AN EDUCATIONAL NOTE ON THE FRIEDMANN EQUATION AND ELEMENTARY SOLUTIONS

David J. Jeffery[1]

ABSTRACT

In this educational note, we derive the Friedmann equation for the cosmic scale factor $a(t)$, the fluid equation and the acceleration equation of cosmology from Newtonian physics plus conventional extra hypotheses. We then derive elementary analytic solutions and approximate solutions for $a(t)$ and $t(a)$ Some of the approximate solutions may be novel. The nearly exact full Λ-CDM model scale factor is presented and some eras for it are derived.

Subject headings: supernovae: cosmology: theory — cosmological parameters — dark energy

1. INTRODUCTION

To continue from the abstract, § 2 gives the Newtonian derivation of the Friedmann equation with the fluid equation and accleration equation being derived, respectively, in §§ 3 and 4. General aspects of solving the Friedmann equation for the cosmic scale factor $a(t)$ are discussed in § 5. Note there is no exact general solution of the Friedmann equation. Elementary families solutions are derived in § 6 (power-of-a solutions: i.e., single power-of-a solutions) § 7 (single power-of-a plus Λ solutions) and § 8 (two-powers-of-a solutions). In § 10, we derive the radiation-matter era solution for $t(a)$ and an approximation for $a(t)$ solution: these apply to the observable universe before Λ (i.e., the cosmological constant or constant dark energy) becomes in important. The nearly exact Λ-CDM model solution is presented § 13 along with an approximate solution. For historical interest and for their continuing relevance in some cases to eras of viable universe models, we present the scale factor solutions for the Einstein universe, the Lemaître universe, and the Lemaître-Eddington universe in, respectively §§ 14, 15, and 16. Conclusions in § 17. § A gives a discussion of behavior of 1st order autonomous ordinary differential equations. This discussion is relevant to the Friedmann equation since it is a differential equation of this kind.

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[1]Department of Physics & Astronomy, University of Las Vegas, Nevada, 4505 S. Maryland Parkway Las Vegas, Nevada 89154, U.S.A.

Note that in this educational note, we refer to classic models of the universe that were important historically at various times before the advent of the Λ-CDM model circa 1995 (e.g., Scott 2018, p. 10) as the standard model of cosmology as "universes": e.g., the Einstein-de Sitter universe (§ 7), the de Sitter universe (§ 7), the Einstein universe (§ 14), and the Lemaître universe (§ 15).

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There is no simple exact analytic solution for $a(t)$ for the Friedmann equation for the flat cosmology model spanning radiation, matter, and Λ (or dark-energy) eras with zero curvature: i.e., the Λ-CDM model. A complex exact solution does exist given by Steiner (2008, p. 9). However, there is a simple exact analytic solution for $a(t)$ for a model with matter and Λ (e.g., Steiner 2008, p. 12; Sazhin 2011, p. 3) and numerical solutions for $a(t)$ for the Λ-CDM model for all eras are straightforward (Cahill 2016, e.g.,). This exact analytic solution for the model with matter and Λ can be extended to include approximately an early-time radiation era. The extended solution can be fitted to the Λ-CDM model with 4 free parameters. The extended solution with the free parameters chosen to give a fit is our analytic fit. The analytic fit is useful for understanding and visualization of the Λ-CDM-model cosmic scale factor. If it transpires that the Λ-CDM model is only a good approximation to the (observable) universe, the analytic fit may be a useful zeroth order cosmic scale factor for fits to the actual universe cosmic scale factor.

In § 2, we present the exact solution to the Friedmann equation for models with Λ and only one mass-energy form obeying an inverse power law. In § 3, we make use of the § 2 results to create our analytic fit. Conclusions are given in § 4. The appendices are given for pedagogical use. Appendix A discusses the formula for the age universe for the Λ-CDM model. Appendix B discusses the exact analytic solution for the closed positive-curvature universe with only matter for mass-energy.

2. THE NEWTONIAN DERIVATION OF THE FRIEDMANN EQUATION

The Friedmann equation of general relativity (GR) cosmology in standard form (e.g., Wikipedia: Friedmann equations: Equations) is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ , \tag{1}$$

where H is the Hubble parameter (which at current cosmic time is the Hubble constant H_0 and has fiducial value 70 (km/s)/Mpc), a is the cosmic scale factor, $\dot{a}$ is the time derivative of the cosmic scale factor with respect to cosmic time t, $G = 6.67430(15) \times 10^{-11}\,\mathrm{J\,m/kg^2}$

is the gravitational constant, ρ is the density of a uniform perfect fluid (in old-fashioned jargon AKA the cosmological substratum: Bo-75–76) which is used to model the universal mass distribution, k is called the curvature (Li-24,28) $k/(c^2a^2)$ is called Gaussian curvature (CL-12,29), and $c = 2.99792458 \times 10^8\,\mathrm{m/s}$ is the vacuum light speed as usual. Note k is often defined with an unabsorbed c^2: i.e., the shown k is replaced by kc^2.

The Friedmann equation is, as one can see, a 1st order nonlinear ordinary differential equation. The fact that is nonlinear means that linear combinations of solutions are not in general solutions though they may be in special cases or approximately. The Friedmann equation is also a homogeneous differential equation at least in the sense that it can be written $\dot{a} = g(a)$. The form $\dot{a} = g(a)$ implies that a must be strictly increasing or decreasing except possibly at $\pm\infty$ and possibly at points where the some order of derivative of g have infinities. Both exceptions do occur for some solutions of the Friedmann equation. For example, the latter exception occurs for the closed universe model (with only matter). The closed universe model solution is closely related to throwing a ball into the air: the maximum size of the closed universe model corresponds to the maximum height of the ball.

The Friedmann equation actually has an interesting nature in that its independent variable is cosmic time t, but the solution the cosmic scale factor $a(t)$ is the factor by which all distances scale with time in expanding universe models.

Let's derive the Friedmann equation from Newtonian physics with extra natural hypotheses as needed. A priori, it not clear that the Newtonian derivation must yield the Friedmann equation with the extra natural hypotheses. But it can be shown that it should (C.G. Wells 2014, ArXiv:1405.1656). Note that the Newtonian derivation can say nothing about the curvature of space and assumes any curvature does not affect the derivation. We will do a long preamble wherein, with any luck, the extra hyptheses are shown to be natural.

First, just as in the GR derivation, we assume for our universe model the cosmological principle which states that the universe has a homogeneous, isotropic mass-energy distribution when averaged on a sufficiently large scale. The cosmological principle is what allows us to approximate the observable universe in our model with a perfect fluid. Observationally, the cosmological principle has been verified to a degree, but some tension remains. The observational scale for the validity of the cosmological principle is 100 Mpc or maybe a factor of a few times that larger (Wikipedia: Cosmological principle: Observations). Note that well beyond the observable universe, the cosmological principle may well fail, but, just as in the GR derivation, we assume this has negligible effect for the observable universe.

As to the perfect fluid of our model, it has uniform rest-frame mass-energy density ρ (uniform in space, not in time). The mass-energy gravitating mass-energy, of course. The perfect fluid has no viscosity and has an isotropic pressure p in its own rest frame (Ca-34). The perfect fluid equation of state (EOS) is $p = p(\rho)$. Actually, the perfect fluid can have internal energy (i.e., thermal energy), but that is counted as part of ρ as follows from $E = mc^2$. Also note that we said "rest-frame mass-energy" which can be the energy of massless particles. In fact, a photon gas is a good realization of the perfect fluid. The actual cosmic background radiation since the recombination era approximates a perfect fluid to high accuracy. Its photons do pass through gravitational wells, scatter off free electrons, and sometime hit planets, etc., but to good approximation the photons act as if they never interacted with anything except gravitationally.

Next, we note a corollory of Birkhoff's theorem (a theorem in GR): a spherical cavity at the center of spherical symmetric mass-energy distribution (static or not, finite or infinite) is a flat Minkowski spacetime (CL-24; We-337–338, 474). The spherical symmetric mass distribution can be, in fact, an unbounded homogeneous, isotropic mass-energy distribution: it can be infinite or finite. Note that if the spherical symmetric mass distribution is finite, it must have positive curvature and be a closed universe model. We assume, just as in the GR derivation, that Birkhoff's theorem applies to good approximation even if the cosmological principle fails well beyond the observable universe. Inside the cavity, we can put mass-energy and it should behave exactly as superimposed on a universe of flat Minkowski spacetime (CL-24; We-337–338, 474) as long as it does not break spherical symmetry significantly, which would cause a significant perturbation of the spherical symmetry of the surroundings. The mass-energy we put in the cavity used for our derivation does not break spherical symmetry.

The situation for the Birkhoff-theorem cavity is analogous to a cavity in spherically symmetric mass distribution in Newtonian physics. Inside the Newtonian cavity, the gravitational field is zero: this is a corollory of the shell theorem first proven by Newton himself. However, what happens if the mass distribution is infinite is not defined by pure Newtonian physics. Analogous to the GR case, inside the cavity, we can put mass-energy and it should behave exactly as superimposed a region where there is no external gravitational field as long as it does not break spherical symmetry significantly which would cause a significant perturbation of the spherical symmetry of the surroundings.

Now consider general relativistic space infinite or finite and unbounded (which would be positive curvature space: Li-33). The space is filled with the aforementioned uniform perfect fluid. The fluid density ρ is a function of cosmic time t in general. The fluid's motions are determined only by gravity (i.e., the geometry of spacetime) and initial conditions, and so each element of the fluid moves along a geodesic in a GR interpretation and in free fall in the Newtonian physics interpretation. Since we demand homogeneity and isotropy, we can only have uniform expansion/contraction of the whole model. Note the fluid can have pressure (positive or negative), but uniformity means the pressure force cancels out everywhere locally. The fluid can also have a formal pressure that does not have to push/pull on anything. However, formal pressure does have a global effect as we will show below.

Now consider a Birkhoff-theorem cavity of radius r for our model which is also filled with the perfect fluid with density ρ. Everything inside the cavity behaves just as everything outside, and so the cosmological principle is maintained. The cavity fluid has total mass M. We assume that gravitational field due to the cavity fluid is asymmptotically Newtonian. This requires

$$\frac{R_{\rm Sch}}{r} = \frac{2GM/c^2}{r} = \frac{8\pi}{3}\frac{G\rho}{c^2}r^2 << 1 \ , \tag{2}$$

where $R_{\rm Sch} = 2GM/c^2$ is the Schwarzschild radius (Wikipedia: Schwarzschild radius). So we just assume r is small enough. Note that Newtonian gravitational field is actually the classical limit of the left-hand side of the Einstein field equations (i.e., the spacetime geometry structure side: We-152), and so it does not itself contribute mass-energy (which comes from the right-hand side of the Einstein field equations and is described by the energy-momentum tensor). So we do not have to worry about the mass-energy contribution of the gravitational field to gravitating mass-energy since it does not contribute.

We also have to assume that r is small enough that the gravitational effects propagate with negligible time delay. Really, they propagate at the vacuum light speed relative to their local inertial frame.

We also have to assume that all relative velocities v of the fluid elements inside the cavity satisfy $v/c << 1$ so that we can employ Newtonian physics. This assumption is also asymptotically valid for small enough cavity radius r since the relative velocities between fluid elements are proportional to their separation distances as shown by Hubble's law which we derive nonrigorously below.

Recall all fluid elements in the perfect fluid are in free fall as aforesaid. This raises an interesting point. Special relativity gives the vacuum light speed c as the highest speed relative to inertial frames, but not between inertial frames. And the strong equivalence principle of GR shows that free-fall frames with uniform external gravity are exact inertial frames. The strong equivalence principle has been verified to very high accuracy (Archibald et al. 2018, arXiv:1807.02059). So the free-fall frames (which we will call comoving frames) of our model can grow apart at faster than c. In fact, Hubble's law shows that they must for large enough separation distances. Note that a light signal between comoving frames can only propagate at the vacuum speed light relative to the comoving frames it propagates through. So the fact that space can grow faster than the vacuum light speed does not imply there is faster-than-light signaling.

To summarize our assumptions for the Newtonian derivation, we require Birkhoff's theorem and that r be sufficiently small so that all relativistic and time-delay effects are small. If the aforesiad effects vanish in the differential limit as $r \rightarrow$, then the Newtonian derivation should be valid. Recall the Friedmann equation holds at every point in the universe model according to the GR derivation. Perhaps, there is some way that the Newtonian proof is still invalid, but it would have to be a very odd way.

Now we are ready to tear into the derivation of the Friedmann equation. We put a test particle of mass m at the surface of our cavity (i.e., at radius r). Given our setup, we have conservation of mechanical energy E:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv^2 - \frac{4\pi G}{3}\rho r^2 m \ , \tag{3}$$

where the first term to the right of the equal signs is the kinetic energy of our test particle and the second is its gravitational potential energy which is also its gravitational field energy in Newtonian physics which as discussed above does not itself contribute to gravitating mass-energy. We now write

$$r = ar_0 \ , \tag{4}$$

where a is the dimensionless cosmic scale factor and r_0 is a time-independent covoming distance. By usual convention the scale factor for the current cosmic time t_0 is defined to be 1: i.e., $a_0 = a(t_0) = 1$. This means that the r_0 are the proper distances for the current cosmic time: i.e., distances that you could measure with a ruler at current instant in cosmic time. Note $v = \dot{a}r_0$. Now defining the Hubble parameter $H = \dot{a}/a$, we get

$$v = Hr \tag{5}$$

which is the general-time Hubble's law. The current cosmic time Hubble's law (with the current Hubble parameter being Hubble's constant) is

$$v_0 = H_0 r_0 \ . \tag{6}$$

The validity of this derivation of Hubble's law follows from the Friedmann equation itself, and so is valid insofar as our Newtonian derivation of the Friedmann equation is valid. A rigorous GR derivation is given by CL-13–14.

Re Hubble's law: it is an exact law for recession velocities (which are velocities between comoving frames: i.e., free-fall frames that are exact inertial frames) and proper distances (which are true physical distances that can be measured at one instant in cosmic time with a ruler). In fact, neither recession velocities nor proper distances are observables, except asymptotically as $r \rightarrow 0$. The exception allows Hubble's constant to be measured from cosmologically nearby galaxies.

We divide the conservation of mechanical energy equation by $-mr_0^2/2$ to get

$$-\frac{2E}{mr_0^2} = -\dot{a}^2 + \frac{8\pi G}{3}\rho a^2 \ . \tag{7}$$

The right-hand side of the second to last equation is independent of E, m, and r_0 and depends only on universal quantities of the universe model, and therefore the constant on the left-hand side must be a universal constant independent of the peculiarities of the test particle: i.e., E, m, and r_0. We use the symbol k for this universal constant: thus,

$$k = -\frac{2E}{mr_0^2} \ . \tag{8}$$

The constant k is called the curvature since GR tells us it describes the curvature of space which we cannot know from Newtonian physics (Li-24, CL-12–13). Note $k > 0$ gives positive curvature (hyperspherical geometry), $k < 0$ gives negative curvature (hyperbolical geometry), and $k = 0$ gives zero curvature (flat or Euclidean geometry): see Wikipedia: Shape of the universe. (As noted above, k is often defined with an unabsorbed c^2: i.e., $kc^2 = -2E/mr_0^2$.) Rearranging the second to last equation gives us the Friedmann equation itself:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} = H_0^2\left[\Omega + \Omega_k\left(\frac{a_0}{a}\right)^2\right] \ , \tag{9}$$

(Li-24), where we have defined

$$\Omega = \frac{\rho}{\rho_{\rm c}} \ , \qquad \rho_{\rm c} = \frac{3H_0^2}{8\pi G} \ , \qquad \Omega_k = -\frac{k}{a_0^2 H_0^2} \tag{10}$$

(Li-51,56).

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$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{kc^2}{a^2} = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{kc^2}{a^2} \tag{11}$$

(e.g., Liddle 2015, p. 55–56). Note that $\rho_\Lambda = \Lambda/(8\pi G)$ is either (a) a parameterization of cosmological constant Λ as a density quantity or (b) is constant dark energy symbolized and often called Λ since its effect in the Friedmann equation is the same case (a). In fact, the concepts of cosmological constant and constant dark energy are often conflated in casual discussion because of their common effect in the Friedmann equation. However, they are in principle very different: the cosmological constant is a modification of gravity as manifested in the Einstein field equations of relativity and constant dark energy is a form of mass-energy with negative pressure. Constant dark energy may have other effects outside of the

Friedmann equation but there is no established theory as to what those are and the simplest hypothesis is that there are none.

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Note Ω is the density parameter (Li-51), $\rho_{\rm c} = 3H_0^2/(8\pi G)$ is the critical density (Li-51), and Ω_k is the curvature density parameter (Li-56). If $\Omega = 1$ at the current cosmic time (or any other cosmic time defined as current cosmic time), one has

$$H_0^2 = H_0^2(1 + \Omega_k) \tag{12}$$

implying $\Omega_k = 0$. So a universe model that is exactly flat at any cosmic time is exactly flat at all times.

There are several interesting points to be made about the Friedmann equation. First, we demanded r be small enough so that we could neglect relativistic and time travel effects. But we would derive the same Friedmann equation no matter what r we choose. So actually, all the effects we have neglected must cancel out for any r due to the conditions we imposed on the universe model: the cosmological principle and the perfect fluid.

A second interesting point is that Friedmann equation allows for mass-energy to appear or disappear as function of a. To explicate, mass-energy that is conserved (which called matter in cosmology jargon) has $\rho_{\rm m} \propto 1/a^3$. We show this below, but is in fact it is somewhat obvious: if the volume of a fluid element scales of up as a^3 and mass-energy is conserved, then density must decrease as $1/a^3$. But we allow other kinds of mass-energy dependence on a. For one example of mass-energy appearance/disappearance is that the cosmic background radiation and cosmic neutrino background (which in cosmology jargon is collectively called radiation) has $\rho_{\rm r} \propto 1/a^4$. The extra power of a is due to the cosmological redshift of extreme relativistic mass-energy which just causes radiation mass-energy to vanish from universe—it's just gone as gravitating mass-energy. Note general relativity cosmology does not have ordinary conservation of mass-energy: it just has the energy-momentum conservation equation $\nabla^\mu T_{\mu\mu} = 0$ (Carroll-120). Another point is that Noether's theorem that gives energy conservation when time invariance applies does not apply in an evolving universe model that does not have time invariance (Carroll-120). Another example of mass-energy appearance/disappearance is that constant dark energy (which is equivalent to the cosmological constant Λ in effect in the Friedmann equation if not otherwise) has ρ_Λ constant. The appearing/disappearing mass-energy contributes both gravitational field energy and, by the conservation of mechanical energy, the kinetic energy of the comoving frames which is sort of energy of expansion. (The disappearance of radiation also removes the kinetic energy of the comoving frames). To make more obvious the way mass-energy appearance/disappearance balances the gravitational field energy and the kinetic energy of the comoving frames , consider the Friedmann equation version

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k \ . \tag{13}$$

Holding a and k fixed, and increasing ρ (mass-energy) proportionally increases $\dot{a}^2$ (kinetic energy of comoving frames). This balanced contribution of gravitational field energy and kinetic energy for appearing/disappearing mass-energy arises only from starting our derivation from the conservation of mechanical energy equation. If we had started from Newton's 2nd law, we would have had no obvious path to include appearing/disappearing mass-energy.

You might ask what if k is a function of time or appearing/disappearing mass-energy is an explicit function of time not merely a function of a which is a function time. We have no guiding theory for these cases, and so far no observational or theoretical need for them.

3. DERIVATION OF THE FLUID EQUATION

We will now derive the fluid equation as it is called in cosmology jargon: i.e., the equation for $\dot{\rho}$. We assume that the perfect fluid obeys the 1st law of thermodynamices (which is actually implicit in the energy-momentum tensor for a perfect fluid: C.G. Wells 2014, ArXiv:1405.1656, p. 4). The 1st law is

$$dE = T\,dS - p\,dV + \mu\,dN \ , \tag{14}$$

where here E is total mass-energy and not mechanical energy as above, T is temperature, S is entropy, p is pressure, V is volume, μ is chemical potential, and N is number of particles. The perfect fluid is adiabatic (i.e., $dS = 0$) and so the 1st law reduces to

$$dE = -p\,dV + \mu\,dN \ , \tag{15}$$

For simplicity, we allow change in number of particles only to a species that is spontaneously created in such a way that N stays proportional to volume V. This means that $N = nV$ where n is the constant density of the spontaneously created particles. The spontaneously created particles are created at rest in the comoving frames, and so their chemical potential is just their rest-mass mass-energy. Given a volume $V \propto a^3$ for an amount of perfect fluid, we have

$$\begin{aligned} E &= \rho c^2 V \\ \dot{E} &= (\dot{\rho}V + \rho\dot{V})c^2 = -p\dot{V} + \mu n\dot{V} \\ \dot{\rho} &= -\frac{\dot{V}}{V}\left(\rho + \frac{p}{c^2} - \frac{\mu n}{c^2}\right) \qquad \text{and using} \qquad \frac{\dot{V}}{V} = \frac{3a^2\dot{a}}{a^3} = 3\frac{\dot{a}}{a} \end{aligned}$$

$$\dot{\rho} \qquad = -3\tfrac{\dot{a}}{a}\left(\rho + \tfrac{p}{c^2} - \tfrac{\mu n}{c^2}\right) \tag{16}$$

(Li-26). At the expense of clutter, we can explicitly allow for different species in the fluid equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}\sum_i\left(\rho_i + \frac{p_i}{c^2} - \frac{\mu_i n_i}{c^2}\right) \ , \tag{17}$$

where $\mu_i = 0$ for those species which are not the spontaneously created particles we allowed for.

We note that in cosmology the equation of state is often parameterized thusly

$$p = \begin{cases} w\rho c^2 & \text{where } w \text{ is constant parameter just called w;} \\ 0 & \text{for matter where } w = 0; \\ \dfrac{1}{3}\rho c^2 & \text{for radiation where } w = 1/3; \\ -\rho c^2 & \text{for constant dark energy where } w = -1; \\ -\dfrac{1}{3}\rho c^2 & \text{for a non-accelerating universe where } w = -1/3. \end{cases} \tag{18}$$

One might well ask what the heck is the negative pressure of constant dark energy. Well for a hypothetical laboratory gas, its something with suction. So expanding it, requires adding internal energy. But the constant dark energy negative pressure may be just formal. There is no reason to require it to couple to anything except maybe itself, and so maybe nothing feels negative pressure, except maybe dark energy itself. In any case, the dark energy is uniform, and so there are no pressure gradients. Where does the mass-energy come from to keep dark energy constant as the universe expands? Well in simplest theory, it just appears as a fundamental fact. However, there are quantum field theory reasons for believing there could be dark energy, but quantum field theory in its simplest prediction gets the size of constant dark energy too big by more than 100 orders of magnitude. So maybe quantum field theory does not know what its talking about.

Why do we allow for constant dark energy? The universal expansion is positively accelerating and constant dark energy supplies a cause. Of course, constant dark energy insofar as it affects Friedmann equation (but perhaps not otherwise) can be replaced by Einstein's cosmological constant Λ with the appropriate positive value. The cosmological constant (if it exists) is a fundamental aspect of gravity and not mass-energy form at all.

The negative pressure for the non-accelerating universe is just a fix to get a non-accelerating universe which has been argued for by some (e.g., Melia 2015, arXiv:1411.5771). So it's just a formal pressure.

Why did we allow for spontaneously created particles? They represent an alternative idea to constant dark energy and the cosmological constant. In the Friedmann equation, they have the same effect as constant dark energy and the cosmological constant Λ with the appropriate positive value. What could such particles be? Very speculatively, dark matter particles, nonrelativistic neutrinos (which can exist even if we have never detected them), and/or baryonic matter (pairs of protons and electrons). All of these would have other effects than just giving a positively accelerating universe. They could clump eventually and affect large-scale structure evolution, and in the case of baryonic matter lead to new star formation. The particles, by the way, certainly have only positive pressure, but to first approximation that is negligible compared to their mass-energy contribution. The case of spontaneous creation of baryonic matter leads to the unlikely hypothesis that the observable universe started with a Big Bang, but is now evolving to the steady-state universe as hypothesized by Bondi, Gold, and Hoyle in 1948. Actually, Einstein anticipated the steady-state universe in unpublished work in 1931.

4. DERIVATION OF THE ACCELERATION EQUATION

5. SOLVING THE FRIEDMANN EQUATION

The Friedmann equation in the standard form that follows from the derivation is not suitable for analytic nor numerical solutions. A suitable form is a scaled Friedmann equation: i.e., one with scaled density, time, and scale factor itself. In § 5.1 below, we introduce the scalings and present the Friedmann equation's relationship to what we call the curvature radius. In § 5.2, we give a general scaled Friedmann equation. There are also special case scaled Friedmann equations which introduce as needed in this educational note.

5.1. Scaled Density, Time, and Scale Factor and the Curvature Radius

Recall the derived form of the Friedmann equation (i.e., eq. (11) in § 2)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{kc^2}{a^2} = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{kc^2}{a^2} \ , \tag{19}$$

where recall $\rho_\Lambda = \Lambda/(8\pi G)$. We specify scaled density (usually called density parameter or just Ω), time, and scale factor, respectively, as follows

$$\Omega = \frac{\rho + \rho_\Lambda}{\rho_{\rm scale}} \ , \qquad \tau = \frac{t}{t_{\rm scale}} \ , \qquad \text{and} \qquad x = \frac{a}{a_0} \ , \tag{20}$$

where

$$t_{\rm scale} = \frac{1}{\sqrt{(8\pi G/3)\rho_{\rm scale}}} \ , \tag{21}$$

and subscript 0 means fiducial time which is not necessarily cosmic present though it usually that for currently viable models of the observable universe. Now $\rho_{\rm scale}$ itself is usually specified as the critical density $\rho_{\rm crit}$ for cosmic present (which we introduce below) for models viable for the observable universe. However, $\rho_{\rm scale}$ can be used as free parameter for Friedmann equation solutions or left unspecified for scaled Friedmann equation solutions.

Why do we need to scale the scale factor $a(t)$ itself. In many cases, $a(t)$ is made dimensionless and its fiducial time value is set by $a_0 = 1$ for cosmic present of the observable universe. In these cases,

$$r = a(t)r_0 \ , \tag{22}$$

r is physical (or proper) distance at cosmic time t, $a(t)$ is the dimensionless scale factor itself, and r_0 is the physical distance at cosmic present and also the comoving distance (i.e., the time-factored-out distance for the expanding universe). However, $a(t)$ can be made a physical distance and for curved spaces this is the natural choice since the general relativity derivation of the Friedmann equation incorporating the Robertson-Walker (RW) metric leads to a physical disance $a(t)$ for curved spaces which we will call the RW $a(t)$ (e.g., Coles & Lucchin 2002, p. 9–13). The RW metric version of the Friedmann equation makes the curvature k a dimensionless quantity with just 3 possible values:

$$k = \begin{cases} 1 & \text{for positive curvature or hyperspherical spaces;} \\ 0 & \text{for Euclidean or flat spaces;} \\ -1 & \text{for negative curvature or hyperbolical spaces.} \end{cases} \tag{23}$$

We now define the curvature density parameter Ω_k (which subscript k for curvature) by

$$\Omega_k = -\frac{kc^2}{(8\pi G/3)\rho_{\rm scale}a^2} = -\frac{kc^2}{(8\pi G/3)\rho_{\rm scale}a_0^2}x^{-2} \ . = \Omega_{k,0}x^{-2} \ , \tag{24}$$

where again the subscript 0 indicates fiducial time (which not in general cosmic present). The negative sign in equation (24) is needed for Ω and Ω_k to appear in a formally consistent way in the general scaled Friedmann equation given below and in § 5.2. However, there is the slight confusion that Ω_k is negative/positive for positive/negative curvature.

We will call the RW a the curvature radius and the RW a_0 the fiducial curvature radius. What is the RW a curvature radius. For negative curvature (i.e., for $k = -1$), its too esoteric to explain in this educational note except that it is characteristic length scale (e.g., Coles & Lucchin 2002, p. 11) for infinite hyperbolic spaces. For positive curvature (i.e., for $k = 1$), there is a simple meaning. An unbounded positive curvature space is finite and has the same geometry as the 3-dimensional surface of a 3-sphere (or hypersphere) which is a sphere in a 4-dimensional Euclidean space. In positive curvature space, the RW a times π is the physical distance from any point along a geodesic in any direction to the antipodal point and the RW a times 2π is the physical distance from any point along a geodesic in any direction back to the point (e.g., Coles & Lucchin 2002, p. 11). That is enough to say in this educational note. By the by, what is is called the Gaussian curvature radius is defined

$$R_{\rm G} = \frac{a}{\sqrt{k}} = \begin{cases} a & \text{for } k = 1 \text{ and positive curvature;} \\ \text{undefined} & \text{for } k = 0 \text{ and flat space;} \\ -ia & \text{for } k = -1 \text{ and negative curvature} \end{cases} \tag{25}$$

(e.g., Coles & Lucchin 2002, p. 12).

The scaled Friedmann equation so far is

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \Omega + \Omega_{k,0} x^{-2} \ , \tag{26}$$

where h is the scaled Hubble paramter (i.e., $h = Ht_{\rm scale}$). Note

$$\begin{cases} \Omega = h^2 & \text{at any time } \tau \text{ implies } k = 0, \text{ and so flat space and } \Omega_k = 0 \text{ at all times;} \\ \Omega > h^2 & \text{implies } \Omega_k < 0, \text{ and so } k = 1 \text{ and positive curvature at all times;} \\ \Omega < h^2 & \text{implies } \Omega_k > 0, \text{ and so } k = -1 \text{ and positive curvature at all times.} \end{cases} \tag{27}$$

A formal solution for the RW a_0 is

$$a_0 = \frac{c}{\sqrt{(8\pi G/3)\rho_{\rm scale}|\Omega_{k,0}|}} = \frac{c}{\sqrt{(8\pi G/3)\rho_{\rm scale}|\Omega_0 - h_0^2|}} \ . \tag{28}$$

Note for $\Omega_{k,0} = 0$ (implying $\Omega_0 = h_0^2$), we have the RW a_0 undefined at all times. In this case, one usually chooses the dimensionless $a_0 = 1$ and uses equation (22) above. In other cases, the RW a_0 is a defined real number as can be seen by comparing equations (28) and (24).

For viable cosmological models circa 2020, one usually chooses $\rho_{\rm scale}$ to be the critical density $\rho_{\rm crit}$ for cosmic present. The critical density $\rho_{\rm crit}$ and the implied scale time $t_{\rm scale}$ are given by

$$\rho_{\rm crit} = \frac{3H_0^2}{8\pi G} \qquad \text{and} \qquad t_{\rm scale} = \frac{1}{\sqrt{(8\pi G/3)\rho_{\rm scale}}} = \frac{1}{H_0} = t_{H_0} \ , \tag{29}$$

where H_0 is the cosmic present Hubble parameter (i.e., the Hubble constant) and t_{H_0} is the

Hubble time. In this case, the RW a_0 specializes to

$$a_0 = \begin{cases} \dfrac{L_{H_0}}{\sqrt{|\Omega_{k,0}|}} = \dfrac{L_{H_0}}{\sqrt{|\Omega_0 - 1|}} & \text{where } L_{H_0} = c/H_0 = ct_{H_0} \text{ is the Hubble length;} \\ \dfrac{[(4.2827\ldots)\,\mathrm{Gpc}]/h_{70}}{\sqrt{|\Omega_{k,0}|}} & \text{where } h_{70} = H_0/[70\,(\mathrm{km/s})/\mathrm{Mpc}] \\ & \quad \text{is the Hubble constant in units of} \\ & \quad \text{fiducial Hubble constant } 70\,(\mathrm{km/s})/\mathrm{Mpc;} \\ (191.53\ldots)\,\mathrm{Gpc} & \text{for } |\Omega_{k,0}| = 0.0005; \\ (19.153\ldots)\,\mathrm{Gpc} & \text{for } |\Omega_{k,0}| = 0.05. \end{cases} \tag{30}$$

Now the Λ-CDM model formally sets $\Omega_{k,0} = 0$ as an approximation to the prediction of inflation cosmology that $\Omega_{k,0}$ should equal zero to high accuracy. Planck (2018, p. 68) in extensions of the Λ-CDM model that allow for curvature find with all data included that $\Omega_{k,0} = 0.0005(40)$ which is consistent with zero curvature. Note that taking $\Omega_{k,0} = 0.0005$ gives RW $a_0 = (191.53\ldots)\,\mathrm{Gpc}$ as shown in equation (30) above. On the other hand, some analyses of observational data circa 2020 suggest that $\Omega_{k,0}$ could be of order -0.05 (e.g., Handley 2019, p. 4) which implies positive curvature and RW $a_0 = (19.153\ldots)\,\mathrm{Gpc}$ as shown in equation (30) above. It remains to be seen if this suggetion remains viable through the 2020s. Note that the Λ-CDM model gives the radius of the observable universe (i.e., the particle horizon) to be $\sim 14.25\,\mathrm{Gpc}$ (e.g., Wikipedia: Observable universe). Since the Λ-CDM model fits the observations so well even it needs revision or replacement, the radius of observable universe not likely to be much different from $14.25\,\mathrm{Gpc}$. Thus, even if universe has positive curvature with RW $a_0 = (19.153\ldots)\,\mathrm{Gpc}$, the antipodal point to the Milky Way is well outside of the observable universe, and so is not observable.

What of ρ_{scale} for non-viable cosmological models circa 2020? In fact, such models can be conveniently left in scaled form since they primarily for educational purposes not for fitting to the observed universe. So there is no need to specify ρ_{scale} and we do not do so in our presentation of them unless $\rho_{\mathrm{scale}} = \rho_{\mathrm{crit}}$. We note however, that many classic models (e.g., positive-curvature-matter universe, the Einstein universe, the Lemaître universe, and the Eddington Lemaitre) have there natural reference a_0 when $H = 0$, and so one does not have $\rho_{\mathrm{scale}} = \rho_{\mathrm{crit}}$ in these cases.

5.2. A General Scaled Friedmann Equation

The common hypothesis is that the density in the Friedmann equation just depends on powers of the cosmic scale a or in our scaled form x. Following this common hypothesis and making use of equation (26) in § 5.1, a general scaled Friedmann equation is

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_p \Omega_{p,0} x^{-p} \ , \tag{31}$$

where the $\Omega_{p,0}$ are various density components including ρ_Λ (cosmological constant or constant dark energy) treated as a density component plus the curvature $\Omega_{k,0}$ (from equation (24) in § 5.1) treated as density component. We only consider commonly considered components with powers p (not to be confused with pressure p) are as follows:

$$\begin{cases} p = 0 & \text{for } \Lambda \text{ (cosmological constant or constant dark energy);} \\ p = 1 & \text{for quintessence (in some theories);} \\ p = 2 & \text{for curvature, cosmic strings (in some theories), or the } R_h = ct \text{ universe} \\ p = 3 & \text{for matter (in the cosmological sense of matter at rest in comoving frames)} \\ & \qquad \text{which includes baryonic matter and dark matter;} \\ p = 4 & \text{for radiation (in the cosmological sense of mass-energy moving} \\ & \qquad \text{at or nearly at the vacuum light speed} \\ & \qquad \text{in comoving frames).} \end{cases} \tag{32}$$

(e.g., Steiner 2008, p. 6–7; e.g., Melia 2014 for the $R_h = ct$ universe).

From the form of equation (31) where τ only appears as $d\tau$ in $\dot{x} = dx/d\tau$, it is obvious that numerical solutions for $\tau(x)$ (i.e., $t(a)$) are much more straightforward than numerical solutions for $x(\tau)$ (i.e., $a(t)$). One then inverts $\tau(x)$ numerically (e.g., from a table of $\tau(x)$) to get $\tau(x)$. An appropriate rearangement of equation (31) for numerical solutions is

$$d\tau = \frac{x\,dx}{\sqrt{\sum_{p=0}^{4} \Omega_{p,0} x^{4-p}}} \ , \tag{33}$$

which has the nice feature that no negative powers of x (i.e., a) appear. Equation (33) can be solved by, e.g., the midpoint method (Wikipedia: Midpoint method) or the Runge-Kutta method (Wikipedia: Runge-Kutta methods)

For analytic solutions from the equation (33) with a single nonzero $\Omega_{p,0}$ (i.e., a single power of a or x), you solve for $\tau(x)$ from

$$d\tau = x^{p/2-1}\,dx \tag{34}$$

and inverts easily to get $x(\tau)$. We call such solutions power-of-a solutions and derive them in § 6. Remarkably if you have only one $\Omega_{p\neq0,0}$ and $\Omega_{p=0,0}$ nonzero, solves for $x(\tau)$ directly most easily and obtains $\tau(x)$ by inversion. We call such solutions power-of-a-Λ solutions and derive them § 7.

Other analytic solutions can be obtained in terms of what is called conformal time symbolized by η and related to τ by $d\eta = d\tau/x$. From equation (33), we see that

$$d\eta = \frac{dx}{\sqrt{\sum_{p=0}^{4} \Omega_{p,0} x^{4-p}}} \tag{35}$$

which because of it nice mathematical appearance is prima facie suggestive that analytic solutions for multiple nonzero $\Omega_{p,0}$ are possible. In fact, equation (35) can be solve analytically for $x(\eta)$ for all five $\Omega_{p,0}$ nonzero though with dependence (Steiner 2008, p. 7–9). on the Weierstrass elliptic function (Wikipedia: Weierstrass elliptic function). We will not present this solution in this educational note.

However, conformal time is not generally useful since physics generally evolves by ordinary time τ. So one must solve for τ via

$$d\tau = x(\eta)\, d\eta \tag{36}$$

which cannot be done analytically in general and then invert to get $\eta(\tau)$ (which also cannot be done analytically in general) in order to get $x[\eta(\tau)]$.

As an example of using conformal time, we will in § 9 we will solve for $x(\eta)$ and $\tau(\eta)$ for the positive-curvature-matter universe: i.e., the model with only $\Omega_{\mathrm{M},0} = \Omega_{3,0}$ and $\Omega_{k,0} = \Omega_{2,0} < 0$ being nonzerio in equation (35).

6. POWER-OF-a SOLUTIONS

Table 1. Power-Law Solutions to the Friedmann Equation

$w\backslash$Quantity	$p = \frac{2}{\gamma}$	$\gamma = \frac{2}{p}$	$a(t)$	$t_0 = \frac{\gamma}{H_0}$	$q_0 = \frac{1}{\gamma} - 1$	ρ
$\left\{ \begin{matrix} w \quad \text{or} \\ w \neq -1 \end{matrix} \right\}$	$3(1+w)$	$\frac{2}{[3(1+w)]}$	$a_0 \left(\frac{t}{t_0}\right)^{\gamma}$	$\gamma \left(\frac{13.968\,\text{Gyr}}{h_{70}}\right)$	$\left\{ \begin{matrix} \frac{1}{2}(1+3w) \\ = \frac{p}{2} - 1 \end{matrix} \right\}$	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w = 0$	$p = 3$	$\gamma = \frac{2}{3}$	$a_0 \left(\frac{t}{t_0}\right)^{2/3}$	$\frac{2}{3}\frac{1}{H_0}$	$\frac{1}{2}$	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w = \frac{1}{3}$	$p = 4$	$\gamma = \frac{1}{2}$	$a_0 \left(\frac{t}{t_0}\right)^{1/2}$	$\frac{1}{2}\frac{1}{H_0}$	1	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w = -1$	$p = 0$	$\gamma = \infty$	$a_0 e^{H_0(t-t_0)}$	∞	-1	ρ_0
$w = -\frac{1}{3}$	$p = 2$	$\gamma = 1$	$a_0 \left(\frac{t}{t_0}\right)$	$\frac{1}{H_0}$	0	$\rho_0 \left(\frac{t_0}{t}\right)^2$

7. POWER-OF-a-Λ SOLUTIONS

8. TWO-POWERS-OF-a SOLUTIONS

9. POSITIVE-CURVATURE-MATTER UNIVERSE

10. THE RADIATION-MATTER ERA SOLUTION

11. MIXED POWER-LAW SOLUTIONS

12. POWER-LAW-LAMBDA SOLUTIONS

The Friedmann equation with one inverse-power dependence on cosmic scale factor $a(t)$ and a Lambda dependence is

$$\left(\frac{\dot{x}}{x}\right)^2 = H_0^2 \left(\Omega_\Lambda + \Omega_{p0} x^{-p}\right) \ , \tag{37}$$

where $x = a/a_0$, the derivative is with respect to cosmic time t and $p > 0$ (so to have inverse-power dependence on cosmic scale factor $a(t)$). We know from § 6, that the solutions for with only one inverse power-law dependence on a (with inverse power p and no Lambda) and only a Lambda dependence are, respectively, a power-law solution and an exponential solution (i.e., the de Sitter solution): i.e.,

$$a = a_0 \left(\frac{t}{t_0}\right)^\gamma = a_0 \left(\frac{t}{\gamma/H_0}\right)^\gamma \qquad \text{and} \qquad a = a_0 e^{H_0(t-t_0)} = a_0 e^{(t-t_0)/t_e} \tag{38}$$

where t_0 is the present cosmic time or any fiducial cosmic time, $\gamma = 2/p$ and $t_e = 1/H_0$ is the e-folding time for the exponential solution. From equation (37), we expect its solution to grow from $a = 0$ like the power-law solution just above for sufficiently small a and eventually to asymptotically become like the exponential solution as $a \to \infty$. Thus, we can guess that the following interpolation solution should be a good approximate solution to equation (37):

$$a = a_\Lambda \sinh^\gamma \left(\frac{t}{\gamma t_\Lambda}\right) \ , \tag{39}$$

where a_Λ is some constant such that $a(t = 0) = a_0$ and here $t_\Lambda = 1/H_\Lambda$ with $H_\Lambda \equiv \sqrt{\Omega_\Lambda} H_0$. The t_Λ is the asymptotic e-folding time for $t \to \infty$ and $t_\gamma = \gamma t_\Lambda$ is the asymptotic power-law scale time as t becomes small.

Recall $\sinh(x)$ is the hyperbolic sine function. Also recall the hyperbolic function definitions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{40}$$

and the hyperbolic function identities

$$\cosh(x) \pm \sinh(x) = e^{\pm x} \qquad \cosh^2(x) - \sinh^2(x) = 1 \tag{41}$$

and the hyperbolic function derivatives

$$\frac{d\sinh(x)}{dx} = \cosh(x) \qquad \frac{d\cosh(x)}{dx} = \sinh(x) \qquad \frac{d\tanh(x)}{dx} = \frac{1}{\cosh^2(x)} \ . \tag{42}$$

In fact, equation (39) is the exact solution to equation (37). We will prove this by proving a more general solution of which equation (39) is a special case. Consider the three identities

$$\begin{cases} (1) \quad \sinh^2(\tau) - \cosh^2(\tau) = -1 \\ (2) \quad \cosh^2(\tau) - \sinh^2(\tau) = 1 \\ (3) \quad \sin^2(\tau) + \cos^2(\tau) = 1 \ , \end{cases} \tag{43}$$

where τ is a scaled time. The identities can be written as one identity which is, in fact, a nonlinear 1st order differential equation with no explicit dependence on τ and three cases:

$$f^2 - g\dot{f}^2 = -h \qquad \begin{cases} g = h = 1, \quad gh = 1 & \text{for identity (1);} \\ g = 1 \quad h = -1, \quad gh = -1 & \text{for identity (2);} \\ g = h = -1, \quad gh = 1 & \text{for identity (3),} \end{cases} \tag{44}$$

where the derivative is with respect to scaled time τ. It ca be rewritten as

$$\dot{f}^2 = gf^2 + gh \ . \tag{45}$$

The only exact solutions yours truly knows of are $\sinh(\tau)$, $\cosh(\tau)$, $\sin(\tau)$ and $\cos(\tau)$. The $\cos(\tau)$ solution is identical to the $\sin(\tau)$ solution other than a shifted time zero and we will not consider it further below.

Now consider the function $y = f(\tau)^\gamma$ and the following differential equation constructed from it:

$$\frac{\dot{y}}{y} = \frac{\gamma f^{\gamma-1}\dot{f}}{f^\gamma} = \gamma\frac{\dot{f}}{f} = \pm\gamma\sqrt{g + ghf^{-2}} = \pm\gamma\sqrt{g + ghy^{-2/\gamma}} \ , \tag{46}$$

where we have used that $f \geq 0$ for all cases we are interested in. Clairvoyance tells us that the cases $\gamma = 0$ and $\gamma = \infty$ are of no interest. The former yields to $p = \infty$ and the latter $p = 0$ which is just the Lambda dependence case all over again.

Now equation (46) looks a lot like the Friedmann equation of interest equation (37). In fact with appropriate scaling and identifications equation (37) can be reduced to equation (46)—or, and this is key for understanding, vice versa. First, we rewrite equation (37) thusly

$$\left(\frac{1}{x}\right)\frac{dx}{dt} = \frac{\sqrt{|\Omega_\Lambda|}H_0}{\gamma}\left(\pm\gamma\sqrt{\ell + \frac{\Omega_{p0}}{|\Omega_\Lambda|}x^{-p}}\right) \ , \tag{47}$$

where $\ell = 1$ for $\Omega_\Lambda > 0$ and $\ell = -1$ for $\Omega_\Lambda < 0$. Now defining

$$d\tau = \frac{\sqrt{|\Omega_\Lambda|}H_0}{\gamma}\,dt \ , \tag{48}$$

we see equations (37) and (46) are the same for $\ell = g$ and

$$\frac{\Omega_{p0}}{|\Omega_\Lambda|}x^{-p} = ghy^{-2/\gamma} \tag{49}$$

with $p = 2/\gamma$.

The upshot is that we have 3 exact solutions for equation (37) derivable from

$$a = a_0\left[\left(\frac{\Omega_{p0}}{\Omega_\Lambda}\right)h\right]^{\gamma/2} f\left[\frac{t}{\gamma/(\sqrt{g\Omega_\Lambda}H_0)}\right]^{\gamma} \tag{50}$$

which correspond to the three cases of differential equation equation (44). Note that we have the physical reality constraints

$$g\Omega_\Lambda > 0 \qquad \text{and} \qquad \left(\frac{\Omega_{p0}}{\Omega_\Lambda}\right)h = \left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}\right)h = \left(\frac{\Omega_{p0}}{1-\Omega_{p0}}\right)h > 0 \tag{51}$$

since $a(t)$ and all powers of it must be real and positive and the sum of the Ω_{p_0} parameters must be 1.

The us briefly describe the three exact solutions:

1. The power-of-hyperbolic-sine solution with $g = h = 1$ (implying $\Omega_\Lambda > 0$ and $\Omega_{p_0} > 0$):

$$a = a_0\left(\frac{\Omega_{p0}}{\Omega_\Lambda}\right)^{\gamma/2}\sinh^\gamma\left[\frac{t}{\gamma t_\Lambda}\right] \ , \tag{52}$$

where $t_\Lambda = 1/\sqrt{\Omega_\Lambda H_0} = 1/H_\Lambda$ is the asymptotic Hubble time as $t \to \infty$ with H_Λ being the asymptotic Hubble constant. Reverting for the moment to the scaled variables, we see

$$\dot{y} = \begin{cases} \gamma\sinh^{\gamma-1}(\tau)\cosh(\tau) > 0 & \text{for } \tau > 0; \\ \gamma\tau^{\gamma-1} & \text{to 1st order in small } \tau; \\ 0 & \text{for } \gamma > 1 \text{ and } \tau = 0; \\ \gamma & \text{for } \gamma = 1 \text{ and } \tau = 0; \\ \infty & \text{for } \gamma < 1 \text{ and } \tau = 0. \end{cases} \tag{53}$$

From the above formulae, we see that $a(t=0) = 0$ and that $a(t)$ strictly increases for $t > 0$ to $a(t=\infty) = \infty$. At $t = 0$, there is a minimun for $\gamma > 1$, a strict increasing for $\gamma = 1$, and an infinite slope for $\gamma < 1$. Recall from § 2 that behavior of $a(t)$ when $a(t) = 0$ is of interest only as limiting behavior since the Friedmann equation becomes invalid as $a(t) \to 0$ where quantum gravity must take over. Note that the only solution with nonzero curvature requires $\Omega_{p_0 - 2} = \Omega_{k_0} = -k/(a_0^2 H_0^2) > 0$ (Li-52) which means $k < 0$ and negative curvature or hyperbolic space (Li-33) and $\gamma = 1$. The power-of-hyperbolic-sine solution has an important special case for $\gamma = 2/3$ (i.e., the matter-Lambda case) which we discuss below.

2. The power-of-hyperbolic-cosine solution with $g = 1$ and $h = -1$ (implying $\Omega_\Lambda > 0$ and $\Omega_{p_0} < 0$):

$$a = a_0 \left(\frac{|\Omega_{p_0}|}{\Omega_\Lambda} \right)^{\gamma/2} \cosh^\gamma \left[\frac{t}{\gamma t_\Lambda} \right] \ , \tag{54}$$

where $t_\Lambda = 1/\sqrt{\Omega_\Lambda H_0} = 1/H_\Lambda$ is the asymptotic Hubble time as $t \to \pm\infty$ with H_Λ being the asymptotic Hubble constant. Reverting for the moment to the scaled variables, we see

$$\dot{y} = \begin{cases} \gamma \cosh^{\gamma-1}(\tau) \sinh(\tau) > 0 & \text{for } \tau > 0; \\ \gamma \cosh^{\gamma-1}(\tau) \sinh(\tau) < 0 & \text{for } \tau < 0; \\ 2\gamma\tau(1+\tau^2)^{\gamma-1} & \text{to 2nd order in small } \tau; \\ 0 & \text{for } \tau = 0. \end{cases} \tag{55}$$

From the above formulae, we see that $a(t)$ strictly decreases from $a(t = -\infty) = \infty$ to a minimum at $t = 0$, and then strictly increases to $a(t = \infty) = \infty$. Note that the only physical case with $\Omega_{p_0} < 0$ is when $\Omega_{p_0 = 2} = \Omega_{k_0} = -k/(a_0^2 H_0^2) < 0$ (Li-52) which means $k > 0$ and positive curvature or hyperspherical space (Li-33), and $\gamma = 1$. If the hypersherical space extended without a boundary, the solution would be a closed universe. At present, the observable universe does not require the power-of-hyperbolic-cosine solution and we will not discuss it further.

3. The power-of-sine solution with $g = -1$ and $h = -1$ (implying $\Omega_\Lambda < 0$ and $\Omega_{p_0} > 0$):

$$a = a_0 \left(\frac{\Omega_{p_0}}{|\Omega_\Lambda|} \right)^{\gamma/2} \sin^\gamma \left[\frac{t}{\gamma t_\Lambda} \right] \ , \tag{56}$$

where

$$H = \frac{\dot{a}}{a} = \frac{1}{t_\Lambda} \frac{1}{\tan[t/(\gamma t_\Lambda)]} \ , \tag{57}$$

and so $t_\Lambda = 1/\sqrt{\Omega_\Lambda H_0} = 1/H_\Lambda$ is the Hubble time and H_Λ the Hubble constant only for $t/(\gamma t_\Lambda) = \pi/2$. Reverting for the moment to the scaled variables, we see

$$\dot{y} = \begin{cases} \gamma \sin^{\gamma-1}(\tau)\cos(\tau) > 0 & \text{for } \tau \in (0, \pi/2); \\ \gamma \sin^{\gamma-1}(\tau)\cos(\tau) < 0 & \text{for } \tau \in (\pi/2, \pi); \\ 0 & \text{for } \tau = \pi/2; \\ \gamma\tau^{\gamma-1} & \text{to 1st order in small } \tau; \\ 0 & \text{for } \gamma > 1 \text{ and } \tau = 0; \\ \gamma & \text{for } \gamma = 1 \text{ and } \tau = 0; \\ \infty & \text{for } \gamma < 1 \text{ and } \tau = 0. \end{cases} \tag{58}$$

From the above formulae, we see that $a(t = 0) = 0$ and that $a(t)$ strictly increases for $t > 0$ to a maximum at $a[t = (\pi/2)\gamma t_\Lambda]$ and then strictly decreases for $t > (\pi/2)\gamma t_\Lambda$. At $t = 0$ ($t = \pi\gamma t_\Lambda$ using symmetry), there is a minimum for $\gamma > 1$, a strict increasing (decreasing) for $\gamma = 1$, and an infinite slope for $\gamma < 1$. Recall again from § 2 that behavior of $a(t)$ when $a(t) = 0$ is of interest only as limiting behavior since the Friedmann equation becomes invalid as $a(t) \to 0$ where quantum gravity must take over. Note that the only solution with nonzero curvature requires $\Omega_{p_0=2} = \Omega_{k_0} = -k/(a_0^2 H_0^2) > 0$ (Li-52) which means $k < 0$ and negative curvature or hyperbolic space (Li-33), and $\gamma = 1$. The power-of-sine solution for matter (i.e., $p = 3$ and $\gamma = 2/3$ and flat space) has a similar functional behavior to the approximate power-of-sine behavior of of the positive-curvature matter solution (see § POSITIVE-CURVATURE-MATTER UNIVERSE). At present, the observable universe does not require the power-of-sine solution and we will not discuss it further.

The power-of-hyperbolic-sine solution with $\gamma = 2/3$ (i.e., the matter-Lambda case with $p = 3$ and flat space) is, in fact, the exact solution for the Λ-CDM model of the observable universe not counting the relatively brief radiation era and earlier (e.g., Steiner 2008, p. 12; Sazhin 2011, p. 3; Universe in Problems 2019, Standard Cosmological Model, Evolution of Universe, problem 13). This fact is well known though apparently not much mentioned in articles on the Λ-CDM model. Thus power-of-hyperbolic-sine solution is very important.

Because its importance, we present in Table 2 some power-law-Lambda-hyperbolic-sine solution results and Λ-CDM model results.

To summarize:

Table 2. Power-Law-Lambda-Hyperbolic-Sine Solution Results and Λ-CDM Model Results

Formula/Quantity	Description
$a = a_0 \left(\frac{\Omega_{p_0}}{\Omega_\Lambda}\right)^{\gamma/2} \sinh^\gamma \left[\frac{t}{\gamma t_\Lambda}\right]$	cosmic scale factor $a(t)$
$\dot{a} = a_0 \left(\frac{\Omega_{p_0}}{\Omega_\Lambda}\right)^{\gamma/2} \left(\frac{\gamma}{\gamma t_\Lambda}\right) \sinh^{\gamma-1} \left[\frac{t}{\gamma t_\Lambda}\right] \cosh \left[\frac{t}{\gamma t_\Lambda}\right]$	1st derivative of $a(t)$
$\ddot{a} = a_0 \left(\frac{\Omega_{p_0}}{\Omega_\Lambda}\right)^{\gamma/2} \left[\frac{\gamma}{(\gamma t_\Lambda)^2}\right] \sinh^{\gamma-2} \left[\frac{t}{\gamma t_\Lambda}\right]$	
$\times \left[\gamma \cosh^2 \left(\frac{t}{\gamma t_\Lambda}\right) - 1\right]$	2nd derivative of $a(t)$
$H = \frac{\dot{a}}{a} = \left(\frac{1}{t_\Lambda}\right) \frac{1}{\tanh[t/(\gamma t_\Lambda)]}$	Hubble parameter
$q = -\frac{\ddot{a}}{aH^2} = \frac{1}{\gamma \cosh^2[t/(\gamma t_\Lambda)]} - 1$	deceleration parameter (Li-53)

13. THE Λ-CDM MODEL SOLUTION

14. THE EINSTEIN UNIVERSE

15. THE LEMAITRE UNIVERSE

16. THE LEMAITRE-EDDINGTON UNIVERSE

17. CONCLUSIONS

Reader, the conclusions are in the abstract and the introduction (i.e., § 1).

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A. FIRST-ORDER AUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

We will digress on a key point about the Friedmann equation. It is a 1st order differential equation with no explicit dependence on the independent variable: i.e., it is of the form

$$x' = g(x) \qquad \text{where the solution is of the form} \qquad x = x(t) \ , \tag{A1}$$

where t is a general independent variable and not time unless it is time. Such differential equations three cases for their stationary point values x_i:

1. Constant solutions where every point is a stationary point.
2. If no order of derivative of $g(x)$ has an infinity for a stationary point value (i.e., $g(x_i)$), then stationary point can only occur at $t = \pm\infty$
3. If some order of derivative of $g(x)$ has an infinity for a stationary point value, than that stationary point will occur at finite t usually.

We explicate the three cases in the subsections below.

A.1. Constant Solutions

A constant solution $x(t) = x_i$ has every point a stationary point. This is easy to prove: $g(x_i) = 0$ implies $x(t) = x_i$.

A.2. $g(x)$ Has No Order of Derivative that is Infinite at x_i

If $g(x)$ has no order of derivative with an infinite value, then the only stationary points are at infinity (meaning either plus or minus) infinity. To prove this expand $g(x)$ around the stationary point x_i first assuming that there is a nonzero 1st order term. One obtains

$$x' = \Delta x g_1 + \frac{\Delta x^2}{2!} g_2 + \dots \ , \tag{A2}$$

where the g_n are derivatives with respect to x evaluated at x_i, $g_0 = 0$ by the assumption that x_i is a stationary point, and $\Delta x = x - x_i$. For small Δx, we approximate the differential equation as

$$\Delta x' = g_0 + \Delta x g_1 \tag{A3}$$

which has solution

$$\Delta x = \Delta x_0 e^{g_1 t} \tag{A4}$$

For g_1 less/greater than zero, x converges/diverges toward the stationary value x_i as t increases. In the convergence case, x only gets to the stationary point at $t = \infty$. Since our 1st order approximation gets better for the convergence behavior, this result is established for the exact $g(x)$. The divergence behavior is to $x = \infty$ as $t \to \infty$, but neglected higher order terms may lead to other behavior as t increases, but, in any case, there never will be convergences to x_i for finite time. If you run the time backward, then for g_1 less/greater than zero, you get the inverse of the behavior just described.

Now what if expansion for $g(x)$ has zero coefficients up to coefficient $n \geq 2$? Alas, this unimportant case is has a tricky discussion. But a human has to do what a human has to do. However, the solution is for $g(x)$ truncated to the nth order is easily found to be

$$\Delta x = \frac{1}{[(g_n t/n!) + \Delta x_0^{-n+1}]^{1/(n-1)}} \ . \tag{A5}$$

The trickiness is because of complications due to number sign, but the solution's general behavior is easy to understand. At t advances in either positive or negative direction, the solution converges to x_i at $t = \pm\infty$ in one direction and diverges to $\pm\infty$ at finite time $t = -n!\Delta x_0^{-n+1}/g_n$ in the other direction. Since our nth order approximation gets better for the convergence behavior, this result is established for the exact $g(x)$. The divergence behavior may be changed by neglected higher terms, but these terms will never lead to convergence x_i for finite or infinite time. If Δx ever gets sufficiently small again for any reason, divergence will resume.

What about the complications due to number sign. First assume $\Delta x_0 > 0$, then clearly

$$A = (g_n t/n!) + \Delta x_0^{-n+1} \qquad \text{and} \qquad \Delta x = \frac{1}{[(g_n t/n!) + \Delta x_0^{-n+1}]^{1/(n-1)}} \tag{A6}$$

will be greater than zero until time has advanced beyond the infinity where $A < 0$, and so

$$A^{1/(n-1)} = |A|^{1/(n-1)} e^{i\pi(2m+1)/(n-1)} \ , \tag{A7}$$

where m can be any integer. If $(n-1)$ is odd (i.e., n is even), then for some m we have $(2m+1)/(n-1) = 1$, but there is no m that can make $(2m+1)/(n-1)$ an even integer since the numerator has no even factor. So the solution going through point $t = -n!\Delta x_0^{-n+1}/g_n$ flips from rising to positive infinity to rising from negative infinity and asymptotically goes x_i as time advances to infinity. If $(n-1)$ is even (i.e., n is odd), then $(2m+1)/(n-1)$ can never be an integer and there is no real solution.

Next assume $\Delta x_0 < 0$ and $(n-1)$ even, then clearly

$$A = (g_n t/n!) + \Delta x_0^{-n+1} \tag{A8}$$

will be greater than zero until time has advance beyond the infinity. Now for the before-the-infinity region

$$A^{1/(n-1)} = |A|^{1/(n-1)} e^{i\pi[2m/(n-1)]} \tag{A9}$$

will be positive for $m = n-1$ and negative $m = (n-1)/2$. The latter choice matches the initial condition of $\Delta x_0 < 0$, and so gives the right solution. Beyond the infinity

$$A^{1/(n-1)} = |A|^{1/(n-1)} e^{i\pi(2m+1)/(n-1)]} \ . \tag{A10}$$

For no choice of m can $(2m+1)/(n-1)$ can ever be an integer for $(n-1)$ even, and so there is no real solution.

Finally, $\Delta x_0 < 0$ and $(n-1)$ odd, then clearly

$$A = (g_n t/n!) + \Delta x_0^{-n+1} \tag{A11}$$

will be less than zero until time has advance beyond the infinity. Now for the before-the-infinity region

$$A^{1/(n-1)} = |A|^{1/(n-1)} e^{i\pi(2m+1)/(n-1)} \tag{A12}$$

will be negative for $2m + 1 = n - 1$ and can never be positive since $(2m + 1)/(n - 1)$ can never be an even integer. So the only real choice allowed matches the initial condition of $\Delta x_0 < 0$ and so gives the right solution. Beyond the infinity,

$$A^{1/(n-1)} = |A|^{1/(n-1)} e^{i\pi[2m/(n-1)]} \tag{A13}$$

and no choice of m makes $2m/(n-1)$ an odd integer, and so there is only a positive solution. So the solution going through point $t = -n!\Delta x_0^{-n+1}/g_n$ flips from falling to negative infinity to falling from positive infinity and asymptotically going x_i as time advances to infinity.

A.3. $g(x)$ Has an Order of Derivative that is Infinite at x_i

If $g(x)$ has some order of derivative with respect to x with an infinite value, then the function x will have a stationary point at finite t. Note $x = f(t)$ is smooth at this stationary point by the assumption that it is a stationary point: i.e., $g(x_0) = 0$. It's $g(x)$ at x_i that has some bad behavior.

To explicate, let's assume the $(n-1)$th derivative $g^{(n)}$ with respect to x at the stationary value x_i is infinite. Now we differentiate the differential equation equation (A1) $(n-1)$ times to get

$$x^{(n)} = (g^{n-1})(x')^{n-1} + \text{other terms} = (g^{n-1})g^{n-1} \ , \tag{A14}$$

where not that x is differentiate with respect to t and g with respect to x. The other terms only include factors of the derivative of $g(x)$ with order less than $(n - 1)$ and factors of derivatives of x of order less than n. By assumption, all the $g^{(k)}$ for $k < (n - 1)$ are finite at x_i and, by an implicit proof by induction, all the $x^{(\ell)}$ for $\ell < n$ are zero at x_i. We now take the limit as $x \to x_i$ and we find that

$$x^{(n)}(x_i) = \lim_{x \to x_i} (g^{n-1})g^n = \text{finite value} \ . \tag{A15}$$

For an interesting case, we assume that the finite value is not itself zero. If we expand $x(t)$ around t_i (the stationary point for stationary value x_i), we find

$$x(t) = \frac{\Delta t^n}{n!} x^n(x_i) + \text{higher order terms} \ . \tag{A16}$$

Thus, equation (A15) gives the lowest order curvature of $x(t)$ at the stationary point.

There is at least one common case where x has a stationary value at finite t. Consider

$$g(x) = h(x)^p \ , \tag{A17}$$

where p is not an integer. We now find

$$x^{(n)} = \{p(p-1)\ldots[p-(n-2)]\} h^{p-(n-1)} (h')^{n-1} h^{p(n-1)} \tag{A18}$$

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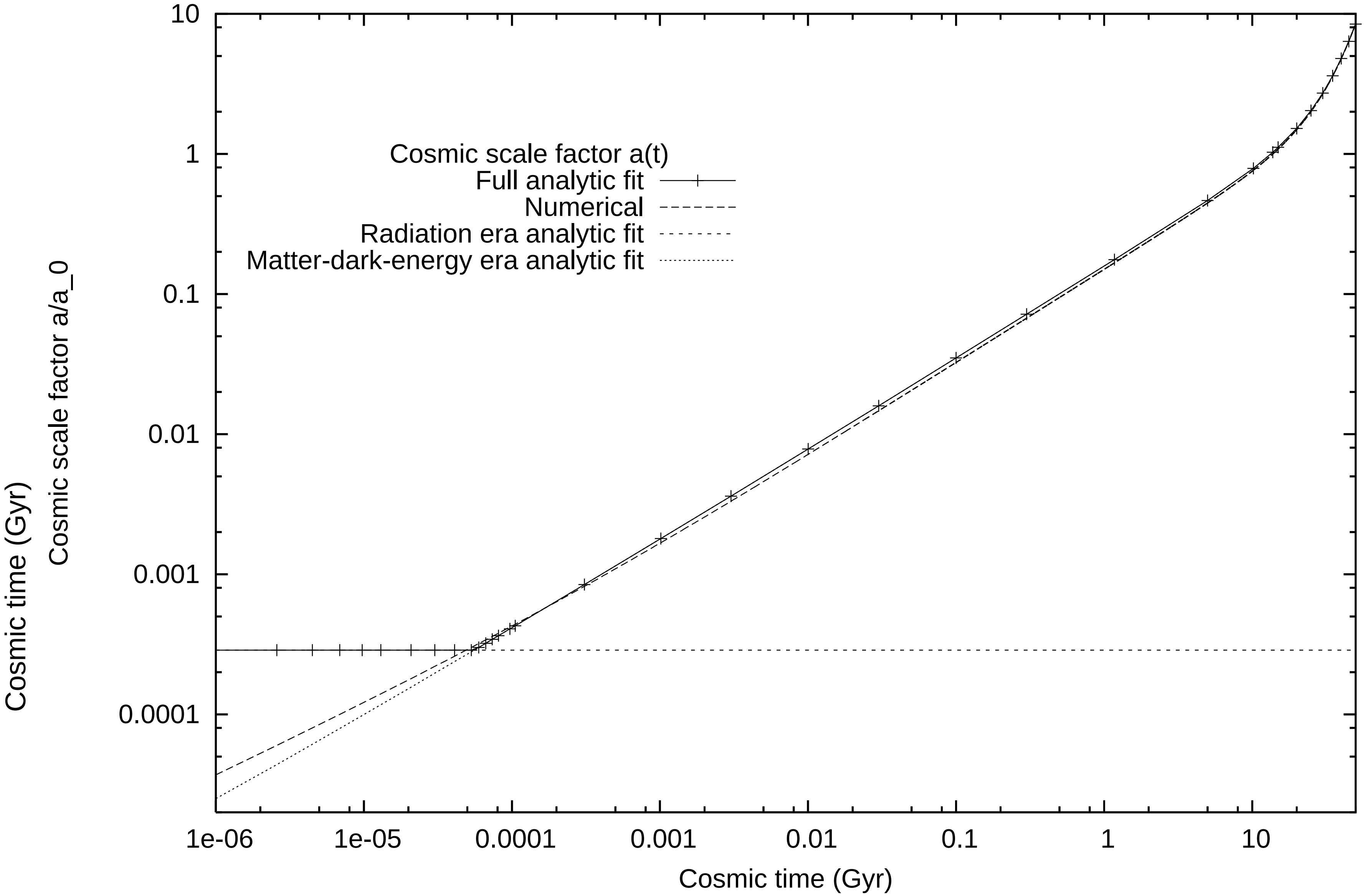
FIGURE CAPTIONS

Fig. 1—The analytic fit to the cosmic scale factor $a(t)$ for the Λ-CDM model (dashed line) compared to the exact $a(t)$ (solid line) for said model calculated using Planck 2015 parmaters Cahill (2016).

Fig. 2—The analytic fit to the cosmic scale factor $a(t)$ for the Λ-CDM model (dashed line) compared to the exact $a(t)$ (solid line) for said model calculated using Planck 2015 parmaters Cahill (2016).

Fig. 1



Cosmic time (Gyr)
Cosmic scale factor a/a_0
10
1
0.1
0.01
0.001
0.0001
Cosmic scale factor a(t)
Full analytic fit
Numerical
Radiation era analytic fit
Matter-dark-energy era analytic fit
1e-06
1e-05
0.0001
0.001
0.01
0.1
1
10
Cosmic time (Gyr)