A NOTE THE EXACT SOLUTION OF THE COSMIC SCALE FACTOR $a(t)$ FOR THE Λ-CDM MODEL

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ABSTRACT

An analytic solution for the cosmic scale factor $a(t)$ for the Λ-CDM model has been given by Galanti (2021) that is exact, except for the qualifcation that early- and late-time have different formulae that agree with negligible error for the bulk of the matter era. In this note, we elucidate some fine points of the inverse solution $t(a)$ and the solution $a(t)$ and present an interpolation formula for $a(t)$ that approximates the exact solution to good accuracy and allows for understanding of its overall behavior.

Subject headings: supernovae: cosmology: theory — cosmological parameters — dark energy

1. INTRODUCTION

An analytic solution for the cosmic scale factor $a(t)$ for the Λ-CDM model has been given by Galanti (2021) (hereafter G&R) that is exact, except for the qualifcation that early- and late-time have different formulae that agree with negligible error for the bulk of the matter era. The solution spans the radiation era, the radiation, matter, and dark energy eras. Note the term dark energy era is conventional. The acceleration of the universe can be attributed to dark energy, but also a cosmological constant (a modification of the Einstein field equations of general relativity), or even the spontaneous creation of matter (with some particle spectrum) at a constant low rate uniformly over the observable universe as in the steady-state universe. In call cases, the cosmological constant symbol Λ is used as shorthand. Note that numerical solutions for $a(t)$ for the Λ-CDM model for all eras are straightforward (Cahill 2016, e.g.,).

In this note, we elucidate some fine points of the inverse solution $t(a)$ and the solution $a(t)$ of Galanti (2021) and present an interpolation formula for $a(t)$ that approximates the exact solution to good accuracy and allows for understanding of its overall behavior.

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We note in passing that an exact solution for $a(\eta)$ (where η is conformal time: $dt = [a(\eta)/c]\,dt$) for a density parameter with terms with any combination of inverse integer powers 0 through 4 is given by Steiner (2008, p. 9).

In § 2, we consider $t(a)$ and in § 3 present the exact solution to the Friedmann equation for models with Λ and only one mass-energy form obeying an inverse power law. In § 3, we make use of the § 2 results to create our analytic fit. Conclusions are given in § 4. The appendices are given for pedagogical use. Appendix A discusses the formula for the age universe for the Λ-CDM model. Appendix B discusses the exact analytic solution for the closed positive-curvature universe with only matter for mass-energy.

2. THE INVERSE $t(a)$

t, τ, t, $\sqcup$, R, T.

(e.g., WolframAlpha: series: $1/sqrt1 + x$)

3. THE EXACT ANALYTIC SOLUTION FOR FRIEDMANN EQUATION FOR MODELS WITH Λ AND ONE OTHER MASS-ENERGY FORM OBEYING AN INVERSE POWER LAW

The Friedmann equation can be written in the standard form

$$H = \frac{\dot{a}}{a} = H_0\sqrt{\Omega_\Lambda + \Omega_k\left(\frac{a_0}{a}\right)^2 + \Omega_{\rm m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{\rm r,0}\left(\frac{a_0}{a}\right)^4}\ , \qquad (1)$$

there H is the Hubble parameter, a is the cosmic scale factor, a_0 is the cosmic scale factor for the current universe (conventionally set to 1), $\dot{a}$ is the time derivative of the cosmic scale factor, H_0 is the Hubble constant (i.e., the Hubble parameter for current universe), Ω_Λ is the density parameter for cosmological constant, Ω_k is the curvature density parameter, $\Omega_{\rm m,0}$ is the density parameter for matter for the current universe, and $\Omega_{\rm r,0}$ is the density parameter for radiation for the current universe. Note that matter includes all non-relativistic mass-energy (i.e., baryonic matter and dark matter) and radiation includes all extreme relativistic mass-energy (i.e., cosmic backgroun radiation and cosmic neutrinos). The non-relativistic mass-energy is the cold dark matter (CDM) of the Λ-CDM model.

The Friedmann equation is, as one can see, a 1st order nonlinear ordinary differential equation. The fact that is nonlinear means that linear combinations of solutions are not in general solutions though they may be in special cases or approximately. The Friedmann equation is also a homogeneous differential equation at least in the sense that it can be written $\dot{a} = g(a)$. The form $\dot{a} = g(a)$ implies that a must be strictly increasing or decreasing except possibly at $\pm\infty$ and possibly at points where the some order of derivative of g have infinities. Both exceptions do occur for some solutions of the Friedmann equation. The latter exception occurs for the closed universe model with only matter (see App. B).

The Friedmann equation with Λ and only one mass-energy form obeying an inverse power law is

$$H = \frac{\dot{a}}{a} = H_0\sqrt{\Omega_\Lambda + \Omega_\gamma\left(\frac{a_0}{a}\right)^\gamma} \ , \tag{2}$$

where γ is integer greater than or equal to 1. The exact solution

$$a = a_\Lambda\left[\sinh\left(\frac{H_\Lambda t}{p} + C\right)\right]^p \ , \tag{3}$$

where

$$H_\lambda = \Omega_\Lambda H_0 \ , \qquad p = \frac{2}{\gamma} \ , \qquad a_\Lambda = \frac{a_0}{[\sinh\left(H_\Lambda t_0/p + C\right)]^p} \ , \tag{4}$$

where t_0 is the current age of the universe and C is a constant. The exact solution is just an obvious generalization of the solutions for $\gamma = 3$ (e.g., Steiner 2008, p. 12; Sazhin 2011, p. 3). Note if $\Omega_\Lambda \to 0$, the exact solution reduces to

$$a = a_0\left(\frac{t}{t_0} + C\right)^p \ , \tag{5}$$

which is the well know exact solution to Friedmann equation the only one mass-energy form being one obeying an inverse power law.

The solution can be proven by direct substition. Note

$$\dot{a} \ = \ a_\Lambda\left[\sinh\left(\frac{H_\Lambda t}{p} + C\right)\right]^p \ , \tag{6}$$

As mentioned in § 1, a complex exact solution does exist given by Steiner (2008, p. 9)

For reference, we note that the radiation era is dominated by photons and neutrinos and has approximately $\rho \propto 1/a^4$ and $a(t) \propto t^{1/2}$; the matter era is dominated by non-relativistic matter (baryons and dark matter) and has approximately $\rho \propto 1/a^3$ and $a(t) \propto t^{2/3}$; the dark-energy era is dominated by dark energy (alternatively a cosmological constant) and has aysmptotically $\rho = \rho_{\Lambda;}$ and $a(t) \propto \exp(H_\Lambda t)$, where $\rho_\Lambda = \Lambda/(8\pi G)$ is the dark-energy density, and $H_\Lambda = \sqrt{\Lambda/3}$ is cosmological-constant Hubble constant (e.g., p. 40–41, 56 Liddle 2015). Following Cahill (2016), we define the transition times to be when the densities of the relevant components are equal: i.e., the radiation era ends at the radiation-matter equality (i.e., when $\rho_{\rm r} = \rho_{\rm m}$) and the matter era ends at the matter-dark-energy equality (i.e., when $\rho_{\rm m} = \rho_\Lambda$). For convenience, we label the three eras by 1, 2, and 3.

The early-time fit and ancillary formulae and remarks are as follows:[2]

$$\begin{cases} a_{\rm E}(t) = a_{2*}\left\{\left(\frac{t+bt_1}{t_{2*}}\right)\left[1-\exp\left(-(t/t_1)^{3/4}\right)\right]\right\}^{2/3} & \text{the general early-time analytic fit. It is accurate to within 11\% from at least } t=10^{-7}\,\text{Gyr until after} \sim 10\,\text{Gyr when it begins diverging from exact solution as the dark-energy era. begins. It is displayed in Fig. 2.} \\ a_{\rm E}(t)|_{\rm r\ limit} = a_{2*}\left(\frac{bt_1}{t_{2*}}\right)\left(\frac{t}{t_1}\right)^{1/2} & \text{for } t << t_1 \text{ where } t_1 \text{ is the radiation-era end time.} \\ a_{\rm E}(t)|_{\rm m\ limit} = a_{2*}\left(\frac{t}{t_{2*}}\right)^{2/3} & \text{for } t >> t_1 \text{ and } t << t_{2*} \end{cases} \tag{7}$$

$$\begin{cases} a_{2*} = 0.764771404147 & \text{is a matter-era characteristic } a \text{ value defined below for the late-time fit.} \\ t_{2*} = 11.57533308010\,\text{Gyr} & \text{is a matter-era characteristic time value defined below for the late-time fit.} \end{cases} \tag{8}$$

[2]We quote the numerical results of this note to 12 decimal places for consistent presentation and to allow for checks of numerical reproducibility. Their physical accuracy is typically a few percent.

$$
\begin{cases}
t_1 = 51,391.05873817\,\mathrm{yr} & \text{is the radiation-matter equality time} \\
\qquad = 5.139105873817 \times 10^{-5}\,\mathrm{Gyr} & \text{determined by fitting the early-time fit} \\
 & \text{to the } a \text{ and } \dot{a} \text{ for that time.} \\
 & \text{It agrees with value 50,953(2,236) of} \\
 & \text{(Cahill (2016) to within his error.} \\
b = 1.676437645262 & \text{is a radiation-matter equality parameter} \\
 & \text{determined by fitting the early-time fit} \\
 & \text{to the } a \text{ and } \dot{a} \text{ for that time.} \\
a_1 = a_0 \left(\dfrac{\Omega_{\mathrm{r0}}}{\Omega_{\mathrm{m0}}}\right) = 2.933182259631 \times 10^{-4} & \text{the cosmic scale factor at the} \\
 & \text{radiation-matter equality.} \\
\dot{a}_1 = \left(\dfrac{a_1}{t_{H_0}}\right)\sqrt{2\Omega_{\mathrm{m0}}\left(\dfrac{a_0}{a_1}\right)^3} & \text{the cosmic scale factor derivative at} \\
\qquad = \left(\dfrac{a_1}{t_{H_0}}\right)\sqrt{2\Omega_{\mathrm{r0}}\left(\dfrac{a_0}{a_1}\right)^4} & \text{the radiation-matter equality.} \\
 & t_{H_0} = 14.434488067328\,\mathrm{Gyr} \\
\qquad = 3.179457626016\,\mathrm{Gyr}^{-1} & \text{is the Hubble time.}
\end{cases}
\tag{9}
$$

$$
\begin{cases}
b = \left(\dfrac{\dot{a}_1 t_{2*}/a_{2*}}{2/3}\right)\left(\dfrac{a_1}{a_{2*}}\right)^{1/2}\left(1 - \dfrac{1}{4}e^{-1}\right)^{-1} & \text{the formula for } b \text{ obtained from equating} \\
\qquad -\dfrac{3}{4}e^{-1} & \text{the derivative of } a_{\mathrm{E}}(t) \text{ to } \dot{a}_1. \\
t_1 = \dfrac{t_{2*}(a_1/a_{2*})}{(1-e^{-1})(1+b)} & \text{the formula for } t_1 \text{ obtained from equating} \\
 & a_{\mathrm{E}}(t) \text{ to } a_1.
\end{cases}
\tag{10}
$$

4. THE LATE-TIME FIT

The late-time fit was given by Steiner (2008) though the present author independently noticed it by analogy to an analytic fit to the closed postive-curvature Friedmann-Lemaître solution given in Appendix B. The fit and ancillary formulae and remarks are as follows:

$$\begin{cases} a_{\rm L}(t) = a_{2*} \sinh^{2/3}\left(\frac{t}{t_{2*}}\right) & \text{the general late-time analytic fit.} \\ & \text{It is accurate to within } 4\,\% \text{ after} \\ & \sim 1\,\text{Gyr to at least } 50\,\text{Gyr before} \\ & \text{which going back in time} \\ & \text{it diverges from exact solution.} \\ & \text{It is displayed in Fig. 2.} \\ a_{\rm L}(t)|_{\rm m\ limit} = a_{2*} \left(\frac{t}{t_{2*}}\right)^{2/3} & \text{for } t >> t_1 \text{ and } t << t_2. \\ a_{\rm L}(t)|_{\Lambda\ {\rm limit}} = a_{\Lambda} \exp\left(\frac{t}{t_{\Lambda}}\right) & \text{for } t >> t_2. \end{cases} \tag{11}$$

$$
\begin{cases}
t_\Lambda = \dfrac{1}{H_\Lambda} = 17.362999620152\,\mathrm{Gyr} & \text{is the dark-energy era asymptotic Hubble time:} \\
 & \text{i.e., the constant Hubble time the} \\
 & \text{Λ-CDM model is approaching as time goes by.} \\
H_\Lambda = 56.314706160907\,(\mathrm{km/s})/\mathrm{Mpc} & \text{is the dark-energy era asymptotic Hubble consta} \\
t_{2*} = \dfrac{2}{3} t_\Lambda = 11.575333080101\,\mathrm{Gyr} & \text{is the characteristic matter-era time mentioned} \\
 & \text{above for the early-time fit.} \\
 & \text{That it is (2/3) of t_Λ allows a fit to} \\
 & \text{dark-energy era. That is fortuitiously approxima} \\
 & \text{equal to t_2, the end time of the matter era,} \\
 & \text{is the late-time fit to transition from matter era} \\
 & \text{to dark-energy era.} \\
a_{2*} = \dfrac{a_0}{\sinh^{2/3}(t_0/t_{2*})} = 0.764771404147 & \text{is the characteristic matter-era a value mentione} \\
 & \text{above for the early-time fit. The $t_0 = 13.799(21)$} \\
 & \text{from Planck 2015.} \\
a_\Lambda = \dfrac{a_{2*}}{2} = 0.382385702073 & \text{is the characteristic dark-energy era a value.} \\
a_2 = a_0 \left(\dfrac{\Omega_{m0}}{\Omega_\Lambda} \right)^{1/3} = 0.764584820798 & \text{is the a value for the end of the matter era.} \\
 & \text{That $a_2 \approx a_{2*}$ is because $t_{2*} \approx t_2$} \\
 & \text{as we show below.} \\
t_2 = t_{2*} \ln(x + \sqrt{x^2+1}) = 10.199197372819\,\mathrm{Gyr} & \text{is the end of the matter era and $x = (a_2/a_{2*})^{3/2}$} \\
 & \text{We have used the inverse of the hyperbolic sine} \\
 & \text{The t_2 value agrees with 10.1928(375) Gyr of Ca} \\
 & \text{within his error.} \\
H_{0,\mathrm{L}} = \dfrac{H_\Lambda}{\tanh(t_0/t_{2*})} = 67.748625572218\,(\mathrm{km/s})/\mathrm{Mpc} & \text{which agrees the with Planck 2015} \\
 & \text{Hubble constant 67.74(46) within their error.} \\
q_{0,\mathrm{L}} = \dfrac{1}{2} - \dfrac{3}{2} \tanh^2\left(\dfrac{t_0}{t_{2*}} \right) = -0.536415562934 & \text{which agrees the with Planck 2015} \\
 & \text{deceleration parameter $-0.5366(93)$ within their}
\end{cases}
\qquad (12)
$$

One can see there is good agreement in results obtained from the analytic fit and the 2015 Planck results and the results of Cahill (2016). Why is this? First, the analytic fit does embody the asymptotic behaviors for the matter and dark-energy dominated eras which are those displayed for, respectively $t/t_{\Lambda b} << 1$ and $t/t_{\Lambda b} >> 1$ in equation ??? above. The second part of the answer requires a short argument. The fiducial transition point for the aysmptotic behaviors is when $t/t_{\Lambda b} = 1$. At later times, the power 2/3 of the sinh function asymptotically cancels the 2/3 factor in $t_{\Lambda b} = (2/3)t_{\Lambda b}$ allowing the sinh function to morph into the correct asymptotic behavior for the dark-energy era $a_\Lambda \exp(t/t_\Lambda)$. No other other fiducial transition point is possible given the nature of the analytic fit. In order for the analytic fit to give even fair agreement to the results cited above, $t_{\Lambda b} = (2/3)t_\Lambda$ has to approximately equal the matter-to-dark-energy-era transition time. It does: $t_{\Lambda b} = (2/3)t_{\Lambda b} =???$ and $t_{\text{matter-to-dark-energy-era transition time}} =???$ (Cahill 2016). Because of this approximate agreement of times, fair agreement to the results could be expected. That the agreement is better than fair seems to be just a fortuitous accident of the nature of the analytic fit.

Another question is why $(2/3)t_\Lambda$ and $t_{\text{matter-to-dark-energy-era transition time}}$ are approximately equal. In the context of Friedmann-Lemaître models this is also just a fortuitous accident. Perhaps, there is a reason in dynamic dark energy theory or from the anthropic principle.

Support for this work has been provided the Department of Physics & Astronomy of the University of Nevada, Las Vegas and the Homer L. Dodge Department of Physics & Astronomy of the University of Oklahoma. I thank Kevin Cahill for answering a question.

A. Appendix A

There is an exact analytic solution for the age of the universe in this case:

$$t = H_0 t_{\text{unscaled}} = \begin{cases} \dfrac{2}{3}\dfrac{1}{\sqrt{1-\Omega_0}} \ln\left[\dfrac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right] & \text{in general;} \\ \dfrac{1}{3}\ln\left(\dfrac{1}{\Omega_0}\right) & \text{asymptotically as } \Omega_0 \to 0; \\ \dfrac{2}{3} + \dfrac{2}{9}(1-\Omega_0) & \text{for } (1-\Omega_0) << 1; \\ \dfrac{1}{3}\ln\left(\dfrac{1}{\Omega_0}\right) - \dfrac{1}{9}(1-\Omega_0) + \dfrac{2}{3} & \text{which is an interpolation formula} \\ & \text{accurate everywhere to } \lesssim 3\,\% \end{cases} \tag{A1}$$

(e.g., Liddle 2015).[3]

[3]Also, e.g., Universe in Problems: Characteristic Parameters and Scales: Problem 9.

A. Appendix B

The Friedmann equations have an exact analytic solution for the closed postive-curvature universe. The solution is in terms of parameter η:

$$a(\eta) = \frac{1}{2}[1-\cos(\eta)] \ , \quad t(\eta) = \frac{1}{2}[\eta-\sin(\eta)] \ , \quad t(a) = \frac{1}{2}\{\cos^{-1}(1-2a)-\sin[\cos^{-1}(1-2a)]\} \ , \tag{A1}$$

where $\eta \in [0, 2\pi]$, $a \in [0, 1]$, and $t \in [0, \pi]$. [4] The variables are scaled as follows:

$$k = \frac{8\pi G\rho_{\rm M}}{3} \ , \qquad t = \sqrt{k}\, t_{\rm unscaled} \ , \qquad a = \frac{a_{\rm unscaled}}{a_{\rm M}} \ , \tag{A2}$$

where k is curvature, $\rho_{\rm M}$ is minimum density, and $a_{\rm M}$ is the maximum cosmic scale factor. There is no analytic formula for $a(t)$. The form of the solution $a(t)$ is a convex-up curve, symmetric about a maximum at the mid-time $t = \pi/2$, and zero at the end points $t = 0$ and $t = \pi$. The special case formulae for $a(t)$ are:

$$a(t) = \begin{cases} \frac{5}{4}\Delta t^{2/3} & \text{for } \Delta t = t << \pi/2 \text{ and } \Delta t = (\pi/2 - t) << \pi/2; \\ \frac{1}{2} & \text{for } t = (1/2)(\pi/2 - 1) = 0.285398\ldots \\ & \qquad \text{and } t = (1/2)(3\pi/2 - 1) = 2.856194\ldots \\ 1 - \frac{1}{4}\left(t - \frac{\pi}{2}\right)^2 & \text{for } (t - \pi/2) << \pi/2. \end{cases} \tag{A3}$$

The simple analytic fit to the exact solution is

$$a_{\rm fit}(t) = \begin{cases} \sin^{2/3}(t) & \text{in general;} \\ \Delta t^{2/3} & \Delta t = t << \pi/2 \text{ and } \Delta t = (\pi - t) << \pi/2; \\ 0.429562\ldots & t = (1/2)(\pi/2 - 1) = 0.285398\ldots \\ & \qquad \text{and } t = (1/2)(3\pi/2 - 1) = 2.856194\ldots \\ 1 - \frac{1}{3}\left(t - \frac{\pi}{2}\right)^2 & \text{for } (t - \pi/2) << \pi/2. \end{cases} \tag{A4}$$

The form of the analytic fit is the same as for the exact solution. However, the analytic fit is lower than the exact soltuion everywhere with maximum deviation $\sim 14\,\%$???????. The parameters of the analytic fit are natural choices, not adjusted free parameters.

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[4]E.g., Universe in Problems: Solutions of Friedman equations in the Big Bang model: Problem 23: closed dusty Universe, exact solution.

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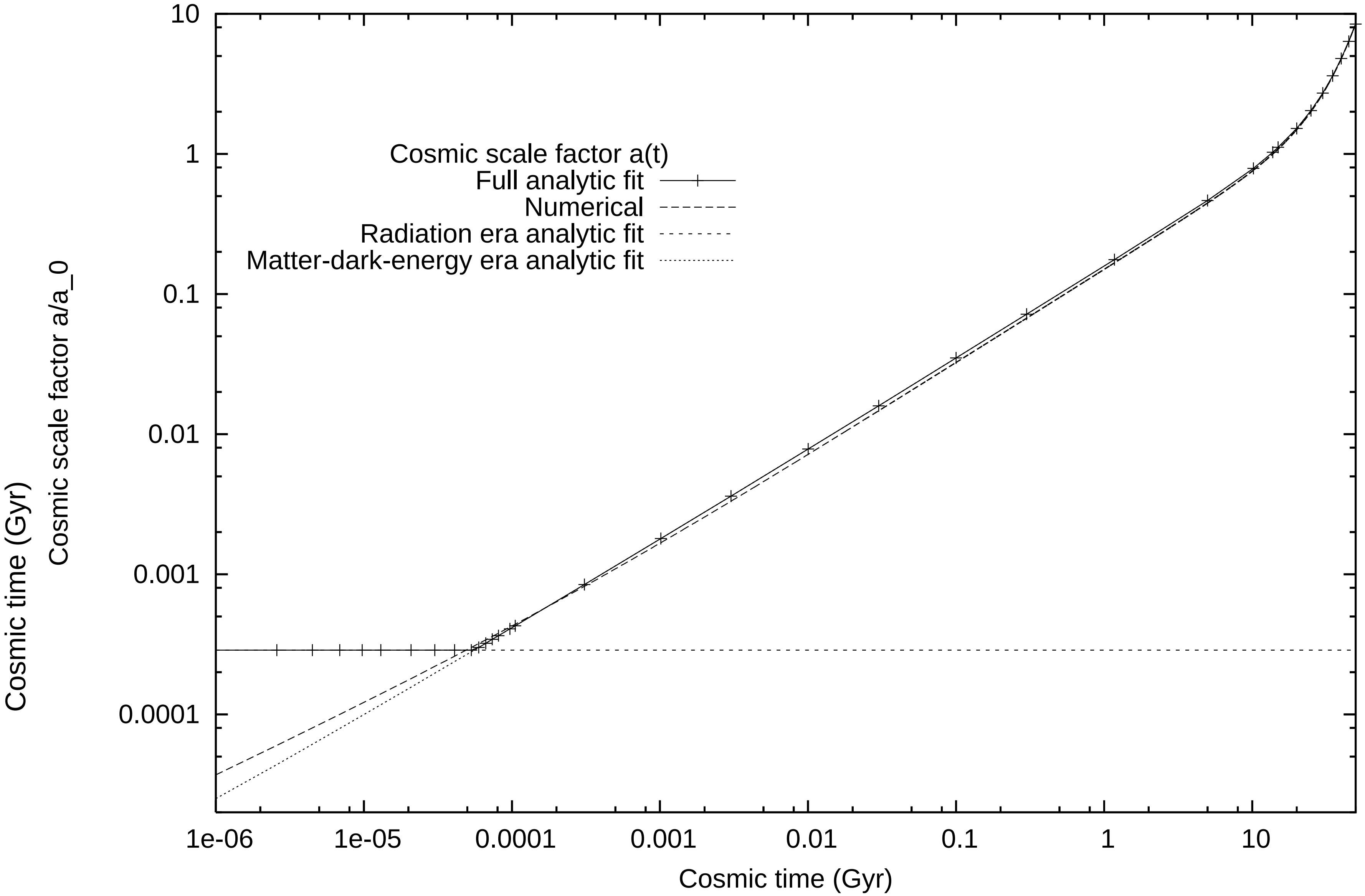
This preprint was prepared with the AAS LaTeX macros v5.2.

FIGURE CAPTIONS

Fig. 1—The analytic fit to the cosmic scale factor $a(t)$ for the Λ-CDM model (dashed line) compared to the exact $a(t)$ (solid line) for said model calculated using Planck 2015 parmaters Cahill (2016).

Fig. 2—The analytic fit to the cosmic scale factor $a(t)$ for the Λ-CDM model (dashed line) compared to the exact $a(t)$ (solid line) for said model calculated using Planck 2015 parmaters Cahill (2016).

Fig. 1


Cosmic scale factor a(t)
Full analytic fit
Numerical
Radiation era analytic fit
Matter-dark-energy era analytic fit
Cosmic scale factor a/a_0
Cosmic time (Gyr)
10
1
0.1
0.01
0.001
0.0001
1e-06
1e-05
0.0001
0.001
0.01
0.1
1
10
Cosmic time (Gyr)