b \text{ giving} \\ \quad \text{increasing/decreasing exponential solutions} \\ \quad \text{with the decreasing exponential solution} \\ \quad \text{going to zero.} \\ \text{Solution } \Gamma_{2-}(\tau)\text{: } b = 0,\ \Delta y_0 > 0 \text{ giving} \\ \quad \text{increasing/decreasing exponential solutions.} \\ \quad \text{An expanding/contracting de Sitter universe.} \\ \text{Solution } \Gamma_{2-}(\tau)\text{: } b < 0,\ \Delta y_0 = 0 \text{ giving} \\ \quad \text{an analogue to the (static) Einstein universe.} \\ \text{Solution } \Gamma_{2-}(\tau)\text{: } b < 0,\ \Delta y_0 > 0 \text{ giving} \\ \quad \text{increasing/decreasing exponential solutions.} \\ \quad \text{An analogue to the Lemaître-Eddington universe:} \\ \text{Solution } \Gamma_{2-}(\tau)\text{: } b < 0,\ \Delta y_0 < 0 \text{ giving} \\ \quad \text{increasing/decreasing exponential solutions.} \\ \quad \text{The negative coefficient } \Delta y_0 \text{ causes} \\ \quad \text{the increasing exponential solution goes to zero.} \\ \quad \text{An analogue to the Lemaître-Eddington universe.} \end{array} \end{cases} \tag{15}$$

$$x = \begin{cases} \left\{ \dfrac{-b \pm \sqrt{b^2-4ac}\cosh[\sqrt{a}\,V^{-1}\tau]}{2a} \right\}^V & \begin{array}{l} \text{Conditions: } a>0,\ b^2-4ac>0. \\ \text{Solution } \Gamma_3(\tau)\text{: upper case,} \\ \quad \tau_{\mathrm{zero}\,x} \text{ exists, } b>0, \\ \quad \text{and } \tau \geq \tau_{\mathrm{zero}\,x}. \\ \text{Solution } \Gamma_{3-}(\tau)\text{: upper case,} \\ \quad \tau_{\mathrm{zero}\,x} \text{ exists, } b>0, \\ \quad \tau \leq -\tau_{\mathrm{zero}\,x}. \\ \text{Solution } \Gamma_{3-}(\tau)\text{: upper case,} \\ \quad \tau_{\mathrm{zero}\,x} \text{ does not exist} \\ \quad \text{and the solution is} \\ \quad \text{physical for all time.} \\ \text{Solution } \Gamma_{3-}(\tau)\text{: lower case,} \\ \quad \tau_{\mathrm{zero}\,x} \text{ exists, } b<0, \\ \quad \text{and the solution is physical for} \\ \quad \tau \in [-\tau_{\mathrm{zero}\,x}, \tau_{\mathrm{zero}\,x}]. \\ \text{No physical solution: lower case,} \\ \quad \tau_{\mathrm{zero}\,x} \text{ does not exist.} \end{array} \\ \tau_{\mathrm{zero}\,x} = \dfrac{V}{\sqrt{a}}\operatorname{arcosh}\left(\left| \dfrac{b}{\sqrt{b^2-4ac}} \right| \geq 1 \right) & \text{Chosen to give } x(\tau=\tau_{\mathrm{zero}\,x})=0. \end{cases} \tag{16}$$

$$x = \begin{cases} \left\{ \dfrac{[(b/2)V^{-1}\tau+\sqrt{c}]^2-c}{b} \right\}^V & \begin{array}{l} \text{Solution } \Gamma_4(\tau)\text{: } a=0,\ b>0,\ c>0. \\ \text{Note } x \text{ grows strictly for } \tau \geq 0. \end{array} \\ (\sqrt{c}\,V^{-1}\tau)^V & \begin{array}{l} \text{Solution } \Gamma_4(\tau)\text{: } a=0,\ b=0,\ c=1. \\ \text{It is the single (inverse) power law density} \\ \quad \text{component solution.} \\ \text{For } p=3 \text{ and } V=2/3, \\ \quad \text{it is the Einstein-de Sitter universe.} \end{array} \\ \left\{ \dfrac{c-[(b/2)V^{-1}\tau]^2}{|b|} \right\}^V & \begin{array}{l} \text{Solution } \Gamma_{4-}(\tau)\text{: } a=0,\ b<0,\ c>0. \\ \text{Note } x^{1/V} \text{ is a parabola with maximum} \\ \quad c/|b| \text{ at } \tau=0 \text{ and } x(\pm\tau_{\mathrm{zero}\,x})=0, \\ \quad \text{where } \tau_{\mathrm{zero}\,x} = V(2/|b|)\sqrt{c}. \\ \text{Note substituting for } \tau \text{ with } \tilde{\tau}-\tau_{\mathrm{zero}\,x} \\ \quad \text{gives the formula for } \Gamma_4(\tilde{\tau}), \\ \quad \text{but now for } b<0. \end{array} \end{cases} \tag{17}$$

$$
x = \begin{cases}
\left\{ \dfrac{b \pm \sqrt{b^2 - 4ac}\,\sin[\sqrt{|a|}\,V^{-1}\tau]}{2|a|} \right\}^V & \text{Conditions: } a < 0,\ b^2 - 4ac \geq 0. \\
 & \Gamma_{5-}(x) \text{ solution: no } \Gamma_5(x) \text{ exists since} \\
 & \quad \text{the } \Gamma_{5-}(x) \text{ solution in all cases has} \\
 & \quad \text{cannot grow to infinity.} \\
\left\{ \dfrac{b + \sqrt{b^2 - 4ac}\,\cos[\sqrt{|a|}\,V^{-1}\tau]}{2|a|} \right\}^V & \text{Choosing a convenient phase.} \\
 & \text{For } b > 0,\ \sqrt{b^2 - 4ac} \leq b, \\
 & \quad \text{(implying } c \geq 0\text{) there is} \\
 & \quad \text{a sinusoidal solution} \\
 & \quad \text{for all time that goes to zero} \\
 & \quad \text{at its minimum, except that} \\
 & \quad \text{for } c > 0,\ x > 0 \text{ always.} \\
 & \text{For } b > 0 \text{ and } b^2 - 4ac = 0, \text{ there} \\
 & \quad \text{is the constant solution } b/(2|a|). \\
 & \text{For } b < 0,\ \sqrt{b^2 - 4ac} \leq |b| \\
 & \quad \text{(implying } c \leq 0\text{),} \\
 & \quad \text{there is no physical solution} \\
 & \quad \text{(i.e., } x^{1/V} \leq 0 \text{ always).} \\
\tau_{\text{zero}\,x} = \pm \dfrac{V}{\sqrt{|a|}} \arccos\left(\dfrac{-b}{\sqrt{b^2 - 4ac}} \right) & \text{All other cases, there are only finite} \\
 & \quad \text{time period solutions with } x = 0 \\
 & \quad \text{at the two endpoints} \\
 & \quad \text{which for the cosine version can be} \\
 & \quad \text{chosen to be at times } \tau_{\text{zero}\,x}.
\end{cases}
\tag{18}
$$

The index $i \leq 3$ of the Γ_i solutions increases with increasing discriminant $b^2 - 4ac$. The Γ_4 solution has discriminant b^2, and so its discriminant may be larger or smaller than that of Γ_3 solution. Given that Λ-CDM model is such a good fit to the observable universe, we expect any viable $b^2 - 4ac$ value to be small and $a > 0$ (ruling out the Γ_4 solution), and so the Γ_1 solution (Equation (14)) is probably the only Γ_i solution of any viability. In fact, the Γ_1 solution is the solution we find in a crude fit to observations (see § 5). However, we are also interest in the the overall behavior of the $\Lambda\Gamma$ model as its parameters are varied, and thus on its flexibility to accommodate observations and not just on the ability of any particular Γ_i solution to do so. We study the overall behavior as a function of the model parameters (i.e., a, b, and c) in § 4.

We should note that the V model solutions (as partially noted above in Equations (14), (15), (16), (17), and (18)) include analogues to the non-analytic standard solutions of the Friedmann equation reviewed by Bondi (1961, esp. p. 80–86). Of course, simple 1-density component solutions with parameters $a = b = 0$ and $c \neq 0$ and $p \geq 0$ follow as special cases of the V model solutions, including the Einstein-de Sitter universe with $p = 3$ (e.g., Bondi 1961, p. 82,166). The analogue solutions permit an analytic understanding of the corresponding non-analytic solutions.

We can remark on some of the noteworthy V solutions. First, the Γ_1 solution (Equation (14)) with $b = 0$ is just the matter-Λ universe solution used for the Λ-CDM model (e.g., Jeffery 2026, § 12.1.1). With $b < 0$, this solution (i.e., the Γ_{1-} solution) is an analogue solution to the non-analytic Lemaître universe solution (a positive-curvature-matter-Λ universe solution) where the analogue to the constant $b < 0$ is the positive curvature density parameter at some fiducial time (e.g., Bondi 1961, p. 82,84–85,120–122,165,168–170,175–176). Since the Lemaître universe has (quasi-static) Einstein universe phase. The analogue solution makes it clear how an Einstein universe phase arises: the length of the analogue Einstein universe phase increases as $\sqrt{4ac - b^2}$ decreases, and so one can create a universe model of any universe age just as for the Lemaître universe.

Second, the parameters of the Γ_{2-} solution (Equation (15)) can be chosen to give an analogue Lemaître-Eddington universe (e.g., Bondi 1961, p. 84–85,117–121,159), an analogue to the (static) Einstein universe (e.g., Bondi 1961, p. 84,98–99,117–121,158–159,171), and the actual de Sitter universe with both exponentially expanding and contracting cases (e.g., Bondi 1961, p. 98–99,105,146–147,154,159,166). From the Γ_{2-} solution, we can see explicitly why the analogue Einstein universe is unstable to global perturbations Δy_0. Such global perturbations could put the universe model on either converging or diverging branches from the analogue Einstein universe. Thus, general global perturbations will always lead to divergence and the analogue Einstein universe is unstable just as is the actual Einstein universe. Of course, uniform global perturbations are not realistic. Dealing with more realistic local perturbations would take more hypotheses to explore.

Third, the Γ_{5-} solution (Equation (18)) is remarkable since it permits an oscillating universe with $x > 0$ always (i.e., true oscillating Friedmann-equation universe model without extra hypotheses). That the Friedmann equation has oscillating solutions has probably been long known, but Bondi (1961) in his review of early cosmological models does not mention them. What he refers to as oscillating models (his Class V universe models) have Friedmann equation solutions that go into the unphysical negative x value range (Bondi 1961, p. 81–86,122). He does hypothesize that such universe models are cyclic: i.e., the positive range of the solution repeats itself after each solution zero (Bondi 1961, p. 82,86) Note the ocscillating Γ_{5-} solution is not limited to the case of the V model with $V = 2/3$. A radiation-negative-curvature-negative-Λ universe (i.e., one with $a < 0$, $b > 0$, and $c > 0$) can also have an ocscillating Γ_{5-} solution just with $V = 1/2$.

4. The Universe Age τ_0

Although the $\Lambda\Gamma$ model solutions (i.e., the Γ_i solutions: Equations (14), (15) (16), and (17)) are elegant analytic solutions, it is not obvious how general variations of parameters a, b, and c will affect their overall behavior. For example, in the Γ_1 solution (Equation (14)), the b parameter (i.e., the Γ parameter) occurs four times (twice implicitly in $\tau_{\mathrm{zero}\,x}$), and so the effect of varying b on the Γ_1 solution is clearly not obvious.

What is needed is a single metric of overall solution behavior for the Γ_i solutions. The universe age (defined in § 1: i.e., time from a Big Bang to cosmic present τ_0) seems a good choice: the faster overall growth of the solution, the smaller τ_0. Now with the $V = 2/3$ explicitly for numerical evaluation, the $\Gamma_{123}^{-1}(x = 1)$ solution (Equation (8) which gives the universe age appropriate for all of the $\Gamma_1(\tau)$, $\Gamma_2(\tau)$, and $\Gamma_3(\tau)$ solutions: i.e., Equations (14), (15), and (16)) is

$$\tau_0 = \frac{2}{3}\frac{1}{\sqrt{a}}\ln\left[\frac{1-c+a+2\sqrt{a}}{1-c-a+2\sqrt{ac}}\right] \tag{19}$$

and the $\Gamma_4^{-1}(x = 1)$ solution (Equation (12) which has $a = 0$ and $b \geq 0$) is

$$\tau_0 = \begin{cases} \dfrac{2}{3}\left(\dfrac{2}{1+\sqrt{1-b}}\right) = \dfrac{2}{3}\left(\dfrac{2}{b}\right)\left(1-\sqrt{1-b}\right) & \text{in terms of } b. \\ \dfrac{2}{3}\left(\dfrac{2}{1+\sqrt{c}}\right) & \text{in terms of } c. \end{cases} \tag{20}$$

Unfortunately, Equation (19) is still too complex to just visualize the behavior of τ_0 as function of a and c. So we will consider a range of special case behaviors. Case 1 has $b = 0$, and so $c = 1 - a$. Case 1 is, in fact, the Λ-CDM model universe age case. The formula is

$$\tau_0 = \begin{cases} \frac{2}{3}\frac{1}{\sqrt{a}}\ln\left(\frac{1+\sqrt{a}}{\sqrt{1-a}}\right) & \text{in general: a well known formula (e.g., Liddle 2015, p. 63).} \\ \frac{2}{3} = 0.6666\ldots & \text{for } a = 0\text{: the Einstein-de Sitter universe age.} \\ \frac{2}{3}\left(\sum_{k=0}^{\infty}\frac{a^k}{2k+1}\right) = \frac{2}{3}\left(1 + \frac{a}{3} + \frac{a^2}{5} + \frac{a^3}{7} + \ldots\right) & \text{for } a \in (0,1). \\ 0.964099381639\ldots & \text{for } a = 0.7,\ c = 0.3\text{: the fiducial Λ-CDM model parameter weights giving the fiducial Λ-CDM model universe age.} \\ \infty & \text{for } a = 1\text{: the infinite-age de Sitter universe age.} \end{cases} \tag{21}$$

In fact, the Λ-CDM model universe age strictly increases with a (e.g., Liddle 2015, p. 63), and so increasing matter parameter c increases the overall growth rate of the solution (i.e., decreases the Λ-CDM model universe age).

Case 2 has $c = 0$, and so $b = 1 - a$. In Case 2, the Γ dark energy is the analogue to matter in the Λ-CDM model. The formula is:

$$\tau_0 = \begin{cases} \frac{2}{3}\frac{1}{\sqrt{a}}\ln\left(\frac{1+a+2\sqrt{a}}{1-a}\right) & \text{in general. In fact, the Case 2 } \tau_0 \text{ formula} \\ \quad = \frac{4}{3}\frac{1}{\sqrt{a}}\ln\left(\frac{1+\sqrt{a}}{\sqrt{1-a}}\right) & \text{is exactly 2 times the Case 1 } \tau_0 \text{ formula.} \\ \frac{4}{3} = 1.3333\ldots & \text{for } a = 0\text{: the pure Γ dark energy universe age.} \\ 1.9281987632789\ldots & \text{for } a = 0.7,\ b = 0.3\text{: the analogue to the fiducial Λ-CDM model universe age.} \\ \infty & \text{for } a = 1\text{: the infinite-age de Sitter universe age.} \end{cases} \tag{22}$$

Comparing Equations (21) and (22), we see that Γ dark energy gives a slower rate of growth than matter (i.e., gives larger universe age for comparable cases). In fact, all the Case 2 universe ages are exactly 2 times those of the corresponding Case 1 universe ages since, as noted above, the Case 2 is τ_0 formula is exactly 2 times the Case 1 τ_0 formula.

Case 3 has $a = 0$, and so $b = 1 - c$. In Case 3, the Γ dark energy is the analogue to Λ dark energy in the Λ-CDM model. Because $a = 0$, the $\Gamma_{123}^{-1}(x = 1)$ solution (Equation (19)) is inappropriate, and so instead, making use Equation (20), we find

$$\tau_0 = \begin{cases} \frac{2}{3}\left(\frac{2}{1+\sqrt{1-b}}\right) = \frac{2}{3}\left(\frac{2}{b}\right)\left(1-\sqrt{1-b}\right) & \text{general in terms of } b. \\ \frac{2}{3}\left(\frac{2}{1+\sqrt{c}}\right) & \text{general in terms of } c. \\ \frac{2}{3} = 0.6666\ldots & \text{for } b = 0,\ c = 1\text{: the Einstein-de Sitter universe age.} \\ \frac{2}{3}\left[\sum_{k=0}^{\infty}\frac{(2k-1)!!}{2^k(k+1)!}b^k\right] = \frac{2}{3}\left(1+\frac{b}{4}+\frac{b^2}{8}+\ldots\right) & \text{for } b \in [0,1]. \\ 0.861480842847\ldots & \text{for } b = 0.7,\ c = 0.3\text{: the analogue to the fiducial } \Lambda\text{-CDM model.} \\ \frac{4}{3}\sum_{k=0}^{\infty}(-1)^k c^{k/2} \\ \quad = \left(1 - c^{1/2} + c - c^{3/2} + c^2 - \ldots\right) & \text{for } c < 1. \\ \frac{4}{3} = 1.3333\ldots & \text{for } b = 1,\ c = 0\text{: the pure } \Gamma \text{ universe age.} \end{cases} \tag{23}$$

Comparing Equations (21) and (23), we see that Γ dark energy gives faster growth than Λ dark energy (i.e., gives smaller universe age for comparable cases).

Case 4 has $a = c$, and so $b = 1 - 2a$. Case 4 is a constrained, and thus simplified, version of the $\Lambda\Gamma$ model useful for a general, but simplified, understanding of its behavior.

The formula is

$$\tau_0 = \begin{cases} \dfrac{2}{3}\dfrac{1}{\sqrt{a}}\ln\left(1+2\sqrt{a}\right) & \text{in general.} \\ \dfrac{2}{3}\sqrt{2}\ln\left(1+\sqrt{2}\right) = 0.83096698685\ldots & \text{for } a=c=1/2,\ b=0\text{: a very non-fiducial } \Lambda\text{-CDM universe age.} \\ \dfrac{2}{3}\sqrt{3}\ln\left(1+\dfrac{2}{\sqrt{3}}\right) = 0.886407892004\ldots & \text{for } a=b=c=1/3\text{: an equal-parameter-weight } \Gamma\Lambda \text{ model universe age.} \\ 0.964070206453\ldots & \text{for } a=c=0.182,\ b=0.636\text{: this universe age is equal to 5 digits to the fiducial } \Lambda\text{-CDM model universe age.} \\ \dfrac{4}{3}\left[\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{(2\sqrt{a}\,)^k}{k+1}\right] & \text{for } 2\sqrt{a}=2\sqrt{c}\leq 1. \\ \quad = \dfrac{4}{3}\left[1-\dfrac{(2\sqrt{a})}{2}+\dfrac{(2\sqrt{a}\,)^2}{3}+\ldots\right] & \\ \dfrac{4}{3} = 1.3333\ldots & \text{for } a=c=0 \text{ and } b=1\text{: the pure } \Gamma \text{ dark energy } \Gamma\Lambda \text{ model universe age.} \end{cases} \tag{24}$$

Cases 1–3 show that increasing the c parameter (i.e., the matter parameter) increases the rate of solution growth relative growth given by the pure a parameter (i.e., the Λ parameter) and increasing the b parameter (i.e., the Γ parameter) does the same, but to a lesser degree. Thus, increasing the b parameter can be compensated for by decreasing the c parameter. But this is just an aspect of the fact that the $\Lambda\Gamma$ model cosmic scale factor solution is more flexible than the Λ-CDM model cosmic scale factor solution since it has two free model parameters (any two of a, b, and c) instead of just one like the Λ-CDM model cosmic scale factor solution (either of a or c). However, the $\Lambda\Gamma$ model is not at all extremely flexible, and so it can be falsified.

Case 4 shows that a constrained, and so simplified, $\Lambda\Gamma$ model can match the fiducial Λ-CDM model universe age which is very probably correct to within a few percent given that Λ-CDM model gives an extremely good fit to cosmic evolution in many respects. However, this matching is done with an implausibly large b parameter weight ($b = 0.636$) and implausibly small a and c parameter weights ($a = c = 0.182$) given the aforesaid overall goodness of the Λ-CDM model. However, it is likely that varying both a and c independently will give a range of relatively small b values that will make the ΛΓ model give a good fit to the fiducial Λ-CDM model universe age. But finding that range in itself is not an interesting project. In § 5, we undertake a more interesting project of trying to match the fiducial Λ-CDM model universe age and give a very limited fit to the DESI DR2 results (Lodha et al. 2025; Abdul Karim et al. 2025).

5. The $Om(z)$ Diagnostic

The $Om(z)$ diagnostic (Sahni et al. 2008) is essentially a way of rewriting the derivative of the cosmics scale factor $x(\tau) = 1/(1+z)$ (where z is cosmic redshift) in a way to emphasize features not otherwise obvious in direct presentations of $x(\tau)$ and its derivative. To slightly generalize from (Sahni et al. 2008) (but with a limitation to density components that evolve as inverse powers of the cosmic scale factor), we write the scaled Friedmann equation thusly

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_{n=0}^{N} a_n(1+z)^{p_n} \ , \tag{25}$$

where the a_n are cosmic present density parameters, $\sum_{n=0}^{N} a_n = 1$ (which give cosmic present $h = 1$), $p_0 = 0$, and $p_{n\geq1} > 0$ are general powers that increase in size with index n implying p_N is the largest power. Using Equation (25), the $Om(z)$ diagnostic becomes

$$Om(z) = \begin{cases} \dfrac{h^2(z)-1}{(1+z)^{p_N}-1} = \dfrac{\left[\sum_{n=0}^{N} a_n(1+z)^{p_n}\right]-1}{(1+z)^{p_N}-1} & \text{general formula.} \\[2ex] \dfrac{\sum_{n=1}^{N} a_n p_n + (1/2)\left[\sum_{n=1}^{N} a_n p_n(p_n-1)\right] z}{p_N + (1/2)[p_N(p_N-1)]z} & \text{1st order} \\ & \quad \text{in small } z \text{ formula.} \\[2ex] \dfrac{\sum_{n=1}^{N} a_n p_n}{p_N} & \text{for } z = 0. \\[2ex] a_N & \text{for } z \to \infty. \end{cases} \tag{26}$$

We now specialize the general formula in Equation (26) for the case of the V model, where $a = a_0$, $b = a_1$, and $c = a_2$, and $p_0 = 0$, $p_1 = q = p/2$, and $p_2 = p$:

$$Om(z) = \begin{cases} \dfrac{a + b(1+z)^q + c(1+z)^p - 1}{(1+z)^p - 1} & \text{the } V \text{ model general formula.} \\ \dfrac{bq + cp + (1/2)[bq(q-1) + cp(p-1)]z}{p + (1/2)p(p-1)z} & \text{the V model 1st order formula.} \\ \dfrac{bq+cp}{p} = \left(\dfrac{1}{2}\right) b + c & \text{for } z = 0. \\ c & \text{for } z \to \infty = 0. \end{cases} \tag{27}$$

We now specialize the general formula in Equation (27) for the case of the $\Lambda\Gamma$ model (i.e., the V model with $p = 3$, $q = 3/2$, and $V = 2/3$):

$$Om(z) = \begin{cases} \dfrac{a + b(1+z)^{3/2} + c(1+z)^3 - 1}{(1+z)^3 - 1} & \text{the } \Lambda\Gamma \text{ model formula} \\ \quad = c + \dfrac{b[(1+z)^{3/2} - 1]}{3z[1 + z + (1/3)z^2]} & \\ c + \dfrac{(b/2)}{[1 + z + (1/3)z^2]}\left[1 + \dfrac{z}{4} - \dfrac{z^2}{24}\right. & \text{the } \Lambda\Gamma \text{ model} \\ \qquad\qquad \left. + \displaystyle\sum_{k=3}^{\infty} \dfrac{(-1)^{k-1}(2k-3)!!}{2^k (k+1)!} z^k \right] & \text{series expansion.} \\ & \text{Absolutely convergent} \\ & \text{for } z \in [-1, 1]. \\ c + \dfrac{(b/2)}{[1 + z + (1/3)z^2]}\left[1 + \dfrac{z}{4}\right. & \text{the } \Lambda\Gamma \text{ model series} \\ \qquad\qquad \left. - \dfrac{z^2}{24} + \dfrac{z^3}{64} + \ldots \right] & \text{expansion truncated} \\ & \text{to 3rd order.} \\ c + \dfrac{b}{2} & \text{for } z = 0. \\ c & \text{for } z \to \infty. \end{cases} \tag{28}$$

In determining Equation (28), we made use of the somewhat difficult to find general expansion formula for $(1+z)^{n/2}$ where n is an odd integer (i.e., $n = \pm 1, \pm 3, \pm 5 \ldots$), and so we give it here for general reference:

$$(1+z)^{n/2} = 1 + \sum_{k=1}^{\infty} (-1)^{\max[0, k-(n+1)/2]} \frac{(2k-n-2)!!}{2^k k!(-n-2)!!} \frac{n!!}{(n-2\ell)!!} z^k \ , \tag{29}$$

where (integer ≤ 0)!! $= 1$ in all cases. For $n > 0$, the series is absolutely convergent for $z \in [-1, 1]$ and for $n < 0$, for $z \in (-1, 1)$. For $z = 1$, there is conditional convergence for $n = -1$, but not for $n \leq -3$.

Lodha et al. (2025) applied $Om(z)$ diagnostic to several cosmological models fitted to various data combinations and to curves generated by Gaussian process regressions (i.e., a non-parametric way of generating curves) applied to various data combinations. Insofar as their models or Gaussian process regressions and data combinations are accurate, their $Om(z)$ curves are a true measure of the evolution of cosmic scale factor emphasizing as aforesaid features not otherwise obvious.

As a preliminary test of the $\Lambda\Gamma$ model, we have done a crude fit to the $Om(z)$ diagnostic curve of Lodha et al. (2025) (shown in their Figure 9) generated using a Gaussian process regression based on the DESI+CMB+Union3 data combination. The fit is crude since we just measured the Gaussian process regression $Om(z)$ curve off of the Figure 9 of Lodha et al. (2025). The criteria for the fit were that the $z = 0$ and $z = 3$ endpoints of Lodha et al. (2025) ($Om(z = 0) \approx 0.45$ and $Om(z = 3) \approx 0.315$) should fit within 2 standard deviations (respectively, ~ 0.07 and ~ 0.01), the matter parameter $c \geq 0.29$ (since Abdul Karim et al. (2025, Fig. 16) strongly disfavor a lower value) and the universe age $t \geq 13.5\,\mathrm{Gyr}$ (since the oldest globular clusters suggest $13.5\,\mathrm{Gyr}$ is a lower limit on the age of the universe in round numbers (e.g., Valcin et al. 2025)).

For the Hubble constant value needed to convert scaled cosmic time τ into unscaled cosmic time t, we used $H_0 = 68.01\,(\mathrm{km/s})/\mathrm{Mpc}$ from Abdul Karim et al. (2025, Table V) since it was based on a model using DESI+CMB+Union3 data combination like the Gaussian process regression $Om(z)$ curve and the model otherwise seemed a reasonable proxy for the Gaussian process regression. We varied the $\Lambda\Gamma$ model parameters a, b, c over all parameter space in steps of 0.01 consistent with the $a + b + c = 1$ constraint, $a > 0$ (since $a = 0$ just gives an approximate power-law dependence of x on τ (see the Γ_4 in Equation (17) and that is remote from the Λ-CDM behavior which is accepted as a good approximation in any case and $a < 0$ has solutions with maxima (see the Equation (18)), and $b \geq 0$ (since we need positive Γ dark energy to achieve the decreasing total dark energy implied by data of Lodha et al. 2025 and Abdul Karim et al. 2025).

Only two $\Lambda\Gamma$ model solutions matched all the constraints and they were nearly the same. We deem the slightly better one to have the Λ parameter $a = 0.53$, the Γ parameter $b = 0.18$, and the matter parameter $c = 0.29$. The quantity $b^2 - 4ac = -0.5824$ which is less than zero, and so the Γ_1 solution (Equation (14)) is the fitted solution. We expected the Γ_1 solution to be best since it is closest the Λ-CDM solution. The universe age $t = 13.54\,\mathrm{Gyr}$ which is significantly lower the Λ-CDM universe age 13.797(23) Gyr favored by Planck Collaboration

(Aghanim et al. 2021, p. 15). The relative deviation from the $\Lambda\Gamma$ model $Om(z)$ from that of Lodha et al. (2025) (whose values were crudely measured) is $-15.6\,\%$ at $z = 0$ rising to $3.8\,\%$ at $z = 1.2$ and then declining to $-1.6\,\%$ at $z = 3$.

Given the crudeness of the fitting procedure, the fit solution achieved by the $\Lambda\Gamma$ model is adequate for the observations so far. However, given that the $\Lambda\Gamma$ model is just a heuristic model, the fit solution can only be considered as interesting.

6. The Deceleration Parameter

The deceleration parameter (symbolized here by $q_{\rm dec}$ to distinguish it for the power q) can also be used as a diagnostic. It is essentially a way of rewriting the 2nd derivative of the cosmics scale factor $x(\tau) = 1/(1+z)$ (where z is cosmological redshift) in a way to emphasize features not otherwise obvious in direct presentations of $x(\tau)$ and its 1st and 2nd derivatives. We do not make use of the deceleration parameter as diagnostic in this paper, but for future reference, we give general and special case formulae below.

Making use of the standard general formula for the deceleration parameter (e.g., Liddle 2015, p. 53), the Friedmann acceleration equation (e.g., Liddle 2015, p. 27), and Equation (3) and Equation (25) (which implies a limitation to density components that evolve as inverse powers of the cosmic scale factor), we obtain

$$q_{\rm dec} = \begin{cases} -\dfrac{\ddot{x}x}{\dot{x}^2} = -\dfrac{\ddot{x}}{x}\dfrac{1}{h(x)^2} = \dfrac{1}{2}\left[\dfrac{\sum_{n=0}^{N} a_n(p_n-2)(1+z)^{p_n}}{\sum_{n=0}^{N} a_n(1+z)^{p_n}}\right] & \text{general formula.} \\ -1+\dfrac{1}{2}\left[\dfrac{\sum_{n=1}^{N} a_n p_n(1+z)^{p_n}}{\sum_{n=0}^{N} a_n(1+z)^{p_n}}\right] & \begin{array}{l}\text{general formula} \\ \quad\text{recalling} \\ \quad\sum_{n=0}^{N} a_n = 1 \\ \quad\text{and } p_0 = 0.\end{array} \\ = -1+\left(\dfrac{1}{2}\right)\displaystyle\sum_{n=1}^{N} a_n p_n & \text{for } z = 0. \\ -1+\dfrac{p_N}{2} & \text{for } z \to \infty. \end{cases} \qquad (30)$$

$$
q_{\rm dec} = \begin{cases}
-1 + \dfrac{1}{2}\left[\dfrac{bq(1+z)^q + cp(1+z)^p}{a + b(1+z)^q + c(1+z)^p}\right] & \text{for the } V \text{ model} \\
 & \quad \text{recalling } q = p/2. \\
-1 + \dfrac{1}{2}(bq + cp) = -1 + \dfrac{p}{2}\left(\dfrac{b}{2} + c\right) & \text{for } z = 0 \\
\quad = -1 + \dfrac{p}{4}(1 - a + c) & \text{and using } b = 1 - (a+c)/2. \\
-1 + \dfrac{p}{2} & \text{for } z \to \infty.
\end{cases} \tag{31}
$$

$$
q_{\rm dec} = \begin{cases}
-1 + \dfrac{1}{2}\left[\dfrac{(3/2)b(1+z)^{3/2} + 3c(1+z)^3}{a + b(1+z)^{3/2} + c(1+z)^3}\right] & \text{for the } \Lambda\Gamma \text{ model.} \\
1 + \dfrac{3}{2}\left(\dfrac{b}{2} + c\right) = -1 + \dfrac{3}{4}(1 - a + c) & \quad z = 0 \text{ and using} \\
 & \quad b = 1 - (a+c). \\
\dfrac{1}{2} & \text{for } z \to \infty \\
 & \quad \text{and also for the} \\
 & \quad \Lambda\text{-CDM model} \\
 & \quad \text{with } z \to \infty.
\end{cases} \tag{32}
$$

$$
q_{\rm dec} = \begin{cases}
-1 + \dfrac{3}{2}\left[\dfrac{c(1+z)^3}{a + c(1+z)^3}\right] & \text{for the } \Lambda\text{-CDM model.} \\
-1 + \dfrac{3}{2}c = \dfrac{1}{2} - \dfrac{3}{2}a & \text{for } z = 0 \text{ and} \\
 & \quad \text{using } c = 1 - a. \\
-0.55\left[\dfrac{0.5 - 1.05 \times (a/0.7)}{-0.55}\right] & \text{for } z = 0 \text{ and} \\
 & \quad \text{fiducial } a = 0.7.
\end{cases} \tag{33}
$$

7. Discussion

It would be interesting project is to see if $\Lambda\Gamma$ model Γ_1 solution (Equation (14)) can give a good fit to all data cited by Lodha et al. (2025) and Abdul Karim et al. (2025) with a least-squares fit done by varying the a parameter (i.e., the Λ parameter: conventionally the constant density parameter Ω_Λ) and the c (i.e., the matter parameter and conventionally the cosmic present matter density parameter $\Omega_{\rm M,0}$). The fact that the Λ-CDM cosmic scale factor solution (i.e, the matter-Λ universe solution) fits all the aforesaid data to the eye very well (Abdul Karim et al. 2025, Fig. 1) suggests that the b parameter (i.e., Γ parameter: the cosmic present Γ dark energy density with the constraint $b = 1 - (a + c)$) will be relatively small in a good fit. However, the crude fit to the $Om(z)$ curve of Lodha et al. (2025) (§ 5) suggests the possibility that Γ dark energy density parameter b might larger than very small.

To recapitulate from the abstract, given that $\Lambda\Gamma$ model is just a heuristic model with no physical reason for its equation of state parameter $w_\Gamma = -1/2$, any expectation that even a good fit of it to observations will be meaningful is very modest.

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