EXACT TWO-DENSITY COMPONENT SOLUTIONS FOR THE COSMIC SCALE FACTOR FROM A GENERAL APPROACH INCLUDING A SIMPLIFIED EXACT SOLUTION FORMULA FOR THE RADIATION-MATTER UNIVERSE

David J. Jeffery[1]

ABSTRACT

We present a general treatment for exact solutions (i.e., exact analytical solutions) for the cosmic scale factor for universes (i.e., universe models) for the Friedmann equation depending on only two density components (which are formulated as density parameters Ω_P) both of which depend on inverse powers of the cosmic scale factor via terms of the form $\Omega_{P,0}(a_0/a)^P$. For the Friedmann equation with only two density components, there are in fact only 3 exact general solutions for a generalized cosmic scale factor z as function a generalized conformal time $\tilde{\eta}$. All other special cases of two-density-component solutions for (ordinary) cosmic scale factor and (ordinary) cosmic time follow from these although complete exact solutions (those for which are available either of exact $a(t)$ and $t(a)$ or both exact a and t in terms conformal time or generalized conformal time) are only available for a restricted class of solutions. We give a fairly detailed explication of the 3 general two-density-component solutions and some examples of interesting special cases of complete exact solutions. However, we usually leave these complete exact solutions in terms of scaled scale factors (symbolized by variables x or y) from which a is easily obtained and scaled cosmic times (symbolized by variables τ or w) from t is easily obtained. For simplicity, we usually just call a, x, and y scale factors and t, τ, and w time. The complete exact solutions we give include all cases with one of the two density components having power $P = 0$ (i.e., the Λ density component) and the radiation-matter universe (which has density components having powers $P = 4$ and $P = 3$). The radiation-matter universe, in fact, has an exact solution formula for the scale factor as function of time (reported as $y(w)$ here) given by Galanti & Roncadelli (2021) which we call the 1st exact formula. We have found another exact solution formula (which we call the 2nd exact formula) which we judge to be simpler.

[1]Department of Physics & Astronomy and Nevada Center for Astrophysics (NCfA), University of Las Vegas, Nevada, 4505 S. Maryland Parkway Las Vegas, Nevada 89154, U.S.A.

The two exact formulae are, of course, mathematically equivalent, but that is not obvious to the eye. They are also numerically equivalent to within machine precision, of course. Both formulae lose numerical accuracy as time goes toward time zero, and so we derive the small time expansion series for them to use for high accuracy at very early times. We also present a moderately??? accurate interpolation formula (which we believe to be novel) for the cosmic scale factor for the Λ-CDM model (i.e., the radiation-matter-Λ universe for a limited range of $\Omega_{P,0}$ values) which is complementary to the nearly exact formula of Galanti & Roncadelli (2021).

For reference, Appendix A presents elementary one-density component exact solutions to the Friedmann equation. In Appendix B, we consider how far we can generalize our two-density-component treatment to a three-density-component treatment. The generalization does give Friedmann equation solutions that apply to the radiation-curvature-Λ universe and the radiation-cosmic-string-Λ universe and these solutions may be of actual cosmological interest for some versions of inflation. Of these solutions, one is for the radiation analogue to the Lemaître universe and another for radiation analogue to the Lemaître-Eddington universe. Both these historic universes are positive-curvature-matter-Λ universes. In Appendix C, we consider stationary points and constants solutions of 1st order autonomous ordinary differential equations since these are relevant to the Friedmann equation (which is a 1st order autonomous ordinary differential equation). In Appendix D, we review the stability of constant solutions to 1st order autonomous ordinary differential equations.

Because we present a fair number of the more elementary special case solutions of the Friedmann equation for the cosmic scale factor (most of which must appear in many places in the literature), this paper also constitutes an educational review of such solutions.

Unified Astronomy Thesaurus concepts: Cosmology (343); Accelerating universe (12); Big Bang theory (152); Closed universe (256); Cosmic background radiation (317); Cosmological constant (334); Cosmological evolution (336); Cosmological models (337); Cosmological parameters (339); Cosmological principle (2363); Dark energy (351); de Sitter universe (361); Density parameter (372); Einstein universe (452); Expanding universe (502); Flat space (543); Friedmann universe (551); Lambda density (898); Lemaître universe (914); Matter density (1014); Open universe (1161)

1. INTRODUCTION

To continue from the abstract, in § 2, we introduce the Friedmann equation and the primary scaled Friedmann equation. In § 4, we discuss general exact solutions to the Friedmann equation and define complete and incomplete exact solutions. In § 5, we derive a two-density-component Friedmann equation in terms of the generalized cosmic scale factor z as function a generalized conformal time $\tilde{\eta}$. We call this equation just the two-density-component Friedmann equation for simplicity. In section § 3, we consider stationary points and constant solutions of the Friedmann equation in general in preparation for considering these items for the two-density-component solutions. Then in § 6 we derive the 3 general formulae for generalized cosmic scale factor z as functions of generalized cosmic time $\tilde{\eta}$. Section 7, we find the all the cases there is a complete exact solution which requires there to be an analytic solution for cosmic time as a function of generalized cosmic time. In § 12, we derive formulae for special complete exact solutions of interest, in most of which generalized cosmic time $\tilde{\eta}$ has been eliminated. These cases include all cases with one of the two density components having power $P = 0$ (i.e., the Λ density component) and the radiation-matter universe (which has density components having powers $P = 4$ and $P = 3$). In § 13, we rederive the well known exact solution for cosmic time for the radiation-matter universe but from a straightforward approach and not our general treatment as a complement to the general treatment and present some further analysis of it. We then compare our formula for the cosmic scale factor for the radiation-matter universe to that of Galanti & Roncadelli (2021) in § 14. In § 15, we present a method for smoothed-piecewise-approximate (SPA) solutions to the Friedmann equation that is educationally useful and present an example SPA solution for the Λ-CDM model. The conclusions (§ 16) are followed by Appendices A, B, C, and Appendix D.

2. THE FRIEDMANN EQUATION AND THE PRIMARY SCALED FRIEDMANN EQUATION

The Friedmann equation written in a standard modern form is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \ , \tag{1}$$

where H is the Hubble parameter, a is the cosmic scale factor, $\dot{a}$ is the time derivative of a, G is the gavitational constant, k is the curvature (positive/negative for a positive/negative curvature universe and zero for a flat universe) which is always time-independent for the Friedmann equation, and Λ is the cosmological constant e.g., Wikipedia: Friedmann equations; Liddle 2015, p. 24,55–56). Note that extended without bound, a positive curvature universe is a finite unbounded hyperspherical space, a negative curvature universe is an infinite hyperbolical space, and the zero curvature universe is an infinite Euclidean (i.e., flat) space (see Wikipedia: Shape of the universe; Wikipedia: n-sphere; Liddle 2015, p. 29–36). For a general treatment, the density ρ is broken up into components that in the most considered cases depend on various inverse powers of a (see below) and the curvature and the Λ terms are written as pseudo density terms to allow a uniform treatment: i.e.,

$$\rho_k = -\frac{3k}{8\pi G a^2} \qquad \text{and} \qquad \rho_\Lambda = \frac{\Lambda}{8\pi G} \ , \tag{2}$$

where the minus sign has been absorbed into ρ_k for uniformity of appearance with the annoying side effect of making ρ_k negative/positive for positive/negative curvature.

All the free parameters to determine the solution of Equation (1) can be set in the density components. However, one can substitute for these by fits to the solution itself. The most considered substitution is of the Hubble constant (i.e., the Hubble parameter at a fiducial cosmic time) to eliminate any one of the density component parameters, usually the curvature k since that sets the whole curvature density component evolution given the solution a itself.

For finding analytic solutions, it is convenient to change the standard modern form Friedmann equation (Equation (1) above) to a scaled form. The primary scaled form we give here where a and t are replaced by the scaled x and τ: see below. We need a secondary scaling to deal with the specific exact solutions we investigate. For these solutions, the scaled y and w replace x and τ; for the specificaltion of y, see § 5; for the specificaltion of w, see § 7. Note the unscaled variables and primary and secondary variables are linearly related: all the scale factor variables are mutually proportional; all the time variables are mutually proportional. To be explicit:

$$a(t) \propto x(\tau) \propto y(w) \qquad \text{and} \qquad t(a) \propto \tau(x) \propto w(y) \ . \tag{3}$$

However, the 3 exact general solutions use generalized cosmic scale factor z and generalized conformal time $\tilde{\eta}$ which are not proportional, respectively, to the ordinary scale factor and time variables. For their specification, see § 5.

The primary scaled Friedmann equation written in terms of density parameters (i.e., density components divided by the critical density $\rho_{\text{critical},0}$: see below) and scaled variables is

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_P \Omega_{P,0} x^{-P} \qquad \text{or} \qquad h = \frac{\dot{x}}{x} = \pm\sqrt{\sum_P \Omega_{P,0} x^{-P}} \ , \tag{4}$$

where 0 indicates the fiducial cosmic scale time t_0 (which is not in general cosmic present in this paper, but it can be), the $\Omega_{P,0} = \rho_{P,0}/\rho_{\mathrm{critical},0}$ are the density parameters at t_0

$$x = \frac{a}{a_0} \tag{5}$$

is a scaled cosmic scale factor, a_0 is the cosmic scale factor at t_0 (which is usually set by $a_0 = 1$ especially for cosmic present, but can be left general), h is the (dimensionless) scaled Hubble parameter, and the implicit time variable τ is a scaled (dimensionless) cosmic time. To explicate the scaled variables and other variables:

$$\begin{cases}
\tau = \dfrac{t}{t_{H_0}} & \text{the scaled time which equals 1} \\
 & \text{at } t = t_{H_0}; \\
t_{H_0} = \dfrac{1}{\sqrt{(8\pi G/3)\rho_{\mathrm{critical},0}}} & \text{is the Hubble time at } t_0; \\
H_0 = t_{H_0}^{-1} & \text{is the Hubble parameter at } t_0. \\
 & \text{(i.e., the Hubble constant for } t_0; \\
\rho_{\mathrm{critical},0} = \dfrac{H_0^2}{(8\pi G/3)} = \dfrac{1}{(8\pi G/3)t_{H_0}^2} & \text{is the critical density at } t_0; \\
\Omega_{P,0} = \dfrac{\rho_{P,0}}{\rho_{\mathrm{critical},0}} & \text{the general density parameter at the } t_0. \\
\sum_P \Omega_{P,0} = 1 & \text{at } t_0 \text{ or } \tau_0\text{, this is required;} \\
H(t_0) = H_0 & \text{is the equation for determining } t_0; \\
h(\tau_0) = 1 & \text{is the equation for determining } \tau_0 \\
 & \text{which, of course, is equivalent to} \\
 & \text{the equation determining } t_0. \\
 & \text{Note } t_0 \text{ and } \tau_0 \text{ are usually} \\
 & \text{part of the solution of the Friedmann} \\
 & \text{and are not taken as free parameters.}
\end{cases} \tag{6}$$

Note that if we fully specify the density components $\rho_{P,0}$ by their values at t_0 the solution is fully determined for the scaled Friedmann equation (Equation (4) above). However, to go from the generic case as mentioned above to the specific case here, one can substitute for these density components by fits to the solution itself. The most considered substitution is of the Hubble constant H_0 (or equivalently $\rho_{\mathrm{critical},0}$ or t_{H_0} to eliminate any one of the density component parameters, usually the curvature k since that sets the whole curvature density component evolution given the solution a itself. For actual cosmological modeling, the value of cosmic present Hubble constant H_0 is often used as one of the free parameters for cosmic scale factor solutions since it is a direct observable. In this paper, we are mostly concerned with analytic solutions in scaled quantities (which have natural parameters), and so do not set free parameters except in § 15 where we analyze the accuracy of smoothed-piecewise-approximate solutions.???

In regard to solutions of Equation (4), we will not consider those that are negative or complex since they cannot be physically real. Also note that the second form of Equation (4) has the $\pm$ cases. What this means is that every solution $x(\tau)$ has a twin solution $x(-\tau)$. One can see just by writing $d\tau_{\rm twin} = -d\tau$ and then the solution for $x(\tau_{\rm twin})$ clearly is the same as $x(\tau)$ and $x(-\tau)$ is $x(\tau)$ mirror imaged around $\tau = 0$ vertical. Another perspective from the $\pm$ cases is that to any solution at a point there is another solution whose slope has the opposite sign to the first solution. If $x(\tau)$ is monotonically increasing, then $x(-\tau)$ is monotonically decreasing. Since the actual universe does not have a monotonically decreasing cosmic scale factor, monotonically decreasing solutions are not very interesting. To avoid pointless generality, we will usually not explicitly give any monotonically decreasing solutions.

Is there a second kind of twin solution? If the right-hand side of Equation (4) depeonds only on even powers of x and $x \geq 0$ is a solution, then $-x \leq 0$ is a solution, but an unphysical one since $x \geq 0$ is required for physical scale factor. We will usually not consider the second kind of twin solutions.

Re: the critical density, Albert Einstein (1879–1955) and Willem de Sitter (1872–1934) introduced the concept of critical density for what we call the Einstein-de Sitter universe (presented 1932: e.g., Wikipedia: Einstein-de Sitter universe; Bondi 1961, p. 166; North 1994, p. 535; Kragh 1996, p. 35,37–38,50,73–75,79,274,286–287; O'Raifeartaigh et al. 2015) which was often considered the standard cosmological model circa 1960–1995 before the start of the dominance of the Λ-CDM circa 1995 (e.g., Bondi 1961, p. 166; Scott 2018, p. 10). The only mass-energy content in the Einstein-de Sitter universe is matter in the cosmological sense of matter at rest or nearly at rest in the comoving frames of the expanding universe :i.e., those inertial frames which participate in the mean expansion of the universe as determined by the cosmic scale factor. The density of the matter is exactly the critical density and therefore the critical denstiy, the Hubble constant, and the Hubble time are information equivalent parameters as seen in Equation (6 above. Einstein and de Sitter introduced the Einstein-de Sitter universe as the simplest universe model and they believed that that was all that was justified by observations circa 1932 (e.g., Kragh 1996, p. 35; O'Raifeartaigh et al. 2015) The Einstein-de Sitter universe has space flat and infinite and the cosmic scale factor and time since the point origin (the older jargon name for the Big Bang singularity:

Bondi e.g., 1961, p. 117) are given by, respectively,

$$a = a_0 \left(\frac{t}{t_0} \right)^{2/3} \qquad \text{and} \qquad t_0 = \frac{2}{3} t_{H_0} = \frac{2}{3} \frac{1}{H_0} \ . \tag{7}$$

In fact, in Einstein and de Sitter's short paper of 1932 did not explicitly give the results in Equation (7) though Einstein did give them explicitly in a review paper of 1933 (e.g., O'Raifeartaigh et al. 2015, p. 15–16). The review shows that as of 1933, Einstein did not believe simple relativistic models could accurately describe the early universe (e.g., O'Raifeartaigh et al. 2015, p. 1). In fact in the 1930s, it seems that only Georges Lemaître (1894–1966) theorized about an early universe that was radically different from the modern universe (e.g., McCrea 1984, p. 13; Kragh 1996, p. 56). We give a brief discussion of Lemaître's theories in Appendix B.1: i.e., the Lemaître universe (as we call it: positive-curvature-matter-Λ universe) and the primeval atom (e.g., Bondi 1961, p. 82,84–85,120–122,165,168–170,175–176; McCrea 1984, p. 7–10; Peebles 1984, p. 23–30; North 1994, p. 528,530–531; Kragh 1996, p. 23–60; Luminet 2011).

3. STATIONARY POINTS AND CONSTANT SOLUTIONS OF THE FRIEDMANN EQUATION IN GENERAL

The Friedmann equation is, in fact, a 1st order autonomous ordinary differential equation (e.g., Wikipedia: Autonomous system (mathematics)). Autonomous differential equations do not depend explicitly on the independent variable and 1st order ones do not in most cases have solutions with stationary points, except at infinity of the independent variable (i.e., either positive or negative infinity) where all orders of derivatives of the solutions are zero and the solutions approach horizontal asymptotes. A horizontal asymptote is actually a constant solution of the 1st order autonomous ordinary differential equation. Constant solutions always accompany solutions with solutions that approach horizontal asymptotes. Hereafter we will refer the stationary points at infinity as asymptotic stationary points to differentiate them from ordinary stationary points. That 1st order autonomous ordinary differential equations usually do not have ordinary stationary points we call the no-ordinary stationary point rule.

But there are exceptions to the no-ordinary stationary point rule as we have hinted above. In Appendix C, we prove the no-ordinary stationary point rule and determine the most obvious exceptions to it.

The Friedmann equation is, in fact, cases where there are exceptions to the no-ordinary stationary point rule, and so can give solutions with ordinary stationary points. We discuss the solutions with ordinary stationary points relevant to this paper in § 3.1 below.

We note as digression that universes where dynamics depended on 1st order autonomous differential equations (generalizing loosely from just ordinary ones) might be rather boring. For example, say that Newton's 2nd law $\vec{F}_{\rm net} = m\vec{a}$ were replaced by $\vec{F}_{\rm net} = m\vec{v}$ and we consider only autonomous cases. It seems there would be no oscillatory behavior and there would have to be absolute states of rest and motion. Perhaps, life would not be possible in such universes. That Newton's 2nd law is a 2nd order autonomous differential equation (when force has no explicit time dependence) might have an anthropic principle explanation if we allow other possibilities could have occurred somehow.

3.1. The Friedmann Equation Exception to the No-Ordinary Stationary Point Rule

As pointed out in § C.2, the Friedmann equation is an obvious possible exception to the no-ordinary stationary point rule since it has $\dot{x}$ equal to the square root of a function of x: see, e.g., Equation (4) in § 2). So it can occur that there are ordinary stationary points and also asymptotic stationary points that also correspond to the constant solutions.

In the case of the two-density-component Friedmann equation, there are, in fact, ordinary stationary points as can be seen from the 3 exact solutions in terms of generalized cosmic scale factor and generalized conformal time: see below, § 6.2 for the hyperbolic sine solution § 6.3 for the hyperbolic cosine solution, and § 6.4 for the sine/cosine solution. The hyperbolic sine solution has no stationary points at all. The hyperbolic cosine solution has a single minimum. Note that the negative value solution is not physically allowed for a cosmic scale factor. The sine/cosine solution has a single maximum. Note that only one half wave of the sine/cosine solution is physically allowed since negative values are not physically allowed for a cosmic scale factor.

What of other cases of the Friedmann equation? The one-component Friedmann equation does have one solution with a stationary point at infinity. This is the exponentially expanding de Sitter universe (presented 1917: e.g., Bondi 1961, p. 98–99,105,146–147,154,159,166; Wikipedia: de Sitter universe)) which we derive in Appendix A. However, the accompanying constant solution has $x = 0$, and so is not physically allowed. No other solutions with stationary points exist for the one-component Friedmann equation.

For 3 or more density components, solutions with ordinary stationary points and asymptotic stationary points (with their accompanying constant solutions) can exist. In Appendix B, we show example solutions for the three-density-component Friedmann equation case of which some have ordinary stationary points and some have asymptotic stationary points with their accompanying constant solutions.

As final word, we note that all constant solutions of the Friedmann equation are unstable to general perturbations. The reason for this is the $\pm$ cases in Equation (4) in § 2). The $\pm$ cases imply that a general perturbation from a static universe model will have expanding and contracting solutions and one or the other of these will diverge forever (unless some other perturbation happens) from the constant solution. So if general perturbations occur, sooner or later one will lead to divergence. Note that we are considering global perturbations of a universe model assumed to remain homogeneous and isotropic (i.e., obey the cosmological principle) no matter what perturbations occur. An actual universe in an overall static state would be probably have local expanding and contracting perturbations. For example consider the Lemaître-Eddington universe (a Friedmann equation expanding positive-curvature-matter-Λ universe) which was considered by some to be a viable universe model prior to circa 1961 (e.g., Bondi 1961, p. 84–85,117–121,159). It starts from a asymptotic static phase (i.e., an Einstein universe phase: for the Einstein universe, presented 1917, see, e.g., Bondi 1961, p. 84,98–99,117–121,158–159,171; O'Raifeartaigh et al. 2017; O'Raifeartaigh 2019; Wikipedia: Einstein's static universe) and then undergoes global expansion that asymptotically approaches the exponentially expanding de Sitter universe . The actual behavior of the universe model with this global behavior was thought to be global expansion with local contractons that became galaxies (e.g., Bondi 1961, p. 118–119).

Note the positive-curvature-radiation universe we derive in Appendix B.2 is the radiation analogue for the positive-curvature-matter universe. The expanding version of the positive-curvature-radiation universe is the radiation analogue of the Lemaître-Eddington universe. The constant positive-curvature-radiation universe we derive in both Appendix B.1 and Appendix B.2 is the radiation analogue of the Einstein universe.

4. GENERAL EXACT SOLUTIONS TO THE FRIEDMANN EQUATION

First, what do we mean by an exact solution? We mean one where we have exact solution (1) $a(t)$, (2) $t(a)$, or (3) both $a(u)$ and $t(u)$ (or the equivalent of these cases in scaled form), where u is some auxiliary parameter: e.g., conformal time (see below), generalized conformal time (§ 5), or some other auxiliary parameter. If we have any of the 3 cases, we can exactly a and t though we may have do a numerical inversion of one of $a(t)$ and $t(a)$ if we do not have an exact analytic inversion formula for one or the other. Note that the using an auxiliary parameter in and of itself is not very useful since it has no direct physical meaning. For example, physical systems do not evolve with conformal time in a direct sense.

What if one has only one $a(u)$ or $t(u)$ and not the other one and needs to calculate its value numerically. We call this an incomplete exact solution. Exact solutions which are not incomplete exact solutions we call complete exact solutions but usually only when we need to distinguish them from incomplete exact solutions.

Only a limited set of the exact solutions of the two-density component Friedmann equation in terms of a a generalized scale factor and generalized conformal time we derive in § 5) can be changed into exact solutions in the sense described above. However, the example cases we consider all can be. All of the some three-component-component solutions in Appendix B are exact solutions, but they are for a limited set of inverse power dependences for density components. One-density-component cases all have exact solutions which, in fact, follow from one general solution that we derive in Appendix A. To conclude this paragraph, we do not investigate solutions that are not complete exact solutions in this paper.

How many exact exact solutions are there? In fact, there must be many, but most will probably have no cosmological relevance. To explicate consider the standard modern form Friedmann equation (Equation (1) in § 2) rewritten in an indefinite integral form

$$\int \frac{da}{a\sqrt{(8\pi G/3)\rho(a)}} = \int dt \ , \tag{8}$$

where $\rho(a)$ is general and can includes pseudo density terms for curvature and the cosmological constant (see § 2). Any exact solution for the indefinite integral (e.g., from a table of integrals: e.g., Wikipedia: List of integrals of irrational functions) gives an exact $t(a)$ which is an exact solution in sense defined above. An exact inverse $a(t)$ will not in general be available. However as indicated above, most of these exact solutions will not have any cosmological relevance.

One can get more exact solutions by using conformal time (which is not our generalized conformal time in general) symbolized by η. One defines conformal time for the standard modern form Friedmann equation by $d\eta = dt/a$ and then one can rewrite this equation to the form

$$\int \frac{da}{\sqrt{(8\pi G/3)\rho(a)a^4}} = \int d\eta \ , \tag{9}$$

Equation (9) eliminates any inverse powers under the radical symbol with inverse power less than or equal to 4. Any exact solution for the indefinite integral (e.g., from a table of integrals: e.g., Wikipedia: List of integrals of irrational functions) gives an exact $\eta(a)$ However, in order to get an exact solution, one needs the inverse solution $a(\eta)$ or $d\eta/da$ (as a function of a) and an exact solution is obtained for one or other of the indefinite integrals in

$$t = \int a(\eta)\, d\eta = \int a\frac{d\eta}{da}\, da \tag{10}$$

if either of the integrals can be solved analytically. If the first indefinite integral can be done, one has exact $t(\eta)$ and $a(\eta)$ and so exact solutions in terms of the auxiliary variable η which perhaps in some cases can be eliminated to give $t(a)$ and/or $a(t)$. If the second indefinite integral can be done, one has exact $t(a)$ and perhaps in some cases this can be inverted to give $a(t)$.

As an example of using conformal time, we rewrite Equation (4) (in § 2) in terms conformal time $d\eta = d\tau/x$ (where we are now using our primary scaled variables from § 2) and limit the inverse powers $P \in [0, 4]$ and obtain

$$d\eta = \pm \frac{dx}{\sqrt{\sum_{P=0}^{4} \Omega_{P,0} x^{4-P}}} \ , \tag{11}$$

which because of it nice mathematical appearance is prima facie suggestive that exact solutions for multiple nonzero $\Omega_{P,0}$ are possible. In fact, Equation (11) has been solved analytically for $x(\eta)$ for all density components simultaneously nonzero for integer $P \in [0, 4]$ by Steiner (2008, p. 7–9) though with dependence on the Weierstrass elliptic function (Wikipedia: Weierstrass elliptic function). We will not present this solution in this paper. A similar and mathematically equivalent solution (but without a $P = 1$ density component) has been given by Boyle & Turok (2022, eq. (12)). However, these solution are incomplete exact solutions since there is no exact solution for cosmic time from conformal time.

The inverse powers P in Equation (4) (in § 2) are general, however, the only widely considered powers seem to be as follows:

$$\begin{cases} P = 0 & \text{for } \Lambda \text{ (either the cosmological constant or constant dark energy);} \\ P = 1 & \text{for quintessence (in some theories);} \\ P = 2 & \text{for curvature, cosmic strings (in some theories), or the } R_h = ct \text{ universe} \\ P = 3 & \text{for matter (in the cosmological sense of matter at rest} \\ & \qquad \text{or nearly at rest in the comoving frames of the expanding universe)} \\ & \qquad \text{which includes baryonic matter and dark matter;} \\ P = 4 & \text{for radiation (in the cosmological sense of mass-energy moving} \\ & \qquad \text{at or nearly at the vacuum light speed} \\ & \qquad \text{in the comoving frames of the expanding universe).} \end{cases} \tag{12}$$

(e.g., Steiner 2008, p. 6–7; e.g., Melia 2014 for the $R_h = ct$ universe).

5. THE TWO-DENSITY-COMPONENT FRIEDMANN EQUATION

We now specialize the Friedmann equation to the case of the Friedmann equation depending on only two powers of a: i.e., powers $P = p$ and $P = q$ with $p > q$ always without loss of generality. Note p and q are not required to be integers. Note that $p = q$ reduces the Friedmann equation to a one-density-component case which is trivially solved by $x(\tau)$ being a power-law in τ or, in the case of $p = 0$, being an exponential. We present elementary one-density-component solutions in in Appendix A. In the following, we carry out the specialization, make transformations to a generalized scale factor z and a generalized conformal time $\tilde{\eta}$, and make some useful definitions:

$$\left(\frac{\dot{x}}{x}\right)^2 = \Omega_{p,0}x^{-p} + \Omega_{q,0}x^{-q}$$

$$x^p\left(\frac{\dot{x}}{x}\right)^2 = \Omega_{p,0} + \Omega_{q,0}x^{p-q}$$

$$\frac{x^p}{|\Omega_{p,0}|}\left(\frac{\dot{x}}{x}\right)^2 = g + h\left[\left(\frac{|\Omega_{q,0}|}{|\Omega_{p,0}|}\right)^{1/2} x^{R/2}\right]^2$$

where we define $g = \dfrac{\Omega_{p,0}}{|\Omega_{p,0}|}$, $h = \dfrac{\Omega_{q,0}}{|\Omega_{q,0}|}$, $R = p - q$

and note $R > 0$ is required, but not R integer

$$\left(\frac{x^{p/2}}{\sqrt{|\Omega_{p,0}|}}\frac{\dot{x}}{x}\right)^2 = g + h\left[\frac{x^{R/2}}{\sqrt{|\Omega_{p,0}|/|\Omega_{q,0}|}}\right]^2 = g + (y^{R/2})^2$$

where define y by $x_{\rm scale}$ by

$$y = \frac{x}{x_{\rm scale}} = \frac{x}{\left(\sqrt{|\Omega_{p,0}|/|\Omega_{q,0}|}\right)^{2/R}}$$

$$\left(\frac{x^{p/2}}{\sqrt{|\Omega_{p,0}|}}\frac{2}{R}\frac{\tilde{z}}{z}\right)^2 = g + hz^2$$

where we define z and y by

$$z = y^{R/2} = \frac{x^{R/2}}{\sqrt{|\Omega_{p,0}|/|\Omega_{q,0}|}} \quad \text{and use} \quad \frac{\tilde{z}}{z} = \frac{R}{2}\frac{\dot{x}}{x}$$

$$\left(\frac{x^{q/2}}{\sqrt{|\Omega_{q,0}|}}\frac{2}{R}\tilde{z}\right)^2 = g + hz^2$$

$$\left(\frac{2}{R}\frac{x^{q/2}}{\sqrt{|\Omega_{q,0}|}}\frac{d\tilde{\eta}}{d\tau}\tilde{z}\right)^2 = g + hz^2$$

where we define $\tilde{z} = \dfrac{dz}{d\tilde{\eta}}$

$$\tilde{z}^2 \;=\; g + hz^2 \;, \tag{13}$$

where the generalized scale factor z increases strictly with x and y and must be real and non-negative since physical x and y must be real and non-negative (but see further discussion below in § 5.2) and we define the generalized conformal time $\tilde{\eta}$ by

$$d\tau = \frac{2}{R}\frac{x^{q/2}}{\sqrt{|\Omega_{q,0}|}}\,d\tilde{\eta} = \frac{2}{R}\frac{\left(\sqrt{|\Omega_{p,0}|/|\Omega_{q,0}|}\right)^Q}{\sqrt{|\Omega_{q,0}|}}z^Q\,d\tilde{\eta}\;, \tag{14}$$

where we note that τ increases strictly with $\tilde{\eta}$ since $z \geq 0$ always and where we define

$$Q = \frac{q}{R} = \frac{q}{p-q}\;. \tag{15}$$

We take up the solution for $\tau(\tilde{\eta})$ and its inverse $\tilde{\eta}(\tau)$ in § 7.

Note that z must monotonically increase with x and y, and vice versa. Note also that

$$g = \begin{cases} \Omega_{p,0}/|\Omega_{p,0}| & \text{in general;} \\ 1 & \text{for } \Omega_{p,0} > 0; \\ -1 & \text{for } \Omega_{p,0} < 0 \text{ which is only true for a positive curvature universe (which gives } p=2\text{),} \end{cases} \quad \text{and} \quad h = \begin{cases} \Omega_{q,0}/|\Omega_{q,0}| & \text{in general;} \\ 1 & \text{for } \Omega_{q,0} > 0; \\ -1 & \text{for } \Omega_{q,0} < 0 \text{ which is only true for a positive curvature universe (which gives } q=2\text{) or for a } \Lambda < 0 \text{ universe (which gives } q=0\text{).} \end{cases} \tag{16}$$

The case of both g and h equal to -1 (i.e., $g = h = -1$) is physically ruled out since that leads to an inconsistency. If $g = h = -1$, the value for $\dot{x}/x$ on the left-hand side of the first line of Equation (13)) would be complex for x real and non-negative on the right-hand side. Recall we require physically real solutions for x to be real and non-negative, and so $\dot{x}$ and $\dot{x}/x$ must both be real and non-negative.

To conclude, the two-density-component Friedmann equation is

$$\tilde{z}^2 = g + hz^2 \qquad \text{and} \qquad \tilde{z} = \pm\sqrt{g + hz^2}\;. \tag{17}$$

Similarly to our discussion in § 2, the $\pm$ cases cause every solution to have a twin solution. Also similarly to our discussion in § 2, we will not usually consider those twin solutions for z

that are monotonically decreasing (implying monotonically decreasing solution x) since since they are uninteresting since the actual universe does not have a monotonically decreasing scale factor.

We address two fine points arising from this section that we address in the subsections below: § 5.1 explores the meaning of $x_{\rm scale}$; § 5.2 shows that complex z solutions are not useful.

5.1. The Meaning of $x_{\rm scale}$

To elucidate the meaning of $x_{\rm scale}$, we take the ratio $\mathcal{R}_{qp}$ of the absolute values of the two density components in the first line of Equation (13) in § 5:

$$\begin{aligned}
\mathcal{R}_{qp} &= \frac{|\Omega_{q,0}|x^{-q}}{|\Omega_{p,0}|x^{-p}} = \frac{|\Omega_{q,0}|}{|\Omega_{p,0}|}x^{p-q} = \left[\frac{x}{(|\Omega_{p,0}|/|\Omega_{q,0}|)^{1/R}}\right]^R = \left[\frac{x}{\left(\sqrt{|\Omega_{p,0}|/|\Omega_{q,0}|}\right)^{2/R}}\right]^R \\
&= \left(\frac{x}{x_{\rm scale}}\right)^R = y^R \ , \qquad (18)
\end{aligned}$$

where we have made other uses of Equation (13) in § 5.

So $x_{\rm scale}$ is the scaled cosmic scale factor when the bsolute values of the two density components in the first line of Equation (13) in § 5 are equal and y is the ratio of the two components to the power $R = p - q$.

We can derive a characteristic time $\tau_{\rm Ch}$ (the characterstic time scale over which x changes by an amount of order $x_{\rm scale}$ when $x = x_{\rm scale}$) in a series of characteristic equations also making use of the first line of Equation (13) in § 5:

$$\begin{aligned}
\left.\frac{\dot{x}}{x}\right|_{x_{\rm scale}} &= \sqrt{2|\Omega_{p,0}|x_{\rm scale}^{-p}} = \sqrt{2|\Omega_{q,0}|x_{\rm scale}^{-q}} \qquad \text{which is actually exact if both } \Omega \text{ parameters are positive} \\
\tau_{\rm Ch} &= \frac{\Delta x}{x_{\rm scale}\sqrt{2|\Omega_{q,0}|x_{\rm scale}^{-q}}} \qquad \text{where } (\dot{x}/x)|_{x_{\rm scale}} \text{ is approximated by } (\Delta x/x_{\rm scale})/\tau_{\rm Ch} \\
\tau_{\rm Ch} &= \frac{1}{\sqrt{2|\Omega_{q,0}|x_{\rm scale}^{-q}}} \qquad \text{where } \Delta x \text{ is approximated by } x_{\rm scale}. \qquad (19)
\end{aligned}$$

Equation (19) is used below in § 7.

5.2. Complex z Solutions Are Not Useful

To explicate a fine point, $x(\tilde{\eta})$ and $y(\tilde{\eta})$ are required to be real and non-negative. But is $z(\tilde{\eta})$ absolutely so required? Let us assume for the moment that z is a complex solution to Equation (13) and write it in polar form with $z_{\rm Re}$ as modulus and $\phi \neq 0$ as argument or phase: thus, $z = z_{\rm Re}e^{i\phi}$. Then from Equation (13), we have

$$y = z_{\rm Re}^{2/R} e^{i\phi(2/R)} \ . \tag{20}$$

In order for y to be real and non-negative, $\phi(2/R) = (2\pi n)$ or $\phi = \pi R n$, where n is a general non-zero integer. Since we cannot physically allow ϕ to make discontinuous jumps by letting n vary as $\tilde{\eta}$ varies, we require ϕ to be a constant for the complex z solution. If we substitute $z = z_{\rm Re}e^{i\phi}$ into the first form of Equation (17), we get

$$\begin{aligned} z_{\rm Re}^2 e^{2i\phi} &= g + h z_{\rm Re}^2 e^{2i\phi} \\ z_{\rm Re}^2 &= g e^{-2i\phi} + h z_{\rm Re}^2 \\ e^{-2i\phi} &= e^{-i\pi(2Rn)} \qquad \text{which must be real to satisfy the differential equation, and so} \\ e^{2i\phi} &= e^{i\pi(2Rn)} = \pm 1 \qquad \text{where } 2Rn \text{ must be an integer.} \end{aligned} \tag{21}$$

There are 2 cases to consider. Case 1: For the upper case in the last line of Equation (21), $2Rn$ must be an even integer and $z = z_{\rm Re}e^{i\pi(Rn)} = \pm z_{\rm Re}$ (upper/lower case for Rn even/odd) and the x solution we obtain from the $z_{\rm Re}$ solution is for the x version of the Friedmann equation with g and h. Case 2: For the lower case, $2nR$ must be an odd integer (which implies Rn is an odd integer divided by 2) and $z = z_{\rm Re}e^{i\pi(Rn)} = \pm i z_{\rm Re}$ (upper case for $Rn = (1/2 + 2k)$ and lower case for $2Rn = 3/2 + 2k$, where k is any integer) and the x solution we obtain from $z_{\rm Re}$ solution is for the x version of the Friedmann equation with $-g$ and h. Since the complex z solution we get for g and h in this case leads to the same solution we get for $-g$ and h from a real z solution, there is nothing to be gained looking for the complex z solution for g and h. We assume that $-g$ and h are both non-negative or there is no real solution as we discussed above.

The conclusion from the 2 cases is that complex z solutions are not useful since all physical solutions can be obtained from real z solutions which recall increase strictly with x and y.

6. TWO-DENSITY-COMPONENT SOLUTIONS FOR THE GENERALIZED COSMIC SCALE FACTOR AS A FUNCTION OF GENERALIZED CONFORMAL TIME

In the following subsections, we derive the 3 exact solutions (and there are no inexat ones) for the two-density-component Friedmann equation for generalized cosmic scale factor and generalized conformal time (i.e., Equation (17) in § 5). All (ordinary) exact cosmic scale factor two-density-component solutions can be derived from these 3, and so we call the 3 general exact solutions. Note only a restricted class of the exact cosmic scale factor two-density-component solutions are complete exact solutions in the sense of § 4. In § 12, we derive formulae for special case solutions of the cosmic scaled factor (using the generalized cosmic scale factor z solutions) in terms of scaled cosmic time (not generalized conformal cosmic time) where possible and conformal cosmic time where not. We do not consider incomplete exact solutions.

6.1. Summary of Hyperbolic Function Identities

For use in this section and other sections below, we summarize here the most useful hyperbolic function identities:

$$
\begin{aligned}
&\cosh^2(x) - \sinh^2(x) = 1 \ , \qquad \cosh^2(x) = 1 + \sinh^2(x) \ , \qquad \sinh^2(x) = \cosh^2(x) - 1 \ , \\
&\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \ , \\
&\cos(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \ , \\
&\frac{d\sinh(x)}{dx} = \cosh(x) \ , \qquad \frac{d\cosh(x)}{dx} = \sinh(x) \ , \\
&\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right) \ , \qquad \operatorname{arccosh}(x) = \pm\ln\left(x + \sqrt{x^2-1}\right) \\
&\sinh^2(x) = \frac{1}{2}[\cosh(2x) - 1] \ , \qquad \text{and} \qquad \cosh^2(x) = \frac{1}{2}[\cosh(2x) + 1] \ ,
\end{aligned}
\tag{22}
$$

(e.g., Wikipedia: Hyperbolic functions: Useful relations; Wikipedia: Hybperbolic functions: Inverse functions as logarithms).

6.2. The Solution for z for $g = h = 1$

First, we note that this general solution has great physical relevance since all special case two-density-component solutions follow from it except those with positive curvature (i.e., those with $\Omega_{2,0} < 0$) and negative Λ (i.e., those with $\Omega_{0,0} = \Omega_{\Lambda,0} < 0$ which are not interesting in modern cosmology at present). In particular the radiation-matter universe and the matter-Λ universe which combine to the make the Λ-CDM model are derivable from this general solution.

For $g = h = 1$, the two-density-component Friedmann equation is

$$\tilde{z}^2 = 1 + z^2 \tag{23}$$

and from the 2nd and 6th members of Equation (22) in § 6.1, it is clear that

$$z = \sinh(\tilde{\eta}) \qquad \text{and} \qquad z = -\sinh(\tilde{\eta}) \ . \tag{24}$$

are solutions to the two-density-component Friedmann equation (i.e., Equation (17) in § 5). Note the first solution is only allowed for $\tilde{\eta} \geq 0$ and the second for $\tilde{\eta} \leq 0$.

As mentioned in § 2, we do not usually consider monotonically decreasing solutions for z and so will not further consider the second solution in Equation (24).

Note that solutions like Equation (24) which expand from a zero of cosmic scale factor or generalized cosmic scale factor (i.e., solutions for any of z, y, x or a), we call point-origin solutions. So Equation (24) is a point-origin solution. Point origin is an old term for what is now called a Big Bang singularity (e.g., Bondi 1961, p. 117). It seems appropriate revive the term point origin and use it rather than use Big Bang singularity since most point-origin solutions are not viable cosmic scale factor or generalized cosmic scale factor solutions for the observable universe, and so there is no need to think of Big Bang like conditions at their origin in time.

Usually, we write point-origin solutions with the point origin at the zero of the time-like variable (i.e., time, conformal time, or generalized conformal time). However, sometimes shifting the point origin to another value of the time-like variable is useful. For example, for Equation (24), we will shift the point origin to $\tilde{\eta}_0$. Using the 4th member of Equation (22) in § 6.1, we obtain the time-like-variable-zero-point-shift formula

$$z = \sinh(\tilde{\eta} - \tilde{\eta}_0) = \cosh(\tilde{\eta}_0)\sinh(\tilde{\eta}) - \sinh(\tilde{\eta}_0)\cosh(\tilde{\eta}) = A\sinh(\tilde{\eta}) \pm \sqrt{A^2 - 1}\cosh(\tilde{\eta}) \ , \tag{25}$$

where the upper case is for $\tilde{\eta}_0) \leq 0$ (since it requires a larger z than for $\tilde{\eta}_0) \geq 0$) and the lower case for $\tilde{\eta}_0) \geq 0$ (since it requires a smaller z than for $\tilde{\eta}_0) \geq 0$), and where we define

$$A = \cosh(\tilde{\eta}_0) \geq 1 \ . \tag{26}$$

We can find a A and $\tilde{\eta}_0$ so that we can drive the solution through any arbitrary point $(\tilde{\eta}_1 \geq \tilde{\eta}_0, z_1 \geq 0)$. The appropriate A follows from solving the quadratic equation

$$0 = A^2 + 2zA\sinh(\tilde{\eta}_1) - [\cosh^2(\tilde{\eta}_1) + z_1^2] \ . \tag{27}$$

The general solution for A and interesting special cases are given by

$$A = \begin{cases} -z_1 \sinh(\tilde{\eta}_1) + \sqrt{z_1^2 \sinh^2(\tilde{\eta}_1) + [\cosh^2(\tilde{\eta}_1) + z_1^2]} & \text{in general and note the other quadratic root for } A \text{ is ruled out since it gives } A < 0; \\ \cosh(\tilde{\eta}_1) & \text{for } z_1 = 0. \text{ This result implies } \tilde{\eta}_0 = \pm\tilde{\eta}_1; \\ \sqrt{1 + z_1^2} & \text{for } \tilde{\eta}_1 = 0; \\ 1 & \text{for } \tilde{\eta}_1 = 0 \text{ and } z_1 = 0 \text{ which is just the unshifted time-like variable value for } A. \end{cases} \quad (28)$$

The value for $\tilde{\eta}_0$ is given by

$$\tilde{\eta}_0 = \operatorname{arccosh}(A) = \pm \ln\left(A + \sqrt{A^2 - 1}\right) \quad (29)$$

where we have used the 9th member of Equation (22) in § 6.1 However the explicit formula for $\operatorname{arccosh}(x)$ is indeterminate with respect to sign. But $\tilde{\eta}_0$ can be fully determined by substituting from $\tilde{\eta}_1$ and z_1 into Equation (25) and rearranging:

$$\tilde{\eta}_0 = \operatorname{arcsinh}\left[\frac{-z_1 + A\sinh(\tilde{\eta}_1)}{\cosh(\tilde{\eta}_1)}\right] . \quad (30)$$

Alternatively, the sign of $-z_1 + A\sinh(\tilde{\eta}_1)$ gives the sign of $\tilde{\eta}_0$ for Equation (29).

The time-like-variable-zero-point-shift formula Equation (25) was used by (Galanti & Roncadelli 2021, p. (3)) in their nearly exact Λ-CDM cosmic scale factor formula for the matter-Λ era.

What of stationary points for the cosmic scale factor solutions derived from $z(\eta)$. From Equation (14) in § 5, we know that $\tilde{\eta}$ increases strictly with τ, and so $z(\tau)$ is increases strictly with τ. From Equation (13) in § 5, we know that x increases strictly with z, and so $x(\tau)$ increases strictly with τ or in unscaled terms $a(t)$ increases strictly with cosmic time t. Consequently, both $x(\tau)$ and $a(t)$ have no ordinary stationary points nor as asymptotic stationary points. Since there are no asymptotic stationary points for these scale factor solutions, there are no constant solutions accompanying them following the discussion given in § 3.

6.3. The Solution for z for $g = -1$, $h = 1$

This general solution is relevant for special case solutions for positive curvature universes which are the only considered universes with $P > 0$ and $\Omega_{P,0} < 0$. Since $p = 2$ for $g = -1$, we must have $q = 0$ and $\Omega_{q,0} > 0$ (i.e., a positive Λ density component) or $q = 1$ for which $\Omega_{q,0} > 0$ are the only considered cases (e.g., a quintessence universe).

For $g = -1$ and $h = 1$, the two-density-component Friedmann equation is

$$\tilde{z}^2 = -1 + z^2 \tag{31}$$

and from the 3rd and 7th members of Equation (22) in § 6.1, it is clear that

$$z = \cosh(\tilde{\eta}) \tag{32}$$

and the $z = -\cosh(\tilde{\eta})$ is not allowed since it is negative for all $\tilde{\eta}$.

We note that the solution $z = \cosh(\tilde{\eta})$ as a single stationary point at $\tilde{\eta} = 0$ which is a minimum point. Analogously to the case of the solution Equation (24) in § 6.2, we see that this implies all special case solutions for cosmic scale factor scaled $y(\tau)$ and consequently unscaled $a(t)$ derived from $z = \cosh(\tilde{\eta})$ have a single stationary point (which is a minimum) at a finite time. Since there is no stationary point at infinity, there are no constant solutions to Equation (23) following the discussion given in § 3.

Universe models with a cosmic scale factor minimum have long been known (e.g., Bondi 1961, p. 84–86) and elaborate variations on them called Big Bounce models have receeved some consideration. (e.g., Wikipedia: Big Bounce). However, we will not consider any special case solutions that follow from $z = \cosh(\tilde{\eta})$ in this paper, except briefly in §§ § 12.2 and § 12.3.

Analogously to the case of the solution Equation (24) in § 6.2, we can shift the point origin of $z = \cosh(\tilde{\eta})$ to any another value of $\tilde{\eta}$. There is no point in repeating the shifting procedure for $z = \cosh(\tilde{\eta})$ since it is straightforward.

6.4. The Solution for z for $g = 1$, $h = -1$

This general solution is relevant for special case solutions for positive curvature universes (which $\Omega_{q=2,0} < 0$) and negative Λ universes (which have ($\Omega_{q=0,0} < 0$).

For $g = 1$ and $h = -1$, the two-density-component Friedmann equation is

$$\tilde{z}^2 = 1 - z^2 \tag{33}$$

and the solutions are immediately seen to be

$$z = \sin(\tilde{\eta}) \qquad \text{and} \qquad z = \cos(\tilde{\eta}) \ . \tag{34}$$

Of course, the two solutions are actually the one and the same solution with a shift of the $\tilde{\eta}$ zero point for the point origin. Also analogously to the case of the solution Equation (24) in § 6.2, we can shift the point origin of either $z = \sin(\tilde{\eta})$ or $z = \cos(\tilde{\eta})$ to any another value of $\tilde{\eta}$. There is no point in repeating the shifting procedure for these solutions since it is straightforward.

A key point about the solutions of Equation (34) is that they are not oscillating solutions. The Friedmann equation in and of itself gives no guidance on how to extend the cosmologically relevant solutions through $z = 0$. Thus, the solution Equation (34) gives only a single half cycle of the sine or cosine behavior. Note then that the solution does have one stationary point, a maximum, and has both a point origin and point end.

We derive the positive-curvature-matter universe solution starting from Equation (34) in § 12.3.1.

6.5. Summary of Solution for z

To summarize the generalized cosmic scale factor solutions and their ranges of physical validity without complicating generality:

$$z = \begin{cases} \sinh(\tilde{\eta}) & \text{for } g = h = 1 \text{ for } \tilde{\eta} \in [0, \infty]; \\ \cosh(\tilde{\eta}) & \text{for } g = -1,\ h = 1 \text{ for } \tilde{\eta} \in [-\infty, \infty]; \\ \sin(\tilde{\eta}) & \text{for } g = 1,\ h = -1 \text{ for } \tilde{\eta} \in [0, \pi]. \end{cases} \tag{35}$$

There are analytic inverses for the solutions $\tilde{\eta}(z)$ that are particularly useful for cases where cosmic time $w(\tilde{\eta})$ has an exact analytic solution as a function of $\tilde{\eta}$ since then one can find exact analytic cosmic time solutions $w(z)$ and $w(y)$. The inverse solutions $\tilde{\eta}(z)$ are:

$$\tilde{\eta} = \begin{cases} \text{arcsinh}(z) = \ln\left(z + \sqrt{z^2+1}\right) & \text{for } g = h = 1 \text{ for } z \in [0, \infty]; \\ \text{arccosh}(z) = \pm\ln\left(z + \sqrt{z^2-1}\right) & \text{for } g = -1,\ h = 1 \text{ for } z \in [1, \infty]; \\ \arcsin(z) \quad \text{and} \quad \pi - \arcsin(z) & \text{for } g = 1,\ h = -1 \text{ for } z \in [0, 1]. \end{cases} \tag{36}$$

Note that the last two cases are double-value functions of z due to the non-monotonic nature of the $\cosh(\tilde{\eta})$ and $\sin(\tilde{\eta})$ solutions.

7. COMPLETE TWO-DENSITY-COMPONENT SOLUTIONS

In order to find all complete two-density-component solutions, we need to find all cases the cosmic time is an exact (analytic) solution of generalized conformal time. To do this and making use of Equation (14) in § 5, we define the new scaled time w and new time scale $\tau_{\rm scale}$

$$F\,dw = \frac{F\,d\tau}{\tau_{\rm scale}} = \frac{F\,d\tau}{F(2/R)\left(\sqrt{|\Omega_{p,0}|/|\Omega_{q,0}|}\right)^{Q}/\sqrt{|\Omega_{q,0}|}} = z^{Q}\,d\tilde{\eta} = y^{q/2}\,d\tilde{\eta}\ , \tag{37}$$

where we define

$$F = \begin{cases} R/2 & \text{for leaving factor } R/2 \text{ explicit;} \\ 1 & \text{for including } R/2 \text{ in the definition of } \tau_{\rm scale}. \end{cases} \tag{38}$$

The reason for the F factor is that depending on the special case solutions for w and $\tilde{\eta}$, one may want either the factor $R/2$ explicit or included in the definition of $\tau_{\rm scale}$. In fact, we find leaving the factor $R/2$ explicit seems the most simplifying and elegant choice since it leaves the formula for $\tau_{\rm scale}$ simple and have used this choise in all our special case complete exact solutions given in §§ 12, 13, and 14.

The meaing of $\tau_{\rm scale}$ can be explicated as follows making use of Equation (13) in § 5 for the definition of $x_{\rm scale}$, Equation (15) in § 5 for the definition of Q, and Equation (19) in § 5.1 for the definition $\tau_{\rm Ch}$:

$$\begin{aligned} \tau_{\rm scale} &= F\left(\frac{2}{R}\right)\frac{\left(\sqrt{|\Omega_{p,0}|/|\Omega_{q,0}|}\right)^{Q}}{\sqrt{|\Omega_{q,0}|}} = F\left(\frac{2}{R}\right)\frac{\left(\sqrt{x_{\rm scale}}\right)^{q}}{\sqrt{|\Omega_{q,0}|}} = F\left(\frac{2}{R}\right)\frac{1}{\sqrt{|\Omega_{q,0}|x_{\rm scale}^{-q}}} \\ &= F\left(\frac{2}{R}\right)\sqrt{2}\frac{1}{\sqrt{2|\Omega_{q,0}|x_{\rm scale}^{-q}}} = F\left(\frac{2}{R}\right)\sqrt{2}\,\tau_{\rm Ch} \approx \tau_{\rm Ch}\ , \end{aligned} \tag{39}$$

where the last expression is an order-of approximation. Equation (39) shows that $\tau_{\rm scale}$ can also be regarded as charateristic time for x to change by of order $x_{\rm scale}$ when $x = x_{\rm scale}$.

The effect of the Q parameter introduced in Equation (15) in § 5 on the solutions can be explicated as follows:

$$Q = \begin{cases} \dfrac{q}{R} = \dfrac{q}{p-q} & \text{in general;} \\ 0 & \text{for } q = 0 \text{ in which case } \tilde{\eta} \text{ is} \\ & \text{just a secondary scaled time.} \\ 1 & \text{for } p = 2q \text{ in which case } \tilde{\eta} \text{ is} \\ & \text{an ordinary conformal time} \\ & \text{relative to the generalized scale factor } z. \\ \text{integer} \geq 0 & \text{for which a complete exact solution exists} \\ & \text{and one can always obtain an exact } w(y) \text{ solution} \\ & \text{as we show below.} \\ 0, 1, 3 & \text{for which one can obtain both } w(y) \text{ and } y(w), \\ & \text{and so } \tilde{\eta} \text{ becomes an unnecessary variable.} \\ & \text{For other integer } Q\text{, one cannot obtain an exact } y(w), \\ & \text{and so } \tilde{\eta} \text{ remains a necessary variable.} \\ & \text{We show this below.} \\ \text{non-integer} > 0 & \text{for which no complete exact solution exists} \\ & \text{since trigonometric and hyperbolic} \\ & \text{functions raised to non-integer powers} \\ & \text{cannot be exactly integrated.} \end{cases} \tag{40}$$

For convenient reference, note:

$$p = \begin{cases} p > 0 & \text{for } q = 0\text{, but otherwise general.} \\ \left(1 + \dfrac{1}{Q}\right) q & \text{for } q > 0. \\ 2q, \dfrac{3}{2}q, \dfrac{4}{3}q, \dfrac{5}{4}q, \ldots & \text{for, respectively, } Q = 1, 2, 3, 4, 5 \ldots \\ 4 & \text{for the radiation-matter universe} \\ & \text{where } q = 3 \text{ and } Q = 4. \end{cases} \tag{41}$$

Why are there always complete exact solutions for all Q integer? First, exact integral solutions exist for integer powers Q of hyperbolic sine, hyperbolic cosine, and sine functions of a variable (see, e.g., Wikipedia: List of integrals of hyperbolic functions: Integrals involving only hyperbolic sine functions; Wikipedia: List of integrals of hyperbolic functions: Integrals involving only hyperbolic cosine functions; Wikipedia: List of integrals of trigonometric functions: Integrands involving only sine). Second, the 3 exact general (two-density-component) solutions $z(\tilde{\eta})$ are of these 3 kinds of functions (see Equation 35 in § 6.5). Therefore one can always find exact $w(\tilde{\eta})$ solutions to complement the exact $z(\tilde{\eta})$, and thus always has complete exact solutions for integer Q.

Why are exact $w(y)$ solutions for all Q integer? As show by Equation 36 in § 6.5, one exact solutions $\tilde{\eta}(z)$, and so one can obtain $w(z)$ solutions and so $w(y)$ solutions.

Why do exact $y(w)$ solutions exist only for Q values 0, 1, and 3? For Q values 0 and 1, one can easily invert the $w(\tilde{\eta})$ solutions as we show in in §§ 8 and 9, and so easily obtain the corresponding $y(w)$ solutions. In § 11, we solve a depressed cubic equation (see Wikipedia: Cubic equation: Depressed cubic) to invert the $w(\tilde{\eta})$ solution, and so obtain the corresponding $y(w)$ solution.

Now for even Q values other than 0, the $w(\tilde{\eta})$ solutions require the exact integrals for even integer powers $Q > 0$ of hyperbolic sine, hyperbolic cosine, and sine functions of a variable see, e.g., Wikipedia: List of integrals of hyperbolic functions: Integrals involving only hyperbolic sine functions; Wikipedia: List of integrals of hyperbolic functions: Integrals involving only hyperbolic cosine functions; Wikipedia: List of integrals of trigonometric functions: Integrands involving only sine). However, these exact solutions always depend on

8. The $Q = 0$ Solutions $w(\tilde{\eta})$, $w(y)$, and $y(w)$

9. The $Q = 1$ Solutions $w(\tilde{\eta})$, $w(y)$, and $y(w)$

10. The $Q = 2$ Solutions $w(\tilde{\eta})$, $w(y)$, and $y(w)$

11. The $Q = 3$ Solutions $w(\tilde{\eta})$, $w(y)$, and $y(w)$

12. SPECIAL COMPLETE TWO-DENSITY-COMPONENT SOLUTIONS

In this section, we derive formulae for special case solutions of the cosmic scale factor (using the generalized cosmic scale factor z solutions) in terms of scaled cosmic time (not generalized conformal cosmic time) where possible and conformal cosmic time where not. But first we need some more formalism. ?????

In the following subsections, where we derive the special case solutions, we consider only integer $Q \in [0, 3]$ since these seem to be the cases of most general interest and are most tractable.

12.1. Solutions for $Q = 0$

Using Equation (37) in § 7 with $Q = 0$, we have

$$Fw = \tilde{\eta} \ , \tag{42}$$

where we have neglected constants of integration since they are a pointless generalization. Thus, our generalized conformal time reduces to ordinary scaled cosmic time aside from the factor of F.

In fact for $Q = 0$ solutions, $q = 0$, and so one of the two density components for the two-density-component solution cases must be the Λ component. The other term is general except that $p > q = 0$.

The general solution for $Q = 0$, $q = 0$, $p > q = 0$, and also $\Omega_{0,0} > 0$ (i.e., positive Λ universes since negative Λ universes are not so interesting in modern cosmology) follows from Equation (24) in § 6.2 (i.e., $z = \sinh(\tilde{\eta})$), Equation (13) in § 5 (i.e., $y = z^{2/R}$), and setting $F = R/2 = p/2$ in Equation (38) in § 7:

$$y = \begin{cases} \sinh^{2/p}\left(\dfrac{p}{2}w\right) & \text{in general;} \\ \left(\dfrac{p}{2}w\right)^{2/p} & \text{in the small } w \text{ limit;} \\ \dfrac{e^w}{2} & \text{in the large } w \text{ limit.} \end{cases} \tag{43}$$

If we descale from y to x (using Equation (13) in § 5) and w to τ (using Equation (37) in § 7) and recalling $\Omega_{\Lambda,0} > 0$ (as aforementioned), Equation (43) above becomes

$$x = \begin{cases} \left(\dfrac{\Omega_{\mathrm{p},0}}{\Omega_{\Lambda,0}}\right)^{1/p} \sinh^{2/p}\left[\left(\dfrac{p}{2}\sqrt{\Omega_{\Lambda,0}}\right)\tau\right] & \text{in general;} \\ \left[\left(\dfrac{p}{2}\sqrt{\Omega_{\mathrm{p},0}}\right)\tau\right]^{2/p} & \text{in the small } \tau \text{ limit;} \\ \left(\dfrac{\Omega_{\mathrm{p},0}}{\Omega_{\Lambda,0}}\right)^{1/p} \dfrac{e^{\left(\sqrt{\Omega_{\Lambda,0}}\right)\tau}}{2} & \text{in the large } \tau \text{ limit.} \end{cases} \tag{44}$$

Note that the 3rd line of Equation (44) is not exactly the exact solution for the pure Λ universe because the $\Omega_{\mathrm{p},0}$ factor has not canceled out. This is a feature of our assumption that there are two powers-of-a terms in the Friedmann equation. If one starts from the Friedmann equation form given by Equation (4) in § 2) with only the $\Omega_{\Lambda,0}$ term, one obtains the pure Λ universe cosmic scale factor: i.e.,

$$x = x_{\rm in} e^{\sqrt{\Omega_{\Lambda,0}}(\tau - \tau_{\rm in})} , \tag{45}$$

where the $x_{\rm in}$ and $\tau_{\rm in}$ are just constants of integration to be set by initial conditions. We note the pure Λ universe has no point origin and the scale factor only goes to 0 asymptotically as $\tau \to -\infty$. The pure Λ universe is, in fact, the de Sitter universe (presented 1917: e.g., Wikipedia: de Sitter universe) which was the first expanding universe model. The 3rd line is a general $p > q = 0$ analogue solution for the de Sitter universe.

12.1.1. Solutions for the Matter-Positive-Λ Universe

An important special case of Equation (44) in § 12.1 is for $p = 3$ (i.e., the p case for matter):

$$x = \begin{cases} \left(\frac{\Omega_{\rm M,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left[\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}\right)\tau\right] & \text{in general;} \\ \left[\left(\frac{3}{2}\sqrt{\Omega_{\rm M,0}}\right)\tau\right]^{2/3} & \text{in the small } \tau \text{ limit;} \\ \left(\frac{\Omega_{\rm M,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \frac{e^{\left(\sqrt{\Omega_{\Lambda,0}}\right)\tau}}{2} & \text{in the large } \tau \text{ limit.} \end{cases} \tag{46}$$

The first line of Equation (46) is the well known Λ-CDM model cosmic scale factor solution for the matter-Λ era (e.g., Steiner 2008, p. 12; Universe in Problems: Evolution of Universe Problem 13), here, of course, in scaled x and τ rather than a and t. We will make use of this cosmic scale factor solution in § 15 where we present a smoothed-piecewise-approximate (SPA) solution for the Λ-CDM model.

The 2nd line of Equation (46) is the exact solution for a pure matter universe and is, in fact, the Einstein-de Sitter universe (presented 1932: e.g., Wikipedia: Einstein-de Sitter universe; O'Raifeartaigh et al. 2015) which was often considered the standard cosmological model circa 1960–1995 before the start of the dominance of the Λ-CDM circa 1995 (e.g., Bondi 1961, p. 166; Scott 2018, p. 10). The pure matter universe cosmic scale factor in unscaled quantities is derived directly in Appendix B.

The 3rd line of Equation (46) is the $p = 3$ analogue solution for the de Sitter universe (presented 1917: e.g., Wikipedia: de Sitter universe). We show the general p analogue above in § 12.1.

12.1.2. Solutions for the Radiation-Positive-Λ Universe

Another important special case of Equation (44) in § 12.1 (though not as important as the $p = 3$ case of § 12.1.1 since it does not apply to the actual observable universe) is for $p = 4$ (i.e., the p case for radiation):

$$x = \begin{cases} \left(\frac{\Omega_{\mathrm{R},0}}{\Omega_{\Lambda,0}}\right)^{1/4} \sinh^{1/2}\left[\left(2\sqrt{\Omega_{\Lambda,0}}\right)\tau\right] & \text{in general;} \\ \left[\left(2\sqrt{\Omega_{\mathrm{R},0}}\right)\tau\right]^{1/2} & \text{in the small } \tau \text{ limit;} \\ \left(\frac{\Omega_{\mathrm{R},0}}{\Omega_{\Lambda,0}}\right)^{1/4} \frac{e^{\left(\sqrt{\Omega_{\Lambda,0}}\right)\tau}}{2} & \text{in the large } \tau \text{ limit.} \end{cases} \tag{47}$$

The first line of Equation (47) (i.e., the exact radiation-positive-Λ universe cosmic scale factor solution) is the analogue of the first line of Equation (46) in § 12.1.1 (i.e., the exact matter-positive-Λ universe cosmic scale factor solution). We will make use of Equation (47) in § 15 where we present a smoothed-piecewise-approximate (SPA) solution for Λ-CDM model.

12.2. Solutions for $Q = 1$

For $Q = 1$ and choosing the integration interval for maximum simplicity, we have for $w(\tilde{\eta})$ solutions

$$Fw = \int_0^{\tilde{\eta}} z\, d\tilde{\eta}' = \begin{cases} \cosh(\tilde{\eta}) - 1 & \text{for } g = h = 1,\ z = \sinh(\tilde{\eta}), \\ & \text{and } \tilde{\eta} \in [0, \infty]; \\ \sinh(\tilde{\eta}) & \text{for } g = -1,\ h = 1,\ z = \cosh(\tilde{\eta}), \\ & \text{and } \tilde{\eta} \in [0, \infty]; \\ 1 - \cos(\tilde{\eta}) & \text{for } g = 1,\ h = -1,\ z = \sin(\tilde{\eta}), \\ & \text{and } \tilde{\eta} \in [0, \pi], \end{cases} \tag{48}$$

where the 3 general solutions for $z(\tilde{\eta})$ are from, respectively, SS 6.2, 6.3, and 6.4. Recall that since $z \geq 0$ always, the $z = \sin(\tilde{\eta})$ is limited to the argument range $\tilde{\eta} \in [0, \pi]$: i.e., this solution has both a point origin and point end.

All the $Fw(\tilde{\eta})$ solutions can be inverted analytically for $\tilde{\eta}(w)$ solutions:

$$\tilde{\eta} = \int_0^{\tilde{\eta}} z\, d\tilde{\eta}' = \begin{cases} \operatorname{arccosh}(Fw+1) & \text{for } g=h=1,\ z=\sinh(\tilde{\eta}), \\ \quad = \ln[(Fw+1)+\sqrt{(Fw+1)^2-1}] & \text{and } Fw \in [0,\infty]; \\ \operatorname{arcsinh}(Fw) = \ln[Fw+\sqrt{(Fw)^2+1}] & \text{for } g=-1,\ h=1,\ z=\cosh(\tilde{\eta}), \\ & \text{and } Fw \in [0,\infty]; \\ \arccos(1-Fw) & \text{for } g=1,\ h=-1,\ z=\sin(\tilde{\eta}), \\ & \text{and } Fw \in [0,2], \end{cases} \tag{49}$$

where we have used two of the hyperbolic function identities of Equation (22) in § 6.1.

Now for $q/R = q/(p-q) = 1$, we must have $p = 2q$. Given that the only widely considered powers are integers $P \in [0,4]$ (see § 6), there may be only two interesting cases of $Q = 1$: i.e., with $p = 2$ and $q = 1$ and with $p = 4$ and $q = 2$. In fact, only the case with $p = 4$ and $q = 2$ (and therefore $R = p - q = 2$ and $F = R/2 = 1$) seems very interesting since this corresponds to the negative-curvature-radiation universe (i.e., which has $\Omega_{q=2,0} > 0$) and the positive-curvature-radiation universe (i.e., which has $\Omega_{q=2,0} < 0$). Recall the general solution for $g = -1$, $h = 1$ can only occur if $p = 2$ since curvature density parameter is the largest density component that can be negative.

For the negative-curvature-radiation universe, one finds from the above formulae, the hyperbolic function identity $\sinh^2(x) = \cosh^2(x) - 1$, and Equation (13) in § 6 (i.e., $y = z^{2/R} = z$ in this case) that

$$\begin{aligned} y &= \sinh(\tilde{\eta}) = \sinh[\operatorname{arccosh}(w+1)] = \sqrt{\{\cosh[\operatorname{arccosh}(w+1)]\}^2 - 1} \\ &= \sqrt{(w+1)^2-1} = \sqrt{w^2+2w} = \sqrt{w}\sqrt{w+2} \end{aligned} \tag{50}$$

for $w \in [0,\infty]$. For the positive-curvature-radiation universe, one finds from the above formulae that

$$\begin{aligned} y &= \sin(\tilde{\eta}) = \sin[\arccos(1-w)] = \sqrt{1-\{\cos[\arccos(1-w)]\}^2} \\ &= \sqrt{1-(1-w)^2} = \sqrt{2w-w^2} = \sqrt{w}\sqrt{2-w} \end{aligned} \tag{51}$$

for $w \in [0,2]$.

12.3. Solutions for $Q = 2$

For $Q = 2$ and choosing the integration interval for maximum simplicity, we have for $w(\tilde{\eta})$ solutions

$$
Fw = \int_0^{\tilde{\eta}} z^2 \, d\tilde{\eta}' = \begin{cases} \displaystyle\int_0^{\tilde{\eta}} \sinh^2(\tilde{\eta}') \, d\tilde{\eta}' & \text{for } g = h = 1,\ z = \sinh(\tilde{\eta}), \\ \quad = \displaystyle\frac{1}{2}\left[\frac{1}{2}\sinh(2\tilde{\eta}) - \tilde{\eta}\right] & \text{and } \tilde{\eta} \in [0, \infty]; \\ \displaystyle\int_0^{\tilde{\eta}} \cosh^2(\tilde{\eta}') \, d\tilde{\eta}' & \text{for } g = -1, h = 1,\ z = \cosh(\tilde{\eta}), \\ \quad = \displaystyle\frac{1}{2}\left[\frac{1}{2}\sinh(2\tilde{\eta}) + \tilde{\eta}\right] & \text{and } \tilde{\eta} \in [0, \infty]; \\ \displaystyle\int_0^{\tilde{\eta}} \sin^2(\tilde{\eta}') \, d\tilde{\eta}' & \text{for } g = 1,\ h = -1,\ z = \sin(\tilde{\eta}), \\ \quad = \displaystyle\frac{1}{2}\left[\tilde{\eta} - \frac{1}{2}\sin(2\tilde{\eta})\right] & \text{and } \tilde{\eta} \in [0, \pi], \end{cases} \quad (52)
$$

where we have used the 3 general solutions for $z(\tilde{\eta})$ from, respectively, SS 6.2, 6.3, and 6.4, hyperbolic function identities from from Equation (22) in § 6.1, and the trigonometric identity $\sin^2(x) = (1/2)[1 - \cos(2x)]$ (e.g., Wikipedia: List of trigonometric identities: Half-angle formulae). Recall that since $z \geq 0$ always, the $z = \sin(\tilde{\eta})$ is limited to the argument range $\tilde{\eta} \in [0, \pi]$: i.e., this solution has both a point origin and point end.

There are no analytic inversion formulae $\tilde{\eta}(w)$ for the formulae in Equation (52). So one can only find solutions for $y = z^{2/R}$ in terms of generalized conformal time $\tilde{\eta}(w)$ or the standard conformal time η as it turns out.

12.3.1. The Negative-Curvature-Matter Universe and the Positive-Curvature-Matter Universe Solutions

In this case for $Q = q/R = q/(p-q) = 2$, we must have $p = (3/2)q$. Given that the only widely considered powers are integers $P \in [0, 4]$ (see § 6), there is only one interesting case of $Q = 2$: i.e., with $p = 3$, $q = 2$, $R = p - q = 1$, and $F = R/2 = 1/2$. This case (which is all we consider below) corresponds to the negative-curvature-matter universe (i.e., which has $\Omega_{q=2,0} > 0$) and the positive-curvature-matter universe (i.e., which has $\Omega_{q=2,0} < 0$). Recall the general solution for $g = -1$, $h = 1$ can only occur if $p = 2$ since curvature density parameter is the largest density component that can be negative.

For $r = 1$ and $F = R/2 = 1/2$, we have

$$y = z^2 \qquad \text{and} \qquad dw = 2y\,d\tilde{\eta} \ , \tag{53}$$

where we have used and Equation (13) in § 6 (i.e., $y = z^{2/R} = z^2$ in this case) and Equation (37) (in § 7: i.e., $F\,dw = z^{q/R}\,d\tilde{\eta} = y\,d\tilde{\eta}$ in this case????). Actually, the (standard) conformal time η is defined by $dw = y\,d\eta$ and so in this case $\eta = 2\tilde{\eta}$. Rewriting Equation (52) for the negative-curvature-matter universe and positive-curvature-matter universe using η instead of $\eta = \tilde{\eta}$

$$w = \begin{cases} \frac{1}{2}\left[\sinh(\eta) - \eta\right] & \text{for } g = h = 1,\ z = \sinh(\eta/2), \\ & \text{and } \eta \in [0, \infty]; \\ \frac{1}{2}\left[\eta - \sin(\eta)\right] & \text{for } g = 1,\ h = -1,\ z = \sin(\eta/2), \\ & \text{and } \eta \in [0, 2\pi], \end{cases} \tag{54}$$

The solutions $y(\eta)$ are

$$y = \begin{cases} \sinh^2\left(\frac{\eta}{2}\right) = \frac{1}{2}[\cos(\eta) - 1] & \text{for } \eta \in [0, \infty]; \\ \sin^2\left(\frac{\eta}{2}\right) = \frac{1}{2}[1 - \cos(\eta)] & \text{for } \eta \in [0, 2\pi], \end{cases} \tag{55}$$

where we have used hyperbolic function identities from from Equation (22) in § 6.1, and the trigonometric identity $\sin^2(x) = (1/2)[1 - \cos(2x)]$ (e.g., Wikipedia: List of trigonometric identities: Half-angle formulae).

Note the Friedmann equation as aforesaid does not in and of itself allow us to extrapolate through zeros of its solutions. So the positive-curvature-matter universe solution is only defined for the η domain $[0, 2\pi]$, and therefore the w domain $[0, \pi]$. At the end points, $y(\eta)$ and $y(w)$ are zero. Note also that

$$\frac{dy}{d\eta} = \frac{1}{2}\sin(\eta) \ , \tag{56}$$

and so $y(\eta = \pi) = y(w = \pi/2) = 1$ is the only maximum of $y(\eta)$.

As noted above, there are no analytic inversion formulae for $w(\tilde{\eta})$ which is the same as saying no analytic inversion formulae for $w(\eta)$. However, one can determine some things about the solutions $y(w)$. For an example, consider the positive-curvature-matter universe. We note the following identities:

$$y(2\pi - \eta) \ = \ \frac{1}{2}[1 - \cos(2\pi - \eta)] = \frac{1}{2}[1 - \cos(2\pi)\cos(\eta) - \sin(2\pi)\sin(\eta)]$$

$$
\begin{aligned}
&= \frac{1}{2}[1-\cos(\eta)] = y(\eta)\ . && (57)\\
\frac{dy}{dw}(\eta) &= \frac{dy/d\eta}{dw/d\eta} = \frac{1}{2}\frac{\sin(\eta)}{y(\eta)} && (58)\\
\frac{dy}{dw}(2\pi-\eta) &= -\frac{1}{2}\frac{\sin(\eta)}{y(\eta)} = -\frac{dy}{dw}(\eta) && (59)\\
w(2\pi-\eta) &= \frac{1}{2}\left[2\pi-\eta-\sin(2\pi-\eta)\right] = \frac{1}{2}\left[2\pi-\eta-\sin(2\pi)\cos(\eta)+\cos(2\pi)\sin(\eta)\right]\\
&= \frac{1}{2}\left[2\pi-\eta+\sin(\eta)\right] = \pi - w(\eta)\ . && (60)
\end{aligned}
$$

From the first and third equation above, we see that $y(\eta)$ at symmetrical points around the point $\eta = \pi$ has the same value and has slopes dy/dw of the opposite sign. Those two points along the η axis by the 4th equation above correspond to symmetrical points on the w axis about the point $w = \pi/2$. The upshot is that $y(w)$ is symmetrical about $w = \pi/2$: i.e.,

$$
y(\pi - w) = y(w)\ , \tag{61}
$$

which is an interesting result to know. Recall that $y(w = \pi/2) = 1$ is the maximum of $y(w)$.

In fact, the symmetry result Equation (61) suggests that $y(w)$ can be approximated by a sine-like function over the w domain $[0, \pi]$. In the following discussion we follow this clue, but we refer only to the w domain $[0, \pi/2]$ since the w domain $[\pi/2, \pi]$ has the same behavior *mutatis mutandis*. For a good approximation, we have to incorporate how $y(w)$ behaves for small w. So now consider, the small w expansions of $w(\eta)$ and $y(\eta)$:

$$
\begin{aligned}
w &= \frac{1}{2}\left[\eta-\sin(\eta)\right] \approx \frac{1}{12}\eta^3 && (62)\\
y &= \frac{1}{2}[1-\cos(\eta)] \approx \frac{1}{4}\eta^2 = \frac{12^{2/3}}{4}w^{2/3} = \left(\frac{3}{2}\right)^{2/3} w^{2/3} = (1.31037\ldots)\times w^{2/3}\ . && (63)
\end{aligned}
$$

In order to incorporate the small w behavior of $y(w)$ in an sine-like approximation for $y(w)$, we first suggest an approximation A:

$$
y_{\rm A}(w) = \sin^{2/3}(w)\ . \tag{64}
$$

Approximation A at the endpoints $w = 0$ and $w = \pi/2$ has the exact values (respectively 0 and 1) and is always an underestimate with maximum relative error of $\sim 24\,\%$ asymptotically as $w \to 0$. Approximation A's major problem is that it has the wrong power coefficient (i.e., 1) as $w \to 0$: the right power coefficient is $(3/2)^{2/3}$, of course. We improve approximation A to approximation B with a 2nd order divisor correction that gives the right power coefficient as $w \to 0$:

$$
y_{\rm B}(w) = \frac{\sin^{2/3}(w)}{[1 - C(w-\pi/2)]^2}\ , \tag{65}
$$

where

$$C = \frac{1 - 1/(3/2)^{2/3}}{(\pi/2)^2} \ . \tag{66}$$

Approximation B at the endpoints $w = 0$ and $w = \pi/2$ has the exact values (respectively 0 and 1) and is always an overestimate with maximum relative error of $\sim 2.25\,\%$ at about $w = (0.11169\ldots) \times \pi/2$. Approximation B is probably as accurate as one would like for educational reasons. Note a comparable 2nd order multiplier correction is always less accurate than the 2nd order divisor correction: they are asymptotically the same as $w \to 0$ and the latter has relative error $\sim 30\,\%$ smaller than the former asymptotically as $w \to \pi/2$.

12.4. Solutions for $Q = 3$

For $Q = 3$, we have only one interesting case: the radiation-matter universe which has $p = 4$, $q = 3$, $R = 1$, and $g = h = 1$. The radiation-matter universe is, of course, Λ-CDM model in the radiation-matter era which is prior to the matter era which is prior to the matter-Λ era where cosmic present is located.

From Equation (13) in § 5 Equation (24) in § 6.2, and Equation (37) in § 7 and Equations (40) and (38) in § 7, we find that

$$y = z^2 = \sinh^2(\tilde{\eta}) \ , \tag{67}$$

$$\frac{1}{2}\, dw = \sinh^3(\tilde{\eta})\, d\tilde{\eta} \ , \tag{68}$$

and

$$w = \frac{\tau}{\tau_{\rm scale}} = \frac{\tau}{\left(\Omega_{\rm R,0}/\Omega_{\rm M,0}\right)^{3/2}/\sqrt{\Omega_{\rm M,0}}} = \frac{\tau}{\Omega_{\rm R,0}^{3/2}/\Omega_{\rm M,0}^2} \ , \tag{69}$$

where we have set $F = R/2 = 1/2$ in Equation (38) from § 7. We now solve for $w(\tilde{\eta})$ requiring $w(\tilde{\eta} = 0) = 0$ so that the zero point of scaled cosmic time and generalized conformal time agree which implies that the cosmic time zero gives the generalized scale factor zero (since $z = \sinh(\tilde{\eta})$), and so the point origin occurs at the fiducial cosmic time zero as one would like. The solution is

$$\begin{aligned}
\frac{1}{2} w &= \int_0^{\tilde{\eta}} \sinh^3(\tilde{\eta}')\, d\tilde{\eta}' \\
&= \int_0^{\tilde{\eta}} \sinh(\tilde{\eta}')[\cosh^2(\tilde{\eta}') - 1]\, d\tilde{\eta}' \\
&= \left[\frac{1}{3}\cosh^3(\tilde{\eta}') - \cosh(\tilde{\eta}')\right]\Bigg|_0^{\tilde{\eta}}
\end{aligned}$$

$$= \frac{1}{3}\cosh^3(\tilde{\eta}) - \cosh(\tilde{\eta}) + \frac{2}{3} \tag{70}$$

which could have been done by a table integral (see, e.g., Wikipedia: List of integrals of hyperbolic functions: Integrals involving only hyperbolic sine functions). We can now find $w(y)$ explicitly:

$$\begin{aligned} w &= \frac{2}{3}[1+\sinh^2(\tilde{\eta})]^{3/2} - 2[1+\sinh^2(\tilde{\eta})]^{1/2} + \frac{4}{3} = \frac{2}{3}(1+y)^{3/2} - 2(1+y)^{1/2} + \frac{4}{3} \\ &= \frac{2}{3}(y-2)\sqrt{1+y} + \frac{4}{3} \ . \end{aligned} \tag{71}$$

We will explicate the solution $w(y)$ in § 13. Here we are interested in finding the inverse solution $y(w)$.

For inverse solution $y(w)$, we first find $\cosh(\tilde{\eta})$ as a function of w. Defining

$$u = \cosh(\tilde{\eta}) \ , \tag{72}$$

we can rearrange Equation (70) as a cubic equation

$$0 = u^3 - 3u + \left(2 - \frac{3}{2}w\right) \tag{73}$$

Equation (73) is, in fact, a depressed cubic equation in that it lacks a u^2 term (e.g., Wikipedia: Cubic equation: Depressed cubic). Depressed cubic equations have simpler solutions than general cubic equations.

The solution of Equation (73) follows from a standard procedure (e.g., Press et al. 1992, p. 179). First we define parameters

$$\tilde{Q} = 1 \qquad \text{and} \qquad \tilde{R} = 1 - \frac{3}{4}w \ , \tag{74}$$

where the tildes are needed to distinguish parameters from the totally different parameters Q and R that we use for other purposes. For

$$\tilde{R}^2 = \left(1 - \frac{3}{4}w\right)^2 \le 1 = \tilde{Q}^3 \tag{75}$$

implying $w \in [0, 8/3]$, there are three real roots written in terms of parameter

$$\theta = \arccos\left(\frac{\tilde{R}}{\sqrt{\tilde{Q}^3}}\right) = \arccos\left(1 - \frac{3}{4}w\right) \tag{76}$$

In fact, the 1st and 3rd of the three real root formulae specified by the procedure (e.g., Press et al. 1992, p. 179) are ruled out since they give some values for $u < 1$ and since $u = \cosh(\tilde{\eta}) \geq 1$ always. That leaves the 2nd root formula which we rearrange thusly

$$\begin{aligned} u &= -2\cos\left\{\frac{\arccos[1-(3/4)w]+2\pi}{3}\right\} = -2\cos\left\{\frac{\pi-\arccos[(3/4)w-1]+2\pi}{3}\right\} \\ &= -2\cos\left\{\pi - \frac{\arccos[(3/4)w-1]}{3}\right\} = 2\cos\left\{\frac{\arccos[(3/4)w-1]}{3}\right\} \ , \end{aligned} \tag{77}$$

where we used the inverse trigonometric identity $\arccos(x) = \pi - \arccos(-x)$ (e.g., Wikipedia: Inverse trigonometric functions: Relationships among the inverse trigonometric functions) and the trigonometric identity $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ (e.g., Wikipedia: Trigonometric functions: Sum and difference formulas). So our formula for allowed root u is

$$u = \begin{cases} 2\cos\left\{\dfrac{\arccos[(3/4)w-1]}{3}\right\} & \text{for } w \in [0, 8/3]; \\ 1 & \text{for } w = 0; \\ \sqrt{3} & \text{for } w = 4/3; \\ 2 & \text{for } w = 8/3. \end{cases} \tag{78}$$

For

$$\tilde{R}^2 = \left(1 - \frac{3}{4}w\right)^2 \geq 1 = \tilde{Q}^3 \tag{79}$$

implying $w \geq 8/3$, there is only one real root

$$u = \begin{cases} A + \dfrac{1}{A} & \text{for } w \geq 8/3; \\ 2 & \text{for } w = 8/3, \end{cases} \tag{80}$$

where

$$A = \left\{\left[\left(\frac{3}{4}\right)w - 1\right] + \sqrt{\left[\left(\frac{3}{4}\right)w - 1\right]^2 - 1}\right\}^{1/3} \ . \tag{81}$$

From Equation (67) and Equation (72) above and the first line of Equation (22) in § 6.1, we find the solution for the scaled cosmic scale factor

$$y(w) = \cosh^2(\tilde{\eta}) - 1 = u^2 - 1 = \begin{cases} \left[2\cos\left\{\frac{\arccos[(3/4)w - 1]}{3}\right\}\right]^2 - 1 & \text{for } w \in [0, 8/3]; \\ \left(A + \frac{1}{A}\right)^2 - 1 & \text{for } w \geq 8/3. \\ \left(\frac{3}{2}w\right)^{2/3} & \text{for } w >> 1 \\ & \text{(i.e., the large } w \text{ asymptotic solution).} \end{cases} \quad (82)$$

We call Equation (82) the 2nd exact formula for $y(w)$. What we call 1st exact formula (Galanti & Roncadelli 2021, p. 3) is given by Equation (99) in § 14.

The functional behavior of $y(w)$ is not obvious (except for the large w asymptotic solution), but it can be easily investigated using the inverse $w(y)$ which we have already found above in Equation (71). However, for a sanity check we rederive it from Equations (73) and (82) and then obtain its 1st and 2nd derivatives:

$$w(y) = \frac{2}{3}u^3 - 2u + \frac{4}{3} = \frac{2}{3}(1+y)^{3/2} - 2(1+y)^{1/2} + \frac{4}{3} = \frac{2}{3}(y-2)\sqrt{1+y} + \frac{4}{3} \quad (83)$$

$$\frac{dw}{dy} = \frac{y}{\sqrt{1+y}} \quad (84)$$

$$\frac{d^2w}{dy^2} = \frac{1 + (1/2)y}{(1+y)^{3/2}} \; . \quad (85)$$

From the derivative dw/dy, the only stationary point of $y(w)$ is a minimum at $y = 0$ (i.e., the point origin) and this implies that $y(w = 0)$ has infinite slope and for $w \geq 0$, $y(w)$ increases strictly. The 2nd order small y solution for $w(y)$ is

$$w = \frac{1}{2}y^2 \qquad \text{implying} \qquad y = \sqrt{2w} \quad (86)$$

is the lowest order solution for $y(w)$. In fact, $y = \sqrt{2w}$ is the pure radiation universe solution for $y(w)$ and it follows from from the small w limit with $p = 4$ (which is exactly a pure radiation universe) of Equation (43) in § 12.1. The asymptotic large y solution for $w(y)$ is

$$w = \frac{2}{3}y^{3/2} \qquad \text{implying} \qquad y = \left(\frac{3}{2}w\right)^{2/3} \quad (87)$$

which is the asymptotic large w solution $y(w)$ again. The $y = [(3/2)w]^{2/3}$ is the pure matter universe solution for $y(w)$ and it follows from from the small w limit with $p = 3$ (which is exactly a pure matter universe) of Equation (43) in § 12.1.

Given that Equation (83) for $w(y)$ has a relatively simple form and has simple limiting behaviors, we conclude that its overall behavior is not complicated. Therefore, we conclude the behavior of $y(w)$ (the inverse of $w(y)$) will not be complicated and will have the simple limiting forms we expect (i.e., for the pure radiation universe and the pure matter universe) despite the complicated appearance of the 2nd exact formula (Equation (82) above). The 1st exact formula also has a complicated appearance (see Equation (99) in § 14).

13. THE RADIATION-MATTER UNIVERSE $t(a)$ SOLUTION

We have have already determined the solution for $t(a)$ for the radiation-matter universe in the scaled form $w(y)$ by Equation (71) in § 12.4. However, for educational reasons we will rederive the solution starting from primary scaled form of the Friedmann equation for the radiation-matter universe (see the first line of Equation (13) in § 5).

The Friedmann equation radiation-matter universe (with radiation labeled by p with $p = 4$ and matter labeled by q with $q = 3$) can be changed into a solvable equation for $w(y)$ for said universe as follows:

$$\begin{aligned}
\left(\frac{\dot{x}}{x}\right)^2 &= \Omega_{p,0}x^{-4} + \Omega_{q,0}x^{-3} \\
\dot{x} &= \sqrt{\Omega_{p,0}x^{-2} + \Omega_{q,0}x^{-1}} \\
d\tau &= \frac{dx}{\sqrt{\Omega_{p,0}x^{-2} + \Omega_{q,0}x^{-1}}} \\
d\tau &= \frac{x\,dx}{\sqrt{\Omega_{p,0} + \Omega_{q,0}x}} \\
dw &= \frac{y\,dy}{\sqrt{1+y}} \ ,
\end{aligned} \tag{88}$$

where the scaled cosmic scale factor is given by

$$y = \frac{x}{x_{\rm scale}} = \frac{x}{\Omega_{p,0}/\Omega_{q,0}} \tag{89}$$

and the scaled time is given by

$$dw = \frac{d\tau}{\tau_{\rm scale}} = \frac{d\tau}{x_{\rm scale}^2/\sqrt{\Omega_{p,0}}} = \frac{d\tau}{(\Omega_{p,0}/\Omega_{q,0})^2/\sqrt{\Omega_{p,0}}} = \frac{d\tau}{(\Omega_{p,0}/\Omega_{q,0})^{3/2}/\sqrt{\Omega_{q,0}}} \ . \tag{90}$$

The $x_{\rm scale}$ is, in fact, the radiation-matter-equality scaled cosmic scale factor x as follows from matter-to-radiation ratio

$$\mathrm{Ratio}_{\rm MR} = \frac{\Omega_{\rm M,0}x^{-3}}{\Omega_{\rm R,0}x^{-4}} = \frac{\Omega_{\rm M,0}}{\Omega_{\rm R,0}}x = y \ . \tag{91}$$

Equation (88). Thus, $y = 1$ is the radiation-matter-equality scaled cosmic scale factor.

Equation (88) fifth line can be solved by a table integral for $w(y)$ (i.e., $t(a)$ when unscaled), where $w(y = 0) = 0$. Additionally, the integrand can be Taylor expanded in small y to obtain series expansion for $w(y)$ that is useful for numerically accurate evaluation of $w(y)$ for small y as we discuss below. We summarize the formulae and some specific values for $w(y)$ as follows:

$$
w = \begin{cases}
\frac{2}{3}(y-2)\sqrt{y+1}+\frac{4}{3} & \text{in general;} \\
0 & \text{for } y=0\text{: i.e., time zero.} \\
w_{\rm RM} = \frac{4}{3}\left(1-\frac{1}{\sqrt{2}}\right) = 0.39052\ldots & \text{for } y=1\text{, where } w_{\rm RM} \text{ is} \\
& \quad \text{the radiation-matter-equality time;} \\
\frac{4}{3} = 1.33333\ldots = (3.41421\ldots)\times w_{\rm RM} & \text{for } y=2\text{: i.e., when the} \\
& \quad \text{matter-to-radiation ratio is 2;} \\
\frac{8}{3} = 2.66666\ldots(6.82842\ldots)\times w_{\rm RM} & \text{for } y=3\text{: i.e., when the} \\
& \quad \text{matter-to-radiation ratio is 3;} \\
\int_0^y dy\left[\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}y^{n+1}\right] & \\
\quad = \int_0^y dy\Big[y-\frac{1}{2}y^2+\frac{3}{8}y^3-\frac{5}{16}y^4 & \\
\qquad +\frac{35}{128}y^5-\frac{63}{128}^{6}+\frac{231}{1024}y^7 & \\
\qquad\quad -\frac{429}{2048}y^8+\ldots\Big] & \text{the integral with } dw/dy \text{ series} \\
& \quad \text{as integrand and note (-1)!!=1;} \\
\sum_{n=0}^{\infty}a_n y^{n+2} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}\frac{y^{n+2}}{n+2} & \\
\quad = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\frac{y^{n+2}}{n+2} & \\
\quad = \frac{1}{2}y^2-\frac{1}{6}y^3+\frac{3}{32}y^4-\frac{1}{16}y^5 & \\
\qquad +\frac{35}{768}y^6-\frac{9}{128}^{7}+\frac{231}{8192}y^8 & \\
\qquad\quad -\frac{143}{6144}y^8+\ldots & \text{the } w(y) \text{ series}
\end{cases}
\tag{92}
$$

The general solution given in Equation (92) is, of course, the exact solution mathematically, but numerically it is subject to significant round-off error as $y \to 0$ because the first term must cancel to give $w(y=0)=0$. Thus, for series solution for $w(y)$ will become numerically more accurate at some point as $y \to 0$. So the series solution is of interest. Below we elucidate its convergence properties in § 13.1 and requirements for switching from the exact to the series solution in order to maintain numerical precision in § 13.2.

13.1. Convergence Properties of the $w(y)$ Series Solution

First, what are the convergence properties of $w(y)$ series solution. Appling the D'Alembert ratio test for absolute convergence (e.g., Arfken 1985, p. 282,294) gives

$$\begin{aligned}
\text{Ratio}_{\text{D'Al.-ra.-test}} &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| = \lim_{n\to\infty}\left|\frac{(2n-1)}{(2n)}\frac{(n+1)}{(n+2)}y\right| \\
&= \lim_{n\to\infty}\left|\left(1-\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)\left(1-\frac{2}{n}\right)y\right| = \lim_{n\to\infty}\left|\left(1-\frac{3}{2n}\right)y\right| \\
&= \begin{cases} y<1 & \text{absolute convergence;} \\ y=1 & \text{indeterminate or no test;} \\ y>1 & \text{absolute divergence.} \end{cases} \qquad (93)
\end{aligned}$$

For the case of $y = 1$, we apply the Raabe test (e.g., Arfken 1985, p. 287,294):

$$\text{Ratio}_{\text{Raabe-test}}(y=1) = \lim_{n\to\infty} n\left|\frac{a_{n-1}}{a_n}-1\right| = \lim_{n\to\infty} n\left|\frac{3}{2n}\right| = \frac{3}{2} > 1 \qquad (94)$$

which implies absolute convergence. Is there conditional converge for some $y > 1$? Note for $y > 1$,

$$\begin{aligned}
\lim_{n\to\infty}|a_n| &= \lim_{n\to\infty}\left|\frac{(2n-1)!!}{(2n)!!}\frac{y^{n+2}}{(n+2)}\right| \\
&\leq \lim_{n\to\infty}\left|\frac{y^{n+2}}{n}\right| = \lim_{n\to\infty}\frac{e^{(n+2)\ln(y)}}{n} = \lim_{n\to\infty}\frac{\ln(y)e^{(n+2)\ln(y)}}{1} = \infty\ , \qquad (95)
\end{aligned}$$

where we have treated n as a real number and used L'Hôpital's rule (see, e.g., Wikipedia: L'Hôpital's rule). Since it is a general sufficient condition for divergence that limit of the terms of an infinite series is nonzero (see, e.g., Wikipedia: Divergent series), we see that the $w(y)$ series diverges in all senses for all $y > 1$. To conclude, the $w(y)$ series converges absolutely for $y \leq 1$ and diverges in all senses for $y > 1$.

13.2. Switching from the Exact to the Series Solution for $w(y)$ in Order to Maintain Numerical Precision

For what y value as $y \to 0$ should one switch from the exact solution to series solution for $w(y)$ in order to maintain numerical precision? Both solutions recall are given by Equation (92) in § 13. Say one keeps terms of numerical signficance to of order precision m: i.e., the smallest term one keeps has size relative to the zero term of order given by

$$10^{-m} = \frac{y^j}{y^{j_0}} = y^{j-j_0} = y^n = 10^{-un} \ , \tag{96}$$

where $j = n+2$ for the w(y) series and where we write $y = 10^{-u}$. We now solve for n to find

$$n = \frac{m}{u} \ . \tag{97}$$

Now in all the calculations done for this paper we use Fortran-95 with machine precision $m = 18$. With $m = 18$, we find for $u = 18, 9, 6, 3, 2, 1, 0.18$ that, respectively, we need to keep terms to approximately $n = 1, 2, 3, 6, 9, 18, 100$. Note $y = 10^{-0.18} \approx 0.66$. Recall the actual number of terms is $n+1$ for the $w(y)$ series.

To study the actual accuracy behavior of the series and exact solutions for $w(y)$, we compare their values for $y \in [10^{-10}, 1]$ computing series coefficients numerically for those of higher order than displayed in Equation (92) in § 13. Recall the series solution is just marginally absolutely convergent $y = 1$ (see S 13.1). For $y = 10^{-10}$, the series solution gives $w = 0.4999999999833333334 \times 10^{-20} \approx 0.5$ as expected and the exact solution fails altogether giving 0 which has infinite relative error. For $y = 10^{-9}$, the series solution and exact solution agree to within relative error 0.08. For $y \in [0.47, 0.70]$, the two solution agree to within machine precision (i.e., their relative discrepancy is less than 10^{-18}) using $n = 99$ for the whole range. The approximate prediction in the last paragraph that $n = 100$ for the series solution gives a machine precision value for w for $y \lessapprox 0.66$ is verified. For $y = 1$, the relative discrepancy between the two solutions, though growing, is still only 0.0007 for $n = 100$.

The range of agreement within machine precision of the two solutions implies that the exact solution reaches machine precision for all $y \gtrsim 0.47$. In fact for $y < 0.50$, the series solution needs only $n = 52$ to give agreement within machine precision with the exact solution. Therefore, we recommend switching to the series solution for $y < 0.50$ using $n = 52$ (i.e., 53 terms in the series solution).

14. COMPARISON OF FORMULAE FOR THE EXACT SOLUTION OF THE RADIATION-MATTER UNIVERSE COSMIC SCALE FACTOR

The exact $w(y)$ formula for the radiation-matter universe given by Equation (92) in § 13 can be rearranged as a cubic equation for y: i.e.,

$$0 = y^3 - 3y^2 + \left(-\frac{9}{4}w^2 + 6w\right) \tag{98}$$

which is not a general cubic equation since it lacks a y term but it is not a depressed cubic equation since it has y^2 term (e.g., Wikipedia: Cubic equation: Depressed cubic). Because Equation (98) is not a depressed cubic equation or a simpler kind of cubic equation, its solution is not as simple as the solution we found for the depressed cubic equation for u: i.e., Equation (73) in § 12.4 (e.g., Press et al. 1992, p. 179). Equation (98) is actually the starting point for the derviation of the radiation-matter universe exact solution for the cosmic scale factor given by Galanti & Roncadelli (2021, eq. (25)). However, they write it in an unscaled form (Galanti & Roncadelli 2021, eq. (25)).

The exact solution formula of Galanti & Roncadelli (2021, eq. (20)) in scaled form (scaled via Equation (6) in § 2, Equations (5), and (13) in § 5, and Equation (37) in § 7 is

$$y(w) = \begin{cases} 1 - 2\sin\left[\dfrac{\arcsin(\tilde{W})}{3}\right] & \text{for } w \in [0, 4/3]; \\[2ex] 1 + 2\cos\left[\dfrac{\arccos(\tilde{W})}{3}\right] & \text{for } w \in [4/3, 8/3]; \\[2ex] 1 + 2\cosh\left[\dfrac{\operatorname{arccosh}(\tilde{W})}{3}\right] & \text{for } w \geq 8/3, \end{cases} \tag{99}$$

where

$$\tilde{W} = 1 - 3w + \frac{9}{8}w^2 \ . \tag{100}$$

Our exact solution formula for the cosmic scale factor (in scaled form) for the radiation-matter universe recapitulated and compacted a bit from Equations (82) and (81) in § 12.4 is

$$y(w) = \begin{cases} \left\{2\cos\left[\dfrac{\arccos(W)}{3}\right]\right\}^2 - 1 & \text{for } w \in [0, 8/3]; \\[2ex] \left(A + \dfrac{1}{A}\right)^2 - 1 & \text{for } w \geq 8/3. \end{cases} \tag{101}$$

where

$$A = \left(W + \sqrt{W^2 - 1}\right)^{1/3} \qquad \text{and} \qquad W = \frac{3}{4}w - 1 \ . \tag{102}$$

It is a remarkable fact that 1st exact formula (i.e., Equation (99)) and the 2nd exact formula (i.e., Equation (101)) are mathematically equivalent given that they look so different. However, they can both be derived exactly from Equation (83) in § 12.4: the former starting with variable y and the latter with the variable u. That there two exact formulae that look so different for the same universe model suggests that there may other cases of this situation for other universe models dervied from the Friedmann equation.

Note we did not need our general treatment of the two-density-component Friedmann equation to find the 2nd exact solution. We just needed Equation (83) in § 12.4 in its form for variable u. However, our general treatment was a natural path to that equation form which otherwise would have been hard to find.

We judge the 2nd exact formula to be the simpler the 1st exact formula. It has only 2 time eras rather than 3. It uses only 2 special functions rather than 6. The innermost argument expression W is linear in scaled cosmic time w rather than quadratic w as is innermost argument expression $\tilde{W}$. However, the A parameter used in Equation (101) is a complication, but we judge that to be less important. Also that the derivation of Equation (101) is simpler than that of Equation (99) as can be seen by a comparison of the derivation of the former in § 12.4 and the derivation of the latter by Galanti & Roncadelli (2021, p. 5–7).

The small and large w cases for the 2 exact formulae are given by

$$y(w) = \begin{cases} \sqrt{2w} & w << 1; \\ \left(\frac{3}{2}w\right)^{2} & w >> 1, \end{cases} \tag{103}$$

where the small w case follows the $w(y)$ series solution in Equation (92) in § 13 (see above) and also Equation (108) in § 14.1 (see below) and the large w case follows directly from Equations (99) and (100) or Equations (101) and (102) above. The small w case is the pure radiation universe and the large w case the pure matter universe. The pure radiation universe and pure matter universe cosmic scale factors in unscaled quantities are derived directly in Appendix A. The pure matter universe (as mentioned above in § 12.1.1) is, in fact, the Einstein-de Sitter universe (presented 1932: e.g., Wikipedia: Einstein-de Sitter universe; O'Raifeartaigh et al. 2015) which was often considered the standard cosmological model circa 1960–1995 before the start of the dominance of the Λ-CDM circa 1995 (e.g., Bondi 1961, p. 166; Scott 2018, p. 10).

We have tested the two exact formulae for numerical accuracy. We first calculate $w(y)$ to machine precision (i.e., $m = 18$ giving relative error $\lesssim 10^{-18}$ for range of y values logarithmically spaced from $y = 10^{-10}$ to $y = 10$. So we know the exact input y values to machine precision. We then calculate the output y values from for obtained w values for both exact formulae. Note the two exact formulae go to zero if evaluated exactly as $w \to 0$ as Equation (103) shows. However, Equations (99) and (101) show that round-off will occur as $w \to 0$ in the numerical evaluation of the exact formulae since an additive factor of 1 must be canceled for $y \to 0$ as $w \to 0$. As expected the exact formulae output values have relative error less than machine precision for all large values of w, but as w becomes small eventually have growing relative error. For the 1st exact formula Equation (99), relative error starts growing for $w \lesssim 0.002$ ($y \lesssim$???) and for $y(w)$ series that we describe in § 14.1 below. The $y(w)$ series yields

14.1. The Series Solution for $y(w)$

The exact solution formulae for $y(w)$ (Equations (99) and (101) in § 14) both seem rather intractable for series expansion for small w. So instead we invert the series solution for $w(y)$ (Equation (92) in § 13) making some notational innovations for convenience as necessary. First, let

$$y = \sum_{k=1}^{\infty} b_k z^k = \sum_{k=0}^{\infty} \tilde{b}_k z^{k+1} \; . \tag{104}$$

where we define $z = \sqrt{w}$ and $\tilde{b}_k = b_{k+1}$ (e.g., $\tilde{b}_0 = b_1$, etc.). Second,

$$w = z^2 = \sum_{n=2}^{\infty} \tilde{a}_n y^n = \sum_{n=2}^{\infty} \tilde{a}_n \left(\sum_{k=1}^{\infty} b_k z^k \right)^n \tag{105}$$

$$= \sum_{n=2}^{\infty} \tilde{a}_n \left(\sum_{k=0}^{\infty} \tilde{b}_k z^{k+1} \right)^n = \sum_{n=2}^{\infty} \tilde{a}_n z^n \left(\sum_{k=0}^{\infty} \tilde{b}_k z^k \right)^n = \sum_{i=2}^{\infty} \left[\sum_{n=2}^{i} c_{m=i-n}(n) \right] z^i \; , \tag{106}$$

where $\tilde{a}_n = a_{n-2}$ (e.g., $\tilde{a}_2 = a_0$, etc.:, the a_n coefficients are those appearing in Equation (92) in § 13) and the $c_{m=i-n}(n)$ coefficients are obtained from the formulae for a power series raised to a power:

$$\begin{aligned} c_{m=0}(n) &= b_0^n \\ c_{m=i-n>0}(n) &= \frac{1}{m b_0} \sum_{k=1}^{m} (kn - m + k) b_k c_{m-k}(n) \end{aligned} \tag{107}$$

(see Wikipedia: Formal power series: Power series raised to powers).

Equation (105) was used to obtain analytic formulae for the first 6 coefficients b_k by hand with increasing labor as k increased and Equations (106) and (107) (using an algorithm discussed below in § 14.2) were used to numerically confirm the analytic formulae and compute numerical values for the coefficients b_k for $k = 7$ to $k = 12$. The coefficients b_k for $k = 1$ to $k = 12$ were sufficient to obtain machine precision values for y ????

$$
b_k = \begin{cases}
\dfrac{1}{\sqrt{a_2}} = \sqrt{2} = 0.141421\ldots & \text{for } k=1; \\
-\dfrac{\tilde{a}_3 b_1^2}{2\tilde{a}_2} = \dfrac{1}{3} = 0.333333\ldots & \text{for } k=2; \\
-\dfrac{\tilde{a}_2 b_2^2 + 3\tilde{a}_3 b_1^2 b_2 + \tilde{a}_4 b_1^4}{2\tilde{a}_2 b_1} = -\dfrac{7}{144}\sqrt{2} = -0.06874649\ldots & \text{for } k=3; \\
\left(\dfrac{3\tilde{a}_2\tilde{a}_3\tilde{a}_4 - 2\tilde{a}_3^3 - \tilde{a}_2^2\tilde{a}_5}{2\tilde{a}_3}\right) b_1^4 = \dfrac{5}{216} = 0.0231481\ldots & \text{for } k=4; \\
-\left(\dfrac{1}{2\tilde{a}_2 b_1}\right)\Big[\tilde{a}_2(2b_1b_4 + b_3^2) & \\
\qquad +\tilde{a}_3(3b_1^2b_4 + 6b_1b_2b_3 + b_2^3) & \\
\qquad +\tilde{a}_4(4b_1^3b_3 + 6b_1^2b_2^2) & \\
\qquad +\tilde{a}_5(5b_2b_1^4) + \tilde{a}_6b_1^6\Big] & \\
\quad = -\dfrac{2275}{345600}\sqrt{2} = -(0.930942\ldots)\times 10^{-2} & \text{for } k=5; \\
-\left(\dfrac{1}{2\tilde{a}_2 b_1}\right)\Big[\tilde{a}_2(2b_2b_5 + 2b_3b_4) & \\
\qquad +\tilde{a}_3(3b_1^2b_5 + 6b_1b_2b_4 & \\
\qquad\qquad +3b_2^2b_3 + 3b_1b_3^2) & \\
\qquad +\tilde{a}_4(4b_1^3b_4 + 12b_1^2b_2b_3 + 4b_1b_2^3) & \\
\qquad +\tilde{a}_5(5b_1b_3 + 10b_1^3b_2^2) & \\
\qquad +\tilde{a}_6(6b_1^5b_2) + \tilde{a}_7b_1^7\Big] & \\
\quad = \dfrac{1}{243} = (0.4115226\ldots)\times 10^{-2} & \text{for } k=6;
\end{cases}
\tag{108}
$$

The reader might wonder if the series solution $y(w)$ and $w(y)$ (given by Equation (92) in § 13) are needed since the exact formulae can be used for high accuracy for, respectively, very small w and y by using extremely high machine precision. We argue that given the great importance of the radiation-matter universe as the ideal limit for the behavior of the actual early observable (given the Big Bang paradigm, of course), one desires the series solutions for complete understanding of the radiation-matter universe. Also the series solutions allow ordinary levels of machine precision to be used without excessive round-off error, and so avoid coding complications in calculating sufficiently high accuracy values of $y(w)$ and its $w(y)$. Sufficiently, high accuracy calculations of $y(w)$ and $w(y)$ from formulae (exact and series) provide stringent comparison tests for any computer code that calculates general cosmic scale factors and cosmic times by numerical integration.

14.2. The Algorithm for the Coefficients for the Series Solution for $y(w)$

Heck:

```
subroutine sub_rad_mat_y_series_coefficients_numerical
use iprecision
use constants
use mod_rad_mat_data
include '/homes/jeffery/jef/aalib/module_implicit.f'
real (kind=np) :: c(0:n_imax-2,2:n_imax)=0
```

Heck.

15. SMOOTHED-PIECEWISE-APPROXIMATE (SPA) SOLUTIONS

????? method for smoothed-piecewise-approximate (SPA) solutions.

16. CONCLUSIONS

The conclusions are in the abstract and the introduction (i.e., § 1).

17. ACKNOWLEDGMENTS

Support for this work was provided the Department of Physics & Astronomy and the Nevada Center for Astrophysics (NCfA) of the University of Nevada, Las Vegas.

A. ELEMENTARY ONE-DENSITY-COMPONENT EXACT ANALYTIC SOLUTIONS

$$H_0 = \left.\frac{\dot{a}}{a}\right|_{t_0} = \left.\frac{a_0\gamma(t/t_0)^{\gamma-1}/t_0}{a_0(t/t_0)^\gamma}\right|_{t_0} = \frac{\gamma}{t_0} \qquad \text{and} \qquad t_0 = \frac{\gamma}{H_0} \ . \tag{A1}$$

Table 1. Power-Law Solutions to the Friedmann Equation

$w\backslash$Quantity	$p = \frac{2}{\gamma}$	$\gamma = \frac{2}{p}$	$a(t)$	$t_0 = \frac{\gamma}{H_0}$	$q_0 = \frac{1}{\gamma} - 1$	ρ
$\left\{\begin{matrix} w \quad \text{or} \\ w \neq -1 \end{matrix}\right\}$	$3(1+w)$	$\frac{2}{[3(1+w)]}$	$a_0 \left(\frac{t}{t_0}\right)^{\gamma}$	$\gamma \left(\frac{13.968\,\text{Gyr}}{h_{70}}\right)$	$\left\{\begin{matrix} \frac{1}{2}(1+3w) \\ = \frac{p}{2} - 1 \end{matrix}\right\}$	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w = -1$	$p = 0$	$\gamma = \infty$	$a_0 e^{H_0(t-t_0)}$	∞	-1	ρ_0
$w = -\frac{1}{3}$	$p = 2$	$\gamma = 1$	$a_0 \left(\frac{t}{t_0}\right)$	$\frac{1}{H_0}$	0	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w = 0$	$p = 3$	$\gamma = \frac{2}{3}$	$a_0 \left(\frac{t}{t_0}\right)^{2/3}$	$\frac{2}{3}\frac{1}{H_0}$	$\frac{1}{2}$	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w = \frac{1}{3}$	$p = 4$	$\gamma = \frac{1}{2}$	$a_0 \left(\frac{t}{t_0}\right)^{1/2}$	$\frac{1}{2}\frac{1}{H_0}$	1	$\rho_0 \left(\frac{t_0}{t}\right)^2$

B. THREE-DENSITY-COMPONENT SOLUTIONS

Can the techniques of generalized cosmic scale factor and generalized conformal time of two-density-component cases of the Friedmann equation (see § 6 be generalized to three-density-component cases? Only to a limited, but educationally useful, extent as we show below. The generalization does give solutions that apply to the radiation-curvature-Λ universe and the radiation-cosmic-string-Λ universe and these solutions may be of actual cosmological interest for some versions of inflation. It is beyond the scope of this paper to consider the significance for these solutions, but we do point out the one likeliest to be of cosmological interest in Appendix B.1.

Rearranging Equation (4) in § 2), we can write the Friedmann equation in scaled cosmic time and scaled cosmic scale factor for a three-density-component case:

$$d\tau = \pm \frac{dx}{x\sqrt{\Omega_{p,0}x^{-p} + \Omega_{q,0}x^{-q} + \Omega_{r,0}x^{-r}}} \ , \tag{B1}$$

where recall the $\Omega_{P,0}$ quantities are the scale-time density parameters (capital P being the general density component symbol) and without loss of generality $p > q > r \geq 0$. Now define a generalized cosmic scale factor by

$$x = z^V \ , \qquad z = x^{1/V} \ , \qquad dx = Vz^{V-1}dz \ , \qquad \text{and} \qquad \frac{dx}{x} = \frac{V\,dz}{z} \ , \tag{B2}$$

where V is to be determined. Note only consider the real positive solution for $x = z^V$ since it is the only physically real one. We substitute for x with z and τ with w in Equation (B1) and multipy through by $1/z^{U/2}$ to obtain three-density-component case:

$$\frac{d\tau}{Vz^{U/2}} = \pm \frac{dz}{\sqrt{\Omega_{p,0}z^{-Vp+2+U} + \Omega_{q,0}z^{-Vq+2+U} + \Omega_{r,0}z^{-Vr+2+U}}} \ , \tag{B3}$$

Now in fact, the only case of Equation (B3) where the radicand permits analytical integration (at least in a simple way) is when the radicand is a quadratic. Since we are only interested in analytic solutions here, we choose U and V to give the z powers of 0 and 1 to, respectively, the $\Omega_{p,0}$ and $\Omega_{q,0}$ in Equation (B3) and these fix what r must be for the z power of $\Omega_{r,0}$ to be 2:

$$V = \frac{1}{p-q} \ , \qquad U = \frac{-p+2q}{p-q} = (-p+2q)\,V \ , \qquad \text{and} \qquad r = \frac{U}{V} = -p+2q \tag{B4}$$

Neither our formalism or anything else we can think of allows us to let r be general.

The upshot is as aforesaid the techniques of generalized cosmic scale factor and generalized conformal time of two-density-component cases of the Friedmann equation are limited for the generalization to three-density-component cases. But also aforesaid, these cases are educationally useful. They allow us to obtain representative (exact analytical) three-density-component solutions.

In fact, the main cases of educational interest are where $r = 0$: i.e., $\Omega_{r,0} = \Omega_\Lambda$, where recall in general Ω_Λ can be positive or negative. This requires $U = 0$, and so $p = 2q$ and $V = 1/q$. We will only consider cases with $r = 0$ (implying $U = 0$, $p = 2q$, and $V = 1/q$) below. The case where all three Ω parameters are nonzero applies to the radiation-curvature-Λ universe (where $\Omega_{q,0}$ can be positive or negative for curvature cases) and the radiation-cosmic-string-Λ universe. These cases are those of interest to some versions of inflation.

To proceed, we will make a notational change since $\Omega_{P,0}$'s are klutzy-looking symbols and are not the symbols used in tables of integrals, and so are hard to mentally keep track of. To conform to common tables of integrals, we define

$$a = \Omega_{r,0} = \Omega_{P=0,0} = \Omega_{\Lambda,0} \ , \qquad b = \Omega_{q,0} \ , \qquad \text{and} \qquad c = \Omega_{p,0} \ , \tag{B5}$$

and so the Friedmann equation (with $U = 0$ and leaving $V = 1/q$) becomes

$$\frac{d\tau}{V} = \pm \frac{dz}{\sqrt{az^2 + bz + c}} \ , \tag{B6}$$

where we refer to $az^2 + bz + c$ as the quadratic below. Note that $a = \Omega_{r,0}$ should not be confused with $a(t)$ the cosmic scale factor.

We note that $a = \Omega_{\Lambda,0}$ can be greater than or less then zero or equal to zero. Of course, $a = 0$ actually gives two-density-component case (assuming b and c are not zero). The b coefficient can only be negative for $q = 2$ for $\Omega_{q,0}$ representing positive curvature and the c coefficient can be negative for $p = 2$ for $\Omega_{p,0}$ representing positive curvature.

The analytic solutions to Equation (B6) with the constant of integration $\tau_{\rm in}$ are

$$\pm\frac{(\tau-\tau_{\rm in})}{V} = \begin{cases} \frac{1}{\sqrt{a}}\text{arcsinh}\left(\frac{2az+b}{\sqrt{4ac-b^2}}\right) & \text{for } a>0 \text{ and discriminant } b^2-4ac<0 \text{ (i.e., the quadratic has no real roots);} \\ \frac{1}{\sqrt{a}}\ln(|2az+b|) & \text{for } a>0 \text{ and discriminant } b^2-4ac=0 \text{ (i.e., the quadratic has one real root);} \\ \frac{1}{\sqrt{a}}\ln(|2\sqrt{a}\sqrt{az^2+bz+c}+2az+b|) & \text{for } a>0 \text{ and discriminant } b^2-4ac>0 \text{ (i.e., the quadratic has two real roots);} \\ \frac{2\sqrt{bz+c}}{b} & \text{for } a=0 \text{ which is actually a two-density-component case, but we include it here for completeness;} \\ \frac{1}{\sqrt{-a}}\text{arcsin}\left(\frac{2az+b}{\sqrt{b^2-4ac}}\right) & \text{for } a<0 \text{ and discriminant } b^2-4ac>0 \text{ (i.e., the quadratic has two real roots). If the discriminant is less than or equal 0, there is no solution since the quadratic is always negative or zero.} \end{cases} \quad \text{(B7)}$$

For the table integrals used for the above solutions, see, e.g., Wikipedia: List of integrals of irrational functions: Integrals involving $R=\sqrt{ax^2+bx+c}$; Integrals involving $S=\sqrt{ax+b}$.

All of $\tau(z)$ solutions (from which the $t(a)$ solutions can be obtained) in Equation (B7) can be inverted to obtain $z(\tau)$ solutions and then the $x(\tau)$ solutions (from which the $a(t)$ solutions can be obtained).

If we limit ourselves to cases $r = 0$ (implying $U = 0$, $p = 2q$, and $V = 1/q$), we have only two cases having integer powers $P \in [0, 4]$ which recall from § 5 are the only widely considered powers. The two cases are case (1) $r = 0$, $q = 1$, $p = 2$, $V = 1$ (e.g., perhaps a curvature-(or cosmic strings)-quintessence-Λ universe solution) and case (2) $r = 0$, $q = 2$, $p = 4$, $V = 1/2$ (e.g., perhaps a curvature-(or cosmic strings)-radiation-Λ universe solution).

Case (2) is more interesting that case (1) since it might apply to universe models of interest in modern cosmology: either some earlier phase of the actual universe or other pocket unverses than our own (if the observable universe is embedded in a pocket universe) in the multiverse theory. (Wikipedia: Multiverse; Wikipedia: Pocket unverse). Also the case discussed in Appendix B.1 with $b < 0$ applies to positive-curvature-radiation-Λ universe which is the an analogue to the Lemaître universe (i.e., a positive-curvature-matter-Λ universe: see, e.g., Bondi 1961, p. 82,84–85,120–122,165,168–170,175–176; McCrea 1984, p. 7–10; Peebles 1984, p. 23–30; North 1994, p. 528,530–531; Kragh 1996, p. 23–60; Luminet 2011).

Note an analogue to the radiation-matter-Λ universe solution (with P value not in the range $[0, 4]$) is the case (3) $r = 0$, $q = 3$, $p = 6$, $V = 1/3$. Case (3) does not a priori look like a promising approximation for the radiation-matter-Λ universe solution (of which the most interesting case is the Λ-CDM model solution), and so we will not consider it further.

Note that the known $x(\tau)$ solutions among the ones given in the subsections below (which are probably all of them) seem hard to find and are probably not given collectively elsewhere.

B.1. Solution for $a > 0$, $b^2 - 4ac < 0$

Inverting the $a > 0$, $b^2 - 4ac < 0$ case of Equation (B7) in Appendix B, setting $V = 1/2$ (for case (2) $r = 0$, $q = 2$, $p = 4$, $V = 1/2$), and transforming from z to x, we find the solution

$$x = \begin{cases} \left\{ \dfrac{-b + \sqrt{4ac - b^2}\,\sinh[2\sqrt{a}(\tau - \tau_{\rm in})]}{2a} \right\}^{1/2} & \text{in general with } \tau_{\rm in} \text{ chosen to give} \\ & x(\tau = 0) = 0 \text{ to avoid pointless} \\ & \text{generality by the } \tau_{\rm in} \text{ formula} \\ \qquad \tau_{\rm in} = \pm \dfrac{1}{2\sqrt{a}} \text{arcsinh}\left(\sqrt{\dfrac{|b|}{4ac - b^2}} \right) & \text{with upper/lower case} \\ & \text{for } b \text{ negative/positive} \\ \qquad\qquad = \pm \dfrac{1}{2\sqrt{a}} \text{arcsinh}\left(\dfrac{1}{\sqrt{\dfrac{4ac}{b^2} - 1}} \right) & \text{(i.e., positive/negative curvature);} \\ \left[\dfrac{\sqrt{4ac}\,\sinh(2\sqrt{a}\,\tau)}{2a} \right]^{1/2} & \text{for } b = 0 \text{ with } \tau_{\rm in} = 0 \text{ to avoid} \\ & \text{pointless generality;} \\ \left(2\sqrt{c}\,\tau \right)^{1/2} & \text{for } b = 0 \text{ and } a \to 0 \text{ with} \\ & \tau_{\rm in} = 0 \text{ to avoid pointless} \\ & \text{generality. This is the pure} \\ & \text{radiation universe solution;} \\ \left(\dfrac{-b}{2a} \right)^{1/2} & \text{for } b < 0 \text{ with } (4ac - b^2) \to 0. \\ & \text{This is a static positive-curvature} \\ & \text{-radiation-}\Lambda \text{ universe It is} \\ & \text{the analogue to the Einstein universe} \\ & \text{(a static positive-curvature} \\ & \text{-matter-}\Lambda \text{ universe);} \\ \left(2\sqrt{c}\,\tau \right)^{1/2} & \text{for } \tau \text{ small for general } a \text{ and } b. \\ & \text{This the pure radiation solution} \\ & \text{again.} \\ \left(\dfrac{\sqrt{4ac - b^2}}{2a} \right)^{1/2} \dfrac{e^{\sqrt{a}(\tau - \tau_{\rm in})}}{2} & \text{for } \tau \text{ large. This is the;} \\ & \text{analogue to the de Sitter universe} \\ & \text{solution (a pure exponentially} \\ & \text{expanding universe solution).} \end{cases} \tag{B8}$$

For the Einstein universe, presented 1917, see, e.g., Bondi (1961, p. 84,98–99,117–121,158–159,171); O'Raifeartaigh et al. (2017); O'Raifeartaigh (2019); Wikipedia: Einstein's static universe. For the de Sitter universe, presented 1917, see, e.g., Bondi (1961, p. 98–99,105,146–147,154,159,166); Wikipedia: de Sitter universe.

For the derivation of the 2nd to last line in Equation (B8), note to 1st order in small τ that

$$\begin{aligned} x_{\rm 1st} &= \left.\left\{\frac{-b+\sqrt{4ac-b^2}\sinh[2\sqrt{a}(\tau-\tau_{\rm in})]}{2a}\right\}^{1/2}\right|_{\rm 1st} \\ &= \left\{\frac{\sqrt{4ac-b^2}\cosh[2\sqrt{a}(-\tau_{\rm in})]\,(2\sqrt{a}\,\tau)}{2a}\right\}^{1/2} \\ &= \left\{\frac{\sqrt{4ac-b^2}\sqrt{1+\sinh^2[2\sqrt{a}(-\tau_{\rm in})]}\,(2\sqrt{a}\,\tau)}{2a}\right\}^{1/2} \\ &= \left[\frac{\sqrt{4ac-b^2}\,\sqrt{1+b^2/(4ac-b^2)}(2\sqrt{a}\,\tau)}{2a}\right]^{1/2} \\ &= \left[\frac{\sqrt{4ac}\,(2\sqrt{a}\,\tau)}{2a}\right]^{1/2} = \left(2\sqrt{c}\,\tau\right)^{1/2}\ , \end{aligned} \tag{B9}$$

where we have used Equation (22) in § 6.1.

Equation (B8) (meaning the general case as we usualy do hereafter) since it has a point origin may be relevant to some eras of nonstandard inflation where curvature and radiation are significant.

Two points can be made about the solution given by Equation (B8). First, the radiation density parameter c/x^4 goes to infinity faster than the curvature density parameter b/x^2 as τ becomes small and the Λ density parameter a is constant, and the radiation density parameter must dominate completely for small enough τ. So it is expected the pure radiation solution is the asymptotic form of the solution for τ small.

Second, the Λ density parameter a completely dominates for τ large as the other two density parameters strictly decrease for finite τ and go to zero as $\tau \to \infty$. To be more precise, the Λ density parameter a completely dominates the derivative of x, not its size. So the relative growth for τ large is exponential as for the de Sitter universe and depends only on a. However, the coefficient for τ large depends on all the density parameter scale values a, b, and c. One can understand this in that these values establish the amount of growth determined by all the density parameters before the Λ density parameter became absolutely dominant in determining the derivative of x.

Another two points about can be made about the solution given by Equation (B8) First, for $b > 0$ (i.e., the negative curvature case) with accompanying $\tau_{\rm in} < 0$, the hyperbolic sine function (which depends on $\Delta\tau = (\tau - \tau_{\rm in})$) in Equation (B8) is asymptotically growing exponentially away from the hyperbolic-sine-function linear region and this growth will be significant unless $|\tau_{\rm in}|$ is sufficiently small.

Second, for $b < 0$ (i.e., the positive curvature case) with accompanying $\tau_{\rm in} < 0$, the hyperbolic sine function must start with a negative value and is asymtotically growing exponentially toward the hyperbolic-sine-function linear region from below. Thus, for $\Delta\tau = (\tau - \tau_{\rm in})$ sufficiently small, the hyperbolic sine function and the solution overall will have a linear growth region. Equation (B8) with $b < 0$ is, in fact, the radiation analogue of Lemaître universe solution (a positive-curvature-matter-Λ universe solution). We will call this analogue the radiation Lemaître universe solution. The Lemaître universe itself is usually considered to have the slope of linear-growth region sufficiently small that there is a long nearly static phase (i.e., an Einstein universe phase). In brief description, Lemaître universe (as usually considered) begins from a point origin (which recall is the old-fashioned name for Big Bang singularity: Bondi e.g., 1961, p. 117), has a decelerating phase, then a nearly static phase (i.e., an Einstein universe phase) which is the small slope region of the solution, and finally grows exponentially. There is no exact solution for the Lemaître universe.

We can show that Equation (B8) is the analogue solution to the Lemaître universe solution (as usually considered) explicitly by writing it thusly

$$x = \begin{cases} \left(\frac{-b}{2a}\right)^{1/2}\left[1 + \left(\frac{4ac}{b^2} - 1\right)^{1/2}\sinh\left(2\sqrt{a}\Delta\tau\right)\right]^{1/2} & \text{where } \Delta\tau = (\tau - \tau_{\rm in}) \\ & \text{and recall } b < 0 \\ & \text{and } \tau_{\rm in} > 0; \\ \left(\frac{-b}{2a}\right)^{1/2}\left[1 + \left(\frac{4ac}{b^2} - 1\right)^{1/2}\left(\sqrt{a}\Delta\tau\right)\right] & \text{for } \Delta\tau \text{ small;} \\ \left(\frac{-b}{2a}\right)^{1/2} & \text{for } 4ac/b^2 = 1 \text{ which is} \\ & \text{the solution for the} \\ & \text{radiation analogue} \\ & \text{of the Einstein universe.} \end{cases} \tag{B10}$$

Note for fixed $\Delta\tau$, one can make the solution as close to the Einstein universe $x = \sqrt{-b/(2a)}$ as you like by making $4ac/b^2$ sufficiently close to 1 which also makes $\tau_{\rm in}$ very large (see the formulae for $\tau_{\rm in}$ in Equation (B8) above). Thus, you can make the Einstein phase as long as required for the radiation Lemaître universe solution.

Can Equation (B8) be interpreted as an approximate radiation-matter-Λ universe solution by taking $b > 0$ to be the scale-time density parameter for matter? Probably yes, but only for cases where matter is never the dominant form of mass-energy since Equation (B8) does not explicitly show that there is any time period where $x \sim \tau^{2/3}$ as required for matter dominance (see, e.g., the solution for the matter-Λ universe Equation (46) in § 12.1.1 and the solution for the radiation-matter universe Equation (82) in § 12.4). Matter since it scales as x^{-3} in the Friedmann equation must eventually dominate radiation which scales as x^{-4}, but for Equation (B8) to be interpreted as an approximate radiation-matter-Λ universe solution this would have to be after Λ has become the dominant density component. To conclude Equation (B8) as an approximate radiation-matter-Λ universe solution might be the case when matter is never dominant. But even so it is probably not an interesting approximation. The smoothed-piecewise approximate (SPA) radiation-matter-Λ universe solution given in § 15 is probably always more interesting.?????

Can Equation (B8) be morphed into an approximate Lemaître universe solution (i.e., positive-curvature-matter-Λ universe solution for which no exact solution exist) that is educationally useful. Yes. First, we interpret c as scale-time matter density parameter and then change appropriate factors of 1/2 into factors 2/3 in Equation (B8) guided by the matter-Λ universe solution (Equation (46) in § 12.1.1) and we obtain

$$x = \begin{cases} \left\{\dfrac{-b+\sqrt{4ac-b^2}\sinh[(3/2)\sqrt{a}(\tau-\tau_{\rm in})]}{2a}\right\}^{3/2} & \text{where recall } \tau_{\rm in} > 0 \text{ for } b < 0 \\ & \text{(i.e., the positive curvature case);} \\ \left(\dfrac{3}{2}\sqrt{c}\,\tau\right)^{3/2} & \text{for } \tau \text{ small, where we have followed} \\ & \text{similar steps to those in} \\ & \text{Equation (B9);} \\ \left(\dfrac{\sqrt{4ac-b^2}}{2a}\right)^{3/2} \dfrac{e^{\sqrt{a}(\tau-\tau_{\rm in})}}{2} & \text{for } \tau \text{ large.} \end{cases} \tag{B11}$$

For small τ, morphed solution does not have the radiation solution behavior (see Equation (B8) above) though it depends on the radiation density parameter c. For large τ, the morphed solution does not have the correct dependence on the matter density parameter as seen from Equation (46) in § 12.1.1. Because of these incorrect behaviors, we conclude the the morphed solution is not educationally useful though it might approximate the radiation-matter-Λ universe solution in some τ regions for some density parameter values.

Another morphed solution is, in fact, of educational interest. is one where we morph Equation (B8) into an approximate solution for the Lemaître universe (which recall as no exact solution). In this, case c is set to be the density parameter for matter, $b < 0$ for negative curvature. and are again guided by matter-Λ universe solution (Equation (46) in § 12.1.1) for large τ. We obtain

$$x = \begin{cases} \left\{\dfrac{-b+\sqrt{4ac-b^2}\sinh[(3/2)\sqrt{a}(\tau-\tau_{\rm in})]}{2a}\right\}^{3/2} & \text{where recall } \tau_{\rm in} < 0 \text{ for } b > 0; \\ \left(\dfrac{3}{2}\sqrt{c}\,\tau\right)^{3/2} & \text{for } \tau \text{ small, where we have followed similar steps to those in Equation (B9);} \\ \left(\dfrac{\sqrt{4ac-b^2}}{2a}\right)^{3/2}\dfrac{e^{a(\tau-\tau_{\rm in})}}{2} & \text{for } \tau \text{ large.} \end{cases} \tag{B12}$$

????Since it is of historical interest, we will digress on the Lemaître universe (e.g., Bondi 1961, p. 82,84–85,120–122,165,168–170,175–176; North 1994, p. 528,530–531; McCrea 1984, p. 7–10; Peebles 1984, p. 23–30; Kragh 1996, p. 23–60; Luminet 2011). Georges Lemaître (1894–1966: e.g., Wikipedia: Georges Lemaître) presented this universe model in 1931 to satisfy at least three desiderata. First, the Einstein universe phase was thought necessary to give a halt to expansion sufficient for local gravitational collapses of matter to form the galaxies. Second, the Einstein universe was used to solve the time-scale problem (e.g., Bondi 1961, p. 120–122). To explicate the time-scale problem of 1931, the age of the Earth from radiometric was believed to be of order 1.6 to 3 gigayears (e.g., Wikipedia: Age of Earth: Arthur Holmes establishes radiometric dating) and the Hubble time at that time was only of order 2 gigayears. If one takes the Hubble time as a characteristic of age of the universe from a point origin, there seemed at most barely enough time for Earth to form and that was the time-scale problem (e.g., Bondi 1961, p. 116). The Einstein universe phase could be adjusted to be as long as needed to solve the time-scale problem. Third, the point origin, but not taken literally, was thought useful for the contents of the Lemaître universe. Lemaître considered that the Lemaître universe tracked into the Lemaître universe Friedmann equation solution a bit after that solution's time zero. The earliest phase of the Lemaître universe was hypothesized to the primeval atom (in French, *L'atome primitive*). The primeval atom was thought of as a giant nucleus consisting of matter quanta which Lemaître may thought of as being just protons in 1931 since the neutron was discovered in 1932 (e.g., Wikipedia: Neutron: Discovery) or a perhaps still undetermined particle The primeval atom was highly unstable and rapidly fragmented during the early deceleration phase of the Lemaître universe to make the hydrogen gas out of which the galaxies later formed. The primeval atom and its fragmentation is a vague precursor of Big Bang nucleosynthesis.

B.2. Solution for $a > 0$, $b^2 - 4ac = 0$

Inverting the $a > 0$, $b^2 - 4ac = 0$ case of Equation (B7) in Appendix B, setting $V = 1/2$ (for case (2) $r = 0$, $q = 2$, $p = 4$, $V = 1/2$), and transforming from z to x, we find the solution

$$x = \sqrt{\frac{-b \pm e^{\pm 2\sqrt{a}(\tau - \tau_{\rm in})}}{2a}} \ , \tag{B13}$$

where the two $\pm$ cases are not the same cases.

B.3. Solution for $a > 0$, $b^2 - 4ac > 0$

The $a > 0$, $b^2 - 4ac > 0$ case of Equation (B7) in Appendix B takes several steps to inverse unlike all the other cases. As a first step, we rewrite the $a > 0$, $b^2 - 4ac > 0$ case in the form

$$\pm C e^{\pm 2\sqrt{a}(\tau - \tau_{\rm in})} = 2\sqrt{a}\sqrt{az^2 + bz + c} + 2az + b \ , \tag{B14}$$

where we have partitioned the original constant of integration $\tau_{\rm in}$ into two constants of integration for convenience (i.e., a new $\tau_{\rm in}$ and C) and here the two $\pm$ cases are not the same cases.

In order to find the solution $z(\tau)$, we need the roots of the quadratic equation $az^2 + bz + c = 0$. They are

$$z_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ . \tag{B15}$$

Since a is greater than zero, the quadratic function $az^2 + bz + c$ is less than zero between the roots and there is no solution between the roots for for z. So between the roots is the forbidden zone which in a Newtonian physics sense corresponds to a zone of negative kinetic energy.

For convenience in the following derivation, let

$$f = \pm C e^{\pm 2\sqrt{a}(\tau - \tau_{\rm in})} \ , \tag{B16}$$

Note at the roots

$$f_\pm = f(z_\pm) = 2az_\pm + b = \pm\sqrt{b^2 - 4ac} \ . \tag{B17}$$

Requiring that $\tau = \tau_{\rm in}$ when $z = z_\pm$ fixes $C = \pm\sqrt{b^2 - 4ac}$, and so gives

$$f = \pm\left(\sqrt{b^2 - 4ac}\right) e^{\pm 2\sqrt{a}(\tau - \tau_{\rm in})} \ , \tag{B18}$$

where as aforesaid the two $\pm$ cases are not the same cases.

Now we solve for z:

$$\begin{aligned}
f &= 2\sqrt{a}\sqrt{az^2+bz+c}+2az+b \\
f-2az-b &= 2\sqrt{a}\sqrt{az^2+bz+c} \\
f^2+4a^2z^2+b^2-4afz-2bf+4abz &= 4a^2z^2+4abz+4ac \\
4afz &= b^2-4ac+f^2-2bf \\
z &= \frac{b^2-4ac+f^2}{4af}-\frac{b}{2a} \\
z &= \pm\frac{\sqrt{b^2-4ac}}{2a}\left[\frac{e^{\mp 2\sqrt{a}(\tau-\tau_{\rm in})}+e^{\pm 2\sqrt{a}(\tau-\tau_{\rm in})}}{2}\right]-\frac{b}{2a} \\
z &= \pm\frac{\sqrt{b^2-4ac}}{2a}\cosh\left[2\sqrt{a}(\tau-\tau_{\rm in})\right]-\frac{b}{2a}\,. \qquad \text{(B19)}
\end{aligned}$$

We explicate the $z(\tau)$ solution in Equation (B19) as follows. The center of the forbidden zone is $-b/(2a)$. There is an upper/lower branch hyperbolic cosine solution that has stationary point minimum/maximum at $\tau_{\rm in}$ with stationary point value $\pm\left(\sqrt{b^2-4ac}\right)/(2a)\leq |b|/(2a)$, where the equality only holds for $c=0$. Recall by hypothesis of this solution is $a>0$ and recall $c\geq 0$ is always assumed since it corresponds a radiation density component. If $b<0$, then at the stationary point both branches are greater than zero. The upper branch is physically allowed for all τ. If $c>0$, the lower branch has a point origin and a point end where $z\to 0$ symmetrically located around the stationary point. If $c=0$, the lower branch reduces to a point with $z=0$ for being physically allowed which is not really a physically allowed solution at all. If $b>0$ and $c>0$, then the upper branch solution breaks into two physically allowed solutions: the one for $\tau<\tau_{\rm in}$ has a point end and the one for $\tau>\tau_{\rm in}$ has a point origin. If $b>0$ and $c=0$, upper branch solution is united again with its minimum being $z=0$. If $b>0$, the lower banch solution is always negative and not physically allowed.

Finally, the $x(\tau)$ solution with $V=1/2$ (for case (2) $r=0$, $q=2$, $p=4$, $V=1/2$) is

$$x=\sqrt{\pm\frac{\sqrt{b^2-4ac}}{2a}\cosh\left[2\sqrt{a}(\tau-\tau_{\rm in})\right]-\frac{b}{2a}}\,. \qquad \text{(B20)}$$

The explication of the $x(\tau)$ solution is similar to that of the $z(\tau)$ solution, *mutatis mutandis*.

B.4. Solution for $a = 0$

Inverting the $a = 0$ case of Equation (B7) in Appendix B, setting $V = 1/2$ (for case (2) $r = 0$, $q = 2$, $p = 4$, $V = 1/2$), and transforming from z to x, we find the solution

$$x = \begin{cases} \sqrt{\dfrac{[b(\tau - \tau_{\rm in})]^2 - c}{|b|}} & \text{for } b > 0; \\ \sqrt{\dfrac{-[b(\tau - \tau_{\rm in})]^2 + c}{|b|}} & \text{for } b < 0, \end{cases} \tag{B21}$$

Recall that the $a = 0$ case is actually a two-density-component Friedmann equation case included for completeness in the solutions of the Friedmann equation depending on the quadratic $az^2 + bz + c$: see Equation (B6) in Appendix B.

B.5. Solution for $a < 0$, $b^2 - 4ac > 0$

Inverting the $a < 0$, $b^2 - 4ac > 0$ case of Equation (B7) in Appendix B, setting $V = 1/2$ (for case (2) $r = 0$, $q = 2$, $p = 4$, $V = 1/2$), and transforming from z to x, we find the solution

$$x = \sqrt{\frac{-b + \sqrt{b^2 - 4ac}\,\sin[2\sqrt{-a}(\tau - \tau_{\rm in})]}{2a}} \ , \tag{B22}$$

C. FIRST ORDER AUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS: STATIONARY POINTS AND CONSTANT SOLUTIONS

The Friedmann equation is, in fact, a 1st order autonomous ordinary differential equation (e.g., Wikipedia: Autonomous system (mathematics)). Autonomous differential equations do not depend explicitly on the independent variable and 1st order ones do not in most cases have solutions with stationary points, except at infinity of the independent variable (i.e., either positive or negative infinity) where all orders of derivatives of the solutions are zero and the solutions approach horizontal asymptotes. A horizontal asymptote is actually a constant solution of the 1st order autonomous ordinary differential equation. Constant solutions always accompany solutions with solutions that approach horizontal asymptotes. Hereafter we will refer the stationary points at infinity as asymptotic stationary points to differentiate them from ordinary stationary points. That 1st order autonomous ordinary differential equations usually do not have ordinary stationary points we call the no-ordinary stationary point rule.

But there are exceptions to the no-ordinary stationary point rule as we have hinted above. The Friedmann equation is, in fact, one of the exceptions. Below, we derive the no-ordinary stationary point rule in § C.1 and determine the most obvious exceptions to the no-ordinary stationary point rule in § C.2.

C.1. The No-Ordinary Stationary Point Rule

Let us prove the rule that there are no ordinary stationary points for 1st order autonomous ordinary differential equations. Consider the 1st order autonomous ordinary differential equation

$$x' = \begin{cases} f(x) & \text{in general;} \\ f(x_0) = 0 & \text{for } x = x_0, \end{cases} \tag{C1}$$

where t is the independent variable, x is assumed to be infinitely differential with respect to t, $x' = dx/dt$, $x = x_0$ is a stationary point value (meaning dependent variable value here and hereafter), and $f(x)$ is assumed to be infinitely differentiable with respect to x. Now note

$$\begin{aligned} x' &= f(x) \\ x'' &= \frac{df}{dx}x' = \frac{df}{dx}f(x) = f_2(x) \\ x^{(k-1)} &= \frac{df_{k-2}}{dx}x' = \frac{df_{k-2}}{dx}f(x) = f_{k-1}(x) \\ x^{(k)} &= \frac{df_{k-1}}{dx}x' = \frac{df_{k-1}}{dx}f(x) \end{aligned} \tag{C2}$$

where the $f_k(x)$ functions are defined iteratively as shown in Equation (C2).

Clearly from Equation (C2), if x_0 is stationary point value at stationary point t_0 giving $x'(t_0) = 0$, then all orders of derivative of $x(t)$ are zero at t_0. Where can the stationary point be? Any finite t_0 implies a constant solution $x = x_0$ rather than a stationary point in an ordinary sense. So the strictly increasing/decreasing $x(t)$ (i.e., strictly before reaching the stationary point) can only have stationary point value x_0 asymptotically as $t \to \pm\infty$ (i.e., at $t_0 = \pm\infty$) since at any finite t, $x(t)$ will have nonzero derivatives of some order. Of course, in general $x(t)$ can have multiple stationary points or no stationary points.

What if there are somehow nonzero derivatives for $x^{(k>1)}$ for $x = x_0$? In this case, the strictly increasing/decreasing $x(t)$ at some point will get sufficiently close to $x = x_0$ that in a finite change in t, it will reach $x = x_0$. So there will be a stationary point not at infinity in this case, and so an exception to the no-ordinary stationary point rule. As aforesaid in Appendix C, we consider the most obvious exceptions in § C.2.

A question that now arises is are the constant solutions stable: i.e., does an infinitesimal displacement of x from x_0 cause asymptotic evolution back to x_0 as t increases or eternally diverging evolution? We take up this question in general in Appendix D.

C.2. The Exceptions to the No-Ordinary Stationary Point Rule

How can one have exceptions to the no-ordinary stationary point rule. There may be exotic exceptions, but the most obvious ones are where x is infinitely differential with respect to t, but $f(x)$ is not infinitely differentiable with respect to x at stationary points and you have zero over zero cancellations arising from zeros in the denominators of derivatives of $f(x)$ being canceled by zeros in x' factors produced by the chain rule. To see how these cancellations arise, consider the 1st order autonomous ordinary differential equation

$$x' = \begin{cases} G(x)[H(x)]^P & \text{in general;} \\ G(x)[H(x)]^P = 0 & \text{for } H(x = x_0) = 0, \end{cases} \tag{C3}$$

where the independent variable is t, x is assumed to be infinitely differential with respect to t, $x' = dx/dt$, $G(x)$ and $H(x)$ are an infinitely differential functions with respect to x, x_0 is a zero of $H(x)$, but not necessarily a stationary point value, and we have $P > 0$ Suppressing all explicit dependence on x for simplicity and defining and redefining $G_{k,\ell}$ functions as needed, we note

$$\begin{aligned}
x' &= GH^P \\
x'' &= PGH^{(P-1)}\frac{dH}{dx}x' + \frac{dG}{dx}x'H^P = PGH^{(P-1)}\frac{dH}{dx}GH^P + \frac{dG}{dx}GH^{2P} \\
x'' &= G_{2,1}H^{(2P-1)} + G_{2,0}H^{2P} \\
x^{(3)} &= G_{3,2}H^{(3P-2)} + G_{3,1}H^{3P-1} + G_{3,0}H^{3P} \\
x^{(k-1)} &= \sum_{\ell=k-2}^{0} G_{k-1,\ell}H^{[(k-1)P-\ell]} \\
x^{(k)} &= \sum_{\ell=k-2}^{0} \left\{ [(k-1)P-\ell]G_{k-1,\ell}H^{[(k-1)P-\ell-1]}GH^P + \frac{dG_{k-1,\ell}}{dx}GH^{(kP-\ell)} \right\} \\
&= \sum_{\ell=k-2}^{0} (\ldots)H^{[kP-\ell-1]} + \sum_{\ell=k-2}^{0} (\ldots)H^{(kP-\ell)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=k-1}^{1} (\ldots) H^{(kP-\ell)} + \sum_{\ell=k-2}^{0} (\ldots) H^{(kP-\ell)} \\
x^{(k)} &= \sum_{\ell=k-1}^{0} G_{k,\ell} H^{(kP-\ell)} \qquad \text{(C4)}
\end{aligned}
$$

In Equation (C4), the lowest power of H is always the first term (i.e., the term $\ell = k-1$). Now note that if

$$
\begin{aligned}
P &\geq 1 \\
kP &\geq k \\
kP-(k-1) &\geq k-(k-1) = 1 \ , \qquad \text{(C5)}
\end{aligned}
$$

and so no singularities occur at $x = x_0$ where $H = 0$. In fact, every $x^{(k)}$ at x_0 is zero since all powers of H are zero there, and so x_0 is a stationary point value at infinity and the no-ordinary stationary point rule holds for all $P \geq 1$ cases. Also note

$$
\frac{d}{dk}[kP-(k-1))] = \begin{cases} P-1 & \text{in general;} \\ P-1=0 & \text{for } P=1; \\ P-1>0 & \text{for } P>1; \\ P-1<0 & \text{for } P<1. \end{cases} \qquad \text{(C6)}
$$

So for $P > 1$, the lowest power of H in Equation (C4) grows as k increases and for $P = 1$, lowest power stays constant at value 1.

The situation is different for $P < 1$. In this case, the lowest power H decreases in general as k grows. Thus in general, the lowest power of H will go negative and x_0 will lead to singularities in the higher order derivatives and not to ordinary nor asymptotic stationary point for $x(t)$. These singularities would in general lead to large and complex oscillations of $x(t)$ as $H(x)$ approached $H(x_0)$ and a singularity in $x(t)$, not a stationary point value. (Note our notation is a bit defective here in that in this case x_0 is never a value of $x(t)$.) However, if the lowest power itself goes to exactly zero at some k, then the power term will in general be finite and nonzero. What is happened is the negative powers of H have been canceled by positive powers of H produced x' factors that arose from the chain rule (which is the result we aimed at in this derivation: see above). In this case, there will be ordinary stationary point. When can the lowest power of H reach zero? Say when $k = n$, we $nP - (n-1) = 0$.

Thus, a stopping P value for the decreasing power of lowest power term is given by

$$P_{\rm stop} = \begin{cases} \dfrac{n-1}{n} = 1 - \dfrac{1}{n} & \text{in general;} \\ 0 & \text{for } n = 1 \text{ which will not give a stationary point} \\ & \text{since } P > 0 \text{ is required for a stationary point.} \\ \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots & \text{for } n = 2, 3, 4, \ldots < \infty; \\ 1 & \text{for } n = \infty \text{ will not give a ordinary stationary point} \\ & \text{as discussed above.} \end{cases} \tag{C7}$$

What is $x^{(k=n)}$? It is

$$x^{(k=n)} = \sum_{\ell=n-1}^{0} G_{n,\ell} H^{(n-1-\ell)} = \sum_{\ell=0}^{n-1} G_{n,\ell} H^{\ell} = G_{n,0} + \sum_{\ell=1}^{n-1} G_{n,\ell} H^{\ell} = G_{n,0} \ . \tag{C8}$$

Note that all the terms in the last summation of the equation above have integer powers of H greater than or equal to 1, and so by the argument after Equation (C6) above they and all higher derivatives of them are zero at the stationary point value $x = x_0$. They can all be disregarded for determining the nonzero values of $x^{(k)}$. But the term

$$G_{n,0} = \{[(n-1)P - (n-2)] \ldots P\} G \left(\frac{dH}{dx} \right)^{(n-1)} \tag{C9}$$

is not zero at the stationary point value $x = x_0$ in general although it might be that coincidentally. Thus, when you have a stopping power $P_{\rm stop}$, the will be a nonzero a value for a derivative of $x(t)$ a the stationary point and thus that point will not be at infinity. So we have proven that there are exceptions to the no-ordinary stationary point rule for 1st order autonomous equations.

Actually, in general there will be infinitely many nonzero derivatives for the exceptions since the derivavitive $G_{n,0}$ function in Equation (C9) gives the next higher derivative equation

$$x^{(k=n+1)} = G_{n,0} x' = G_{n,0} G H^P + \ldots \tag{C10}$$

where $G_{n,0} G H^P$ has the same functional behavior as GH^P in Equation (C3). So there will in general be nonzero higher order derivatives for $(k = n, 2n, 3n, \ldots)$ although some of them might be zero coincidentally. There is, in fact, a cycle of nonzero derivatives of x at the stationary point value $x = x_0$.

The most important case exception to the no-ordinary stationary point rule is very likely for $n = 2$, and so $P_{\rm stop} = 1/2$ since this case is the most likely one to occur in physics.

This important case obviously occurs for the Friedmann equation since it has $\dot{x}$ equal to the square root of a function of x: see, e.g., Equation (4) in § 2). We will discuss the case of the Friedmann equation in § 3.1. However, we can mention the elementary and important case of vertical motion in a constant gravity field g that is pointed down. This case is that for objects in ballistic motion near the Earth's surface and actually has a close relation to the Friedmann equation as seen from the Newtonian physics derivation of the Friedmann equation. (e.g., Liddle 2015, p. 22–24). Using y for the vertical coordinate, the conservation of mechanical energy gives

$$\begin{aligned} E &= \frac{1}{2}mv^2 + mgy \\ \frac{dy}{dt} &= v = \sqrt{2\left(\frac{E}{m} - gy\right)} \, . \end{aligned} \tag{C11}$$

which last equation can be regarded as a 1st order autonomous equation. Obviously, there is a stationary point at the maximum of the trajectory where $y_{\text{max}} = E/(mg)$. The solution to Equation (C11) is

$$y = \frac{1}{2}g(t - t_0)^2 + y_{\text{max}} \, , \tag{C12}$$

where t starts below t_0 for the rising phase, reaches t_0 at the maximum, and then increases above t_0 for the falling phase.

D. STABILITY OF CONSTANT SOLUTIONS OF FIRST ORDER AUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

REFERENCES

Ade, P.A.R., et al. 2015, A&A, 594, A13, arXiv:1502.01589 (Planck 2015)

Aghanim, N., et al. 2020, A&A, 641, A6, arXiv:1807.06209 (Planck 2018)

Arfken, G. 1985, Mathematical Methods for Physicists, 3rd edition (New York: Academic Press, Inc.)

Bondi, H. 1961, Cosmology (Cambridge: Cambridge University Press)

Boyle, L., & Turok, N. 2022, arXiv:2210.01142

Cahill, K. 2016, arXiv:1606.08865

Fixsen, D. J. 2011, ApJ, 707, 916, arXiv:0911.1955

Galanti, G., & Roncadelli, M. 2021, arXiv:2102.01637

Kragh, H. Cosmology and Controversy: The Historical Development of Two Theories of the Universe (Princeton, New Jersey: Princeton University Press)

Liddle, A. 2015, An Introduction to Modern Cosmology (Chichester, United Kingdom: John Wiley & Sons, Ltd.) (Li)

Luminet, J.-P. 2011, General Relativity and Gravitation, 43, 2911, arXiv:1105.6271

McCrea, W. H. 1984, in The Big Bang and Georges Lemaître, Proceedings of a Symposium in Honour of G. Lemaître Fifty Years After His Initiation of Big-Bang Cosmology, Louvain-la-Neuve, Belgium, 10–13 October 1983 (Dordrecht: D. Reidel Publishing Company), 3

Melia, F. 2015, arXiv:1411.5771

North, J. 1994, The Norton History of Astronomy and Cosmology (New York: W.W. Norton & Company)

Peebles, P. J. E. 1984, in The Big Bang and Georges Lemaître, Proceedings of a Symposium in Honour of G. Lemaître Fifty Years After His Initiation of Big-Bang Cosmology, Louvain-la-Neuve, Belgium, 10–13 October 1983 (Dordrecht: D. Reidel Publishing Company), 23

Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1992, Numerical Recipes in Fortran (Cambridge: Cambridge University Press)

O'Raifeartaigh, C. 2019, arXiv:1906.08678v1

O'Raifeartaigh et al. 2015 O'Raifeartaigh, C., O'Keeffe, M., Nahm, W, & Mitton, S. 2015,

O'Raifeartaigh, C., O'Keeffe, M., Nahm, W, & Mitton, S. 2017, arXiv:1701.07261

Scott, D. 2018, arXiv:1804.01318

Steiner, F. 2008, Proceedings of the Int. Conference "Evolution Equations — The State of the Art", Reisensburg/Germany, Dec. 2007, ed. W. Arendt et al., Ulmer Seminare 2008/Sonderausgabe, Heft 13, 231 (Ulm Reports and Publications: Ulm report ULM-TP/08-7)

This preprint was prepared with the AAS LaTeX macros v5.2.