An Educational Note on Quasi-Equilibrium Dark Matter Halo Physics

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ABSTRACT

This note is naive theorizing to see how far it can go in understanding the physics of the quasi-equilibrium dark matter halos. Many of the conundrums that come up are probably resolved by a study a modern treatment of point-mass gravitating bodies (e.g., Hamilton 2024). The presentation is likely to stay fragmentary for some time.

Unified Astronomy Thesaurus concepts: Center of mass (216); Cosmology (343); Cosmological constant (334); Dark energy (351); Dark matter (353); Dark matter density (354); Dark matter distribution (356); Galaxy dark matter halos (1880); Galaxy rotation curves (619) Gravitation (661); Gravitational fields (667); Isothermal sphere profile (866); Large-scale structure of the universe (902); Lambda density (898); Matter density (1014); Navarro-Frenk-White profile (1091)

1. Drift Velocity

Over a mean free path in the radial direction a particle loses all the velocity it gains from the acceleration due the gravitation of the interior mass.

$$\ell = \frac{1}{2}at^2 \ , \tag{1}$$

and thus has drift velocity

$$v = \frac{\ell}{\sqrt{2\ell/a}} = \sqrt{\frac{1}{2}\ell a} \ , \tag{2}$$

where

$$g = \frac{GM(r)}{r^2} - \frac{\Lambda}{3}r \tag{3}$$

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is the graviational field due the interior mass $M(r)$ and the cosmological constant force and we have counted downward as positive.

A vague equipartition theorm tells us that the total average kinetic energy for the radial and two perpendicular direction is

$$KE = 3\left(\frac{1}{2}mv^2\right) \ . \tag{4}$$

The mean free path satisfies

$$\frac{1}{\ell} = n\sigma \ , \tag{5}$$

where

$$n = \frac{1}{L^3} \qquad \text{and} \qquad \rho = \frac{m}{L^3} \ , \tag{6}$$

and thus

$$L = \left(\frac{m}{\rho}\right)^{1/3} \ , \tag{7}$$

Now we estimate

$$\sigma = K^{-1}L^2\left[Q\frac{(L/t_{\rm orbit})}{v}\right]^{\beta} \ , \tag{8}$$

where the velocity ratio (also the momentum ratio) is of order the amount of velocity (or momentum) removed from an impacting particle of mass M, K is a parameter of order unity that can be used to improve on simplifying assumptions in the choice of length and other scales,

$$t_{\rm orbit} = \frac{2\pi}{\sqrt{G(m_1+m)}}\left(\frac{L}{2}\right)^{3/2} = \frac{2^{-1/2}\pi}{\sqrt{G(m_1+m)}}L^{3/2} \tag{9}$$

is the Kepler 3rd law orbital period formula with $L/2$ being the semi-major axis of the relative orbit, and

$$Q = \frac{m}{m_1+m} \tag{10}$$

is the factor that corrects from the relative orbit length scales to center of mass frame of frame length scales for the impacting particle with mass m_1. The mean free path formula is

$$\frac{1}{\ell} = n\sigma = K^{-1}\left(\frac{m}{\rho}\right)^{-(1+\beta/2)/3}\left[Q\left(\frac{\sqrt{m}}{v}\right)\frac{\sqrt{G(m_1+m)/m}}{2^{-1/2}\pi}\right]^{\beta} \tag{11}$$

and so

$$\ell \quad = \quad \frac{1}{n\sigma} = Kv^{\beta}\rho^{-(1+\beta/2)/3}m^{(1+\beta/2)/3-\beta/2}\left[Q^{-1}\frac{2^{-1/2}\pi}{\sqrt{G(m_1+m)/m}}\right]^{\beta}$$

$$= \quad \frac{1}{n\sigma} = Kv^{\beta}\rho^{-(1+\beta/2)/3}m^{(1+\beta/2)/3-\beta/2}\left[\sqrt{\frac{m_1+m}{m}}\frac{(2^{-1/2}\pi)}{\sqrt{G}}\right]^{\beta} \tag{12}$$

If the ℓ is not to depend on mass in the case where $m_1 = m$ as N-body simulations show, we require

$$1) \quad 0 = \frac{(1+\beta/2)}{3} - \frac{\beta}{2} \qquad\qquad 2) \quad \beta = 1 \ . \tag{13}$$

which is the most natural choice since N-body simulations show the macroscopic properties of dark matter halos are independent of the dark matter particle mass. Thus,

$$\ell = 2^{-1/2}\pi Kv\left(\sqrt{\frac{m_1+m}{m}}\right)\frac{1}{\sqrt{G\rho}} \tag{14}$$

For the case that $m_1 = m_{\rm b} = \rho_{\rm b}L^3$, we assume that $m_1 >> m$ and that the mean free path for baryonic matter goes to infinity to 1st order and so to 1st order the baryonic matter just responds to the bulk gravitational field of the dark matter. We now specialize to the case of interest $m_1 = m$: i.e., the impacting particle is a dark matter particle. Thus

$$\ell = \pi Kv\left(\frac{1}{\sqrt{G\rho}}\right) \ . \tag{15}$$

Now

$$1) \quad v = \sqrt{\frac{1}{2}\ell a} \propto \sqrt{v} \qquad\qquad 2) \quad v = \frac{\pi}{2}K\left(\frac{1}{\sqrt{G\rho}}\right)\left[\frac{GM(r)}{r^2} - \frac{\Lambda}{3}r\right] \ . \tag{16}$$

This is our most general result for the drift velocity in one direction (more exactly, drift speed in one dimension). Using our vague energy equipartition assumption, the average drift speed is $\sqrt{3}v$ which can be used to find the kinetic energy of the particles.

2. Power-Law Density Profiles

Assuming

$$\rho = \rho_{\rm s}\left(\frac{r}{r_s}\right)^{-\alpha} = \rho_{\rm s}x^{-\alpha} \ , \tag{17}$$

we obtain interior mass

$$M(r) = 4\pi r_{\rm s}^3 \rho_{\rm s} \int_{x_{\rm in}}^{x} \bar{x}^{-\alpha} \bar{x}^2 \, d\bar{x} = 4\pi r_{\rm s}^3 \rho_{\rm s} \begin{cases} \dfrac{x^{3-\alpha}}{3-\alpha} & \text{for } \alpha = 3 \text{ and } x_{\rm in} = 0; \\ \ln\left(\dfrac{x}{x_{\rm in}}\right) & \text{for } \alpha = 3 \text{ and } x_{\rm in} > 0; \\ \dfrac{x^{3-\alpha} - x_{\rm in}^{3-\alpha}}{3-\alpha} & \text{for } \alpha > 3 \text{ and } x_{\rm in} > 0. \end{cases} \qquad (18)$$

3. Drift Velocity for Power-Law Density Profiles

If we assume

$$\rho = \rho_{\rm s} \left(\frac{r}{r_{\rm s}}\right)^{-\alpha} = \rho_{\rm s} x^{-\alpha} \qquad (19)$$

then

$$M(r) = 4\pi r_{\rm s}^3 \rho_{\rm s} \int_0^x \bar{x}^{-\alpha} \bar{x}^2 \, d\bar{x} = 4\pi r_{\rm s}^3 \rho_{\rm s} \frac{x^{3-\alpha}}{(3-\alpha)} = M_{\rm s} x^{3-\alpha} \ , \qquad (20)$$

where $x \neq 3$ and we require $\alpha < 3$ to prevent divergence as $x \to 0$ and $\alpha > 3$ to prevent divergence as $x \to \infty$. When $\alpha = 3$, there is logarithmic divergence both for $x \to 0$ and $x \to \infty$. In the following, we assume halo cut-off radius R and negligible cosmological constant force, and restrict our consideration cases of $\alpha < 3$. For the halo acceleration (i.e., due to the bulk dark matter mass),

$$1) \quad a = g = \frac{GM(r)}{r^2} = \frac{GM_{\rm s}}{r_{\rm s}^2} x^{1-\alpha} \qquad\qquad 2) \quad v_{\rm cir} = \sqrt{\frac{GM_{\rm s}}{r_{\rm s}}} x^{1-\alpha/2} \ , \qquad (21)$$

which is the circular orbit velocity of matter not subject to dark matter particle drag: i.e., baryonic matter. Thus

$$\begin{aligned} v &= \frac{\pi}{2} K \left(\frac{1}{\sqrt{G\rho}}\right) \frac{GM(r)}{r^2} = \frac{\pi}{2} K \left(\frac{1}{\sqrt{G\rho_{\rm s}}}\right) x^{\alpha/2} \frac{GM_{\rm s}}{r_{\rm s}^2} x^{1-\alpha} = \frac{\pi}{2} K \left(\frac{1}{\sqrt{G\rho_{\rm s}}}\right) \frac{GM_{\rm s}}{r_{\rm s}^2} x^{1-\alpha/2} \\ &= \frac{\pi}{2} K \sqrt{\frac{4\pi}{3-\alpha}} \left(\frac{1}{\sqrt{GM_{\rm s}/r_{\rm s}^3}}\right) x^{\alpha/2} \frac{GM_{\rm s}}{r_{\rm s}^2} x^{1-\alpha} = K \sqrt{\frac{\pi^3}{3-\alpha}} \sqrt{\frac{GM_{\rm s}}{r_{\rm s}}} x^{1-\alpha/2} \\ &= \ = K \sqrt{\frac{\pi^3}{3-\alpha}} v_{\rm cir} x^{1-\alpha/2} \ . \end{aligned} \qquad (22)$$

To have v constant and $v_{\rm cir}$ constant requires $\alpha = 2$.

Note that the drift velocity v and the circular velocity $v_{\rm cir}$ have the same dependence on x, but differ in scale. But this is not a problem. The drift velocity v is the velocity of particles going in random directions due to local particle interactions as well as the global fields (i.e., gravitational and cosmological constant). On the other hand the circular velocity $v_{\rm cir}$ is for matter that is just responding to the global field at radius r and in a cirular orbit by some initial condition. The circular velocity applies baryonic matter in circular orbits in the ideal limit that the baryonic matter does not feel the dark matter drag force.

Note also that the drift velocity v goes to infinity as $\alpha \to 3$. However, this an unphysical limit since it also implies a logarithmic divergence of the mass of the halo.

Now the empirical power index 2 for density profiles (i.e., $\rho \propto r^{-2}$ which is called singular isothermal sphere profile: e.g., Wikipeda: Singular isothermal sphere profile) has long been known to give a good fit to the large middle radius range of galaxy rotation curves where the curve is flat since it yields a constant velocity for circular orbital speed for baryonic matter in that range. Note the baryonic matter tends to orbit in circular orbits no matter how the dark matter is orbiting and beyond the inner region of galaxies the baryonic matter is just a small component of the matter. Our simple derivation is based on the idea that dark matter is largely rising and sinking.

Also a vast N-body simulation study of dark matter halos has found the power index 2 holds over the large middle radius range for dark matter halo masses spanning 20 orders of magnitude (Wang et al. 2020).

4. A Dark Matter Halo Profile from the Drift Velocity Result

We assume the dark matter acts like a classical gas with pressure which only coupled to dark matter and neglect the cosmological constant force. Therefore

$$P = \frac{2}{3}\left[\frac{1}{2}\rho(3)v^2\right] = \left(\frac{\pi}{2}K\right)^2 \left\{\left(\frac{1}{\sqrt{G\rho}}\right)\left[\frac{GM(r)}{r^2}\right]\right\}^2 \tag{23}$$

We assume hydrostatic equilibrium on the gross scale for the dark matter halo

$$\begin{aligned} \frac{dP}{dr} &= \frac{GM(r)}{r^2}\rho \\ \left(\frac{d}{dr}\right) W\rho \left\{\left(\frac{1}{\sqrt{G\rho}}\right)\left[\frac{GM(r)}{r^2}\right]\right\}^2 &= \frac{GM(r)}{r^2}\rho \end{aligned} \tag{24}$$

where

$$W = 2\left(\frac{\pi}{2}K\right)^2 \left(\frac{\rho}{\rho_{\rm s}}\right)^\delta . \tag{25}$$

Then

$$\left(\frac{d}{dr}\right) W \left[\frac{GM(r)}{r^2}\right]^2 = \frac{GM(r)}{r^2}\rho \tag{26}$$

5. Dark Matter Halo Profiles from the Expansion of Specific Volume

Hypothesize the specific volume $V_{\rm sp} = 1/\rho$ is a smooth function of the radius. Then it can be expanded around $r = 0$ with some radius of convergence:

$$V_{\rm sp} = V_{\rm sp,0} + rV_{\rm sp,1} + r^2 V_{\rm sp,2} + r^3 V_{\rm sp,3} + \ldots \tag{27}$$

It then follows the density for that radius of convergence is

$$\begin{aligned} \rho(r) &= \frac{1}{V_{\rm sp,0} + rV_{\rm sp,1} + r^2 V_{\rm sp,2} + r^3 V_{\rm sp,3} + \ldots} \\ \rho(r) &= \frac{\rho_{\rm s}}{V_{\rm sp,0} + cr + br^2 + ar^3 + \ldots} \ , \end{aligned} \tag{28}$$

where $\rho_{\rm s}$ is some scale density and the second set of expansion coefficients for the powers of r have been given symbols that clairvoyance tells agree with those needed for common tables of integrals, except $V_{\rm sp,0}$ So smoothness of functions gives us a natural density profile form.

We argue $V_{\rm sp,0}$ is zero for dark matter halos in the pure N-body simulation limit. Recall

$$a = g = \frac{GM(r)}{r^2} = \frac{GM_{\rm s}}{r_{\rm s}^2} x^{1-\alpha} \ , \tag{29}$$

where $a \to 0$ as $x \to 0$ if $\alpha = 0$. But recall $v \propto a$ for the drift velocity. If $a \to 0$, then $v \to 0$ and this seems unlikely for the quasi-equilibrium case of pure dark matter since that suggests dark matter can continuously accumulate near the origin preventing quasi-equilibrium from occurring. It seems likely that dark matter falling to the center will not be able to get rid of all its kinetic energy, and so will come to quasi-equilibrium with $a(r = 0) \neq 0$. Note N-body simulations suggest asymptotically that the zeroth order coefficient is indeed zero.

The NFW profile terminates the coefficients at the 3rd order. Say that 3 coefficients are sufficient to characterize the profile, then we need 3 contraints. Total mass provides one constraint. The virial theorem provides a second contraint. But some extra physics is needed for a third constraint.

6. NFW Profile for Dark Matter Halos

The NFW profile was originally found by being good fit to quasi-equilibrium dark matter halos based on N-body simulations (Navarro et al. 1996). It has been confirmed to be accurate almost everywhere to within 10 % over 20 orders of dark halo mass (Wang et al. 2020). The formula is

$$\rho(r) = \begin{cases} \dfrac{4\rho_{\rm s}}{(r/r_{\rm s})(1+r/r_{\rm s})^2} = \dfrac{4\rho_{\rm s}}{(r/r_{\rm s})+2(r/r_{\rm s})^2+(r/r_{\rm s})^3} & \text{in general;} \\ \dfrac{4\rho_{\rm s}}{(r/r_{\rm s})} & \text{for } r/r_{\rm s} << 1; \\ \rho_{\rm s} & \text{for } r/r_{\rm s} = 1; \\ \dfrac{4\rho_{\rm s}}{(r/r_{\rm s})^3} & \text{for } r/r_{\rm s} >> 1, \end{cases} \tag{30}$$

where $\rho_{\rm s}$ and $r_{\rm s}$ are scale values chosen so that the logarithmic slope of $\rho(r)$ is -2 at $r/r_{\rm s} = 1$. To confirm the logarithmic slope result using $x = r/r_{\rm s}$, note

$$\begin{aligned} \frac{d\ln(\rho)}{d\ln(r)} &= \frac{d\ln(\rho)}{d\ln(x)} = \frac{x}{\rho}\frac{d\rho}{dx} = -\frac{x}{\rho}\frac{4\rho_{\rm s}}{(x+2x^2+x^3)^2}(1+4x+3x^2) \\ &= -x\left(\frac{1+4x+3x^2}{x+2x^2+x^3}\right) \qquad (31) \\ \left.\frac{d\ln(\rho)}{d\ln(r)}\right|_{x=1} &= -2 \ . \qquad (32) \end{aligned}$$

However, what is also needful is the NFW profile expressed in terms of $y = r/R$ where R is the cut-off radius of the atmosphere as determined either by the turnaround radius (i.e., zero-global-acceleration radius) or otherwise. To find this we first write the NFW profile in the form

$$\rho(y) = \frac{4\rho_{\rm s}}{cy + by^2 + ay^3} \ , \tag{33}$$

where the particular coefficient symbols are chosen by clairvoyance to match the common usage in tables of integrals. Now

$$\begin{aligned} \frac{d\ln(\rho)}{d\ln(r)} &= \frac{d\ln(\rho)}{d\ln(y)} = \frac{y}{\rho}\frac{d\rho}{dy} = -\frac{y}{\rho}\frac{4\rho_{\rm s}}{(cy+by^2+ay^3)^2}(c+2by+3ay^2) \\ &= -y\left(\frac{c+2by+3ay^2}{cy+by^2+ay^3}\right) \\ 2 &= \frac{cy+2by^2+3ay^3}{cy+by^2+ay^3} \end{aligned}$$

$$
\begin{aligned}
2cy + 2by^2 + 2ay^3 &= cy + 2by^2 + 3ay^3 \\
cy &= ay^3 \\
y_{\rm s} &= \left(\frac{c}{a}\right)^{1/2} \ ,
\end{aligned} \tag{34}
$$

where $y_{\rm s}$ is the value of y with the logarithmic slope is -2. But what are a, b, and c? To make the $\rho_{\rm s}$ the density at $y_{\rm s}$, we require

$$
\begin{aligned}
&1) \quad cy_{\rm s} = 1 && c = a^{1/3} \\
&2) \quad by_{-2}^2 = 2 && b = 2\left(\frac{c}{a}\right)^{-1} = 2a^{2/3} \\
&3) \quad ay_{-2}^3 = 1 && a\left(\frac{c}{a}\right)^{3/2} = 1 \qquad c = a^{1/3}
\end{aligned} \tag{35}
$$

We need now constrain the value of a. First we impose the total mass M constraint:

$$
\begin{aligned}
M &= (4\rho_{\rm s})(4\pi R^3)\int_0^1 \frac{y^2}{cy + by^2 + ay^3}\,dy \\
&= (4\rho_{\rm s})(4\pi R^3)\int_0^1 \frac{y}{c + by + ay^2}\,dy \\
&= (4\rho_{\rm s})(4\pi R^3)\left[\frac{1}{2a}\ln(ay^2 + by + c) - \left(\frac{b}{2a}\right)\left(-\frac{2}{2ay + b}\right)\right]\Bigg|_0^1 \\
&= (4\rho_{\rm s})(4\pi R^3)\left[\frac{1}{2a}\ln(ay^2 + 2a^{2/3}y + a^{1/3}) + \left(\frac{1}{a^{4/3}y + a}\right)\right]\Bigg|_0^1 \\
&= (4\rho_{\rm s})(4\pi R^3)\left[\frac{1}{2a}\ln(a^{2/3} + 2a^{1/3} + 1) - \frac{1}{a(1 + a^{-1/3})}\right] \\
\rho_{\rm ave} &= \frac{M}{(4/3)\pi R^3} = 12\rho_{\rm s}\left[\frac{1}{2a}\ln(a^{2/3} + 2a^{1/3} + 1) - \frac{1}{a(1 + a^{-1/3})}\right] \ .
\end{aligned} \tag{36}
$$

However, something becomes obvious now. Even if we know M and R or even just the ration M/R^3, we still have two parameters $\rho_{\rm s}$ and a to determine. Now those using the NFW profile to fit N-body simulations or observed data have many data points and can to a least-squares fit to the NFW profile. However, trying to fit it from pure physics requires another constraint. That constraint is provided by the virial theorem, but that requires a determination of the kinetic energy as a function of radius. But that is what our drift velocity analysis provides. But can we impose the constraint analytically? In fact, it looks impossible.

However, say we have $\rho_{\rm s} = f\rho_{\rm ave}$, where f is some known factor. Then we can solve for a from an iteration function:

$$a = 12f\left[\frac{1}{2}\ln(a^{2/3} + 2a^{1/3} + 1) - \frac{1}{(1 + a^{-1/3})}\right] \ . \tag{37}$$

The iteration with this iteration function will converge with a reasonable initial value since the slope of the right-hand side is always less than 1 ???? for $a > 0$ which is the only allowed case. As an example, for $f = 3$ is plausibly of the right order to within a factor of 3 and using 18 digit precision, we find

$$a = 21.3543073\ldots \qquad b = 15.3940779\ldots \qquad c = 2.774353793\ldots \qquad y_{\rm s} = 0.3604442960\ldots \ . \tag{38}$$

7. The Virial Theorem

To prove the virial theorem, we first specify the scalar moment of inertia and take its first and second time derivatives:

$$\begin{aligned}
I &= \sum_i m_i \vec{r}_i \cdot \vec{r}_i \\
\frac{dI}{dt} &= 2\sum_i m_i \vec{v}_i \cdot \vec{r}_i = 2\sum_i \vec{p}_i \cdot \vec{r}_i \\
\frac{d^2 I}{dt^2} &= 2\left(\sum_i \vec{F}_i \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \vec{v}_i\right) = 2\left(\sum_i \vec{F}_i \cdot \vec{r}_i + 2\sum_i T_i\right) \\
&= 2\left(\sum_i \vec{F}_i \cdot \vec{r}_i + 2T\right) \ ,
\end{aligned} \tag{39}$$

where the sum is over all particles of a system and T is the total kinetic energy. If the system is, in fact, stationary (i.e., in equilibrium) at the macroscopic level on average $\langle I\rangle$ is constant all time-averaged derivatives of I are zero. Thus, the general virial theorem

$$\langle T\rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle \ , \tag{40}$$

where the sum on the right hand side (which must always be negative or zero) is the virial itself.

An important special case is when all the forces are derivable from potentials depending on power-law interparticle forces: i.e., the force of particle j on particle i is given by

$$\vec{F}_{ji} = -\sum_k \nabla U_{k,ji} r_{ji}^k = -\sum_k k U_{k,ji} r_{ji}^{k-1} \hat{r}_{ji} \ . \tag{41}$$

In this case, we find

$$\begin{aligned}\sum_i \vec{F}_i \cdot \vec{r}_i &= \sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_i = -\sum_{j,i,j\neq i} \vec{F}_{ij} \cdot \vec{r}_i = -\sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_j \\ &= \frac{1}{2}\left[\left(\sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_i\right) - \left(\sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_j\right)\right] \\ &= \frac{1}{2}\sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_{ji} = -\frac{1}{2}\sum_{k,i,j,j\neq i} (kU_{k,ji} r_{ji}^{k-1} \hat{r}_{ji}) \cdot \vec{r}_{ji} = -\frac{1}{2}\sum_{k,i,j,j\neq i} kU_{k,ji} r_{ji}^k \\ &= -\sum_k kU_k \end{aligned} \tag{42}$$

where the 1/2 was introduced to avoid double counting on the indexes ij and disappeared when we counted over all particles. Now the virial becomes

$$\langle T \rangle = \frac{1}{2}\sum_k k \langle U_k \rangle \ . \tag{43}$$

In fact, a stationary system must be bound since otherwise some of the particles will travel to infinity. Thus, the total energy must be negative and some of the potential energies must be negative, but in order for the virial theorem to hold then some of the powers k must be negative. In large-scale structure astronomy, the important cases are $k = -1$ for gravitation and $k = 2$ for the cosmological constant force. In this case,

$$\langle T \rangle = -\frac{1}{2} \langle U_{-1} \rangle + \langle U_2 \rangle \ . \tag{44}$$

Support for this work has been provided the Department of Physics & Astronomy of the University of Nevada, Las Vegas and the Nevada Center for Astrophysics (NCfA).

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This preprint was prepared with the AAS LaTeX macros v5.2.