The Born Rule and Projective Measurements

In the text we cited the Born rule

> **Postulate III (Born's rule)** The act of measurement associated with Hermitian operator **A** results in one of its eigenvalues. The probability for obtaining a nondegenerate eigenvalue a is given by the expression $|\langle a|\Psi\rangle|^2$ where $|a\rangle$ is an eigenvector of **A** that corresponds to eigenvalue a. If the eigenvalue a is degenerate, the probability to find that value is $\sum_i |\langle a_i|\Psi\rangle|^2$ where the sum is over all i in which $a_i = a$.

and according to Theorem 1.2, the states $|\, a_i \rangle$ are orthogonal for each state index a_i. According to the collapse hypothesis, the act of measurement can be represented by the following

$$|\,\psi\rangle \mapsto \; |\, a_i\rangle, \quad \text{where} \quad \langle a_j \,|\, a_i\rangle = \delta_{i\,j} \tag{1}$$

if the result of the measurement leads to eigenvalue a_i. The above mapping is also called also called a projection. That is, defining the operator

$$M_i \;\equiv\; |\, a_i\rangle \langle a_i \,| \tag{2}$$

and using Dirac's multiplication rule we find that

$$\mathrm{M}_i |\, \psi\rangle = (|\, a_i\rangle\langle a_i \,|)|\, \psi\rangle \; = |\, a_i\rangle \; \langle a_i |\, \psi\,\rangle \,.$$

M_i projects the state along the direction $|\, a_i\rangle$. Note that $M_i M_i = M_i$ and so satisfies the definition for a projection operator. Thus we can equate the expectation values

$$\langle \psi |\, M_i^\dagger \, M_i \,|\, \psi\rangle = \langle \psi \;|\, a_i\rangle\langle a_i \,|\, \psi\rangle = \; |\,\langle a_i \,|\, \psi\rangle \;|^2 \,.$$

which is simply a re-statement of the Born rule for the measurement probabilities. In addition, using the fact that $M_i^\dagger \, M_i \; = \; M_i M_i = M_i$ we construct the following state

$$|\, \psi'\rangle \; = \; M_i |\, \psi\,\rangle \Big/ \sqrt{\langle \psi \,|\, M_i^\dagger \; M_i \;|\, \psi\rangle} \tag{3}$$

It is clear that $\langle \psi' |\, \psi' \rangle = 1$, and using definition (2) we can also express $|\, \psi'\rangle$ as

In[1]:=

$$|\, \psi'\rangle = \; |\, a_i\,\rangle \, \langle a_i \,|\, \psi\rangle \,/\, |\,\langle a_i \,|\, \psi\rangle \,| \; = \; e^{i\,\alpha_i} \; |\, a_i\rangle$$

where we defined $e^{i\,\alpha_i} = \langle a_i \,|\, \psi\rangle \,/\, |\,\langle a_i \,|\, \psi\rangle \,|$. Thus, Eq. (3) is equivalent, up to an arbitrary phase factor, of the collapse mapping stated by postulate (1).

The projection approach is useful when considering a measurement that has degenerate eigenvalues. Suppose we take measurements with device A, whose eigenvalues

a are n – fold degenerate. That is, the orthonormal states $|a_1\rangle$, $|a_2\rangle$ ... $|a_{i3}\rangle$, are all eigenstates of A, with eigenvalue a. We now define, as before, the set

$$M_i = |a_i\rangle\langle a_i|$$

but now construct

In[2]:=

$$M_a = \sum_{i=1}^{n} |a_i\rangle\langle a_i|$$

thus

$$\langle\psi|\, M_a{}^{\dagger}\, M_a\, |\psi\rangle = \sum_{i=1}^{n} \langle\psi|a_i\rangle\langle a_i|\psi\rangle = \sum_{i=1}^{n} |\langle a_i|\psi\rangle|^2$$

which, again, conforms to the Born rule.

Also,

$$|\psi_a\rangle = M_a\,|\psi\rangle \Big/ \sqrt{\langle\psi|\, M^{\dagger}\, M\, |\psi\rangle} = \sum_{i=1}^{n} c_i\,|a_i\rangle \qquad c_i \equiv \frac{|\langle a_i|\psi\rangle|}{\sqrt{\sum_{i=1}^{n} |\langle a_i|\psi\rangle|^2}}$$

Note that

$$\sum_{i=1}^{n} |c_i|^2 = \sum_{i=1}^{n} \Big| \langle a_i|\psi\rangle|^2 \Big/ \sqrt{\sum_{k=1}^{n} |\langle a_k|\psi\rangle|^2} = 1$$

and so $\langle\psi_a|\psi_a\rangle = 1$.

Because, from our definition of measurement device $A = \Sigma\, a_i\, M_i$, the projection operators are orthogonal, i.e.

$$M_i\, M_j = \delta_{ij} \tag{4}$$

and complete

$$\sum M_i^{\dagger}\, M_i = \mathbf{1}$$

These measurements are called projective, or von Neumann, measurements. However, measurement devices in which relation (4) is relaxed can also be defined. The latter are called POVM (Positive Operator - Valued Measure) [1]

Generalized Born Rule[2]

Often, we are concerned with a measurement that is restricted to a subspace of the system Hilbert space. For example, consider a 2-qubit system for which we measure only the second qubit with measurement operator

$$A = \sigma_Z \otimes 1$$

Suppose the 2-qubit system is in the state

$$|\psi\rangle = c_1\ \alpha|00\rangle + c_1\beta|01\rangle + c_2\ a|10\rangle + c_2 b|11\rangle$$

$$|c_1|^2 + |c_2|^2 = 1 \qquad |a|^2 + |b|^2 = 1 \qquad |\alpha|^2 + |\beta|^2 = 1$$

We calculate the probability that measurement with A results in the value +1. Since the following states are eigenstates of A

State	Eigenvalue
$\|00\rangle$	$+1$
$\|01\rangle$	$+1$
$\|10\rangle$	-1
$\|11\rangle$	-1

we construct the projection operator $M_+ = |00\rangle\langle 00| + |01\rangle\langle 10|$ and so

$$p(+1) = \langle\psi|M_+|\psi\rangle = |\langle\psi|00\rangle|^2 + |\langle\psi|01\rangle|^2 = |c_1|^2\alpha^2 + |c_1|^2\beta^2 = |c_1|^2$$

and the system collapse into state (up to an overall phase factor)

$$M_+|\psi\rangle\Big/\sqrt{\langle\psi|M_+|\psi\rangle} = \alpha|00\rangle + \beta|01\rangle$$

Now, we can express the above state in the form

$$|\psi\rangle = c_1|0\rangle\otimes|\psi_a\rangle + c_2|1\rangle\otimes|\psi_b\rangle$$

$$|\psi_a\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\psi_b\rangle = a|0\rangle + b|1\rangle$$

So measurement of the 2nd qubit leads to collapse into state $|0\rangle\otimes|\psi_a\rangle$, with probability $|c_1|^2$, in agreement with the results obtained above. Likewise the probability for measurement value (-1) and collapse into state $|1\rangle\otimes|\psi_a\rangle$ is given by $|c_2|^2$.
Note that $|\psi_a\rangle$, $|\psi_b\rangle$ are not orthogonal.

1 Michael Nielsen and Isaac L. Chuang, Quantum Computation and Quantum Information, Cambridge U Press, 2011

2 N. David Mermin, Quantum Computer Science, Cambridge U Press, 2007