A REVIEW OF MISCELLANEOUS PLANCK LAW AND RELATED TOPICS RELEVANT TO ASTROPHYSICS

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ABSTRACT

The Planck funtion

Subject headings: methods: data analysis — methods: statistical

1. INTRODUCTION

The Planck function

$$x = \frac{1}{n} \sum x_i \tag{1}$$

2. THE FREQUENCY GRID

Logarithmically equally spaced frequency points is probably best for the most general treatment of radiative transfer frequency gridding per the following reasoning We want frequency points equally spaced in some sense so that complete redistribution over the emissivity (CRDE) can be implemented effectively (see § **??**) and also for mental convenience. Now for line-of-sight velocity width w (here in units of c) for a Gaussian line profile due to Doppler broadening in the comoving frame, the equivalent frequency width $\Delta\nu$ is given by

$$\frac{\Delta\nu}{\nu} = -w \ , \tag{2}$$

where we have used the 1st order Doppler shift formula (which is extremely adequate) and the negative sign merely accords with the usual convention that a positive velocity between source and receiver is a redshift. So in frequency space, the line widths are proportionl to the line center frequency. Now $d\ln(\nu) = d\nu/\nu$, and so equally revolving lines can be done with equal logarithmic spacing: i.e., $\Delta\ln(\nu)$ constant.

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Consider the frequency range ν_0 to ν_n, where the index labels the frequency point and $n+1$ is the number of frequency points. The number of frequency bins for integration purposes is then n and n follows from the following:

$$\begin{aligned} \nu_n &= \nu_0(1+f)^n \\ n &= \frac{\ln(\nu_n/\nu_0)}{\ln(1+f)} \approx \frac{\ln(\nu_n/\nu_0)}{f} \end{aligned} \qquad (3)$$

Now the standard Doppler width w is

$$w = \sqrt{2}\sigma = \sqrt{\frac{2kT}{m}} = \frac{(12.895319\ldots)\mathrm{km/s}}{c}\sqrt{\frac{T_4}{A}} = [(4.3014154\ldots)\times 10^{-5}]\times\sqrt{\frac{T_4}{A}}\ , \qquad (4)$$

where σ is the standard deviation of the Gaussian profile, A is the atomic weight of the species, and $T_4 = T/(10^4\,\mathrm{K})$ (e.g., **?**, p. 279). Defining $f = w/g$ where g is the number of frequency points per line width, we have

$$n = \frac{\ln(\nu_n/\nu_0)}{\ln(1+w/g)} \qquad (5)$$

$$n \approx [(5.35308699\ldots)\times 10^5]\times\log\left(\frac{\nu_n}{\nu_0}\right)\left(\frac{g}{10}\right)\sqrt{\frac{A}{T_4}} \qquad (6)$$

$$n \approx (5.35308699\ldots)\times 10^5]\times\log\left(\frac{\lambda_n}{\lambda_0}\right)\left(\frac{g}{10}\right)\sqrt{\frac{A}{T_4}} \qquad (7)$$

where $\lambda_n = 1/\nu_0$ and $\lambda_0 = 1/\nu_n$ and where we have chosen g to have fiducial value 10 for resolving a line profile. Since reasonable values for SNe Ia are $T_4 = 1$, $A = 30$, and wavelength range 300–$30,000$ Å, we can expect n to be of order 5×10^6.

Can the frequency bin number be reduced. Maybe using $g = 1$ and replacing all line profiles with square profiles and artificially centering all lines in each frequency bin. All continuous opacity could also be made flat in each frequency bin. This would reduce frequency bin number by about 10 to of order 5×10^5. It would also simplify integrations for the occupation numbers. This is a gross approximation in one sense, but one expects a lot of cancellation of errors. A stronger argument can be given for SNe. In SNe, there are strong velocity gradients and as beams propagate through space they propagate through frequency. Now for isolated, the beams redshift through a whole line profile and only the integrated effect of the line has an effect, not its shape: so Gaussian or square profile, the result is the same. This effect is exploited by Sobolev method for radiative transfer in moving atmospheres (e.g., **?**, p. 471–490; **?**, p. 149–194). Now actually lines and continuous opacity do overlap everywhere, and so there are no exact isolated lines. But strong lines rarely overlap strongly and dominate any continuous opacity in them. So they are approximately isolated lines. As for weak lines, once again one can invoke a lot of cancellation of errors.

For continuous opacity, coarse gridding is fine, of course.

3. VAGUE QUESTIONS, PLAUSIBLE ANSWERS

4. THE PLANCK-LIKE FUNCTIONS

4.1. The Endpoints of the Planck-Like Functions

4.2. The Maxima of the Planck-Like Functions

Do Planck-like functions (for domain $x \in [0, \infty]$) have stationary points other than at the endpoints of their domain (i.e., in the domain $x \in (0, \infty)$? First, for the special case $z = 0$, we find

$$\begin{aligned} f(x) &= \frac{1}{e^x - 1} \\ \frac{df}{dx} &= -\frac{e^x}{(e^x - 1)^2} \end{aligned} \tag{8}$$

which is always less than 0 for domain $x \in (0, \infty)$. So the $z = 0$ case has no stationary points and $f(z)$ in domain $x \in (0, \infty)$ strictly decreases from infinity at $x = 0$ to 0 at $x = \infty$ (see § The Endpoints of the Planck-Like Functions).

Second, for $z \neq 0$, we find

$$\begin{aligned} f(x) &= \frac{x^z}{e^x - 1} \\ \frac{df}{dx} &= \frac{zx^{z-1}}{e^x - 1} - \frac{x^z e^x}{(e^x - 1)^2} \\ \frac{df}{dx} &= x^{z-1} \left[\frac{ze^x - z - xe^x}{(e^x - 1)^2} \right] \\ \frac{df}{dx} &= x^{z-1} e^x \left[\frac{z(1 - e^{-x}) - x}{(e^x - 1)^2} \right] \begin{cases} < 0 & \text{for } x \in (0, \infty) \text{ for } z < 0; \\ = 0\,? & \text{for } x \in (0, \infty) \text{ for } z > 0; \end{cases} \\ x &= z(1 - e^{-x}) \ , \end{aligned} \tag{9}$$

where the last equation is the stationary-point equation and it may have solutions for $z > 0$.

To see if the stationary-point equation $x = z(1 - e^{-x})$ has solutions, consider functions

$$\text{LHS}(x) = x \tag{10}$$

$$\text{RHS}(x) = \begin{cases} z(1 - e^{-x}) & \text{In general;} \\ zx & \text{For } x << 1; \\ z & \text{For } x \to \infty; \end{cases} \tag{11}$$

$$\frac{d\text{RHS}}{dx} = \begin{cases} ze^{-x} & \text{In general;} \\ z & \text{For } x << 1; \\ 0 & \text{For } x \to \infty; \end{cases} \ , \tag{12}$$

where from the $d\mathrm{RHS}/dx$ expression, we see the slope of $\mathrm{RHS}(x)$ strictly decreases from z at $x = 0$ to a minimum of 0 at $x = \infty$. In the graph in our minds of $\mathrm{LHS}(x)$ and $\mathrm{RHS}(x)$, there is always a solution (i.e., an intersection of $\mathrm{LHS}(x)$ and $\mathrm{RHS}(x)$) at the $x = 0$ In fact, We already know the behavior of the Planck-like functions $f(x)$ at $x = 0$ from § The Endpoints of the Planck-Like Functions:

$$f(x) = \begin{cases} \infty & \text{For } z < 1 \text{ and } x = 0 \text{ is a maximum;} \\ 1 & \text{For } z = 1 \text{ and } x = 0 \text{ is not a stationary point;} \\ 0 & \text{For } z > 1 \text{ and } x = 0 \text{ is a minimum.} \end{cases} \quad (13)$$

For $z > 1$ ($z \leq 1$), there are is a solution (no solution) at $x > 0$ since the $\mathrm{RHS}(x)$ rises faster (slower) than $\mathrm{LHS}(x)$ and its slope strictly decreases as x increases and it will (will not) intersect $\mathrm{LHS}(x)$ before it asymptotically reaches a maximum of z at $x = \infty$.

So the Planck-like functions $f(x)$ for $z \leq 1$ have no stationary points between their endpoints and strictly decrease in the domain $x \in (0, \infty)$. On the other hand, the Planck-like functions $f(x)$ for $z > 1$ have one stationary point in the domain $x \in (0, \infty)$ and it must be a maximum since the endpoints are both minima.

The maximum-point equation $x = z(1 - e^{-x})$ (for $z > 1$) has no exact analytic solution. However, we see that as maximum point x_{max} increases eventually we must have $x_{\mathrm{max}} \approx z$, but with x_{max} a bit smaller than z. Actually, even for $z = 2$, the result $x_{\mathrm{max}} \approx z$ is not so bad: the actual $x_{\mathrm{max}}(z = 2) = 1.59362\ldots$, in fact. Thus for x_{max} sufficiently large, we find

$$f(x_{\mathrm{max}}) = \frac{x_{\mathrm{max}}^z}{e^{x_{\mathrm{max}}} - 1} \approx z^z e^{-z} = \exp[z\ln(z) - z] \approx \exp(z!) \ , \quad (14)$$

where for the last expression we have used Stirling's approximation (e.g., Arfken 1985, p. 556). For better values of $f(x_{\mathrm{max}})$, we need better values of x_{max}. We show how to obtain those in the following subsections.

4.3. The Iteration Function Solution for Maximum Point of the Planck-Like Functions

The maximum-point equation $x = z(1 - e^{-x})$ (for $z > 1$) was derived in § 4.2. This equation can be used as iteration function for the maximum point x_{max}. One inputs an initial estimate for x_{max} into the right-hand side and the left-hand side result is a better estimate and input that better estimate into the right-hand side, and so on to convergence to the exact value to within machine precision. The maximum-point equation expressed as in iteration function is

$$x_{i+1} = z(1 - e^{-x_i}) \quad (15)$$

where the initial iterate is x_0. Convergence occurs when $x_{i+1} = x_i$ to within machine precision. As we explicate below, there is always converge for any $x_0 > 0$ and converge is faster as z gets bigger and slower as z gets smaller. Recall there only are solutions for $z > 1$. The Newton-Raphson iteration to the solution for $x_{\rm max}$ (see § 4.4) is much more rapid than the iteration function solution. Of course, convergence for both the iteration function and the Newton-Raphson solution converge faster the closer x_0 is to $x_{\rm max}$. Numerical tests reported in § 4.9 quantify the rates of convergence just discussed.

Actually one would guess that the iteration should converge for sufficiently large values of the maximum point since the right-hand side dependence on x is rather weak and the insensitivity of iteration function to input values is a usual sign of good convergence. Zero sensitivity is the ideal: every input to the iteration function gives the exact solution as an output In fact, the iteration with $x = z(1 - e^{-x})$ converges for all $z > 1$ as aforesaid. To prove this, we need some general results.

First, imagine rotating the 4 quadrants of the Cartesian plane by 45° clockwise. For the nonce, we call these rotated quadrants the convergence quadrants. Say we have interation function $f(x)$, where $x = f(x)$ is satisfied by the exact solution $x_{\rm exact}$ and there is only one such solution. Center the convergence quandrants on $x_{\rm exact}$ so that the quandrant boundaries are lines with slopes 1 and -1 passing through $x_{\rm exact}$.

For mental clarity, define $x' = x - x_{\rm exact}$ and $g(x') = f(x) - x_{\rm exact}$. Now the solution is the origin. If $g(x')$ has a domain (including the origin) where $g(x')$ lies entirely in the 1st and 3rd quadrants, then convergence is guaranteed for any inital iterate x'_0 in the domain. Say iterate $x'_{i-1} > 0$, then we have

$$\begin{aligned} -x'_{i-1} &< g(x'_{i-1}) < x'_{i-1} \\ |g(x'_{i-1})| &< |x'_{i-1}| \\ |x'_i| &< |x'_{i-1}| \end{aligned} \qquad (16)$$

and similarly if iterate $x'_{i-1} < 0$, we have

$$\begin{aligned} x'_{i-1} &< g(x'_{i-1}) < -x'_{i-1} \\ |g(x'_{i-1})| &< |x'_{i-1}| \\ |x'_i| &< |x'_{i-1}| \ . \end{aligned} \qquad (17)$$

So either way the iterate x'_i is closer to 0 than the iterate x'_{i-1} and convergence follows as the iteration continues.

Note that since there is only one solution by hypothesis, $g(x')$ changes sign only once and this is when it passes $x' = 0$. If $g(x')$ is goes positive (negative) at $x' = 0$ as x' increases, then iterate x'_i has the same sign (opposite sign) as x'_{i-1} and there is a non-alternating sign (alternating sign) convergence for the iteration.

Now if $g(x')$ has a domain (including the origin) where where $g(x')$ lies entirely in the 2nd and 4th quadrants, then divergence is guaranteed for any inital iteration value x'_0 in the domain. Say $g(x')$ goes positive through the origin as x' increases. If the iterate $x'_{i-1} > 0$ ($x'_{i-1} < 0$), then we have $x'_i = g(x_{i-1}) > x'_{i-1}$ ($x'_i = g(x_{i-1}) < x'_{i-1}$) and we have a non-alternating divergence. Say $g(x')$ goes negative through the origin as x' increases. If the iterate $x'_{i-1} > 0$ ($x'_{i-1} < 0$), then we have $x'_i = g(x_{i-1}) < -x'_{i-1}$ ($x'_i = g(x_{i-1}) > -x'_{i-1}$) and we have an alternating divergence.

To return to our specific case. We showed in § 4.2, that iteration function $x = f(x) = z(1 - e^{-x})$ for $x < x_{\rm max}$ was entirely in the 3rd convergence quadrant and for $x > x_{\rm max}$ was entirely in the 1st convergence quadrant for solution $x_{\rm max}$. (Note we did not use the convergence quadrant terms in § 4.2.) So convergence is guaranteed for the whole domain $x \in (0, \infty)$. Since iteration function increases going through $x_{\rm max}$, the convergence is a non-alternating sign convergence.

Recall that $x = 0$ is also a solution to $x = f(x) = z(1 - e^{-x})$. However, the iteration function is the 2nd quandrant for the exact solution $x = 0$. Thus, an iteration for will diverge from the $x = 0$ solution and, in fact, converge to the soltuion $x_{\rm max}$ as we already know.

How does convergence depend on z? Note (suppressing the subscript max for clarity for the moment) that

$$\begin{aligned}
x &= z(1 - e^{-x}) \\
z &= \frac{x}{1 - e^{-x}} \\
\frac{\partial z}{\partial x} &= \frac{1}{1 - e^{-x}} - \frac{x e^{-x}}{(1 - e^{-x})^2} \\
\frac{\partial z}{\partial x} &= \frac{1 - (1 + x)e^{-x}}{(1 - e^{-x})^2} > \frac{1 - e^{x} e^{-x}}{(1 - e^{-x})^2} = 0 \\
\frac{\partial x}{\partial z} &= \frac{(1 - e^{-x})^2}{1 - (1 + x)e^{-x}} > 0 \\
\frac{\partial x}{\partial z} &> 0 \qquad (18)
\end{aligned}$$

for $x > 0$. So $x_{\rm max}$ increases with z which is actually clear from the graphical picture we considered in § 4.2.

Now as z goes small, $x_{\rm max}$ goes small and the curve $f(x) = z(1 - e^{-x})$ comes closer to the $y = x$ line for $x < x_{\rm max}$ which is the boundary between convergence and divergence. If you imagine the $f(x) = z(1 - e^{-x})$ as becoming coincident with the boundary, then input x_{i-1} yields output $x_i = x_{i-1}$ and convergence stops. So having the curve approach the divergence bounary should slow convergence. The numerical tests (see § 4.9) show this true as aforementioned. As z decreases, convergence slows down. But as long as $z > 1$, there is still convergence.

4.4. The Newton-Raphson Solution for Maximum Point of the Planck-Like Functions

Much more efficient than the iteration function solution for the maximum point of the Planck-Like Functions (see § 4.3) is the Newton-Raphson solution.

The Newton-Raphson iteration function is determined as follows:

$$\begin{aligned}
f(x) &= z(1 - e^{-x}) - x \\
\frac{df}{dx}(x) &= ze^{-x} - 1 \\
0 &= f(x) = f(x_i) + (x_{i+1} - x_i)\frac{df}{dx}(x_i) \\
x_{i+1} &= x_i - \frac{f(x_i)}{(df/dx)(x_i)} \\
x_{i+1} &= x_i + \frac{z(1 - e^{-x_i}) - x_i}{1 - ze^{-x_i}} \ ,
\end{aligned} \tag{19}$$

where x_0 is the initial iterate.

4.5. The Small z Series for the Maximum Point of Planck-Like Functions

The solution for maximum point for small z can be obtained in a power series in $\Delta z = (z - 1)$. Behold:

$$\begin{aligned}
z &= \frac{x}{1 - e^{-x}} = \frac{x}{1 - [1 - x + (1/2)x^2 - (1/6)x^3 + \ldots]} = \frac{1}{1 - [(1/2)x - (1/6)x^2 + \ldots]} \\
z &= 1 + \left[\frac{1}{2}x - \frac{1}{6}x^2 + \ldots\right] + \left[\frac{1}{2}x - \frac{1}{6}x^2 + \ldots\right]^2 + \ldots \\
\Delta z &= z - 1 = \frac{1}{2}x - \frac{1}{6}x^2 + \frac{1}{4}x^2 + \ldots = \frac{1}{2}x + \frac{1}{12}x^2 + \ldots \\
x_{\max} &= 2\Delta z - \frac{1}{6}x_{\max}^2
\end{aligned}$$

$$
\begin{aligned}
x_{\rm max} &= 2\Delta z - \frac{1}{6}[x_{\rm max,1st} + O(\Delta z^2) + \ldots]^2 \\
x_{\rm max,1st} &= 2\Delta z \\
x_{\rm max,2nd} &= 2\Delta z - \frac{2}{3}\Delta z^2 \ , \qquad (20)
\end{aligned}
$$

where we used the geometric series expansion (e.g., Arfken 1985, p. 279) in the 2nd line.

To go beyond the 2nd order formula, one needs to be more systematic. First let's try just submitting a power series in Δz (with 0th term zero as we already know) into the maximum point equation:

$$
\begin{aligned}
x &= z(1 - e^{-x}) \\
\sum_{\ell=1}^{\infty} a_\ell \Delta z^\ell &= (1 + \Delta z)\left[1 - \exp\left(\sum_{\ell=1}^{\infty} a_\ell \Delta z^\ell\right)\right] \\
\sum_{\ell=1}^{\infty} a_\ell \Delta z^\ell &= (1 + \Delta z)\left[\left(\sum_{\ell=1}^{\infty} a_\ell \Delta z^\ell\right) - \frac{1}{2}(\ldots)^2 + \ldots\right] \\
0 &= \Delta z(\ldots) + (1 + \Delta z)\left[-\frac{1}{2}(\ldots)^2 + \ldots\right] \ . \qquad (21)
\end{aligned}
$$

Well, equation (21) is rather hopeless for solving for the coefficients of the power series in general. However, note that the first term gives a term $a_\ell \Delta z^{\ell+1}$ and the second term will give terms with a_ℓ with factors $\Delta z^{\ell+1}$ or higher exponents. Similarly, $a_{\ell+1}$ can only be in terms with factors $\Delta z^{\ell+2}$ or higher exponents. So equation (21) does show that the coefficient a_ℓ is solved for from a sum of coefficients times factors involving only coefficients $a_{k\leq\ell}$ times $\Delta z^{\ell+1}$. Thus, one must solve the coefficients in order of increasing order and the term with Δz^1 gives no solution for coefficients.

To actually solve for the coefficients, we can use the complete exponential Bell polynomial expansion (Wikipedia: Bell polynomials: Generating function). Behold:

$$
\begin{aligned}
\exp\left[\sum_{\ell=1}^{\infty}(y_\ell = -a_\ell \ell!)\frac{\Delta z^\ell}{\ell!}\right] &= \sum_{\ell=0}^{\infty} B_\ell(y_1, \ldots, y_\ell)\frac{\Delta z^\ell}{\ell!} = 1 + \sum_{\ell=1}^{\infty} B_\ell(y_1, \ldots, y_\ell)\frac{\Delta z^\ell}{\ell!} \\
\sum_{\ell=1}^{\infty} y_\ell \frac{\Delta z^\ell}{\ell!} &= (1 + \Delta z)\left[1 - \exp\left(\sum_{\ell=1}^{\infty} y_\ell \frac{\Delta z^\ell}{\ell!}\right)\right] \\
\sum_{\ell=1}^{\infty} y_\ell \frac{\Delta z^\ell}{\ell!} &= (1 + \Delta z)\left[1 - \sum_{\ell=0}^{\infty} B_\ell(y_1, \ldots, y_\ell)\frac{\Delta z^\ell}{\ell!}\right] \\
&= (1 + \Delta z)\left[-\sum_{\ell=1}^{\infty} B_\ell(y_1, \ldots, y_\ell)\frac{\Delta z^\ell}{\ell!}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{\ell=1}^{\infty} B_\ell(y_1,\ldots,y_\ell)\frac{\Delta z^\ell}{\ell!} - \sum_{\ell=1}^{\infty} B_\ell(y_1,\ldots,y_\ell)\frac{\Delta z^{\ell+1}}{\ell!} \\
&= -B_1(y_1)\frac{\Delta z}{1!} - \sum_{\ell=2}^{\infty}\left[\frac{B_\ell(y_1,\ldots,y_\ell)}{\ell!} + \frac{B_{\ell-1}(y_1,\ldots,y_{\ell-1})}{(\ell-1)!}\right]\Delta z^\ell \\
a_1\Delta z + \sum_{\ell=2}^{\infty} y_\ell \frac{\Delta z^\ell}{\ell!} &= a_1\Delta z - \sum_{\ell=2}^{\infty}\left[\frac{B_\ell(y_1,\ldots,y_\ell)}{\ell!} + \frac{B_{\ell-1}(y_1,\ldots,y_{\ell-1})}{(\ell-1)!}\right]\Delta z^\ell \\
\sum_{\ell=2}^{\infty} y_\ell \frac{\Delta z^\ell}{\ell!} &= -\sum_{\ell=2}^{\infty}\left[\frac{B_\ell(y_1,\ldots,y_\ell)}{\ell!} + \frac{B_{\ell-1}(y_1,\ldots,y_{\ell-1})}{(\ell-1)!}\right]\Delta z^\ell \qquad (22)
\end{aligned}
$$

where we used $-B_1(y_1) = -(-a_1 \times 1!) = a_1$ and where as expected from the discussion of equation (21) the Δz^1 term gives no solution since it cancels out.

Using the expression for the 1st and 2nd complete Bell polynomials, we find

$$
\begin{aligned}
a_2 &= -\frac{1}{2!}(y_1^2 + y_2) - \frac{y_1}{1!} = -\frac{1}{2}(a_1^2 - 2a_2) + a_1 \\
0 &= -\frac{1}{2}a_1^2 + a_1 \\
a_1 &= 2 \qquad (23)
\end{aligned}
$$

confirming what we already knew. Likewise with more labor solving by hand, we find $a_2 = -2/3$ (which we also already knew) and $a_3 = 4/9$ (which has been confirmed numerically). Higher order coefficients can be determined with increasing labor.

To conclude this subsection, we find

$$x_{\rm max} = \sum_{\ell=1}^{\infty} a_\ell \Delta z^\ell \qquad (24)$$

$$x_{\rm max,1st} = 2\Delta z \qquad (25)$$

$$x_{\rm max,2nd} = 2\Delta z - \frac{2}{3}\Delta z^2 \qquad (26)$$

$$x_{\rm max,3rd} = 2\Delta z - \frac{2}{3}\Delta z^2 + \frac{4}{9}\Delta z^3 \ . \qquad (27)$$

4.6. The Large z Series for the Maximum Point of Planck-Like Functions

4.7. A Good Interpolation Formula for the Maximum Point of Planck-Like Functions

4.8. An Excellent Interpolation Formula for the Maximum Point of Planck-Like Functions

The Newton-Raphson iteration formula evaluated with $x_i = z$

$$f(z) = z - \frac{ze^{-z}}{1 - ze^{-z}} \tag{28}$$

is formally a 2nd order good in small ze^{-z} series formula for $x_{\rm max}$. It is actually a slight improvement over the fiducial 2nd order small ze^{-z} series formula (i.e., large z series formula: see 4.6) for z less than about 2 and slightly worse for z greater than about 3.??? That it is a slight improvement for z less than about 2 is explicable (though not predictable) since the Newton-Raphson iteration formula is expected to work everywhere to some degree and the fiducial 2nd order small ze^{-z} series formula is expected to get worse as z gets small.

An interesting fact about the factor $1/(1-ze^{-z})$ equation (28) is that the denominator is sum of an infinite geometric series (e.g., Arfken 1985, p. 279). This expansion suggests that we create a geometric series sum formula that approximates the small ze^{-z} series formula for $x_{\rm max}$ to second order in ze^{-z} and that also approximates the small Δz series formula for $x_{\rm max}$ in Δz. This geometric series sum formula would have a smooth transition from the small Δz behavior to the large z behavior like the good interpolation formula given in § 4.7, but the good formula only is 1st order goodness for small ze^{-z} and small Δz. It turns out that the geometric series sum formula is an excellent interpolation formula.

This excellent interpolation formula is

$$x_{\rm max} = z - \frac{ze^{-z}}{1 - [a + \Delta z(b + c\Delta z)]e^{-z}} , \qquad \text{where} \tag{29}$$

$$\Delta z = z - 1 \tag{30}$$

$$a = e^1 - 1 = 1.7182817\ldots \tag{31}$$

$$b = a - 1 = e^1 - 1 = 0.7182817\ldots \tag{32}$$

$$c = \frac{e^1}{2} - \frac{4}{3} = 0.025807\ldots . \tag{33}$$

In equation (33), we have written the expression $[a+\Delta z(b+c\Delta z)]$ in a numerically accurate form and in terms of Δz rather than z for simplicity in determining the expressions for the coefficients a, b, and c. The coefficients are chosen so that for small Δz the small Δz series formula $x_{\rm max} = 2\Delta z - (2/3)\Delta z^2$ to 2nd order is recovered.

The excellent interpolation formula, of course, goes asymptotically to the exact solution as $\Delta z \to 0$ and $z \to \infty$, both to 2nd order goodness. It is a slight overestimate for $z < 8$ approximately and a slight underestimate for $z \geq 8$ approximately. Its maximum error of $\sim 2.8/10^4$ (or 0.028 %) occurs at $z \approx 1.8$. This is much smaller than the good interpolation formula's maximum error $\sim 3\,\%$ which occurs at $z \approx 1.5$. Obviously, the excellent interpolation formula is an excellent interpolation formula.

Now for some details of getting the coefficients a, b, and c. To recover the small Δz series formula 0th order term, we need the coefficient a obtained by setting $x_{\rm max}(\Delta z = 0, z = 1) = 0$. Behold:

$$
\begin{aligned}
0 &= x_{\rm max}(\Delta z = 0) = 1 - \frac{e^{-1}}{1 - ae^{-1}} \\
\frac{e^{-1}}{1 - ae^{-1}} &= 1 \\
1 - ae^{-1} &= e^{-1} \\
a &= e^{-1} - 1 \ .
\end{aligned}
\tag{34}
$$

To recover the small Δz series formula 1st order term, we need the coefficient b obtained by setting $(dx_{\rm max}/dz)(\Delta z = 0, z = 1) = 2$. Behold:

$$
\begin{aligned}
\frac{dx_{\rm max}}{dz} &= 1 - \left\{ \frac{e^{-z}}{(\ldots)} - \frac{ze^{-z}}{\ldots)} - \frac{ze^{-z}}{(\ldots)^2}[-(b + 2c\Delta z) + (a + b\Delta z + c\Delta z^2)]e^{-z} \right\} \\
&= 1 - \left\{ -\frac{x}{z} + 1 + x - z + \frac{(x - z)}{(\ldots)}[a - b + (b - 2c)\Delta z + c\Delta z^2]e^{-z} \right\} \\
2 &= \frac{dx_{\rm max}}{dz}(\Delta z = 0, z = 1) = 1 - \left[0 + 1 + 0 - 1 + \frac{1}{e^1}(a - b)e^{-1} \right] = 1 - (a - b) \\
(a - b) &= 1 \\
b &= a - 1 = e^{-1} - 2
\end{aligned}
\tag{35}
$$

To recover the small Δz series formula 2nd order term, we need the coefficient c obtained by setting $(d^2x_{\rm max}/dz^2)(\Delta z = 0, z = 1) = -4/3$. Behold:

$$
\begin{aligned}
\frac{d^2x_{\rm max}}{dz^2} &= \frac{(dx/dz)}{z} - \frac{x}{z^2} - \frac{dx}{dz} + 1 - \left\{ \frac{[(dx/dz) - 1]}{(\ldots)}[\ldots]e^{-z} - \frac{(x - z)}{(\ldots)^2}[\ldots]^2 e^{-2z} \right. \\
&\qquad \left. + \frac{(x - z)}{(\ldots)}[b - 2c + 2c\Delta z - 1 - (b - 2c)\Delta z - c\Delta z^2]e^{-z} \right\} \\
&= \frac{(dx/dz)}{z} - \frac{x}{z^2} - \frac{dx}{dz} + 1 - \left\{ \frac{[(dx/dz) - 1]}{(\ldots)}[\ldots]e^{-z} - \frac{(x - z)}{(\ldots)^2}[\ldots]^2 e^{-2z} \right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \frac{(x-z)}{(\ldots)}[b - 2c - 1 + (-b+4c)\Delta z + \Delta z^2]e^{-z}\Big\} \\
-\frac{4}{3} &= \frac{d^2 x_{\rm max}}{dz^2}(\Delta z = 0, z = 1) = 2 - 0 - 2 + 1 - \Big\{\frac{1}{e^{-1}}[1+0+0]e^{-1} + \frac{1}{e^{-2}}[1+0+0]^2 e^{-2} \\
&\qquad - \frac{1}{e^{-1}}[b - 2c - 1]e^{-1}\Big\} \\
-\frac{4}{3} &= 1 - 1 - 1 + (b - 2c - 1) = -2 + b - 2c \\
2c &= -\frac{2}{3} + b \\
c &= -\frac{1}{3} + \frac{b}{2} = \frac{e^1}{2} - \frac{4}{3}\ . \qquad (36)
\end{aligned}
$$

4.9. Summary of the Solutions for the Maximum Point of Planck-Like Functions

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