Cosmology **NAME:**

Homework 4: The Geometry of the Universe

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004 qmult 00120 1 4 1 easy deducto-memory: factoring the curvature term

1. The Friedmann equation written in terms of density parameter components with some specializations is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\Omega + \Omega_k + \Omega_\Lambda\right)$$

where H is the Hubble parameter, H_0 is the Hubble constant, Ω is the sum of all density parameter components (excluding the curvature and Λ components),

$$\Omega_k = -\frac{kc^2}{H_0^2 a^2}$$

is the curvature density parameter component, and

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2} = \frac{\Lambda/(8\pi G)}{3H_0^2/(8\pi G)} = \frac{\rho_\Lambda}{\rho_{\text{crit},0}}$$

is the Λ density parameter component (i.e., the cosmological constant component). At the fiducial cosmic present,

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2 a_0^2}$$

and we are free to factorize k/a_0^2 as we like. In fact, the Robertson-Walker metric choice is to make $k = 1$ for positive curvature space (i.e., hyperspherical space with $\Omega_{k,0} < 0$), $k = 0$ for flat space (i.e., Euclidean space), and $k = -1$ for negative curvature space (i.e., hyperbolical space with $\Omega_{k,0} > 0$). For non-flat space, this implies a definite physical scale for a_0:

$$a_0 = \frac{c/H_0}{\sqrt{k\Omega_k}} = \frac{c/H_0}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\text{Gpc})/h_{70}}{\sqrt{|\Omega_k|}} = \frac{(13.968\ldots\,\text{Gly})/h_{70}}{\sqrt{|\Omega_k|}}$$

(where $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$) which can be called the curvature radius of the universe. Note, formally the Gaussian curvature radius is defined

$$R_G = \frac{a_0}{\sqrt{k}}$$

which is imaginary for $k = -1$ (CL-12). Note also, that k and Ω_k have annoyingly opposite signs, and so positive/negative curvature k corresponds to negative/positive density parameter component Ω_k.

The Friedmann equation as written has 3 free parameters for cosmic present which we can choose to be H_0, Ω_0, and Ω_Λ. This means we have the constraint $\Omega_0 + \Omega_{k,0} + \Omega_\Lambda = 1$, and so $\Omega_{k,0} = 1 - \Omega_0 - \Omega_\Lambda$, and so $\Omega_{k,0}$ follows if all other density parameters are known by assumption or a fit to data. Calabrese et al. (2025, p. 45) give $\Omega_{k,0} = 0.0019(15)$ consistent with 0 within of order 1σ, and so consistent with flat space.

Assuming $\Omega_k = 16 \times 10^{-4}$, what is the approximate curvature radius and how does that compare with the radius of the observable universe according to the Λ-CDM model 14.25 Gpc which must be approximately true whatever the correct universe model is (Wikipedia: Observable universe).

a) 107 Gpc; large. b) 1070 Gpc; large. c) 107 Gpc; small. d) 1070 Gpc; small.
e) 0.107 Gpc; small.

SUGGESTED ANSWER: (a) Behold:

$$a_0 = \frac{(4.2827\ldots\,\text{Gpc})h_{70}}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\text{Gpc})h_{70}}{4 \times 10^{-2}} \approx 107\text{Gpc}\ .$$

Wrong answers:

b) You've divided by 0.004.

Redaction: Jeffery, 2008jan01

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004 qmult 00150 1 1 2 easy memory: proper distance to the antipodal point

2. For a positive curvature space (i.e., $k = 1$ space with a_0 being dimension curvature radius), the proper distance to the antipodal point (i.e., the antipode of the observer) according to the Robertson-Walker metric formulation at cosmic present is

   a) a_0. b) πa_0. c) $2\pi a_0$. d) $a_0/2$. e) $a_0/4$.

SUGGESTED ANSWER: (b) The Robertson-Walker metric is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] ,$$

where $ds^2 = d\tau^2$ is the spacetime interval squared (CL-10; Wikipedia: Spacetime: Spacetime interval) and also the proper time differential squared $d\tau^2$ in the convention adopted here (CL-10). The $a(t)$ is the physical curvature radius and r is the conventional dimensionless comoving coordinate and t is cosmic time. The r coordinate is proportional to tangential proper distance at any time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. The χ is proportional to the radial proper distance at any time. Note,

$$r = \begin{cases} \sin(\chi) & \text{for } k = 1 \text{ (positive curvature space);} \\ \chi & \text{for } k = 0 \text{ (flat space);} \\ \sinh(\chi) & \text{for } k = -1 \text{ (negative curvature space)} \end{cases}$$

and

$$dr = \begin{cases} \cos(\chi)\,d\chi = \sqrt{1-\sin^2(\chi)}\,d\chi = \sqrt{1-r^2}\,d\chi & \text{for } k = 1 \text{ (positive curvature space);} \\ d\chi & \text{for } k = 0 \text{ (flat space);} \\ \cosh(\chi)\,d\chi = \sqrt{1+\sinh^2(\chi)}\,d\chi = \sqrt{1+r^2}\,d\chi & \text{for } k = -1 \text{ (negative curvature space),} \end{cases}$$

where we have used the hyperbolic identity $\cosh^2(\chi) - \sinh^2(\chi) = 1$ (Wikipedia: Hyperbolic functions: Useful relations). We now find

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} .$$

Note, if $dt = 0$, then ds corresponds to a proper distance (i.e., a spacetime interval that is measurable at one instant in cosmic time with a ruler).

There are two ways of seeing for positive curvature space (i.e., $k = 1$ space) that the antipodal point is at proper distance πa_0 at cosmic present. First, any circumference about the origin is perpendicular to a radius from the origin, and thus can be calculated from the tangential proper distance for $d\phi = 0$. One obtains $2\pi r a_0 = 2\pi a_0 \sin(\chi)$. At the antipodal point all the radii converge, and so the circumference there is 0. Thus $\chi = \pi$.

Second, the surface area of a 2-sphere (which is just an ordinary sphere: see Wikipedia: n-sphere) surrounding the origin $\pi a_0^2 r^2 = \pi a_0^2 \sin^2(\chi)$ goes to zero when $\sin(\pi) = 0$. Thus, $\chi = \pi$ must give the antipodal point from the origin.

By either way, πa_0 is the proper distance to the origin at cosmic present.

Wrong answers:

a) A nonsense answer.

Redaction: Jeffery, 2008jan01

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004 qmult 00180 1 1 4 easy memory: geodesic is a stationary path

3. A geodesic is a ____________ between two points in a general geometry. It is not in general a global minimum path nor a global maximum ____________. However, a sufficiently small segment of a geodesic is always has the shortest distance between the endpoints in that segment.

   a) non-stationary path b) straight line c) great circle d) stationary path
   e) small circle

SUGGESTED ANSWER: (d)

Wrong answers:
a) A nonsense answer.

Redaction: Jeffery, 2008jan01

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004 qmult 00200 1 1 3 easy memory: general metric

4. The spacetime interval squared (which in relativity is also called the metric) in general is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\mu$$

where $g_{\mu\nu}$ is the ____________ or sometimes just the metric in another meaning of the term. Note, Einstein summation on repeated indices is used.

a) Minkowski tensor b) geodesic c) metric tensor d) gravity tensor
e) stress-energy tensor

SUGGESTED ANSWER: (c) Wikipedia makes it clear that in pure differential geometry the metric is the metric tensor (Wikipedia: Metric tensor), but in relativity the metric can also be the (differential) spacetime interval squared (Wikipedia: Friedmann-Lemaire-Robertson-Walker metric: Concept).

Wrong answers:
a) This is a special case which is usually called the Minkowski metric.

Redaction: Jeffery, 2008jan01

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004 qmult 00210 1 1 3 easy memory: Minkowski metric tensor tests

5. The ____________ is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(CL-24). Some authors define ____________ with an overall negative sign compared to the definition above.

a) Robertson Walker metric tensor b) geodesic tensor c) Minkowski metric tensor
d) gravity tensor e) stress-energy tensor

SUGGESTED ANSWER: (c)

Wrong answers:
a) As Lurch would say AAAaargh.

Redaction: Jeffery, 2008jan01

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004 qmult 00220 1 4 5 easy deducto-memory: Robertson-Walker metric identified

6. "Let's play *Jeopardy*! For \$100, the answer is:

$$ds^2 = c^2\, dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right] .$$

What is the ____________ metric, Alex?

a) Einstein-Hilbert b) de-Sitter-Schwarzschild c) Eddington-Lemaître
d) Milne-McCrea e) Robertson-Walker

SUGGESTED ANSWER: (e)

Wrong answers:
a) As Lurch would say AAAARGH.
c) Alexander Friedmann and Georges Lemaître independently derived the Robertson-Walker metric in the 1920s and it is sometimes called the Friedmann-Lemaître-Robertson-Walker

metric (FLRM metric), but that is too longwinded to say. Robertson and Walker in the 1930s generalized the derivation.

Redaction: Jeffery, 2008jan01

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001 qmult 00240 1 1 3 easy memory: radial and transverse proper distances

7. The Robertson-Walker metric is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] ,$$

where $ds^2 = d\tau^2$ is the spacetime interval squared (CL-10; Wikipedia: Spacetime; Spacetime interval) and also the proper time differential squared $d\tau^2$ in the convention adopted here (CL-10). The $a(t)$ is the physical curvature radius and r is the conventional dimensionless comoving coordinate and t is cosmic time. The r coordinate is proportional to tangential proper distance at any cosmic time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. The χ is proportional to the radial proper distance at any time. Note

$$r = \begin{cases} \sin(\chi) & \text{for } k = 1 \text{ (positive curvature space);} \\ \chi & \text{for } k = 0 \text{ (flat space);} \\ \sinh(\chi) & \text{for } k = -1 \text{ (negative curvature space)} \end{cases}$$

and

$$dr = \begin{cases} \cos(\chi)\,d\chi = \sqrt{1-\sin^2(\chi)}\,d\chi = \sqrt{1-r^2}\,d\chi & \text{for } k = 1 \text{ (positive curvature space);} \\ d\chi & \text{for } k = 0 \text{ (flat space);} \\ \cosh(\chi)\,d\chi = \sqrt{1+\sinh^2(\chi)}\,d\chi = \sqrt{1+r^2}\,d\chi & \text{for } k = -1 \text{ (negative curvature space),} \end{cases}$$

where we have used the hyperbolic function identity $\cosh^2 - \sinh^2 = 1$ (Wikipedia: Hyperbolic functions: Useful relations). We now find

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} .$$

The differential radial proper distance is

$$dD_{\text{proper,radial}} = a(t)\left(\frac{dr}{\sqrt{1-kr^2}}\right) = a(t)\,d\chi .$$

The differential transverse proper distance $dD_{\text{proper,transverse}}$ is:

a) $4\pi[a(t)r]^2$. b) $a(t)r$. c) $a(t)r\sqrt{d\theta^2 + \sin^2\theta\,d\phi^2}$. d) $\pi a(t)$. e) $2\pi a(t)$.

SUGGESTED ANSWER: (c)

Wrong answers:

a) A nonsense answer.

Redaction: Jeffery, 2008jan01

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004 qfull 00350 1 3 0 easy math: some of the geometry of Robertson-Walker metric

8. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] ,$$

where ds is the (differential) spacetime interval (also equal to $d\tau$ the differential proper time in the present convention: CL-10), dt is the differential cosmic time interval, the coordinates are for an arbitrary origin in the homogeneous and isotropic spacetime of the Robertson-Walker metric, θ and ϕ are the ordinary polar coordinates, r a dimensionless (i.e., unitless) comoving coordinate for the tangential direction, t is cosmic time, $a(t)$ is the cosmic scale factor with dimensions of length, and $k = 1$ for hyperspherical space (i.e., positive curvature space with the geometry of the surface of a 3-sphere which is sphere in 4-dimensional Euclidean space: see Wikipedia: n-sphere), $k = 0$ for Euclidean space (i.e., flat space), and $k = -1$ for hyperbolical space (i.e., negative curvature space). Note, an ordinary sphere is a 2-sphere in math jargon. For $ds^2 > 0$ / $ds^2 = 0$ / $ds^2 < 0$, the interval is time-like / light-like (or null) / space-like (CL-10; Carroll-9).

For non-flat space, the Robertson implies a definite physical scale for a_0:

$$a_0 = \frac{c/H_0}{\sqrt{k\Omega_k}} = \frac{c/H_0}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\mathrm{Gpc})/h_{70}}{\sqrt{|\Omega_k|}} = \frac{(13.968\ldots\,\mathrm{Gly})/h_{70}}{\sqrt{|\Omega_k|}}$$

(where $h_{70} = H_0/[70\,(\mathrm{km/s})/\mathrm{Mpc}]$) which can be called the curvature radius of the universe. Note, formally the Gaussian curvature radius is defined

$$R_G = \frac{a_0}{\sqrt{k}}$$

which is imaginary for $k = -1$ (CL-12).

The Friedmann equation as written has 3 free parameters for cosmic present which we can choose to be H_0, Ω_0, and Ω_Λ. This means we have the constraint $\Omega_0 + \Omega_{k,0} + \Omega_\Lambda = 1$, and so $\Omega_{k,0} = 1 - \Omega_0 - \Omega_\Lambda$, and so $\Omega_{k,0}$ follows if all other density parameters are known by assumption or a fit to data. Calabrese et al. (2025, p. 45) give $\Omega_{k,0} = 0.0019(15)$ consistent with 0 within of order 1σ, and so consistent with flat space. For $k = 0$, there is no physically determined a_0 value and one can set it for convenience: e.g., $a_0 = 1\,\mathrm{Gpc}$ or $a_0 = c/H_0 = [4.2827\ldots)/h_{70}]\,\mathrm{Gpc}$ which is the Hubble length. However, for flat universe models, one usually makes $a(t)$ dimensionless and sets $a_0 = 1$. In these models, the comoving coordinates are dimensioned, given length units (e.g., Gpc), and are usually chosen to the be proper distances at cosmic present.

The r coordinate is the tangential comoving coordinate since it is proportional to tangential proper distance at any time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. The χ is proportional to the radial proper distance at any time.

The radial proper distance $D_{\mathrm{P,radial}}$ is given by

$$D_{\mathrm{P,radial}} = a(t)\begin{cases} \chi & \text{for } k = 1 \text{ with } \chi \in [0, \pi]; \\ \chi & \text{for } k = 0 \text{ with } \chi \in [0, \infty]; \\ \chi & \text{for } k = -1 \text{ with } \chi \in [0, \infty], \end{cases}$$

The r coordinate is related to the χ by

$$r = \begin{cases} \sin(\chi) & \text{for } k = 1 \text{ (positive curvature space);} \\ \chi & \text{for } k = 0 \text{ (flat space);} \\ \sinh(\chi) & \text{for } k = -1 \text{ (negative curvature space)} \end{cases}$$

and

$$dr = \begin{cases} \cos(\chi)\,d\chi = \sqrt{1 - \sin^2(\chi)}\,d\chi = \sqrt{1 - r^2}\,d\chi & \text{for } k = 1 \text{ (positive curvature space);} \\ d\chi & \text{for } k = 0 \text{ (flat space);} \\ \cosh(\chi)\,d\chi = \sqrt{1 + \sinh^2(\chi)}\,d\chi = \sqrt{1 + r^2}\,d\chi & \text{for } k = -1 \text{ (negative curvature space),} \end{cases}$$

where we have used the hyperbolic function identity $\cosh^2(\chi) - \sinh^2(\chi) = 1$ (Wikipedia: Hyperbolic functions: Useful relations). We now find

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}}\ .$$

The transverse proper distance $D_{\mathrm{P,transverse}}$ is given by

$$D_{\mathrm{P,transverse}} = a(t)r\sqrt{d\theta^2 + \sin^2\theta\,d\phi^2}\ .$$

The general differential proper distance $D_{\rm P}$ formula is

$$dD_{\rm P}^2 = a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right]$$
$$= a(t)^2\left[d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \ .$$

NOTE: There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c. The parts can be done independently, and so do not stop if you cannot do a part.

a) For the $k = 1$ case, what directions from the origin do radial geodesics lead to the antipodal point (i.e., the antipode)? How far in proper distance is it from the origin to the antipodal point along a radial geodesic? How far in proper distance to make the geodesic round trip from origin to origin?

b) What is the general formula for circumference C in proper distance for a circle at r in terms of r and χ? Sketch a plot of C as a function of χ for all cases of k.

c) Integrate over all solid angle to find the proper surface area A of the curved-space 2-sphere surrounding the origin at comoving coordinate r. This area is analogous to the circumference of a small circle on a ordinary sphere at polar angle θ. Sketch a plot of A as a function of χ for all cases of k. **HINT:** The integration is really easy and $d\theta^2 + \sin^2\theta\,d\phi^2$ is a differential path distance created using the differential Pythagorean theorem and not a differential piece of solid angle.

d) The differential volume for the sphere is $dV = A(\chi)a\,d\chi$. For all k, determine explicit formulae for $V(\chi)$ small χ and then for general χ. What is the total volume for space for $k = 1$? **HINT:** You will need the identities $\sin^2(x) = (1/2)[1-\cos(2x)]$ and $\sinh^2(x) = (1/2)[\cosh(2x)-1]$.

SUGGESTED ANSWER:

a) Radial geodesics from the origin lead to the antipodal point for all directions: all roads lead to Rome. This behavior is analogous to following meridians from the pole of an ordinary sphere (i.e., a 2-sphere: see Wikipedia: n-sphere) to the antipodal pole. The proper distance along a geodesic from the origin is

$$D_{\rm P} = \begin{cases} a\chi & \text{in general for } \chi \in [0,\pi]; \\ \pi a & \text{for } \chi = \pi; \\ 2\pi a & \text{for a round trip from the origin to the origin.} \end{cases}$$

So the proper distance to the antipodal point is πa and the proper distance for the round trip is $2\pi a$. These results are analogous to the distances on a ordinary sphere (i.e., a 2-sphere).

b) Behold:

$$C = 2\pi a(t)r = 2\pi a(t)\begin{cases} \sin(\chi) & k = 1 \text{ with } \chi \in [0,\pi]; \\ \chi & k = 0 \text{ with } \chi \in [0,\infty]; \\ \sinh(\chi) & k = -1 \text{ with } \chi \in [0,\infty]. \end{cases}$$

You will have imagine the plot. However, all three curves will grow linearly for small χ with χ starting from $\chi = 0$ with slope $2\pi a(t)$. The $k = 1$ curve will become sinusoidal and reach a maximum a $\chi = \pi/2$ and then fall sinusoidally to zero at the antipodal point where $\chi = \pi$. The $k = 0$ curve just continues as a straight line with χ to $\chi = \infty$. The $k = -1$ curve steepens into an exponential with exponential factor $e^\chi/2$.

c) The differential piece of solid angle is $d\theta\,\sin\theta\,d\phi$ which integrates immediately to 4π just as in ordinary space. The differential piece of proper area is $(ar)^2 d\theta\,\sin\theta\,d\phi$. Therefore, the surface area of a sphere surrounding the origin is

$$A(r) = A(\chi) = 4\pi(ar)^2 = 4\pi a^2\begin{cases} \sin^2(\chi) & k = 1 \text{ with } \chi \in [0,\pi]; \\ \chi^2 & k = 0 \text{ with } \chi \in [0,\infty]; \\ \sinh^2(\chi) & k = -1 \text{ with } \chi \in [0,\infty]. \end{cases}$$

You will have imagine the plot. However, all three curves will initially grow quadratically with χ (i.e., as parabola χ^2) starting from $\chi = 0$. The $k = 1$ curve will become sinusoidal squared and reach a maximum a $\chi = \pi/2$ and then fall squared sinusoidally to zero at the antipodal point where $\chi = \pi$. Near the two endpoints, its shape is parabolic. The $k = 0$ curve just continues as parabola χ^2 to $\chi = \infty$. The $k = -1$ curve steepens into an exponential with exponential factor $e^{2\chi}/4$.

d) For small χ,

$$V(\chi << 1) = \int_0^\chi A(\chi')a\,d\chi' = 4\pi a^3 \int_0^\chi \chi'^2\,d\chi' = \frac{4\pi}{3}(a\chi)^3$$

which is just what you would get for flat space for all χ. For general χ,

$$V(\chi) = \int_0^\chi A(\chi')a\,d\chi' = 4\pi a^3 \int_0^\chi d\chi' \begin{cases} \sin^2(\chi') & k=1 \text{ with } \chi\in[0,\pi]; \\ \chi'^2 & k=0 \text{ with } \chi\in[0,\infty]; \\ \sinh^2(\chi') & k=-1 \text{ with } \chi\in[0,\infty]; \end{cases}$$

$$= 4\pi a^3 \int_0^\chi d\chi' \begin{cases} \frac{1}{2}[1-\cos(2\chi')] & k=1 \text{ with } \chi\in[0,\pi]; \\ \chi^2 & k=0 \text{ with } \chi\in[0,\infty]; \\ \frac{1}{2}[\cosh(2\chi')-1] & k=-1 \text{ with } \chi\in[0,\infty]; \end{cases}$$

$$= 4\pi a^3 \begin{cases} \frac{1}{2}\left[\chi - \frac{1}{2}\sin(2\chi')\right] & k=1 \text{ with } \chi\in[0,\pi]; \\ \frac{1}{3}\chi^3 & k=0 \text{ with } \chi\in[0,\infty]; \\ \frac{1}{2}\left[\frac{1}{2}\sinh(2\chi')-\chi\right] & k=-1 \text{ with } \chi\in[0,\infty]; \end{cases}$$

$$= 4\pi a^3 \begin{cases} \frac{\pi}{2} \\ \frac{\pi^3}{3} \\ \frac{1}{2}\left[\frac{1}{2}\sinh(2\pi)-\pi\right] \end{cases} = \begin{cases} 2\pi^2 a^3 = (19.7392\ldots)a^3 & k=1 \text{ with } \chi=\pi; \\ \frac{4\pi^4}{3}a^3 = (129.878788\ldots)a^3 & k=0 \text{ with } \chi=\pi; \\ (821.406\ldots)a^3 & k=-1 \text{ with } \chi=\pi. \end{cases}$$

Note, the derivative for the $k = 1$ solution is just $\sin^2(\chi) \geq 0$ which implies volume grows monotically with χ and which has stationary points $\chi = 0$ (which is a minimum for the allowed range) and $\chi = \pi$ (which is a maximum for the allowed range). However, if you extended the range in both directions, then both points would actually be inflection points since $d\sin^2(\chi)/d\chi = 2\sin(\chi)\cos(\chi) = 0$ for all $\chi = n\pi$ where n is an integer. Clearly, for the hyperspherical space (i.e., the $k = 1$ space), the total volume is $2\pi^2a^3$. The Fortran-95 code for the numerical calculation is:

```
      pi=acos(-1.0_np)
      pi=3.14159265358979323846264338327950288419716939937510
           23456789a123456789b123456789c123456789d123456789e1
      v1=2.0_np*pi**2
      v0=(4.0_np/3.0_np)*pi**4
      vn=4.0_np*pi*0.5_np*(0.5_np*sinh(2.0_np*pi)-pi)
      print*,'v1,v0,vn'
      print*,v1,v0,vn
! 19.739208802178717239 129.87878804533658300 821.40618335325637295
```

Redaction: Jeffery, 2018jan01

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004 qfull 00400 1 3 0 easy math: prove Hubble's law from the RW metric

9. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] .$$

Note, r is the radial comoving coordinate chosen so that r is proportional to proper distance in the transverse direction (i.e., the direction perpendicular to the radial direction).

Prove Hubble's law in general form from the Robertson-Walker metric: i.e., prove

$$v_{\rm R} = HD_{\rm P} \ ,$$

where $v_{\rm R} = \dot{D}_{\rm P}$ is the recession velocity, $H = \dot{a}/a$ is the Hubble parameter, and $D_{\rm P}$ is proper (radial) distance. Note, proper distance is distance that can be measured at one instant in cosmic time using a ruler: i.e., with $dt = 0$, proper distance is

$$D_{\rm P} = \int \sqrt{-ds^2} \ .$$

The general form of Hubble's law is an exact result, but alas containing two quantities that are not direct observables, $v_{\rm R}$ and $D_{\rm P}$, except asymptotically as $z \to 0$ or, in other words, in the limit where the 1st-order-in-small-z formulae can be treated as exact. The observational Hubble's law is

$$v_{\rm red} = H_0 D_{\rm P,1st} \ ,$$

where $v_{\rm red} = zc$ is redshift velocity (a direct observable) and $D_{\rm P,1st}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance (which are direct observables). The observational Hubble's law is very plausible a priori, but a formal proof is left to a later problem.

SUGGESTED ANSWER:

For a proper distance along a radial direction (for which $d\theta = 0$ and $d\phi = 0$), we have

$$D_{\rm P} = \int \sqrt{-ds^2} = a(t) \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} = a(t)\chi(r) \ ,$$

where $\chi(r)$ is just the displayed integral which is, in fact, time independent. Thus

$$v_{\rm R} = \dot{D}_{\rm P} = \dot{a}\chi(r) \ .$$

Dividing the second by the first expression and rearranging, we get

$$v_{\rm R} = \frac{\dot{a}}{a} D_{\rm P} = HD_{\rm P} \ , \qquad \text{or, compactly,} \qquad v_{\rm R} = HD_{\rm P} \qquad \text{QED.}$$

Redaction: Jeffery, 2018jan01

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004 qfull 00500 1 3 0 easy math: cosmological time dilation and cosmological redshift

10. The Robertson-Walker metric (RW metric) in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \right] \ .$$

Note, r is the radial comoving coordinate chosen so that r is proportional to proper distance in the transverse direction (i.e., the direction perpendicular to the radial direction).

NOTE: There are parts a,b,c,d. The parts can be done independently, so don't stop if you can't do one.

a) For a light-like interval $ds^2 = 0$ for a light source at comoving coordinate r distant from an observer, starting from the RW metric prove that

$$\chi(r) = \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} = \int_t^{t_0} \frac{c\,dt'}{a(t')} = \eta(t_0) - \eta(t) = \Delta\eta \ ,$$

where $\chi(r)$ is just r integral and $\eta(t)$ is conformal time (which has differential definition $d\eta = c\,dt/a(t)$). Note, $\Delta\eta$ is the conformal time change for light to travel from the light source at comoving coordinate r to the observer. What does the proven result imply about $\Delta\eta$ relative to t and t_0?

b) For light signals coming from comoving coordinate r to the observer, prove the differential cosmological time-dilation formula (CL-16,19):

$$\frac{dt_0}{a_0} = \frac{dt}{a(t)} \qquad \text{or} \qquad \frac{dt_0}{dt} = \frac{a_0}{a(t)} \ ,$$

where t is the cosmic time of emission, t_0 is the cosmic time of observation (i.e., the cosmic present), $a_0 = a(t_0)$, dt is the differential time between two emitted light signals, and dt_0 is the differential time between the corresponding two observed signals. **HINT:** For fixed r, the comoving frame coordinate $\chi(r)$ from part (a) is independent of both t and t_0 with each of these depending only on the other. Take the derivative $\partial\chi(r)/\partial t$ and make use of the chain rule.

c) Prove the cosmological redshift formula $1 + z = a_0/a(t)$. **HINT:** You will have to use the part (b) answer to relate frequency of emission to frequency of reception assuming the inverse frequencies are adequately approximated as differential time periods. Recall, observationally $z = (\lambda_0 - \lambda)/\lambda$.

d) The cosmological redshift formula is a very useful connecting the direct observable cosmological redshift z and the scaling up of the universe since a light signal was emitted $a_0/a(t)$. Why can't it be used to directly determine the cosmic scale factor function of time $a(t)$?

SUGGESTED ANSWER:

a) Behold:

$$1)\quad \frac{dr^2}{1-kr^2} = \frac{c^2\,dt^2}{a(t)^2} \qquad 2)\quad \pm\frac{dr}{\sqrt{1-kr^2}} = \frac{c\,dt}{a(t)} \qquad 3)\quad \pm[\chi(0)-\chi(r)] = \pm\int_r^0 \frac{dr'}{\sqrt{1-kr'^2}} = \int_t^{t_0} \frac{c\,dt'}{a(t')}$$

$$4)\quad -[0-\chi(r)] = -\int_r^0 \frac{dr'}{\sqrt{1-kr'^2}} \qquad 5)\quad \chi(r) = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_t^{t_0} \frac{c\,dt'}{a(t')} = \eta(t_0) - \Delta(t) = \Delta\eta \ ,$$

where we have taken the arrival point (i.e., $\chi(0) = 0$ where the observer is located) to be the origin and chosen the physically correct root for this arrival point since the conformal time integral must be positive. The proven result implies that $\Delta\eta$ is independent of t and t_0. No matter when a light signal starts or arrives from the light source, it always takes the same conformal time change: i.e., $\Delta\eta = \chi(r)$.

b) Behold:

$$1)\quad \frac{\partial\chi(r)}{\partial t} = 0 = \frac{d\eta}{dt_0}\frac{dt_0}{dt} - \frac{d\eta}{dt} \qquad 2)\quad 0 = \frac{c}{a(t_0)}\frac{dt_0}{dt} - \frac{c}{a(t)} \qquad 3)\quad \frac{dt_0}{a_0} = \frac{dt}{a(t)} \qquad 4)\quad \frac{dt_0}{dt} = \frac{a_0}{a(t)} \ .$$

c) Behold:

$$1)\quad \frac{dt_0}{a_0} = \frac{dt}{a(t)} \qquad 2)\quad \frac{1}{\nu_0 a_0} = \frac{1}{\nu a(t)} \qquad 3)\quad \frac{\lambda_0}{a_0} = \frac{\lambda}{a(t)} \qquad 4)\quad \frac{\lambda_0}{\lambda} = \frac{a_0}{a(t)}$$

$$5)\quad z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{a_0}{a(t)} - 1 \qquad 6)\quad 1 + z = \frac{a_0}{a(t)} \qquad \text{QED.}$$

d) The cosmic time of emission t is not a direct observable. It would be great if galaxies had clock faces showing cosmic time, but they don't.

Redaction: Jeffery, 2018jan01

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004 qfull 00610 1 3 0 easy math: Robertson-Walker metric and observables

11. The basic Λ-CDM model has its cosmic scale factor $a(t)$ fully specified via the Friedmann equation (FE) by the Hubble constant H_0 and three density parameter constants: i.e., $\Omega_{\rm r,0}$ ("radiation": i.e., all extremely relativistic particles including photons and extremely relativistic neutrinos), $\Omega_{\rm m,0}$ ("matter": i.e., all rest mass neglecting classical kinetic energy mass-energy as negligible), and Ω_Λ (cosmological constant or constant dark energy). Obtaining the parameters is a major observational goal. In principle, only 2 of the density parameter constants are independent, but observational uncertainties make obtaining all 3 somewhat independently a useful goal.

If the FE model is not flat, the Friedmann equation (in its derivation from general relativity) plus Robertson-Walker metric tells us that the physical scale of the FE models at cosmic present t_0 is given by

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_0 - 1|}} = \frac{c/H_0}{\sqrt{|\Omega_{k,0}|}} = \frac{(4.2827\ldots\,\mathrm{Gpc})/h_{70}}{\sqrt{|\Omega_{k,0}|}} = \frac{(13.968\ldots\,\mathrm{Gly})/h_{70}}{\sqrt{|\Omega_{k,0}|}} \ ,$$

where a_0 is curvature radius (but formally $a_0/\sqrt{k}$ is the Gaussian curvature radius which is imaginary for $k = -1$: CL-12), Ω_0 is the sum of all density parameters, except $\Omega_{k,0}$, and $h_{70} = H_0/[70\,(\mathrm{km/s})/\mathrm{Mpc}]$ is the reduced Hubble constant which must be 1 to within a few percent. If the FE model is flat, there is no physical scale for the model and a_0 can be chosen arbitrarily or set to dimensionless 1 in which case the comoving distances r have length units and are equal to the proper distances at cosmic present. In all cases, the proper distance to an object at comoving distance r is

$$D_{\mathrm{P}} = a_0 \chi(r) = a_0 \int_0^r \frac{dr}{\sqrt{1 - kr^2}} \ ,$$

where r is the radial comoving coordinate (chosen so that r is proportional to proper distance in the transverse direction (i.e., the direction perpendicular to the radial direction), $\chi(r)$ is the alternative comoving coordinate proportional to proper distance, and $k = 1$ for hyperspherical space, $k = 0$ for Euclidean space (i.e., flat space in which case $\chi(r) = r$ and whether you assign units of length to a_0 or $\chi(r) = r$ is arbitrary), and $k = -1$ for hyperbolical space. The variable k is called the curvature (Li-24).

One way to test a FE model or fit it to observations is to plot an observable cosmic distance measure for objects versus their cosmological redshifts z (which are the only easily obtained direct observables) and then compare to the theoretical cosmic distance measure plotted as a function of z. The two best known observable cosmic distance measures (other than cosmological redshift z itself) are the luminosity distance D_{L} and the angular diameter distance D_{A} both of which have explicit dependence on z, but also depend on z via the comoving coordinate $r(z)$ whose z dependence due to the evolution of the cosmic scale factor $a(t)$ with cosmic time not an intrinsic dependence.

NOTE: There are parts a,b,c,d. On exams, omit part d. Use minimal words. Some of the parts can be done independently, and so not stop if you cannot do one.

a) Recall the Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right] \ .$$

For a light signal traveling from a source at comoving coordinate r, time t, and cosmological redshift z to the origin at cosmic present t_0 (i.e., us) along a radial path, derive an equation from the Robertson-Walker metric relating $\chi(r)$ to conformal time $\eta(t)$ (whose differential definition is $d\eta = c\,dt/a(t)$). The left-hand side should depend only on parameters r and k and the right-hand side only on t and t_0 through the expression $\Delta\eta = \eta(t_0) - \eta(t)$. **HINT:** The interval is light-like for a light signal: i.e., $ds = 0$.

b) Formal expressions for r, t, and lookback time t_{LB} for a light signal are, respectively,

$$r = r\,[\chi(\Delta\eta)] = r\,\{\chi[\eta(t_0) - \eta(t)]\} = r\,[\chi(z)] \ , \qquad t = t(a) = t\left(\frac{a_0}{1+z}\right) = t(z) \ ,$$

and

$$t_{\mathrm{LB}} = -\Delta t = -[t(a) - t_0] = t_0 - t(z) \ ,$$

where we have used the cosmological redshift formula

$$1 + z = \frac{a_0}{a(t)} \qquad \text{and} \qquad r = \begin{cases} \sin(\chi) & \text{for } k = 1 \text{ (positive curvature space);} \\ \chi & \text{for } k = 0 \text{ (flat space);} \\ \sinh(\chi) & \text{for } k = -1 \text{ (negative curvature space).} \end{cases}$$

We can only obtain theoretical formulae cosmic distance measures (i.e., r, χ, t_{LB}, and others, e.g., proper distance D_{P}, luminosity distance D_{L}, angular-diameter distance D_{A}, and recession velocity $v_{\mathrm{R}}(z)$) in special cases of the cosmic scale factor $a(t)$ and not always all of them for specific special cases. In other cases, one needs numerical solutions. The simplest special case that allows theoretical formulae for all of the standard cosmic distance measures is the de-Sitter universe. For the de-Sitter universe (with $k = 0$ for its most standard form),

$$a(t) = a_0 e^{H_0 \Delta t} ,$$

where the Hubble constant $H_0 = \sqrt{\Lambda/3}$ is time-independent and $\Delta t = t - t_0$ is the already-specified time relative to cosmic present.

For the de-Sitter universe, determine in order the explicit formulae for $\Delta t(z)$, $t_{\rm LB}(z)$, $\chi(z)$, radial proper distance $D_{\rm P}(z)$, and recession velocity $v_{\rm R}(z)$.

What is odd about lookback time $t_{\rm LB}$ and $D_{\rm P}$ as $z \to \infty$ relative to the case of a cosmological model with a point origin (AKA Big Bang singularity)?

c) What is the explicit expression for the deceleration parameter $q_0 = -(\ddot{a}_0 a_0/\dot{a}_0^2)$ for the de Sitter universe?

d) The formal expressions for the standard cosmological distance measures (expressed in observational forms if they exist and are distinct from theoretical forms and then in the theoretical forms) are as follows:

Cosmological redshift: $z = \dfrac{\lambda_0 - \lambda}{\lambda} = \dfrac{a_0}{a(t)} - 1$

Lookback time: $t_{\rm LB} = t_0 - t(a) = -\Delta t$

Comoving coordinate r: $r = r\left[\chi(z)\right]$

Comoving coordinate χ: $\chi(z) = \displaystyle\int_t^{t_0} \frac{c\,dt'}{a(t')} = \eta(t_0) - \eta(t) = \Delta\eta$

Radial proper distance: $D_{\rm P} = a_0 \chi(z)$

Recessional velocity: $v_{\rm R} = H_0 D_{\rm P}$

Redshift velocity: $v_{\rm red} = zc$

Luminosity distance: $D_{\rm L} = \sqrt{\dfrac{L}{4\pi f}} = a_0 r(1+z)$

Angular diameter distance: $D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1+z)}$

Distance-duality relation: $\dfrac{D_{\rm L}}{D_{\rm A}} = (1+z)^2 ,$

where the distance-duality relation is also called the Etherington reciprocity relation.

Determine special case expressions (if they exist) for the cosmological distance measures above as a functions of z for the de Sitter universe for which recall $k = 0$, and so $r = \chi$. Note, some were already determined in part (b) and some already functions of z. What is odd about $D_{\rm A}$ as z goes to infinity in the case of $k = 0$?

SUGGESTED ANSWER:

a) Behold:

$$1) \quad ds^2 = c^2\,dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \qquad 2) \quad \pm\, a(t)\left(\frac{dr}{\sqrt{1-kr^2}}\right) = c\,dt$$

$$3) \quad -\left[\chi(0) - \chi(r)\right] = -\int_r^0 \frac{dr}{\sqrt{1-kr^2}} = \int_t^{t_0} \frac{c\,dt'}{a(t')} \qquad 4) \quad \chi(r) = \int_0^r \frac{dr}{\sqrt{1-kr^2}} = \eta(t_0) - \eta(t)$$

$$5) \quad \chi(r) = \Delta\eta = \eta(t_0) - \eta(t) .$$

b) Behold:

$$1) \quad a(t) = a_0 e^{H_0 \Delta t} \qquad 2) \quad \Delta t = \frac{1}{H_0}\ln\left(\frac{a}{a_0}\right) = -\frac{1}{H_0}\ln(1+z)$$

$$3) \quad t_{\rm LB} = -\Delta t = \frac{1}{H_0}\ln(1+z)\ .$$

There is more than one way to find $\chi(z)$. The most general way is using the Friedmann equation for density parameters that depend only inverse powers of $x = a/a_0$: i.e.,

$$1) \quad \left(\frac{\dot a}{a}\right) = H_0\sqrt{\sum_p \Omega_{p,0}x^{-p}} \qquad 2) \quad dt = \frac{1}{H_0}\left(\frac{dx}{x\sqrt{\sum_p \Omega_{p,0}x^{-p}}}\right)$$

In the present case, there is only $\Omega_\Lambda = \Omega_{0,0} = 1$. Thus, we find

$$\chi = \int_t^{t_0}\frac{c\,dt'}{a(t')} = \frac{c}{H_0a_0}\int_x^1\frac{dx}{x^2\sqrt{\sum_p \Omega_{p,0}x^{-p}}} = \frac{c}{H_0a_0}\int_0^z\frac{dz}{\sqrt{\Omega_\Lambda(1+z)^0}} = \frac{zc}{H_0a_0}\ .$$

where we have used

$$1) \quad x = \frac{1}{1+z} \qquad 2) \quad dx = -\frac{dz}{(1+z)^2} = -x^2\,dz\ ,$$

Less smart ways to determine $\chi(z)$ are

$$\chi = \int_{\Delta t}^0\frac{c\,d\Delta t'}{a} = \int_a^{a_0}\frac{c\,da'}{a'\dot a'} = \frac{c}{H_0}\int_a^{a_0}\frac{da'}{a'^2} = -\frac{c}{H_0}\left(\frac{1}{a_0}-\frac{1}{a}\right) = -\frac{c}{H_0a_0}\left(1-\frac{a_0}{a}\right) = \frac{zc}{H_0a_0}\ .$$

and (using $\Delta t = t - t_0$, $\Delta t_0 = 0$, and $d\Delta t = dt$)

$$\chi = \frac{c}{a_0}\int_{\Delta t}^0 e^{-H_0\Delta t'}\,d\Delta t' = -\frac{c}{H_0a_0}\left(1-e^{-H_0\Delta t}\right) = -\frac{c}{H_0a_0}\left(1-\frac{a_0}{a}\right) = \frac{zc}{H_0a_0}\ .$$

The end result is always

$$\chi = \frac{zc}{H_0a_0}\ .$$

Thus,

$$D_{\rm P} = a_0\chi(z) = \frac{zc}{H_0} \qquad \text{and} \qquad v_{\rm R} = H_0D_{\rm P} = zc = v_{\rm red}\ .$$

In this special case, the recession velocity equals the redshift velocity defined by $v_{\rm red} = zc$.

For cosmological models with a point origin, lookback time $t_{\rm LB}$ and $D_{\rm P}$ go to finite values as $z \to \infty$. However, for the de Sitter universe $t_{\rm LB}$ and $D_{\rm P}$ go to infinity as $z \to \infty$ as the formulae above show explicitly.

c) Behold:

$$q_0 = -\frac{\ddot a_0a_0}{\dot a_0^2} = -\frac{H_0^2}{H_0^2} = -1 \qquad \text{or, compactly,} \qquad q_0 = -1\ .$$

The deceleration parameter is negative because for de Sitter universe, the expansion is positively accelerating.

d) Behold:

Cosmological redshift: $\quad z = \dfrac{\lambda_0-\lambda}{\lambda} = \dfrac{a_0}{a(t)} - 1$

Lookback time: $\quad t_{\rm LB} = t_0 - t(a) = -\Delta t = \dfrac{1}{H_0}\ln(1+z)$

Comoving coordinate r: $\quad r = \chi(z) = \dfrac{zc}{H_0a_0}$

Comoving coordinate χ: $\quad \chi(z) = \dfrac{zc}{H_0a_0}$

Radial proper distance: $\quad D_{\rm P} = a_0\chi(r) = a_0\chi(z) = \dfrac{zc}{H_0}$

Recessional velocity: $v_{\rm R} = H_0 D_{\rm P} = zc$

Redshift velocity: $v_{\rm red} = zc$

Luminosity distance: $D_{\rm L} = \sqrt{\dfrac{L}{4\pi f}} = a_0 r(1+z) = \left(\dfrac{zc}{H_0}\right)(1+z)$

Angular diameter distance: $D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1+z)} = \left(\dfrac{zc}{H_0}\right)\dfrac{1}{(1+z)}$

Distance-duality equation: $\dfrac{D_{\rm L}}{D_{\rm A}} = (1+z)^2$.

The odd thing about $D_{\rm A}$ as $z \to \infty$ for the de Sitter universe is that it goes to a constant c/H_0 which is, in fact, the Hubble length. The proof is

$$\lim_{z\to\infty} D_{\rm A} = \lim_{z\to\infty} \left(\frac{zc}{H_0}\right)\frac{1}{(1+z)} = \frac{c}{H_0} \ .$$

This means the standard ruler goes to a constant angular diameter as z goes to infinity. The constancy, your truly thinks, is mostly because you are seeing the ruler sort of where it was in the past. But note, the luminosity distance continues to increase, and so that the ruler keeps getting fainter if it is also a standard candle. Note also, the angular diameter distance is based on the small angle approximation and that might fail in some way if the angular diameter distance starts getting smaller (meaning the ruler is bigger on the sky) with z as, in fact, it does for the Λ-CDM model.

Redaction: Jeffery, 2018jan01

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004 qfull 00650 1 3 0 easy math: chi(z), conformal time, cosmological redshift

12. The alternative comoving coordinate (i.e., the comoving coordinate proporptional to radial proper distance at any cosmic time) is

$$\chi = \int_t^{t_0} \frac{c\,dt}{a(t)} = \eta(t_0) - \eta_t) \ ,$$

where $\eta(t)$ is conformal time whose differential definition is $d\eta = c\,dt/a(t)$

NOTE: There are parts a,b,c,d,e,f. Omit part d,e,f during exams.

a) Starting from the scaled Friedmann equation form

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\sum_p \Omega_{p,0} x^{-p}\right)$$

(where $x = a/a_0$ and $p \geq 0$) derive without words an integral formula for $\chi(x)$ with all a quantities eliminated, but with one a_0 left. **HINT:** Isolate dt on the left-hand side and multiply both sides by appropriate factors. The integral should have limits x to 1.

b) Transform the the integral formula of part (a) so that we have $\chi(z)$.

c) Assume there is a single nonzero density parameter constant $\Omega_{p,0} = 1$ and derive the exact solution for $\chi(z)$. Note, $p = 2$ is a special case and needs its own derivation.

d) Give the formula for radial proper distance $D_{\rm P}$ with $\chi(z)$ expanded into the integral form. Does $D_{\rm P}$ depend on a_0? Give the formula for $a_0 r$ (where r is the comoving distance coordinate for proper distances perpendicular to the radial direction) for $\chi(z)$ all cases of k with $\chi(z)$ unexpanded. Does $a_0 r$ depend on a_0?

e) For the solution found in part (c), write out the special cases for (i) $p < 2$, $p = 2$, and $p > 2$ (which only looks different from the $p < 2$ case for niceness), (ii) 1st order in small z, (iii) asymptotically large for $p < 2$, $p = 2$, and $p > 2$, (iv) $p > 2$ in the limit that $z \to \infty$, (v) $p = 0$, and (vi) $p = 3$.

f) Expand the solution found in part (b) in small z and integrate to get a series solution for $\chi(z)$ good to 3rd order in small z.

SUGGESTED ANSWER:

a) Behold:

$$1)\quad dt = \frac{1}{H_0}\left(\frac{da}{a\sqrt{\sum_p \Omega_{p,0}x^{-p}}}\right) \qquad 2)\quad \frac{c\,dt}{a} = \frac{c}{H_0}\left(\frac{da}{a^2\sqrt{\sum_p \Omega_{p,0}x^{-p}}}\right) = \frac{c}{H_0}\left(\frac{a_0\,dx}{a_0^2x^2\sqrt{\sum_p \Omega_{p,0}x^{-p}}}\right)$$

$$3)\quad \frac{c\,dt}{a} = \frac{c}{H_0a_0}\left(\frac{dx}{x^2\sqrt{\sum_p \Omega_{p,0}x^{-p}}}\right) \qquad 4)\quad \chi(x) = \frac{c}{H_0a_0}\int_x^1 \frac{d\tilde{x}}{\sqrt{\sum_p \Omega_{p,0}\tilde{x}^{-p+4}}}\ .$$

Note for the set of p consisting of $\{4,3,2\}$ (i.e., set of $4-p$ consisting of $\{0,1,2\}$), an exact solution exists for the integral. This case is the radiation-matter-curvature universe which has some theoretical interest and may even apply to the observable universe (if that has some non-zero curvature) before the phase when dark energy becomes important. There may be some other theoretically interesting cases.

b) Behold:

$$1)\quad \frac{a_0}{a} = 1+z \qquad 2)\quad \frac{1}{x} = 1+z \qquad 3)\quad z = \frac{1}{x}-1 \qquad 4)\quad x = \frac{1}{1+z} \qquad 5)\quad dx = -\frac{dz}{(1+z)^2} = -x^2\,dz\ .$$

$$6)\quad \int_x^1 \frac{d\tilde{x}}{\tilde{x}^2\sqrt{\sum_p \Omega_{p,0}\tilde{x}^{-p}}} = \int_0^z \frac{d\tilde{z}}{\sqrt{\sum_p \Omega_{p,0}(1+\tilde{z})^p}} \qquad 7)\quad \chi(z) = \frac{c}{H_0a_0}\int_0^z \frac{d\tilde{z}}{\sqrt{\sum_p \Omega_{p,0}(1+\tilde{z})^p}}\ .$$

c) For $p \neq 2$,

$$1)\quad \chi(z) = \frac{c}{H_0a_0}\int_0^z \frac{d\tilde{z}}{\sqrt{\Omega_{p,0}(1+\tilde{z})^p}} = \frac{c}{H_0a_0}\frac{1}{\sqrt{\Omega_{p,0}}}\int_0^z (1+\tilde{z})^{-p/2}\,d\tilde{z} = \frac{c}{H_0a_0}\frac{1}{\sqrt{\Omega_{p,0}}}\left[\frac{(1+\tilde{z})^{-p/2+1}}{-p/2+1}\right]\Bigg|_0^z$$

$$2)\quad \chi(z) = \frac{c}{H_0a_0}\left[\frac{(1+z)^{-p/2+1}-1}{-p/2+1}\right]\ .$$

For $p = 2$,

$$3)\quad \chi(z) = \frac{c}{H_0a_0}\int_0^z (1+z)^{-1}\,d\tilde{z} = \frac{c}{H_0a_0}\ln(1+z)\ .$$

d) Behold:

$$D_{\mathrm{P}} = a_0\chi(z) = \frac{c}{H_0}\int_0^z \frac{d\tilde{z}}{\sqrt{\sum_p \Omega_{p,0}(1+\tilde{z})^p}}\ .$$

The radial proper distance has no dependence on a_0. Behold:

$$a_0r = \begin{cases} a_0\sin[\chi(z)] & \text{for } k=1; \\ a_0\chi(z) & \text{for } k=0; \\ a_0\sinh[\chi(z)] & \text{for } k=-1. \end{cases}$$

For $k \neq 0$, the a_0r does depend on a_0 except in the limit of z small. For $k=0$, the a_0 cancels out just as for D_{P} and in this case $D_{\mathrm{P}} = a_0r = a_0\chi(z)$.

e) Behold:

$$\chi(z) = \frac{c}{H_0 a_0} \begin{cases} \left[\dfrac{(1+z)^{-p/2+1}-1}{-p/2+1}\right] & \text{for } p<2. \\ \ln(1+z) & \text{for } p=2. \\ \left[\dfrac{1-(1+z)^{-p/2+1}}{p/2-1}\right] & \text{for } p>2. \\ z & \text{for all } p \text{ to 1st order in small } z. \\ \left[\dfrac{z^{-p/2+1}}{-p/2+1}\right] & \text{for asymptotically large } z \text{ for } p<2. \\ \ln(z) & \text{for asymptotically large } z \text{ for } p=2. \\ \left[\dfrac{1-z^{-p/2+1}}{p/2-1}\right] & \text{for asymptotically large } z \text{ for } p>2. \\ \left[\dfrac{1}{p/2-1}\right] & \text{for } z\to\infty \text{ for } p>2. \\ z & \text{for } p=0 \text{ which is the de Sitter universe case.} \\ 2\left[1-(1+z)^{-1/2}\right] & \text{for } p=3 \text{ which is the Einstein-de Sitter universe case.} \end{cases}$$

f) Behold:

$$\begin{aligned} \frac{1}{\sqrt{\sum_p \Omega_{p,0}(1+z)^p}} &= \frac{1}{\sqrt{\sum_p \Omega_{p,0}[1+pz+p(p-1)z^2+\ldots]}} \\ &= 1-\frac{1}{2}\left(\sum_p \Omega_{p,0}p\right)z-\frac{1}{2}\left[\sum_p \Omega_{p,0}p(p-1)\right]z^2+\frac{3}{8}\left(\sum_p \Omega_{p,0}p\right)^2 z^2+\ldots \\ &= 1-\frac{1}{2}\left(\sum_p \Omega_{p,0}p\right)z+\left\{-\frac{1}{2}\left[\sum_p \Omega_{p,0}p(p-1)\right]+\frac{3}{8}\left(\sum_p \Omega_{p,0}p\right)^2\right\}z^2+\ldots \end{aligned}$$

where we have used the fact that $\sum_p \Omega_{p,0}=1$. Now

$$\begin{aligned} \chi(z) &= \frac{c}{H_0 a_0}\int_0^z \frac{d\tilde{z}}{\sqrt{\sum_p \Omega_{p,0}(1+\tilde{z})^p}} \\ &= \frac{c}{H_0 a_0}\left[\left[z-\frac{1}{4}\left(\sum_p \Omega_{p,0}p\right)z^2+\frac{1}{3}\left\{-\frac{1}{2}\left[\sum_p \Omega_{p,0}p(p-1)\right]+\frac{3}{8}\left(\sum_p \Omega_{p,0}p\right)^2\right\}z^3+\ldots\right]\right] . \end{aligned}$$

Since the coeffcients are likely of order 1, the relative error in the series is estimated to be

$$\text{relerr} = \frac{z^4}{z} = \begin{cases} z^3 & \text{in general.} \\ 10^{-3} & \text{for } z=0.1. \\ \dfrac{1}{8} & \text{for } z=1/2. \end{cases}$$

Whether the small z series for $\chi(z)$ has any use in fitting data is unkown to yours truly.

Redaction: Jeffery, 2018jan01

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004 qfull 00700 1 3 0 easy math: deceleration parameter

13. The theoretical cosmological distance measures to 2nd order in small cosmological redshift z are conventionally written in terms of the Hubble constant $H_0 = \dot{a}_0/a_0$ and the deceleration parameter $q_0 = -(\ddot{a}_0 a_0/\dot{a}_0^2)$ (which is unitless or rather has natural units). In fact in the 1970s, cosmology was

sometimes comically oversimplified as a search for two numbers: H_0 and q_0 (see A.R. Sandage, 1970, Physics Today, 23, 34, *Cosmology: A search for two numbers*). Nowadays, q_0 has lost some of its glamor. It is now not regarded as a basic parameter of cosmological models, but just one of the derived parameters and its peculiar definition just a historical convention. The fact that the universal expansion is accelerating makes the deceleration parameter negative which is an incongruity.

There are parts a,b,c,d. On exams, omit part d.

a) Taylor expand $a(t)$ in small $\Delta t = t - t_0$ to 2nd order and rewrite the coefficients in terms of H_0 and q_0. The rewritten expansion should begin $a(t) = a_0[1 + \ldots$.

b) Recalling the cosmological redshift formaula $1 + z = a_0/a$, rewrite the formula from the part (a) answer as an expansion for z to 2nd order small Δt. **HINT:** You will need the geometric series:

$$\frac{1}{1-x} = \sum_{\ell=0}^{\infty} x^{\ell} ,$$

which converges for $|x| < 0$ (Ar-279).

c) Now we need to invert the power series for z to find lookback time $t_{\rm LB} = t_0 - t = -\Delta t$ to 2nd order in small z. We will need the power series inversion cofficients. Given

$$\Delta y = \sum_{\ell=1}^{\infty} a_{\ell} \Delta x^{\ell} \qquad \text{and} \qquad \Delta x = \sum_{\ell=1}^{\infty} b_{\ell} \Delta y^{\ell} ,$$

where the inversion coefficients b_i run $b_1 = 1/a_1$, $b_2 = -a_2/a_1^3$, ... (Ar-316–317).

d) The Friedmann equation for (inverse) power law density parameters in scaled form can be used to obtain a useful expression for the deceleration parameter q_0:

$$\frac{\dot{x}}{x} = \sqrt{\sum_p \Omega_{p,0} x^{-p}} \qquad \text{where } \dot{x} = dx/d\tau = dx/d(H_0 t).$$

$$\dot{x} = x\sqrt{\sum_p \Omega_{p,0} x^{-p}} = \sqrt{\sum_p \Omega_{p,0} x^{-p+2}}$$

$$\ddot{x} = \left(\frac{1}{2}\right)\left[\frac{\sum_p \Omega_{p,0} x^{-p+1}(-p+2)}{\sqrt{\sum_p \Omega_{p,0} x^{-p+2}}}\right]\dot{x}$$

$$= \sum_p \Omega_{p,0} x^{-p+1}(-p/2+1)$$

$$q = \begin{cases} -\dfrac{\ddot{x}x}{\dot{x}^2} & \text{in general.} \\ \dfrac{\sum_p \Omega_{p,0} x^{-p+1}(p/2-1)x}{\left(x\sqrt{\sum_p \Omega_{p,0} x^{-p}}\right)^2} = \dfrac{\sum_p \Omega_{p,0} x^{-p}(p/2-1)}{\sum_p \Omega_{p,0} x^{-p}} & \text{for power law density parameters.} \\ q_0 = \dfrac{\sum_p \Omega_{p,0}(p/2-1)}{\sum_p \Omega_{p,0}} = \sum_p \Omega_{p,0}(p/2-1) & \text{for cosmic present} \\ & \quad \text{recalling } \sum_p \Omega_{p,0} = 1. \\ q_0 = -0.55\left[\dfrac{0.15(\Omega_{\rm m,0}/0.3) - 0.7(\Omega_\Lambda/0.7)}{-0.55}\right] & \text{for the } \Lambda\text{-CDM model.} \\ & \quad \text{written in terms of fiducial values.} \end{cases}$$

Before 1998, people mostly thought $\Omega_\Lambda = 0$ which with $\Omega_{\rm M} = 0.3$ (which was what it seemed then as well as now) gives $q_0 = 0.15$. However, some people then hoped that $\Omega_{\rm M} = 1$ which would give $q_0 = 1/2$ which many thought was the great good value. Why?

SUGGESTED ANSWER:

a) Behold:

$$a(t) = a_0 + \Delta t \dot{a}_0 + \frac{1}{2}\Delta t^2 \ddot{a}_0 + \ldots = a_0 \left[1 + \Delta t H_0 + \frac{1}{2}\Delta t^2 \frac{\ddot{a}_0}{a_0} + \ldots\right]$$
$$= a_0 \left[1 + H_0 \Delta t - \frac{1}{2} q_0 H_0^2 \Delta t^2 + \ldots\right]$$

b) Behold:

$$z = -1 + \frac{a_0}{a(t)} = -1 + \frac{1}{1 + H_0\Delta t - (1/2) q_0 H_0^2 \Delta t^2 + \ldots} = -1 + \left[1 - H_0 \Delta t + \left(\frac{1}{2}\right) q_0 H_0^2 \Delta t^2 + H_0^2 \Delta t^2 + \ldots\right]$$
$$= -H_0 \Delta t + \left(1 + \frac{1}{2} q_0\right) H_0^2 \Delta t^2$$

c) Behold:

$$t_{\rm LB} = t_0 - t = -\Delta t = -\left\{\frac{z}{-H_0} + \frac{-[1 + (1/2)q_0] H_0^2 z^2}{(-H_0)^3}\right\} = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2} q_0\right) z + \ldots\right] .$$

d) The value $q_0 = 1/2$ made the universe geometry flat (which makes it simpler to understand) and didn't need a cosmological constant. It is also true that nearly exact flatness was a prediction of inflation which was thought of as a promising theory since circa 1980. However, the fact that $\Omega_{\rm M}$ kept turning out to be ~ 0.3 suggested to some even before the discovery of the acceleration of the universal expansion that maybe we needed a cosmological constant if inflation was going to be maintained.

An alternative and probably useless derivation of the deceleration parameter q_0 in this context is

$$\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda}{3} \\
\frac{\ddot{a}a}{\dot{a}^2} H^2 &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2} + \rho_\Lambda + 3\frac{p_\Lambda}{c^2}\right) \\
-qH^2 &= -\frac{4\pi G}{3}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{4\pi G}{3H^2}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{1}{2}\frac{1}{\rho_{\rm critical}}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{1}{2}\left[\Omega_{\rm M}(1+3w) + \Omega_\Lambda(1+3w_\Lambda)\right] \\
q &= \frac{1}{2}\left[\Omega_{\rm M} - 2\Omega_\Lambda\right] = \frac{\Omega_{\rm M}}{2} - \Omega_\Lambda \qquad \text{with } w = 0 \text{ and } w_\Lambda = -1 \text{ as per usual} \\
q &= \frac{1}{2}\left[0.3\alpha_{\rm M} - 2\times(0.7\alpha_\Lambda)\right] = \frac{1}{2}\left[0.3\alpha_{\rm M} - 1.4\alpha_\Lambda\right] = 0.15\alpha_{\rm M} - 0.7\alpha_\Lambda \ .
\end{aligned}$$

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004 qfull 00710 1 3 0 easy math: small z expressions for the cosmological distance measures

14. To get the small cosmological redshift z formulae for cosmological distance measures one expands $a(t)$ around current time t_0 to 2nd order in $\Delta t = t - t_0$, parameterizes the first expansion coefficients with the Hubble constant $H_0 = \dot{a}_0/a_0$ and the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$, substitutes for $a(t)$ with z (and thereby assuming t is the start time for a light signal coming from z), and inverts the power series to get lookback time $t_{\rm LB}$ to 2nd order in small z:

$$t_{\rm LB} = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2} q_0\right) z + \ldots\right] .$$

One then uses the $t_{\rm LB}$ formula with the Robertson-Walker metric applied to the light signal to get the comoving coordinate r to 2nd order in z:

$$r = \frac{zc}{H_0 a_0}\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right] .$$

There are parts a,b,c,d. The parts can be done be at least semi-independently, so don't stop necessarily if you can't do a part.

a) Use the 2nd-order-in-z formulae given in the preamble to get the **2nd-order-in-z** formulae (simplified so that there is only one second order term appearing) and **1st-order-in-z** formulae (expressed just one term appearing) for the following standard cosmological distance measures (expressed in observational form if it exists and then theoretical form), except for expression for z itself included for completeness:

Cosmological redshift: $z = \dfrac{\lambda_0 - \lambda_{\rm e}}{\lambda_{\rm e}} = \dfrac{a_0}{a(t)} - 1$ and $1 + z = \dfrac{a_0}{a(t)}$

Lookback time: $t_{\rm LB} = t_0 - t(a)$

Comoving coordinate r: $r = \chi^{-1}\left\{A\left[t_0, t\left(\dfrac{a_0}{1+z}\right)\right]\right\}$

Proper distance: $D_{\rm P} = a_0\chi(r)$

Recessional velocity: $v_{\rm R} = H_0 D_{\rm P}$

Redshift velocity: $v_{\rm red} = zc$

Luminosity distance: $D_{\rm L} = \sqrt{\dfrac{L}{4\pi F}} = a_0 r(1+z)$

Angular diameter distance: $D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1+z)}$.

b) Under what conditions are the cosmological distances measures direct observables to 1st and 2nd order given that one can measure z?

c) Prove that all the standard cosmological distance measures are the same to 1st order in small z aside from constants. Show what they are in terms of quantity zc/H_0, where $c/H_0 = (13.968\ldots\,\text{Gly})/h_{70} = (4.2827\ldots\,\text{Gpc})/h_{70}$ is the Hubble length with $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$.

d) Prove the observational Hubble's law:

$$v_{\rm red} = H_0 D_{\text{P-1st}} ,$$

where $D_{\text{P-1st}}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance.

e) Given that $|q_0| \lesssim 1$, at what z values would one expect the standard cosmological distance measures (with constants applied as needed to make them all all equal to 1st order in z) to diverge by of order or less than 1 %, 10 %, 30 %, 50 %, and 100 %.

SUGGESTED ANSWER:

a) Behold:

Cosmological redshift: $z = \dfrac{\lambda_0 - \lambda_{\rm e}}{\lambda_{\rm e}} = \dfrac{a_0}{a(t)} - 1 \approx \dfrac{a_0}{a(t)}$ for $z >> 1$ $\quad 1 + z = \dfrac{a_0}{a(t)}$

Lookback time: $$t_{\rm LB} = t_0 - t(a) = \frac{z}{H_0}\left[1 - (1 + \frac{1}{2}q_0)z + \ldots\right] = \frac{z}{H_0} + \ldots$$

Comoving coordinate r: $$r = \chi^{-1}\left\{A\left[t_0, t\left(\frac{a_0}{1+z}\right)\right]\right\} = \frac{zc}{a_0 H_0}\left[1 - \frac{1}{2}(1 + q_0)z + \ldots\right] = \frac{zc}{a_0 H_0} + \ldots$$

Proper distance: $$D_{\rm P} = a_0\chi(r) = \frac{zc}{H_0}\left[1 - \frac{1}{2}(1 + q_0)z + \ldots\right] = \frac{zc}{H_0} + \ldots$$

Recessional velocity: $$v_{\rm R} = H_0 D_{\rm P} = zc\left[1 - \frac{1}{2}(1 + q_0)z + \ldots\right] = zc + \ldots$$

Redshift velocity: $$v_{\rm red} = zc$$

Luminosity distance: $$D_{\rm L} = \sqrt{\frac{L}{4\pi f}} = a_0 r(1+z) = \frac{zc}{H_0}\left[1 + \frac{1}{2}(1 - q_0)z + \ldots\right] = \frac{zc}{H_0} + \ldots$$

Angular diameter distance: $$D_{\rm A} = \frac{D_{\rm ruler}}{\theta} = \frac{a_0 r}{(1+z)} = \frac{zc}{H_0}\left[1 - \left(\frac{3}{2} + \frac{1}{2}q_0\right)z + \ldots\right] = \frac{zc}{H_0} + \ldots \ .$$

b) All the standard cosmological distance measures are direct observables to 1st order in small z if H_0 is known and to 2nd order in small z if H_0 and q_0 are known.

c) By inspection from part (a) to 1st order in small z:

$$\begin{aligned} ct_{\rm LB} = a_0 r = D_{\rm P} = \frac{v_{\rm R}}{H_0} = \frac{v_{\rm red}}{H_0} = D_{\rm L} = D_{\rm A} = \frac{zc}{H_0} \\ = z\left(\frac{13.968\ldots\ {\rm Gly}}{h_{70}}\right) = z\left(\frac{4.2827\ldots\ {\rm Gpc}}{h_{70}}\right) \ . \end{aligned}$$

d) By inspection from part (a), we find the observational Hubble's law

$$v_{\rm red} = H_0 D_{\rm P,1st} \ ,$$

where $D_{\rm P,1st}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance.

e) By z equal to 0.01, 0.1, 0.3, 0.5 and 1.

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