

## Cosmology & Galaxies

NAME:

### Homework 3: The Friedmann Equation

003 qmult 00100 1 4 5 easy deducto-memory: Friedmann equation derivation

1. "Let's play *Jeopardy!* For \$100, the answer is: It was derived from general relativity in 1922 with the assumptions of a homogeneous and isotropic universe and that all mass-energy in the universe could be modeled by a perfect fluid. A Newtonian derivation (which required extra natural hypotheses) was given in 1934.

What is the \_\_\_\_\_, Alex?

- a) Einstein equation    b) Milne-McCrea equation    c) Synge equation    d) Bondi equation  
e) Friedmann equation

**SUGGESTED ANSWER:** (e)

**Wrong answers:**

- a) Albert Einstein (1879–1955) missed it.  
b) Milne and McCrea gave the Newtonian derivation (Bondi 1961, p. 75).

**Redaction:** Jeffery, 2008jan01

003 qmult 00110 1 4 3 easy deducto-memory: Why Newtonian derivation of FE not found

**Extra keywords:** in the 19th century.

2. A Newtonian derivation of the Friedmann equation (with extra natural hypotheses) could easily have been done in the 19th century, but it wasn't. There were probably 3 reasons why 19th century astronomers did not think of such a derivation. First, many were still thinking of a universe that was static on average even though dynamic equilibrium seemed hard to arrange, even though the universe was obviously not in thermodynamic equilibrium (and so why should be in dynamic equilibrium), and even though idea existed that the Milky was held up by rotation around its center of mass located somewhere. Second, they did not know that other galaxies existed though some believed this and they had not observed the general redshifts of the objects they thought might be other galaxies. Third, they thought in terms of Newton's absolute space (i.e., a single fundamental inertial frame) and did not think of the alternative idea completely compatible with their data that all \_\_\_\_\_ unrotating with respect to the observable universe were elementary inertial frames (i.e., frames with respect to which Newtonian physics and all other known physics could be referenced to). The elementary inertia frames could be incorporated into more general inertial frames (e.g., center-of-mass inertial frames) and the whole observable universe could organized into the more general inertial frames. Going beyond what 19th century astronomers probably could have thought of, there is whole hierarchy of general inertial frames that tops out with the comoving frames of the expanding universe.

What is the \_\_\_\_\_, Alex?

- a) star frames    b) planet frames    c) free-fall frames    d) thermodynamics frames  
e) gravity frames

**SUGGESTED ANSWER:** (c)

**Wrong answers:**

- e) As Lurch would say AARrrrgh.

**Redaction:** Jeffery, 2008jan01

003 qmult 00250 1 1 2 easy memory: shell theorem to point masses interaction

3. "Let's play *Jeopardy!* For \$100, the answer is: This theorem (originally proven by Newton by primitive means) allows one to show by means of a **COROLLARY** that spherically symmetric masses should interact gravitationally as though they are point masses as long as they are do not interpenetrate.

What is the \_\_\_\_\_, Alex?

- a) Newton theorem    b) shell theorem    c) point-mass theorem    d) sphere theorem  
e) waste book theorem

**SUGGESTED ANSWER:** (b)

**Wrong answers:**

- e) In fact, Newton had a waste book (which essentially meant notebook to him) in which he developed much of his version of calculus. So this answer is more reasonable that it might seem at first.

**Redaction:** Jeffery, 2008jan01

003 qmult 00410 1 4 5 easy deducto-memory: Bertrand's theorem, the inverse-square and linear forces

**Extra keywords:** (Go3-92)

4. "Let's play *Jeopardy!* For \$100, the answer is: The theorem that states that the only attractive central forces that give closed orbits for all bound orbits are the inverse-square law force and the attractive linear force (AKA Hooke's law force or the radial harmonic oscillator force). All attractive central forces give closed **CIRCULAR** orbits, of course."

What is \_\_\_\_\_, Alex?

- a) the virial theorem      b) Euler's theogonic proof      c) the brachistochrone problem  
d) Schubert's unfinished symphony      e) Bertrand's theorem

**SUGGESTED ANSWER:** (e)

Newton himself showed that the inverse-square law force and attractive linear force (AKA radial harmonic oscillator force) give closed orbits for all bound orbits and that these orbits were elliptical orbits. In the former case, the center of force was at one focus of the elliptical orbit (the other focus being just empty point in space of no obvious significance) and in the latter case, the center of force is the geometric center of the elliptical orbit (see Isaac Newton: *Principia* translated and annotated by Ian Bruce, Book I, Section 2 introduction). But it was beyond the mathematical technique of Newton's day to show that those two force laws were the only two that give closed orbits for all bound orbits.

**Wrong answers:**

- a) Out of two plausible answers, that's the wrong one.  
b) I seem to recall that when confronted with that notorious Deist and scoffer Voltaire, Euler presented him with several pages of advanced math ending with "and so God exists, QED."  
d) Was it Schubert who died in mid symphony?

**Redaction:** Jeffery, 2001jan01

003 qmult 01100 1 1 1 easy memory: cosmological and Hubble quantities

5. The solutions of the Friedmann equation have characteristic cosmological quantities some of which are called Hubble quantities since the Hubble constant is one of their ingredients. The table below displays some the cosmological quantities. Since the currently determined values of the quantities always fluctuate a bit depending on whose analysis is used, we have written the quantities as fiducial values with correction factors that are 1 to within a few percent:  $h_{70}$  is the Hubble constant divided by 70 (km/s)/Mpc (i.e.,  $H_0/(70 \text{ (km/s)/Mpc})$ ),  $\omega_{m,0} = \Omega_{m,0}/0.3$ , and  $\omega_\Lambda = \Omega_\Lambda/0.7$ . The asymptotic Hubble quantities are those that will be the Hubble quantities as cosmic time goes to infinity if the  $\Lambda$ -CDM model is correct.

Table: Cosmological Quantities

Cosmic scale factor for the present cosmic time  $a_0 = 1$  by convention

Hubble constant  $H_0 = 70h_{70}$  (km/s)/Mpc

Hubble time  $t_H = 1/H_0 = (13.968 \dots)/h_{70}$  Gyr

Hubble length  $\ell_H = c/H_0 = (13.968 \dots)/h_{70}$  Gly =  $(4.2827 \dots)/h_{70}$  Gpc

Critical density  $\rho_{\text{critical}} = [3H_0^2/(8\pi G)] = (9.2039 \times 10^{-27})h_{70}^2 \text{ kg/m}^3$   
 $= (1.3599 \times 10^{11})h_{70}^2 \text{ M}_\odot/\text{Mpc}^3$

AKA Hubble density (i.e., the density implied by the Hubble constant at cosmic present)

Cosmological constant matter density parameter  $\Omega_{m,0} = 0.3\omega_{m,0}$

Cosmological constant  $\Lambda$  density parameter  $\Omega_\Lambda = 0.7\omega_\Lambda$

Asymptotic  $\Lambda$  Hubble parameter  $H_\Lambda = H_0\sqrt{\Omega_\Lambda} = \sqrt{\Lambda/3} = (58.566 \dots)h_{70}\sqrt{\omega_\Lambda}$  (km/s)/Mpc

Asymptotic  $\Lambda$  Hubble time  $t_{H_\Lambda} = (16.6955 \dots)/(h_{70}\sqrt{\omega_\Lambda})$  Gyr

Given that the  $\Lambda$ -CDM model is correct, to 1st order, the observable universe is already expanding like a cosmological-constant universe with  $a = a_0 \exp(\Delta t/t_{H_\Lambda})$  (where  $\Delta t = t - t_0$ ) and this formula becomes more correct as time advances. On what time scale  $\Delta t$  will the matter mass-energy density of the observable universe fall to of order 2% of the total mass-energy? Note, you have to solve for  $a/a_0$  from

$$\Omega_m = \Omega_{m,0} \left(\frac{a_0}{a}\right)^3 \approx 0.02\Omega_\Lambda$$

and then solve for  $\Delta t$ .

- a)  $t_{H_\Lambda}$ .   b)  $2t_{H_\Lambda}$    c)  $3t_{H_\Lambda}$ .   d)  $4t_{H_\Lambda}$ .   e)  $5t_{H_\Lambda}$ .

**SUGGESTED ANSWER:** (a)

Behold:

$$\begin{aligned} \Delta t &\approx t_{H_\Lambda} \times \ln\left(\frac{a}{a_0}\right) \approx t_{H_\Lambda} \times \frac{1}{3} \ln\left(\frac{\Omega_{m,0}}{0.02\Omega_\Lambda}\right) = \frac{t_{H_\Lambda}}{3} \ln\left(\frac{0.3}{0.02 \times 0.7}\right) = \frac{t_{H_\Lambda}}{3} \ln\left(\frac{50 \times 3}{7}\right) \\ &\approx \frac{t_{H_\Lambda}}{3} \ln(21) \approx \frac{t_{H_\Lambda}}{3} \times 3.0455 \approx t_{H_\Lambda} . \end{aligned}$$

If the  $\Lambda$ -CDM model is correct, it won't be long in cosmic time before matter is rather negligible by comparison to constant dark energy if that is what the cosmological constant signifies.

**Wrong answers:**

- b) Nah.

Fortran-95 Code

```

      print*
      print*, 'Cosmological/Hubble quantities for fiducial values.'
      !      pi=acos(-1.0_np)
      pi=3.14159265358979323846264338327950288419716939937510_np
      !
      !!23456789a123456789b123456789c123456789d123456789e123456789f123456789g12
      !      ! https://en.wikipedia.org/wiki/Pi#Approximate_value_and_digits 51
digits
      daysec=86400.0_np
      xjy=365.25_np
      xjy_s=xjy*daysec      ! Julian year in seconds
      grav=6.67430e-11_np
      ! http://en.wikipedia.org/wiki/Gravitational_constant MKS error (15):
      !      ! so 4 digit accurate, but there is controversy
      clight=2.99792458e8_np      ! light speed in m/s
      !      ! https://en.wikipedia.org/wiki/Speed_of_light
      !      pc_m=(1.49597870700e11_np/(pi/(180.0_np*3600.0_np)))
      pc_m=9.6939420213600000e+16_np/pi
      !
      !
https://en.wikipedia.org/wiki/Parsec#Calculating_the_value_of_a_parsec
      !      ! http://en.wikipedia.org/wiki/Astronomical_unit
      xly_m=9.460730472580800e15_np ! exact value by definition,
      !      ! light distance traveled in 1 Jyr
      !      ! https://en.wikipedia.org/wiki/Light-year#Definitions
      pc_ly=pc_m/xly      ! parsec in lyr: ! Also conversion _pc_ly
      !      ! = (x m/pc)/(y m/ly) =(x/y) ly/pc = factor of unity = 1
      xmpc_km=pc_m*1.e+6_np*1.e-3_np ! Also the conversion Mpc to km
      gpc_m=pc_m*1.e+9_np      ! Also the conversion Gpc to m
      print*, 'pc_m,pc_ly,xmpc_km,gpc_m'
      print*, pc_m,pc_ly,xmpc_km,gpc_m
      ! 30856775814913672.789      3.2615637771674335622
30856775814913672788.

```



7. In this problem, we consider the general Gauss law from which are derived the inverse-square-law Gauss' law (i.e., Gauss' law without qualification) and the linear-force Gauss' law.

**NOTE:** There are parts a,b,c,d. On exams, omit parts d,e and use minimal words.

- a) Determine the most general function  $f(r)$  satisfying

$$\nabla \cdot \vec{f} = \frac{1}{r^2} \frac{\partial[r^2 f(r)]}{\partial r} = 3C ,$$

where  $\vec{f} = f(r)\hat{r}$ ,  $C$  is a constant, and the divergence is for a spherically symmetric vector field in spherical coordinates (Ar-104). What problem does this general function have for  $r = 0$ ?

- b) To deal with the problem found in part (a), consider a general central force obeying a power law

$$\vec{f} = f(r)\hat{r} = r^p \hat{r} ,$$

where  $p$  is general: it can be positive, negative, and zero, and integer and noninteger. Imagine a small spherical Gaussian surface of radius  $R$  centered on the origin. For this surface determine in all cases

$$\lim_{R \rightarrow 0} \oint \vec{f} \cdot d\vec{A} ,$$

where the integral is over the whole surface and  $d\vec{A}$  is a differential surface area vector. Given Gauss' theorem (AKA the divergence theorem)

$$\oint \vec{g} \cdot d\vec{A} = \int \nabla \cdot \vec{g} dV$$

(where  $\vec{g}$  is a general field) and part (a), what must be

$$\nabla \cdot (r^{-2}\hat{r}) ?$$

- c) Given parts (a) and (b), what is the most general form of Gauss' law: i.e., the most general form of the equation

$$\oint \vec{f} \cdot d\vec{A} = \text{Constant} ?$$

Note, the constant is not completely independent of position. **HINT:** Consider two kinds of charge  $Q_1$  and  $Q_{(-2)}$ . The point charges for these two kinds of charge each give rise to central forces.

- d) So how many kinds of Gauss' law are there? Give their formulae.  
 e) Determine the differential equation form of the two kinds of Gauss' law. **HINT:** Make use of Gauss' theorem.

### SUGGESTED ANSWER:

- a) Behold:

$$1) \quad \nabla \cdot \vec{f} = \frac{1}{r^2} \frac{\partial[r^2 f(r)]}{\partial r} = 3C \quad 2) \quad \frac{\partial[r^2 f(r)]}{\partial r} = 3Cr^2$$

$$3) \quad r^2 f(r) = Cr^3 + K \quad 4) \quad f(r) = Cr + \frac{K}{r^2} ,$$

where  $K$  is a constant of integration. At  $r = 0$ , the second term of  $f(r)$  is undefined and ordinary integration, and so differentiation fails at  $r = 0$ .

- b) Behold:

$$\lim_{R \rightarrow 0} \oint \vec{f} \cdot d\vec{A} = \lim_{R \rightarrow 0} \oint r^{p+2} d\Omega = \lim_{R \rightarrow 0} 4\pi R^{p+2} = \begin{cases} 0 & \text{for } p > -2. \\ 4\pi & \text{for } p = -2. \\ \infty & \text{for } p < -2. \end{cases}$$

To satisfy Gauss' theorem, we must have

$$\nabla \cdot (r^{-2}\hat{r}) = 4\pi\delta(\vec{r}) ,$$

where  $\delta(\vec{r})$  is a Dirac delta function.

- c) From parts (a) and (b), the most general constant divergence for a central force for  $r > 0$  is

$$\vec{f}(r) = Cr\hat{r} + \frac{K}{r^2}\hat{r} :$$

i.e., a linear combination of linear force and inverse-square law force. Thus, the most general central force for a general Gauss' law is

$$\vec{f}(r) = Q_1r\hat{r} + \frac{Q_{(-2)}}{r^2}\hat{r}$$

which gives

$$\nabla \cdot \vec{f}(r) = 3Q_1 + 4\pi\delta(\vec{r})Q_{(-2)}$$

Therefore, the most general Gauss' law for a point charge is

$$\oint \vec{f} \cdot d\vec{A} = 3Q_1V + 4\pi Q_{(-2),\text{enclosed}} ,$$

where  $V$  is the total volume of the Gaussian surface and  $Q_{(-2),\text{enclosed}}$  is zero if the charge is outside the Gaussian surface, and so this term is not completely independent of position. Now, we have assumed the two charges are for the same point charge. But they can be for different point charges and we can generalize the above equation to any array of point charges. Thus, the most general Gauss' law without qualification is

$$\oint \vec{f} \cdot d\vec{A} = 3Q_1V + 4\pi Q_{(-2),\text{enclosed}} .$$

It is remarkable that the  $Q_1$  can be spread anywhere across the universe. But recall the linear force goes to infinity as  $r \rightarrow \infty$ , and so it is a very strong long-range force. Recall also, we are only counting the charges that contribute to the vector field we are considering on the Gaussian surface we are considering. There, of course, can be other charges not appearing on the right-hand side of the general Gauss' law, but we are not considering their contribution to the field on the left-hand side of the general Gauss' law.

- d) There seem to be only two kinds of Gauss' law. Gauss' law (without qualification) for the inverse-square law force

$$\oint \vec{f} \cdot d\vec{A} = 4\pi Q_{(-2),\text{enclosed}}$$

and Gauss' law for the linear force

$$\oint \vec{f} \cdot d\vec{A} = 3Q_1V .$$

Carl Friedrich Gauss (1777-1855) probably knew this more than 200 years ago.

Because the inverse-square law force and the linear force seem to be the only two forces that can be incorporated into the Newtonian derivation of the Friedmann equation, yours truly thinks of them as the only two possible cosmological forces. The linear force appears as the cosmological constant force or  $\Lambda$  force. There may be also dark energy, but yours truly thinks the cosmological constant force exists because it can exist—OK, not a convincing reason, but it's still interesting that it can exist.

Another interesting point is that classically the inverse-square law force of gravity and the  $\Lambda$  force are long-range forces. Note, gravity only goes to zero only asymptotically as distance from source goes to infinity and  $\Lambda$  force acts as if it goes to infinity as one goes to infinity from any point. However, in the Einstein field equations (which are differential equations posited as true everywhere in spacetime), both forces arise from terms and their long-range effects (which dictates the curvature of space) follow from the integration of the Einstein field equations over boundless space at an one time. So the classical long-range nature of gravity and the  $\Lambda$  force are not mysterious in the context of the Friedmann equation solutions which are derived from the Einstein field equations with cosmological constant assumption: i.e., the boundless universe is

homogeneous and isotropic at every instant in cosmic time. Of course, the Friedmann equation solutions may apply only over some finite region of spacetime: i.e., our pocket universe if that exists. But we are so deeply embedded in our pocket universe that boundary effects are not seen. However, the whole paradigm of pocket universes existing in a background universe is very unguided, and so many theories are possible and which one if any is right is unknown and may stay that way for a long time, maybe forever.

Actually, the linear force Gauss' law may just be an interesting mathematical result, except for its use in the Newtonian derivation of the Friedmann equation.

e) For Gauss' law without qualification,

$$1) \oint \vec{f} \cdot d\vec{A} = 4\pi Q_{(-2),\text{enclosed}} \quad 2) \oint \vec{f} \cdot d\vec{A} = \int \nabla \cdot \vec{f} dV = 4\pi \int \rho_{(-2)} dV$$

$$3) \nabla \cdot \vec{f} = 4\pi \rho_{(-2)} ,$$

where the the integral is over the volume of the Gaussian surface and the last equation follows since the volume is general, there can be no dependence on it. For the linear force Gauss' law,

$$1) \oint \vec{f} \cdot d\vec{A} = 3Q_1 V \quad 2) \oint \vec{f} \cdot d\vec{A} = \int \nabla \cdot \vec{f} dV = 3Q_1 \int dV$$

$$3) \nabla \cdot \vec{f} = 3Q_1,$$

where the the integral is over the volume of the Gaussian surface and the last equation follows since the volume is general, there can be no dependence on it.

Note, quantum mechanical point particles are spread out in a continuum superposition of position by the wave function and so really charge is never infinite at point. However, we derived integral Gauss' laws assuming classical point particles which do not actually exist. But the integral Gauss' laws make no reference to specific points, and so in a non-rigorous way we can imagine smearing the charge out from classical point particles into a continuum and we effectively do this in deriving the differential equation Gauss' law for the inverse-square law force. For the differential equation Gauss' law for the linear force there is no spatial dependence on charge location at all to worry about. For the differential equation Gauss' law for the inverse-square law force, that the actual complexly varying charge density can be replaced average charge density seems intuitively clear for macroscopic determinations of  $\vec{f}$ : i.e.,

$$\nabla \cdot \vec{f} = 4\pi \rho_{(-2)} \quad \text{can be replaced by} \quad \nabla \cdot \vec{f}_{\text{averaged}} = 4\pi \rho_{(-2),\text{averaged}} .$$

But yours truly does know of a completely convincing argument for replacement other than it has always been verified by experiment.

It might seem a odd for the linear force Gauss' law that  $\nabla \cdot \vec{f} = 3Q_1$ . For a sanity check, imagine a set of point charges for the linear force  $Q_j$  located at points  $x_{ji}$  where  $j$  labels the charge and  $i$  the coordinate in 3-dimensional Cartesian coordinates. The total force in direction  $i$  is

$$f_i = \sum_j Q_j (x_i - x_{ji})$$

with divergence

$$\nabla \cdot \vec{f} = \sum_{ij} Q_j = 3 \sum_j Q_j = 3Q ,$$

where  $Q$  is just the sum of all charge. Thus, we have recovered the differential equation linear force Gauss' law and sanity is verified this time.

**Redaction:** Jeffery, 2018jan01

- a) The shell theorem applies to a spherically symmetric mass distributions and is given by

$$\vec{g}(r) = -\frac{GM(r)}{r^2} \hat{r} ,$$

where  $\vec{g}(r)$  is the gravitational field,  $r$  is the radius from the origin, and  $M(r)$  is the mass enclosed by a sphere of radius  $r$ . Prove the theorem given Gauss' law for gravity

$$\oint \vec{g} \cdot d\vec{A} = -4\pi GM_{\text{enclosed}} ,$$

where the integral is over a Gaussian surface with differential surface vector  $d\vec{A}$  and  $M_{\text{enclosed}}$  is the mass enclosed by the Gaussian surface.

- b) First corollary: Determine the formula for the gravitational field outside of a spherically symmetric mass distribution with radius  $R$  and total mass  $M$ . What formula is this formula identical too? This formula was very hard for Isaac Newton (1643–1727) to prove with his primitive mathematical techniques, but the formula was vital for his pioneering work in celestial mechanics.
- c) Second corollary: Determine the formula for the gravitational field in a spherically symmetric cavity of radius  $R$  inside a spherically symmetric mass distribution.
- d) Third corollary: First show that Newton's 3rd law holds explicitly for two general gravitating mass systems of point masses. Then show that two non-overlapping spherically symmetric mass distributions gravitationally interact like point masses: i.e., show  $\vec{F}_{1,2} = \vec{F}_{1_{\text{point}},2_{\text{point}}}$ , where the numbers label the two systems and the numbers subscripted label their point mass replacements. This result is the third corollary itself and it may seem intuitively obvious, it still needs a proof.
- e) The fourth corollary is actually more of interesting feature than a corollary. The feature is the local gravitational field contribution  $\vec{g}_{\text{local}}$  to the total gravitational field of an infinitely thin (mass) shell (radius  $R$  and mass  $M$ ) by an infinitesimal piece of the shell infinitesimally close to the shell. First, find the gravitational field of the shell for all radius  $r$ . Note, the gravitational field at  $r = R$  is actually indeterminate without specifying a limiting process to obtain it. Second, consider an infinitesimal Gaussian pillbox (a cylindrical Gaussian surface) that straddles the shell with its symmetry axis aligned with radial direction. To the Gaussian pillbox, the shell surface is an infinite plane. Determine the gravitational field  $\vec{g}_{\text{local}}$  on the top and bottom of the pillbox due to the mass contained in the pillbox using Gauss' law and then the gravitational field  $\vec{g}_{\text{remote}}$  on the top and bottom of the pillbox due to the remote parts of the shell. Is  $\vec{g}_{\text{remote}}$  continuous at  $r = R$ ? How do the local and remote gravitational contributions just inside the shell compare in this ideal limit?

#### SUGGESTED ANSWER:

- a) Behold:

$$1) \quad \oint \vec{g} \cdot d\vec{A} = -4\pi GM_{\text{enclosed}} \quad 2) \quad -4\pi r^2 g(r) = -4\pi GM(r)$$

$$3) \quad g(r) = \frac{GM(r)}{r^2} \quad 4) \quad \vec{g}(r) = -\frac{GM(r)}{r^2} \hat{r} ,$$

where we have exploited spherical symmetry and required  $\hat{r} \cdot d\vec{A} < 0$  for the negative signs to cancel.

- b) For the first corollary by inspection of the shell theorem,

$$\vec{g}(r) = -\frac{GM}{r^2} \hat{r} ,$$

for  $r \geq R$ . This formula identical to that of a point mass. So as long as  $r \geq R$ , the mass distribution gravitates like a point mass. In fact, one can say that point mass gravitation is just the ideal limit of the formula since classical point masses do not exist.

- c) For the second corollary by inspection of the shell theorem,

$$\vec{g}(r) = 0 .$$

This is a remarkable result that like all of the corollaries traces back to the inverse-square law nature of gravity.

d) For two general systems of point masses

$$F_{1,2} = \sum_{i,j} \left( -\frac{Gm_i m_j}{r_{ij}} \hat{r}_{ij} \right) = - \sum_{i,j} \left( -\frac{Gm_i m_j}{r_{ij}} \hat{r}_{ji} \right) = -F_{2,1} ,$$

index  $i$  labels the particles in distribution 1, index  $j$  labels the particles in distribution 2,  $\hat{r}_{ij}$  points from particle  $i$  to particle  $j$  with  $\hat{r}_{ij} = -\hat{r}_{ji}$ . Thus, the 3rd law explicitly holds for two general gravitating systems of point masses. For the third corollary itself, the proof is

$$\vec{F}_{1,2} = \vec{F}_{1\text{point},2} = -\vec{F}_{2,1\text{point}} = -\vec{F}_{2\text{point},1\text{point}} = \vec{F}_{1\text{point},2\text{point}} ,$$

where we have used Newton's 3rd law and the first corollary as needed.

e) Behold:

$$1) \oint \vec{g} \cdot d\vec{A} = -4\pi G M_{\text{enclosed}} \quad 2) (\vec{g}A)_{\text{local top}} \cdot \hat{r} + [(\vec{g}A)_{\text{local bottom}}] \cdot (-\hat{r}) = -4\pi G \rho A = -4\pi G \frac{M}{4\pi R^2} A$$

$$3) g_{\text{local}} = \frac{1}{2} \frac{GM}{R^2} \quad 4) \vec{g}_{\text{local}} = \mp \frac{1}{2} \frac{GM}{R^2} \hat{r} ,$$

where  $\rho$  is the surface density of the shell,  $A$  is the top/bottom area of the pillbox, and the upper case is for  $r > R$  and the lower case for  $r < R$ . Note for our small pillbox,  $\vec{g}_{\text{local}}$  has no dependence on  $r$ . Now very near the pillbox,

$$1) \vec{g}(r) = \vec{g}_{\text{remote}} + \vec{g}_{\text{local}} \quad 2) \vec{g}_{\text{remote}} = \vec{g}(r) - \vec{g}_{\text{local}}$$

$$3) \vec{g}_{\text{remote}} = \begin{cases} -\frac{GM}{r^2} \hat{r} - \left( -\frac{1}{2} \frac{GM}{R^2} \hat{r} \right) & \text{for } r > R. \\ 0 - \frac{1}{2} \frac{GM}{R^2} \hat{r} = -\frac{1}{2} \frac{GM}{R^2} \hat{r} & \text{for } r < R. \\ -\frac{1}{2} \frac{GM}{R^2} \hat{r} & \text{for } r = R. \end{cases}$$

We see that  $\vec{g}_{\text{remote}}$  is continuous across the shell, but has a kink at  $r = R$ . The local and remote gravitational contributions just inside the shell are equal in this ideal limit: a remarkable fact.

Nearly half a century ago, an intro physics textbook Tipler asserted to yours truly that the gravitational field just in tiny hole in an infinitely thin shell (like the one we are considering) should be

$$\vec{g}_{\text{remote}} = -\frac{1}{2} \frac{GM}{R^2} \hat{r}$$

since there is no local contribution. Like any good first year student, yours truly just believed. But now it seems to yours truly that that gravitational field is still indeterminate without specifying a limiting process. For the setup above, the remote gravitational field  $\vec{g}_{\text{remote}}$  is continuous, but the local field  $\vec{g}_{\text{local}}$  is still discontinuous and exactly what happens when you remove it by making a hole is not clear a priori.

Let's do a little analysis considering a realistic thin shell. Say the shell has mass  $M$ , inner radius  $R_1$ , outer radius  $R_2$ , and uniform density  $\rho = M/[(4\pi/3)(R_2^3 - R_1^3)]$ . We define  $fM$  to be the enclosed mass midpoint (with  $f = 1/2$ ) where we will consider the gravitational field. We now solve for the radius  $r$  of this mass point:

$$1) fM = \int_{R_1}^r \rho(r)(4\pi r^2) dr \quad 2) fM = \frac{(4\pi/3)(r^3 - R_1^3)M}{(4\pi/3)(R_2^3 - R_1^3)} \quad 3) f = \frac{r^3 - R_1^3}{R_2^3 - R_1^3}$$

$$4) r = [f(R_2^3 - R_1^3) + R_1^3]^{1/3} \quad 5) \Delta r_{1st} = [(3f\Delta R)R_1^2 + R_1^3]^{1/3} - R_1 \quad 6) \Delta r_{1st} = f\Delta R = \frac{1}{2}\Delta R ,$$

where  $\Delta r = r - R_1$ ,  $\Delta R = R_2 - R_1$ , subscript 1st indicates 1st order expansion, and we have expanded to first order in small  $\Delta R/R$  twice. Thus, the gravitational field at the midpoint  $\vec{g}_{\text{mid},1\text{st}}$  in the shell to 1st order  $\vec{g}_{\text{mid},1\text{st}}$  is

$$\vec{g}_{\text{mid},1\text{st}} = \vec{g} \left( R_1 + \frac{1}{2} \Delta R \right)_{1\text{st}} = - \frac{GM/2}{[R_1 + (1/2)\Delta R]^2} = - \frac{1}{2} \frac{GM}{[R_1 + (1/2)\Delta R]^2} .$$

If we now imagine two Gaussian pillboxes forming a cylinder through the shell with their meeting place at the midpoint, then by symmetry their summed gravitational field at the midpoint is zero to 1st order and they make no contribution to the gravitational field at the midpoint to 1st order which we just believe without detailed analysis. So if we removed the mass in the Gaussian pillboxes leaving a hole, the gravitational field in the hole at the midpoint to 1st order would be

$$\vec{g}_{\text{mid,hole},1\text{st}} = \vec{g}_{\text{mid},1\text{st}} = - \frac{1}{2} \frac{GM}{[R_1 + (1/2)\Delta R]^2} .$$

If we now take the limit as  $\Delta R \rightarrow 0$  and relabel  $R_1$  to  $R$ , we get

$$\vec{g}_{\text{mid,hole}} = - \frac{1}{2} \frac{GM}{R^2}$$

just as Tipler assured yours truly a jillion years ago. However, yours truly is not quite sure all the specifications are right. Note, the result is the ideal limit for a realistic system that one could actually build easily. It's not the ideal limit for all possible systems.

Now although one can actually build a very exact thin shell with a tiny hole, measuring its gravitational field in the hole or anywhere would be difficult. However, such very precise measurements may be possible in a spacecraft in free fall in orbit (e.g., Wikipedia: Gravity Probe B: note, Gravity Probe B was actually used to measure general relativity effects). One might guess that measurements on the equivalent electrostatic setup (where the inverse-square law Coulomb force applies) would be easier: i.e., measurements on a charged spherical conducting shell. However, the setup would not be equivalent in fact: the net electric charge on the shell would be in a microscopic layer on the outer surface of the shell and not uniformly distributed in the conductor. So the electric field  $\vec{E} = 0$  just inside outer surface for ideal spherical symmetry and not a nonzero value in the bulk conductor and in any case a hole would cause the charge accumulate somewhat near the hole breaking the spherical symmetry somewhat. Nevertheless, maybe very sensitive electric field measurements could measure the change in electric field through the hole on the outer surface. But the experiment may be pointless.

**Redaction:** Jeffery, 2018jan01

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003 qfull 00240 1 3 0 easy math: Gauss' law derivation with specializations

9. In this problem, we will derive the general Gauss' law (for the inverse-square law) in its integral form and then specialize to the gravity and Coulomb force cases.

**NOTE:** There are parts a,b,c,d. Some of the parts can be done independently, and so do not stop if you cannot do a part.

a) Consider the general inverse-square law central force

$$\vec{f} = \frac{q}{r^2} \hat{r} ,$$

where  $q$  is a general charge for the force located at the origin (which is the center of force),  $r$  is the distance to a point where the force is evaluated, and  $\hat{r}$  is the direction to that point. Now consider a differential surface area vector  $d\vec{A}$  for a Gaussian surface (i.e., a **CLOSED** surface) Note,  $d\vec{A}$  is defined as pointing outward from the Gaussian surface. Let  $d\vec{A}$  and  $\hat{r}$  define a  $r$ - $y$  plane that is picture as rotated about a  $z$  axis. We take the surface area defined by  $d\vec{A}$  as a square of area  $\Delta z \Delta y$  (where both  $\Delta z$  and  $\Delta y$  are defined as positive) since  $d\vec{A}$  is differentially small. The area components of  $d\vec{A}$  in the  $\hat{r}$  and  $\hat{y}$  directions are, respectively,

$$\hat{r} \cdot d\vec{A} = A_r = \Delta z \Delta y \cos(\theta) \quad \text{and} \quad \hat{y} \cdot d\vec{A} = A_y = \Delta z \Delta y \sin(\theta) ,$$

where  $\theta$  is the angle of  $d\vec{A}$  measured from the  $\hat{r}$  direction. Note,  $A_r$  is positive for  $\theta \in [-\pi/2, \pi/2]$  and negative for  $\theta \in [\pi/2, 3\pi/2]$ , and the area subtended from the origin is always  $|A_r|$ . Thus, we find

$$\hat{r} \cdot d\vec{A} = A_r = r^2(\pm d\Omega) ,$$

where  $d\Omega$  is the differential solid angle subtended by  $|A_r|$  and the upper/lower case is for positive/negative  $A_r$  (i.e., for the solid angle cone from the origin going outward/inward through the differential surface area  $d\vec{A}$ ).

Prove

$$\vec{f} \cdot d\vec{A} = q(\pm d\Omega) ,$$

where the upper/lower cases are for the solid angle cone going outward/inward through the differential surface area. Note, the charge could be inside or outside the closed surface. **HINT:** This is easy.

- b) Consider a differentially small cone extending from the origin. It intersects the Gaussian surface  $n$  times. Note, Gaussian surface is finite, and so the cone must exit Gaussian surface forever at some point. We form the sum

$$\sum_{i=1}^n \vec{f} \cdot d\vec{A}_i ,$$

where sum is over all intersections. What is the sum equal to in terms of solid angle for all cases? **HINT:** A few words of explanation and a diagram are needed.

- c) Say you had multiple charges  $q_i$  with total charge  $Q$  and total charge  $Q_{\text{enclosed}}$  inside a closed surface. Evaluate

$$\oint \vec{f} \cdot d\vec{A} .$$

The result is the general Gauss' law in its integral form. Specialize the result for the cases of gravity and the Coulomb force.

- d) What is the necessary condition for a force to obey Gauss' law?

**SUGGESTED ANSWER:**

- a) Behold:

$$\vec{f} \cdot d\vec{A} = \frac{q}{r^2} \hat{r} \cdot d\vec{A} = \frac{q}{r^2} A_r = \frac{q}{r^2} r^2(\pm d\Omega) = q(\pm d\Omega) .$$

- b) Behold:

$$\sum_{i=1}^n \vec{f} \cdot d\vec{A}_i = \begin{cases} q d\Omega & \text{for the source inside the Gaussian surface.} \\ 0 & \text{for the source outside the Gaussian surface.} \end{cases}$$

Note, if the charge is inside the closed surface, it must exit one more time than it enters, and thus by the part (a) answer all contributions must cancel pairwise, except that from the last exit which yields the net contribution to the sum  $q d\Omega$ . On the other hand, if the charge is outside the closed surface, cone must exit as often as it enters, and thus by the part (a) answer, the contributions to the sum must all cancel out pairwise. You will have to imagine the diagram.

- c) From part (b), by inspection,

$$\oint \vec{f} \cdot d\vec{A} = \begin{cases} 4\pi Q_{\text{enclosed}} & \text{the general Gauss' law} \\ & \text{in integral form: QED;} \\ -4\pi G M_{\text{enclosed}} & \text{the gravity case;} \\ 4\pi k q_{\text{enclosed}} = \frac{q_{\text{enclosed}}}{\epsilon_0} & \text{the Coulomb force case,} \end{cases}$$

where we identify the charge for gravity as  $-GM_{\text{enclosed}}$  (where  $M$  is the general symbol for mass), the charge for the Coulomb force as  $kq_{\text{enclosed}}$  (where  $q$  is the common symbol for electric charge),  $G$  is the gravitational constant,  $k = 1/(4\pi\epsilon_0)$  is the Coulomb constant, and  $\epsilon_0$  is the vacuum permittivity.

- d) The force has to be an inverse-square law force. If it wasn't, the derivation of Gauss' law given above would not work. It is the cancellation of the  $1/r^2$  factor in the force law with the implicit  $r^2$  factor in the differential surface area that allows the simple integration over all solid angle.

Note: The old a ridiculously complex solution to part (a) is as follows. To transparently show in pure mathematical formalism what can be shown with fewer mathematical steps and diagram (e.g., on an exam),

$$\begin{aligned}
 \vec{f} \cdot d\vec{A} &= \frac{q}{r^2} \hat{r} \cdot d\vec{A} = \frac{q}{r^2} dA (\hat{r} \cdot \hat{n}) = \frac{q}{r^2} dA \cos \theta \\
 &= \frac{q}{r^2} dA \times \begin{cases} |\cos \theta| & \text{for } \theta \in [0, \pi/2]; \\ -|\cos(\theta)| = -|-\cos(\theta)| = -|\cos(\pi - \theta)| = -\cos(\pi - \theta) & \text{for } \theta \in [\pi/2, \pi]; \end{cases} \\
 &= \frac{q}{r^2} \times \begin{cases} dA |\cos \theta| & \text{for the radius from the origin exiting the closed surface;} \\ -dA \cos(\pi - \theta) & \text{for the radius from the origin entering the closed surface;} \end{cases} \\
 &= \frac{q}{r^2} \times \begin{cases} dA_{\perp} & \text{for the radius from the origin exiting the closed surface;} \\ -dA_{\perp} & \text{for the radius from the origin entering the closed surface;} \end{cases} \\
 &= \frac{q}{r^2} \times \begin{cases} r^2 d\Omega & \text{for the radius from the origin exiting the closed surface;} \\ -r^2 d\Omega & \text{for the radius from the origin entering the closed surface;} \end{cases} \\
 &= q(\pm d\Omega) ,
 \end{aligned}$$

where we have used the fact that the differential area perpendicular to a radius from the origin (located at the charge  $q$ ) is given by

$$dA_{\perp} = r^2 d\Omega = \begin{cases} dA \cos \theta & \text{for the radius from the origin} \\ & \text{exiting the closed surface;} \\ dA \cos(\pi - \theta) & \text{for the radius from the origin} \\ & \text{entering the closed surface.} \end{cases}$$

For the diagrammatic obviousness for the  $\pm d\Omega$  product, draw a differential area vector  $d\vec{A} = dA\hat{n}$  for an exiting/entering radius radiating from the origin.

**Redaction:** Jeffery, 2018jan01

003 qfull 00250 1 3 0 easy math: linear-force Gauss's law and shell theorem (reform or cannabilize)

10. Remarkably the linear force obeys analogues to Gauss's law and shell theorem for the inverse-square law force. Let the linear-force field (force per unit charge) for a point charge be

$$\vec{f} = kqr\hat{r} ,$$

where  $k$  is a constant which could be positive or negative,  $q$  is the charge (of some unspecified kind), and  $r$  is the distance from the point charge. We assume Newtonian physics, and so to maintain Newton's 3rd law, we require

$$\vec{F}_{1,2} = kq_1q_2r_{1,2}\hat{r}_{1,2} ,$$

where  $\vec{F}_{1,2}$  is the force of point charge 1 on point charge 2.

There are parts a,b,c,d,f. Some of the parts can be done independently, and so do not stop if you cannot do a part. Omit part (f) during exams.

- a) Without words, for a close surface derive the linear-force Gauss' law

$$\oint \vec{f} \cdot d\vec{A} = kQ ,$$

where  $\vec{f}$  is the field due to the entire charge distribution, the integral is over the whole close surface, and  $Q$  is the total charge of the charge distribution wherever it is in space. **HINT:** Recall the divergence theorem (AKA Gauss' theorem)

$$\oint \vec{Y} \cdot d\vec{A} = \int \nabla \cdot \vec{Y} dV ,$$

where  $Y$  is a general vector field and the volume integral is over all volume  $V$  inclosed by the closed surface (Wikipedia: Divergence theorem). Recall also the divergence operator for spherically symmetric system in spherical coordinates obeys

$$\nabla \cdot \vec{Z} = \frac{1}{r^2} \frac{\partial(r^2 Z_r)}{\partial r} ,$$

where  $Z$  is spherically symmetric, but otherwise general, and  $Z_r$  is the radial component of  $\vec{Z}$  (Arfken-104).

- b) For what symmetries can the linear-force field be easily solved for directly from the linear-force Gauss' law?
- c) Without words, solve for the linear-force field for a spherically symmetric charge distribution. What simple charge distribution would give an equivalent linear-force field for all radius  $r$ ? What can this result be called? How is this equivalent linear-force field different from the analogue result with the inverse-square-law force?
- d) Without words, show for a general charge distribution 1 and a spherical symmetric charge distribution 2 that the force of distribution 1 on distribution 2 is exactly the same as when distribution 2 is replaced point-charge 2. If charge distribution 1 were also spherically symmetric, what be the force between them be equal to and what would it be if their centers coincided exactly?
- e) Say you had a charge distribution that maintained spherically symmetry no matter what, that had its center of mass at its center, and the only external forces that acted on it were external linear forces. How would described its motion? Recall Newton's 2nd law:

$$\vec{F}_{\text{net external}} = m \vec{a}_{\text{cm}} ,$$

where  $\vec{F}_{\text{net external}}$  is the net external force on a body of mass  $m$  and  $a_{\text{cm}}$  is the center of mass of the body. Given the result of part (d) Without words, show for two spherically symmetric distribution charges that the force of distribution 1 on distribution 2 is exactly the same **HINT:** Recall the part (d) answer.

- f) Is the linear force for spherically symmetric mass distribution with mass as its charge consistent with linear force that occurs in the Newtonian derivation of the Friedmann equation:

$$\vec{F} = \frac{\Lambda}{3} m r \hat{r} ,$$

where  $m$  is a test particle mass. There is no right answer. This is a discussion question.

### SUGGESTED ANSWER:

- a) For a point charge,

$$\nabla \cdot \vec{f} = \frac{1}{r^2} \frac{\partial(r^2 f_r)}{\partial r} = 3kq ,$$

which remarkably is constant throughout space. Therefore

$$\oint \vec{f} \cdot d\vec{A} = \int \nabla \cdot \vec{f} dV = 3kQV ,$$

where  $\vec{f}$  is the net field of that charge distribution and we have summed over all charge in the charge distribution to get the total charge  $Q$ . Note, the charges outside of the closed surface all contribute. There is no cancellation of their flux from the sum of the differential bits of surface area subtended by differential solid angle emanating from a point change. This cancellation only happens for inverse-square law forces. Finally, the linear-force Gauss' law is

$$\oint \vec{f} \cdot d\vec{A} = 3kQV .$$

- b) For spherical, cylindrical, and planar symmetries.

c) Behold:

$$1) \quad \oint \vec{f} \cdot d\vec{A} = 4\pi r^2 f = 3kQV = 3kQ \frac{4\pi r^3}{3} \quad 2) \quad f = kQr\hat{r} .$$

A point charge at the center of the distribution would give a equivalent linear-force field for all radius  $r$ . This result can be called the linear-force shell theorem even though there is no containing shell. For the inverse-square-law force, the analogue would be the same only for outside the whole charge distribution. Inside only the charge within radius  $r$  would contribute to the field. This result is a remarkable feature of inverse-square-law forces. The inverse-square-law force analogue is the shell theorem.

d) Behold:

$$\vec{F}_{1,2} = -\vec{F}_{2,1} = -\vec{F}_{2\text{-point},1} = \vec{F}_{1,2\text{-point}} ,$$

where we have used in order Newton's 3rd law, the linear-force shell theorem, and Newton's 3rd law again. Note, this result holds no matter where the center of charge distribution 1 is long as it stays spherically symmetric even if it changes in time or interpenetrates the charge distribution 1. If charge distribution 1 were also spherically symmetric, then

$$\vec{F}_{1,2} = \vec{F}_{1\text{-point},2\text{-point}} .$$

If the centers coincided exactly, the force between them would be zero.

- e) The net external force on the body is exactly force as if the body were a point charge at the center of mass. The center of mass moves exactly as if the body were a point mass. Therefore the center of mass moves exactly as point charge. This is true even if the external force comes from charges that penetrate the spherically symmetric charge distribution.
- f) Hm, tricky. If one had a finite boundless hyperspherical universe where every point could be treated as a center of spherical symmetry and mass-energy were conserved, then consistently  $kQ = kM = \Lambda/3$ , where  $M = Q$  is the total mass-energy of the universe. However, none of these conditions may hold. In fact, radiation mass-energy is not conserved since it scales down as  $1/a(t)$ . However, maybe  $kM$  magically is constant even for infinite mass-energy or changing mass-energy and every point could be treated as a center of spherical symmetry. In this case, the linear force between mass-energy may be a valid Newtonian explanation of the cosmological constant in the Friedmann equation, but not in Einstein field equations since the cosmological constant there does not need mass-energy. In any case, that the linear force is suggestive of cosmological constant force in the Newtonian derivation of the Friedmann equation is an interesting curiosity.

**Redaction:** Jeffery, 2018jan01

003 qfull 00260 1 3 0 easy math: the linear force or cosmological force in cosmology

11. The (Newtonian) cosmological constant force **PER UNIT MASS** is given by

$$\vec{f} = \frac{\Lambda}{3} \vec{r} ,$$

where  $\Lambda$  is the cosmological constant, the  $1/3$  factor is for consistency with cosmological constant as it appears in the Einstein field equations, and  $\vec{r}$  is the displacement vector from any point in space. In an extra Newtonian hypothesis, one can hypothesize that  $\Lambda$  is set somehow by a universal force charge density that is constant in space and time and the Newtonian-3rd-law equal-and-opposite force caused by the cosmological constant force on a particular mass is exerted by the particular mass on on this charge throughout the universe. But this may be a useless hypothesis.

Consider a system of point masses  $m_i$  at displacements  $\vec{r}_i$  relative to an external origin. The total mass of the system is  $m = \sum_i m_i$ . The center of mass of the system is  $\vec{r}_{\text{cm}}$  and the relative displacements are  $\Delta\vec{r}_i = \vec{r}_i - \vec{r}_{\text{cm}}$ .

**NOTE:** There are parts a,b,c.

- a) Write down the cosmological constant force  $\vec{F}_i$  on point mass  $m_i$  relative to the **ORIGIN** both in terms of  $\vec{r}_i$  and  $\Delta\vec{r}_i$ . **HINT:** This is easy.

- b) Determine the net cosmological constant force  $\vec{F} = \sum_i \vec{F}_i$  on the system and simplify as much as possible. **HINT:** Recall the definition of center of mass.
- c) What simplifying conclusion can you draw from the part (b) answer?

**SUGGESTED ANSWER:**

- a) Behold:

$$\vec{F}_i = \frac{\Lambda}{3} m_i \vec{r} = \frac{\Lambda}{3} m_i \vec{r}_{\text{cm}} + \frac{\Lambda}{3} m_i \vec{\Delta} \vec{r}_i .$$

- b) Behold:

$$\vec{F} = \sum_i \vec{F}_i = \frac{\Lambda}{3} m \vec{r}_{\text{cm}} + 0 ,$$

where the second term is zero by the definition of center of mass since the displacement of center of mass relative to itself is zero.

- c) The simplifying conclusion is that the net cosmological constant force on a system of point masses is the same as if all the mass of the system were concentrated at the center of mass of the system. Since any object can be considered as made of point masses, it follows that the net cosmological constant force on an object is the same as if all the mass of the object were concentrated at the center of mass of the object.

**Redaction:** Jeffery, 2018jan01

003 qfull 00700 1 3 0 easy math: Friedmann equation and Hubble law derivations

12. The Friedmann equation of general relativity (GR) cosmology in its most standard form (e.g., Wikipedia: Friedmann equations: Equations) is

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3} ,$$

where  $H$  is the Hubble parameter (which at current cosmic time is the Hubble constant  $H_0$  and has fiducial value 70 (km/s)/Mpc),  $a$  is the cosmic scale factor,  $\dot{a}$  is the time derivative of the cosmic scale factor with respect to cosmic time  $t$ ,  $G = 6.67430(15) \times 10^{-11}$  J m/kg<sup>2</sup> is the gravitational constant,  $\rho$  is the density of a uniform perfect fluid (in old-fashioned jargon AKA the cosmological substratum: Bo-75-76) which is used to model the universal mass distribution,  $k$  is called the curvature (Li-24,28)  $k/(c^2 a^2)$  is called Gaussian curvature (CL-12,29),  $c = 2.99792458 \times 10^8$  m/s is the vacuum light speed as usual. and  $\Lambda$  is the cosmological constant which is the simplest form of the dark energy even though is only a form of energy in one interpretation. Note,  $k$  is often defined with an unabsorbed  $c^2$ : i.e., the shown  $k$  is replaced by  $kc^2$ .

There are parts a,b,c. Some of the parts can be done independently, and so do not stop if you cannot do a part. During exams do **ONLY** parts a,b,c,d.

- a) Without words prove the Friedmann equation starting from the work-energy theorem

$$E_{\text{mechanical}} = \frac{1}{2} m v^2 - \frac{GMm}{r} - \frac{1}{2} \frac{\Lambda}{3} m r^2 ,$$

where  $m$  is the mass of a test particle.

- b) Without words prove the general Hubble law  $v = Hr$ , where  $v$  is recession velocity (i.e., the velocity between comoving frames) and  $r$  is proper distance (i.e., the distance measurable in with a ruler at one instant in cosmic time).
- c) What is the asymptotic Hubble law (i.e., Hubble law valid in the limit  $z \rightarrow 0$ )?

**SUGGESTED ANSWER:**

- a) Behold:

$$\begin{array}{ll} 1) \quad U_g = \frac{GMm}{r} = \frac{4\pi G \rho r^2 m}{3} & 2) \quad r = ar_0 \\ 3) \quad \frac{2E_{\text{mechanical}}}{m r_0^2 a^2} = \left( \frac{\dot{a}}{a} \right)^2 - \frac{8\pi G \rho}{3} - \frac{\Lambda}{3} & 4) \quad H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} . \end{array}$$

b) Behold:

$$1) \quad r = ar_0 \qquad 2) \quad \dot{r} = \dot{a}r_0 \qquad 3) \quad v = \left(\frac{\dot{a}}{a}\right) r \qquad 4) \quad v = Hr .$$

c) As  $z \rightarrow 0$ ,  $v \rightarrow zc$  and  $r \rightarrow r_{\text{luminosity}}$ , and so

$$zc = Hr_{\text{luminosity}} .$$

**Redaction:** Jeffery, 2018jan01

003 qfull 00820 1 3 0 easy math: 1-component solutions to the scaled Friedmann equation

13. The scaled Friedmann equation for multi-component (power-law) density components is

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_p \Omega_{p,0} x^{-p} ,$$

where 0 indicates the fiducial time which may be cosmic present,  $h = H/H_0$  is the scaled Hubble parameter with  $H_0$  being the Hubble constant,  $x = a/a_0$  is the scaled cosmic scale factor,  $x_0 = 1$ ,  $\dot{x} = dx/d\tau$  is the rate of change of the scaled cosmic scale factor,  $\tau = H_0 t = t/t_{H_0}$  is the scaled time with  $t_{H_0}$  being the Hubble time, the  $\Omega_{p,0}$  are the density parameters for the density components at the fiducial time with their sum being 1, and  $p$  are the powers of the power-law density components.

**NOTE:** There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c,d.

- a) Without words, derive the general asymptotic solution  $\tau(x)$  and its inverse  $x(\tau)$  for the leading density component as  $\tau \rightarrow 0$  (i.e., the density component with highest  $p$ ). As a shorthand, this solution can be called the early universe solution. Assume  $p > 0$ . To avoid pointless generality, assume  $x(\tau = 0) = 0$  (i.e., there is a point origin in time at time zero).
- b) Without words, derive early universe formula for  $\Omega_p(\tau) = \Omega_p[x(\tau)]$  for  $p > 0$ .
- c) Without words, derive the special case early universe solutions for  $p = 1, 2, 3, 4, 5$ .
- d) Without words, derive the Hubble parameter  $h = \dot{x}/x$  and the deceleration parameter  $q = -\ddot{x}x/(\dot{x})^2 = -\ddot{x}/(xh^2)$  for the general early universe with  $p > 0$ . Simplify the latter as much as possible. For what  $p$  values is the universe in positive/zero/negative acceleration?
- e) We now assume the universe has only one density component with power  $p > 0$ . Without words, derive the general age of the universe formula (which we assume to the fiducial time where  $x = 1$ ) for  $\tau$  and  $t$  and give the fiducial value version for  $t$  with the Hubble time  $t_{H_0} = (13.968 \dots \text{Gyr})/h_{70}$ , where  $h_{70} = H_0/[70 \text{ (km/s)/Mpc}]$ .
- f) We assume the universe has only one density component with power  $p = 0$ . Without words, derive  $x(\tau)$  and  $x(t)$  assuming  $x(0) = 1$ . Note, this universe is the de Sitter universe and the Hubble constant  $H_0 = \sqrt{\Lambda/3}$ .
- g) Students are now welcome to view a table in the answer to this part that presents the single density component solutions plus relevant features for powers  $p = 5, 4, 3, 2, 1, 0$ . Note, if we assume that the dependence of the density components on the scale factor is due to a perfect fluid pressure obeying the equation of state  $p_{\text{pressure}} = w\rho c^2$  where  $w$  is a constant parameter (with no special name), then power

$$p = 3(1 + w) .$$

The  $w$  values are included in the table.

**SUGGESTED ANSWER:**

a) Behold:

$$1) \quad d\tau = \frac{dx}{\sqrt{\Omega_{p,0} x^{-p/2+1}}} \qquad 2) \quad d\tau = \frac{1}{\sqrt{\Omega_{p,0}}} x^{p/2-1} dx$$

$$3) \quad \tau = \frac{1}{\sqrt{\Omega_{p,0}}} \left(\frac{2}{p}\right) x^{p/2} \qquad 4) \quad x = \left[\sqrt{\Omega_{p,0}} \left(\frac{p}{2}\right) \tau\right]^{2/p} .$$

b) Behold:

$$\Omega_p = \Omega_{p,0} x^{-p} = \Omega_{p,0} \left[ \sqrt{\Omega_{p,0}} \left( \frac{p}{2} \right) \tau \right]^{-2} = \left[ \left( \frac{p}{2} \right) \tau \right]^{-2} .$$

Remarkably, no matter what the  $p > 0$  value, the density falls as  $1/\tau^2$  and there is no dependence on  $\Omega_{p,0}$ .

c) Behold:

$$x = \begin{cases} \left[ \sqrt{\Omega_{p,0}} \left( \frac{p}{2} \right) \tau \right]^{2/p} & \text{in general for } p > 0. \\ \left[ \sqrt{\Omega_{1,0}} \left( \frac{1}{2} \right) \tau \right]^2 & \text{for } p = 1 \text{ which could be an early quintessence universe.} \\ \left[ \sqrt{\Omega_{2,0}} (1) \tau \right]^1 & \text{for } p = 2 \text{ which could be an early} \\ & \text{cosmic string universe} \\ & \text{or an early } R_h = ct \text{ universe.} \\ \left[ \sqrt{\Omega_{3,0}} \left( \frac{3}{2} \right) \tau \right]^{2/3} & \text{for } p = 3 \text{ which is an early matter universe.} \\ \left[ \sqrt{\Omega_{4,0}} (2) \tau \right]^{1/2} & \text{for } p = 4 \text{ which is an early radiation universe.} \\ \left[ \sqrt{\Omega_{5,0}} \left( \frac{5}{2} \right) \tau \right]^{1/2} & \text{for } p = 5 \text{ which is for the impossible case} \\ & \text{of an early universe dominated by classical kinetic energy.} \\ & \text{Impossible because rest-mass mass-energy must dominate} \\ & \text{if there were classical kinetic energy: i.e.,} \\ & \text{the universe would have to be an early matter universe.} \end{cases}$$

d) Behold:

$$h = \frac{\dot{x}}{x} = \left( \frac{2}{p} \right) \left( \frac{1}{\tau} \right)$$

and

$$q = -\frac{\ddot{x}x}{(\dot{x})^2} = -\frac{\ddot{x}}{xh^2} = -\left( \frac{2}{p} \right) \left( \frac{2}{p} - 1 \right) \left( \frac{1}{\tau^2} \right) \left( \frac{p}{2} \right)^2 \tau^2 = -\left( \frac{p}{2} \right) \left( \frac{2}{p} - 1 \right) = \frac{p}{2} - 1 ,$$

where for the universe, we find

$$\begin{cases} p > 2 & \text{for negative acceleration.} \\ p = 2 & \text{for zero acceleration.} \\ p < 2 & \text{for positive acceleration.} \end{cases}$$

e) Behold:

$$\tau = \frac{2}{p} \quad t = \frac{2}{p} t_{H_0} = \frac{2}{p} \left[ \frac{(13.968 \dots \text{Gyr})}{h_{70}} \right] .$$

f) Behold:

$$1) \quad d\tau = \frac{dx}{x} \quad 2) \quad \tau - \tau_0 = \ln \left( \frac{x}{x_0} \right) \quad 3) \quad x = e^\tau \quad 4) \quad x = e^{(\sqrt{\Lambda/3})t} .$$

The scaled Hubble parameter and deceleration parameter are, respectively

$$h = 1 \quad \text{and} \quad q = -1$$

and the unscaled versions are

$$H = \sqrt{\frac{\Lambda}{3}} \quad \text{and} \quad q = -1 .$$

g) The table is below.

Table: Single Density Component Solutions for the Friedmann Equation Plus Relevant Features for Powers $p = 5, 4, 3, 2, 1, 0$						
$p \backslash$ Quantity	$w$	$\gamma = \frac{2}{p}$	$a(t)$	$t_0 = \frac{\gamma}{H_0}$	$q_0 = \frac{1}{\gamma} - 1$	$\rho$
$3(1+w)$	$\left\{ \begin{array}{l} w \text{ for} \\ w \neq -1 \end{array} \right\}$	$\frac{2}{[3(1+w)]}$	$a_0 \left( \frac{t}{t_0} \right)^\gamma$	$\gamma \left( \frac{13.968 \text{ Gyr}}{h_{70}} \right)$	$\left\{ \begin{array}{l} \frac{1}{2}(1+3w) \\ = \frac{p}{2} - 1 \end{array} \right\}$	$\rho_0 \left( \frac{t_0}{t} \right)^2$
$p = 5$	$w = \frac{2}{3}$	$\gamma = \frac{2}{5}$	$a_0 \left( \frac{t}{t_0} \right)^{2/5}$	$\frac{2}{5} \frac{1}{H_0}$	$\frac{3}{2}$	$\rho_0 \left( \frac{t_0}{t} \right)^2$
$p = 4$	$w = \frac{1}{3}$	$\gamma = \frac{1}{2}$	$a_0 \left( \frac{t}{t_0} \right)^{1/2}$	$\frac{1}{2} \frac{1}{H_0}$	1	$\rho_0 \left( \frac{t_0}{t} \right)^2$
$p = 3$	$w = 0$	$\gamma = \frac{2}{3}$	$a_0 \left( \frac{t}{t_0} \right)^{2/3}$	$\frac{2}{3} \frac{1}{H_0}$	$\frac{1}{2}$	$\rho_0 \left( \frac{t_0}{t} \right)^2$
$p = 2$	$w = -\frac{1}{3}$	$\gamma = 1$	$a_0 \left( \frac{t}{t_0} \right)$	$\frac{1}{H_0}$	0	$\rho_0 \left( \frac{t_0}{t} \right)^2$
$p = 1$	$w = -\frac{2}{3}$	$\gamma = 2$	$a_0 \left( \frac{t}{t_0} \right)^2$	$\frac{2}{H_0}$	$-\frac{1}{2}$	$\rho_0 \left( \frac{t_0}{t} \right)^2$
$p = 0$	$w = -1$	$\gamma = \infty$	$a_0 e^{H_0(t-t_0)}$	$\infty$	-1	$\rho_0$

**Redaction:** Jeffery, 2018jan01

003 qfull 01130 1 3 0 easy math: perfect fluid solutions

14. The differential equation (DE) for the perfect fluid of Friedmann equation cosmology is

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) ,$$

where  $\rho$  is mass-energy in the comoving frames of Friedmann equation cosmology and  $p$  is isotropic pressure in those frames (in some sense) (Liddle 26). The perfect fluid DE can be derived rigorously from general relativity (Carroll 333–334) and, perhaps somewhat fudgily, from classical thermodynamics and special relativity. Remarkably, this equation does not guarantee conservation of energy in the ordinary sense of classical physics: it does embody the general relativity feature that the covariant derivative of the energy-momentum tensor is zero (Carroll 117,120): i.e., the energy-momentum conservation equation. General relativity may or may not in some sense conserve energy for cosmology, but certainly gravitating mass-energy is allowed to appear and disappear by the perfect fluid DE.

Multiple perfect fluids can exist and if they are assumed to act independently (which is the usual cosmological assumption), then they all obey their own perfect fluid DE: i.e., for perfect fluid  $i$ ,

$$\dot{\rho}_i = -3 \frac{\dot{a}}{a} \left( \rho_i + \frac{p_i}{c^2} \right) .$$

In current standard cosmology (i.e., the  $\Lambda$ CDM model or simple variations thereof), it is assumed that the perfect fluid equation of state (EOS) is of the form

$$p = w \rho c^2 ,$$

where  $w$  is a constant parameter that seems to have no special name. Most standard/interesting values

of  $w$  are given by

$$w = \begin{cases} 2/3 & \text{for nonrelativistic kinetic energy of free-streaming nonrelativistic particles} \\ & \text{(e.g., late-time cosmic neutrinos).} \\ 1/3 & \text{for extreme relativistic (ER) mass-energy (AKA "radiation"): most obviously photons,} \\ & \text{but also the ER neutrinos of the Big Bang era} \\ & \text{and to some later not perfectly known cosmic time;} \\ 0 & \text{for nonrelativistic (NR) rest mass-energy (AKA "matter"} \\ & \text{or "dust": Liddle-40);} \\ -1/3 & \text{for zero-acceleration (or constant } \dot{a}) \text{ universes such as} \\ & \text{Fulvio Melia's } R_h = ct \text{ universe, cosmic strings or a universe with cosmic scale} \\ & \text{determined only by negative curvature } k. \\ -2/3 & \text{for some kinds of quintessence.} \\ -1 & \text{for cosmological constant or equivalently (constant) dark energy;} \end{cases}$$

Solve for the formula for  $\rho(a)$  for general  $w$  and the 6 special cases of  $w$  listed above. Assume  $a_0$  and  $\rho_0$  for cosmic present values.

**SUGGESTED ANSWER:**

Behold:

$$1) \quad \dot{\rho} = -3\frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) = -3\frac{\dot{a}}{a}(1+w)\rho \quad 2) \quad \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \quad 3) \quad \ln\left(\frac{\rho}{\rho_0}\right) = -3(1+w)\ln\left(\frac{a}{a_0}\right)$$

$$\rho = \begin{cases} \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} & \text{in general.} \\ \rho_0 \left(\frac{a_0}{a}\right)^4 & \text{for } w = 2/3. \\ \rho_0 \left(\frac{a_0}{a}\right)^4 & \text{for } w = 1/3. \\ \rho_0 \left(\frac{a_0}{a}\right)^3 & \text{for } w = 0. \\ \rho_0 \left(\frac{a_0}{a}\right)^2 & \text{for } w = -1/3. \\ \rho_0 \left(\frac{a_0}{a}\right)^1 & \text{for } w = -2/3/. \\ \rho_0 & \text{for } w = -1. \end{cases}$$

**Redaction:** Jeffery, 2018jan01

003 qfull 01260 1 3 0 easy math: quick derivation Friedmann, fluid, and acceleration equations

15. Here we do the quick derivations of the Friedmann equation, the fluid equation, the Friedmann acceleration equation, and some other results.

**NOTE:** There are parts a,b,c,d,e. On exams, do **ONLY** parts a,b,c,d. Some of the parts can be done independently, and so do not stop if you cannot do a part.

- a) Without words, derive the Friedmann equation in standard form (with the cosmological constant force  $F_\Lambda = (\Lambda/3)mr$  included) from classical physics with the hypotheses that all free-fall frames are elementary inertial frames (as told to us by general relativity) and that the shell theorem for a spherically symmetric mass distribution can be extended to infinite distance (which is validated by Birkhoff's theorem from general relativity). The derivation makes use of the classical conservation of mechanical energy. You should end up with a  $-k/a^2$  term among other things. You can draw a diagram if you like. **HINT:** Start with the conservation of mechanical energy of a test particle of mass  $m$ :

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} - \left(\frac{1}{2}\right)\frac{\Lambda}{3}mr^2 .$$

- b) Without words and starting from the 1st law of thermodynamics

$$dE = T dS - p dV + \mu dN ,$$

derive the cosmological fluid equation in standard form (which means with  $dS = 0$  and  $dN = 0$ ) and in a form with  $\dot{\rho}a/\dot{a}$  equal to something for use in part (d). Recall the rest-frame energy is  $E = \rho c^2 V$ .

- c) Specialize the fluid equation to the special case where the equation of state is  $p = w\rho c^2$  where  $w$  is the equation-of-state constant (which seems to have no special name). Determine the explicit solution  $\rho(a)$  for the special case where  $\rho_0 = \rho(a_0)$ . **HINT:** You will have to eliminate the time derivative.
- d) Without words, derive the acceleration equation (or Friedmann acceleration equation) in standard form using parts (a) and (b). A subtle point is that you have to assume that the gravitational potential energy formula continues to be valid (though perhaps with a different meaning) for cases where mass is not conserved. There is an argument why it should, but that is beyond the scope of this question.
- e) Without words, derive from the Friedmann equation the de Sitter universe solution which has  $\rho = 0$  and  $k = 0$ , but  $\Lambda \neq 0$ .

**SUGGESTED ANSWER:**

- a) Behold:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} - \left(\frac{1}{2}\right) \frac{\Lambda}{3}mr^2 = \frac{1}{2}mv^2 - \frac{4\pi Gmr^2\rho}{3} - \left(\frac{1}{2}\right) \frac{\Lambda}{3}mr^2$$

$$E = \frac{1}{2}m\dot{a}^2r_0^2 - \frac{4\pi G\rho ma^2r_0^2}{3} - \left(\frac{1}{2}\right) \frac{\Lambda}{3}ma^2r_0^2 = \left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3} + \frac{\Lambda}{3}\right] \left(\frac{1}{2}ma^2r_0^2\right)$$

$$\frac{2E}{ma^2r_0^2} = \left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3}$$

$$-\frac{k}{a^2} = \left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} \quad \text{with} \quad k \equiv -\frac{2E}{mr_0^2}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}.$$

- b) Behold:

$$dE = T dS - p dV + \mu dN \quad d(\rho c^2 V) = -p dV \quad V d\rho + \rho dV = -\frac{p}{c^2} dV$$

$$\dot{\rho} = -\frac{\dot{V}}{V} \left(\rho + \frac{p}{c^2}\right) \quad \text{with} \quad \frac{\dot{V}}{V} = \frac{3a^2\dot{a}}{a^3} = 3\frac{\dot{a}}{a}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2}\right) \quad \text{and for use in part (d)} \quad \dot{\rho}\frac{a}{\dot{a}} = -3 \left(\rho + \frac{p}{c^2}\right).$$

- c) Behold:

$$1) \quad \dot{\rho} = -3\frac{\dot{a}}{a}(1+w)\rho \quad 2) \quad \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$3) \quad \ln\left(\frac{\rho}{\rho_0}\right) = -3(1+w)\ln\left(\frac{a}{a_0}\right) \quad 4) \quad \rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$$

- d) Behold:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k + \left(\frac{\Lambda}{3}\right)a^2$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\dot{a}) + \frac{\Lambda}{3}(2a\dot{a})$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3} \left(\dot{\rho}\frac{a}{\dot{a}} + 2\rho\right) + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3} \left( -3\rho - \frac{3p}{c^2} + 2\rho \right) + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + 3\frac{p}{c^2} \right) + \frac{\Lambda}{3} .$$

e) Behold:

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\Lambda}{3} \quad \frac{da}{a} = \sqrt{\frac{\Lambda}{3}} dt \quad a = a_0 e^{\left[ \sqrt{(\Lambda/3)}(t-t_0) \right]} .$$

Note, the Hubble parameter  $H^2$  is actually a constant for the de Sitter universe: i.e.,  $H = \sqrt{(\Lambda/3)}$ .

**Redaction:** Jeffery, 2018jan01

003 qfull 01270 1 3 0 easy math: scaled Friedmann equation and scaled acceleration equation

16. The Friedmann equation in the most standard form is

$$H^2 = \left( \frac{dx/dt}{x} \right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{x^2} + \frac{\Lambda}{3} ,$$

where we use  $x$  rather than  $a$  as the cosmic scale factor since yours truly finds  $a$  intolerable as a variable.

**NOTE:** There are parts a,b,c,d,f,g. On exams, omit parts d,e,f,g.

- a) Write the right-hand side of the Friedmann equation as  $H_0$  times a bracketed factor. The quantity  $H_0$  is the Hubble constant for a fiducial time  $t_0$  which for expanding solutions with a point origin is the time since the point origin and is usually taken as cosmic present, but it could be any time. What is the bracketed factor equal to at fiducial time  $t_0$ ?
- b) Define scaled time by  $d\tau = H_0 dt$  and let  $\dot{x} = dx/d\tau$ . Now derive the scaled Friedmann equation

$$h^2 = \left( \frac{\dot{x}}{x} \right)^2 = \Omega_{\text{ME}} + \Omega_k + \Omega_\Lambda ,$$

where  $\Omega_{\text{ME}}$  is the mass-energy density parameter,  $\Omega_k$  is the curvature density parameter,  $\Omega_\Lambda$  is the cosmological constant or  $\Lambda$  density parameter, and the scalings are  $x = a$ ,  $d\tau = H_0 dt$ ,  $\dot{x} = dx/d\tau$ ,  $h = H/H_0$ , and  $\rho_c = 3H_0^2/(8\pi G)$ . Note, the subscript 0 indicates fiducial time  $t_0$  where  $H = H_0$  and which is often chosen to be cosmic present. Note, the density parameters defined to sum to 1 at the fiducial time  $t_0$ . Give the explicit expressions for the critical density  $\rho_c$  defined by  $\Omega_{\text{ME}} = \rho/\rho_c$ ,  $\rho_k = \rho_c \Omega_k$ , and  $\rho_\Lambda = \rho_c \Omega_\Lambda$ . What is the curvature formula at the fiducial time  $t_0$  or  $\tau_0$ : i.e., the formula for  $\Omega_{k,0}$ ? What does the value  $\Omega_{k,0}$  imply as we know from general relativity?

- c) We assume the pressure of a cosmological density component follows from a simple equation of state  $P = w\rho c^2$  where  $w$  is a constant which seems to have no special name. Given the density evolution formula

$$\dot{\rho} = -3\frac{\dot{x}}{x} \left( \rho + \frac{P}{c^2} \right) ,$$

determine the  $w$  values for the commonly consider powers  $p = 0, 1, 2, 3, 4, 5$  for (inverse) power-law density parameter evolution

$$\rho = \rho_0 x^{-p} ,$$

where set  $x = 1$  at fiducial time  $t_0$  (which is a usual, but not necessary assumption). Note,  $p = 0$  for the cosmological constant (or constant dark energy)  $p = 1$  for some kinds of quintessence  $p = 2$  for curvature, maybe cosmic strings, and the  $R_c = ht$  universe,  $p = 3$  for matter (i.e., non-relativistic rest mass),  $p = 4$  for radiation (i.e., photons or extreme relativistic particles), and  $p = 5$  for nonrelativistic kinetic energy of free-streaming particles (e.g., nonrelativistic cosmic neutrinos).

- d) Write the scaled Friedmann equation for a general set of density components obeying (inverse) power laws including those for the cosmological constant and curvature. Assume  $x = 1$  as the fiducial time  $t_0$ .
- e) Given part (d), derive the formula for  $\ddot{x}$ .

- f) Given parts (d) and (e), derive the formula for the deceleration parameter whose general formula is

$$q = -\frac{\ddot{x}x}{\dot{x}^2}$$

(Li-53), where the annoying minus sign is because people expected a decelerating universe once and wanted a positive diagnostic. The deceleration parameter is a sort of scaled acceleration. Which terms tend to acceleration, coasting, deceleration?

- g) Given the single density component elementary solution for power  $p > 0$

$$\tau = \frac{1}{\sqrt{\Omega_{p,0}}} \left(\frac{2}{p}\right) x^{p/2},$$

determine the fiducial time  $t_0$  in terms of  $p$  and  $H_0$ , in terms of  $p$  and Hubble time  $t_{H_0}$ , in terms of  $q$  and  $H_0$ , and  $q$  and Hubble time  $t_{H_0}$ . What is the condition for  $t_0 = t_{H_0}$ ?

**SUGGESTED ANSWER:**

- a) Behold:

$$H^2 = \left(\frac{dx/dt}{x}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{x^2} + \frac{\Lambda}{3}$$

$$H^2 = \left(\frac{dx/dt}{x}\right)^2 = H_0^2 \left(\frac{8\pi G\rho}{3H_0^2} - \frac{k}{a^2 H_0^2} + \frac{\Lambda}{3H_0^2}\right).$$

At time  $t_0$ , the bracketed factor must equal 1.

- b) Behold:

$$H^2 = \left(\frac{dx/dt}{x}\right)^2 = H_0^2 \left(\frac{8\pi G\rho}{3H_0^2} - \frac{k}{a^2 H_0^2} + \frac{\Lambda}{3H_0^2}\right)$$

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \Omega_{ME} + \Omega_k + \Omega_\Lambda$$

where

$$1) \quad \rho_c = \frac{3H_0^2}{8\pi G} \quad 2) \quad \rho_k = \rho_c \Omega_k = \frac{3H_0^2}{8\pi G} \left(-\frac{k}{x^2 H_0^2}\right) = -\frac{3k}{x^2(8\pi G)} \quad 3) \quad \rho_\Lambda = \frac{3H_0^2}{8\pi G} \frac{\Lambda}{3H_0^2} = \frac{\Lambda}{8\pi G}.$$

At the fiducial time  $t_0$  where  $h = h_0 = 1$ , the curvature formula is

$$1 - \Omega_{ME,0} - \Omega_\Lambda = \begin{cases} \Omega_{k,0} > 0 & \text{implies } k < 0 \text{ and hyperbolic space.} \\ \Omega_{k,0} = 0 & \text{implies } k = 0 \text{ and Euclidean or flat space.} \\ \Omega_{k,0} < 0 & \text{implies } k > 0 \text{ and hyperspherical space.} \end{cases}$$

- c) Behold:

$$1) \quad \dot{\rho} = -3\frac{\dot{x}}{x} \left(\rho + \frac{P}{c^2}\right) = -3\frac{\dot{x}}{x} \rho(1+w) \quad 2) \quad \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{x}}{x} \quad 3) \quad \ln(\rho) = -3(1+w)\ln(x) + C$$

$$4) \quad \rho = \rho_0 x^{-3(1+w)} \quad 5) \quad w = \begin{cases} p/3 - 1 & \text{in general.} \\ -1 & \text{for } p = 0. \\ -2/3 & \text{for } p = 1. \\ -1/3 & \text{for } p = 2. \\ 0 & \text{for } p = 3. \\ 1/3 & \text{for } p = 4. \\ 2/3 & \text{for } p = 5. \end{cases}$$

- d) Behold:

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_p \Omega_p x^{-p}.$$

e) Behold:

$$1) \quad \dot{x}^2 = \sum_p \Omega_p x^{-p+2} \quad 2) \quad 2\dot{x}\ddot{x} = \sum_p \Omega_p x^{-p+1}(-p+2)\dot{x} \quad 3) \quad \ddot{x} = \sum_p \Omega_p x^{-p+1}(-p/2+1) .$$

f) Behold:

$$q = -\frac{\ddot{x}x}{\dot{x}^2} = \frac{\sum_p \Omega_p x^{-p}(p/2-1)}{\sum_p \Omega_p x^{-p}} .$$

Clearly,  $p < 2$  tends to acceleration,  $p = 2$  to coasting, and  $p > 2$  to deceleration. For  $p < 3$ , one sort of has an antigravity effect which in the classical Friedmann equation picture can be interpreted as terms of decreasing potential energy as  $x$  increases. To balance the Friedmann equation, the decreasing potential energy with  $x$  means the classical kinetic energy must increase with  $x$ .

g) Behold:

$$1) \quad \tau_0 = \frac{2}{p} = \frac{2}{2q+2} = \frac{1}{q+1} \quad 2) \quad t_0 = \frac{2}{pH_0} = \frac{2t_{H_0}}{p} = \frac{1}{(q+1)H_0} = \frac{t_{H_0}}{q+1} .$$

The condition for  $t_0 = t_{H_0}$  is  $p = 2$  or  $q = 0$ . This elementary solution is a purely coasting solution.

**Redaction:** Jeffery, 2018jan01

002 qfull 00510 1 3 0 easy math: simple 1st order DE solution

17. Consider the following linear 1st order autonomous differential equation (DE):

$$x' = A - kx ,$$

where  $t$  is the independent variable,  $A > 0$  is a constant, and  $k > 0$  is the rate constant. Note, the DE is autonomous because there is no explicit dependence on the independent variable  $t$ .

There are parts a,b,c,d. On exams, omit parts b,d.

a) Solve for the constant solution  $x_A$ . **HINT:** This is easy.

b) We can now write the DE as

$$x' = k(x_A - x) .$$

Without solving for non-constant solution describe what it must look like as a function of  $t$  for arbitrary initial value  $x_0 = x(t=0) \neq x_A$ . Note,  $x_0$  can be greater or less than  $x_A$ . In particular, where are the solutions stationary points if there are any? **HINT:** Consider the continuity of all orders of derivative of  $x$ .

c) Given  $x_0 = x(t=0)$ , solve for the solution  $x(t)$  from the differential equation form  $x' = k(x_A - x)$  and give its 1st order in small  $t$  solution  $x_{1st}(t)$  and its asymptotic solution as  $t \rightarrow \infty$ . Verify that the  $x(t)$  is monotonic with its only stationary point at  $t = \infty$ . **HINT:** You can use an integrating factor, but there is a more straightforward way.

d) What is the  $e$ -folding time  $t_e$  of your solution and what does it signify? What is remarkable about  $x_{1st}(t_e)$ ?

**SUGGESTED ANSWER:**

a) The constant solution has  $x' = 0$  everywhere. Therefore

$$x_A = \frac{A}{k} .$$

b) Consider intelligently

$$x' = k(x_A - x) .$$

If  $x_A$  is greater/less than  $x_0$ , then  $x'$  is greater/less than 0, and then  $x$  must increase/decrease until there is a maximum/minimum at  $x = x_A$ , where  $x' = 0$ . Now since  $x^{(n)} = -kx^{(n-1)}$  for

all  $n \geq 2$ , all orders of derivative must go to zero at the same time  $t$  without discontinuities. But for any finite time, there must be a discontinuity in some derivative for them all to go to zero at the same time since the function goes perfectly flat at that time. Therefore,  $x'$  can only go to zero at infinity: i.e., asymptotically as  $t \rightarrow \infty$ . It follows at once that the only stationary point is at infinity: it's a maximum/minimum for  $x_0$  is less/greater than  $x_A$ .

c) Behold:

$$1) \quad x' = k(x_A - x) \quad 2) \quad \frac{dx}{x_A - x} = k \quad 3) \quad -\ln(x_A - x) = kt + C$$

$$4) \quad x_A - x = (x_A - x_0)e^{-kt} \quad 5) \quad x = \begin{cases} x_0 e^{-kt} + x_A(1 - e^{-kt}) & \text{in general.} \\ x_0(1 - kt) + x_A kt & \text{to 1st order in small } t. \\ x_A & \text{the asymptotic solution for } t \rightarrow \infty. \end{cases}$$

We find

$$x = k(x_A - x_0)e^{-kt}$$

which is monotonic increasing/decreasing for  $x_A$  greater/less than  $x_0$  and the only stationary point is clearly at  $t = \infty$ . Note, the solution  $x(t)$  matches the description of part (b).

Just for completeness, using an integrating factor, one obtains the solution thusly:

$$1) \quad x' = A - kx \quad 2) \quad x' + kx = A \quad 3) \quad gx' + gkx = gA \quad 4) \quad (gx)' = gx' + g'x$$

$$5) \quad g' = gk \quad 6) \quad g = e^{kt} \quad 7) \quad (gx)' = gA \quad 8) \quad e^{kt}x|_{t=0}^t = (A/k)e^{kt}|_{t=0}^t$$

$$9) \quad e^{kt}x - x_0 = x_A(e^{kt} - 1) \quad 10) \quad x = x_0 e^{-kt} + x_A(1 - e^{-kt}) .$$

d) Behold:  $t_e = 1/k$  and  $t = t_e$  is the fiducial limit time for validity of the 1st order solution. At  $t = t_e$ , we have

$$x_{1st}(t = t_e) = x_0(1 - kt_e) + x_A kt_e = x_A .$$

Remarkably,  $x_{1st}(t = t_e)$  is independent of  $x_0$  and equals the asymptotic value  $x_A$ .

**Redaction:** Jeffery, 2018jan01

002 qfull 00640 1 3 0 easy math: 1st order autonomous DE and stationary points, definitive version

**Extra keywords:** This is the definitive version as of 2025mar09

18. First order autonomous ordinary differential equations (FAODEs), linear or nonlinear, only have solutions with stationary points at infinity (SPIs), (except for special cases which are not all that rare) and constant solutions. Actually, each SPI corresponds to a constant solution which could also be viewed as a continuum of stationary points. Note, an autonomous differential equation depends only on functions of the dependent variable, and so has no explicit dependence on the independent variable.

To investigate the SPI behavior of FAODEs consider the (somewhat general) FAODE

$$x^{(1)} = [f(x)]^{1/k} ,$$

where  $t$  (not necessarily time) is the independent variable, the superscript (1) means 1st derivative with respect to  $t$ ,  $f(x)$  is an infinitely differentiable function with zeros at set of values  $\{x_i\}$ , and  $k > 0$ . We limit  $k$  to being greater than zero to avoid uninteresting generality. Since  $f(x)$  is infinitely differentiable at (general)  $x_i$ , we can expand  $f(x)$  about  $x_i$  with some radius of convergence: i.e.,

$$f(\Delta x) = \sum_{j=\ell}^{\infty} \Delta x^j f_j = \Delta x^\ell f_\ell + \dots ,$$

where  $\Delta x = x - x_i$ , the  $f_j$  are expansion constants, and  $\ell > 0$  is the lowest (nonzero) order in the expansion. Note,  $\ell \neq 0$  since we have assumed  $x_i$  is a zero of  $f(x)$ : i.e.,  $f(x_i) = 0$ .

We will primarily be examining the lowest order solutions in  $\Delta x$ , and so we will be dealing with  $\Delta x^{\ell/k} f_\ell^{1/k}$  and related expressions. Mathematically, if  $\ell/k$  is not an integer, complex numbers can arise in these expressions. However, we are only interested FAODEs and their solutions corresponding to physical systems involving real numbers. In these systems, the solutions just never evolve into

the complex number realm. So we are not going to concern ourselves with question what happens mathematically if some our expressions can give rise to complex numbers. They never give rise to complex numbers physically.

**NOTE:** There are parts a,b,c,d,e,f,g,h,i,j,k. On exams, do **ONLY** parts i,j.

- What is the behavior of  $x$  as a function of  $t$  between the points in the set  $\{x_i\}$ .
- In this question we are only interested in the SPI behavior and constant solution behavior, and so we are only interested in the behavior of  $x(t)$  when it is arbitrarily close to  $x_i$  where SPI and constant solutions occur. Therefore expand the FAODE about  $x_i$  with dependent variable  $\Delta x$  to lowest order in the exponent.
- Determine the formula  $p(n)$  for the exponent of  $\Delta x$  in the  $n$  derivative of  $\Delta x$  (for the lowest order of the FAODE) with respect to  $t$ . **HINT:** Drop all constants that turn up in the differentiations.
- What is behavior of the  $t$  derivatives of  $\Delta x$  when  $x = x_i$  for  $\ell/k \geq 1$ ? What solutions  $x(t)$  are implied by  $\ell/k \geq 1$ ?
- What is behavior of the  $t$  derivatives of  $\Delta x$  for  $f(x_i)$  for  $\ell/k < 1$  assuming the formula  $p(n)$  never equals zero? What solution  $x(t)$  behavior is implied by  $\ell/k < 1$  in this case? Only a short answer is expected to the last question.
- If  $\ell/k < 1$  and the formula  $p(n)$  goes to zero for a stopping  $n_{st}$ , what is the formula for  $\ell/k$  as a function of  $n_{st}$  and what are the values of  $\ell/k$  for the set  $n_{st} = 1, 2, 3, \dots, \infty$  and what do the  $n_{st} = 1$  and  $n_{st} = \infty$  cases mean? What is the formula  $n_{st}$  as a function of  $\ell/k$ ? What is this formula good for?
- What is implied by a stopping  $n_{st} \in [2, \infty)$  (i.e., an actual integer  $n_{st}$  in this range)? Give the solution for small  $\Delta x(t)$  with with initial condition  $\Delta x(t = 0) = 0$ . Describe the function behavior at  $\Delta x(t = 0) = 0$ : i.e., maximum or minimum stationary point or rising or falling inflection point.
- What would you expect the two likeliest values for  $\ell$  to be for physically relevant FAODEs? What would you expect the two likeliest value for  $k \neq 1$  to be for physically relevant FAODEs?
- Now we intuited for the case of  $\ell/k \geq 1$  that the stationary point would be a stationary point at infinity (i.e., an SPI), but we did not prove this directly. To prove directly, we need to show that the small  $\Delta x$  (meaning small in absolute value) solutions of

$$\Delta x^{(1)} = \Delta x^{\ell/k} f_\ell^{1/k}$$

that go to zero only do so as  $t \rightarrow \infty$ . Solutions that go to zero are convergent solutions. This means that the constant solutions they correspond to are stable solutions: small perturbations from the constant solutions damp out. Those that do not go to zero are divergent solutions. This means that the constant solutions they correspond to are unstable solutions: small perturbations from the constant solutions cause non-stopping divergence from the constant solutions.

Here consider the  $\ell/k = 1$  case and the solutions for  $\Delta x(t)$  starting from  $t = t_0$  and  $\Delta x = \Delta x_0$  as initial conditions. Determine the solutions and under what conditions they are convergent/divergent. Does the convergent solution, in fact, have a SPI? **HINT:** Let  $y = \pm \Delta x$  where the upper/lower case is for positive/negative  $\Delta x_0$ .

- Repeat part (i) for the case of  $\ell/k > 1$ .
- An optional continuation of the discussion of the part (h) answer.

#### SUGGESTED ANSWER:

- Since  $f(x)$  has no zeros between points in the set  $\{x_i\}$ ,  $x(t)$  has no stationary points there and must either increase or decrease always. Since  $f(x)$  is infinitely differentiable, it seems intuitively clear that  $x(t)$  cannot reach any  $x_i$ , except as  $t \rightarrow \infty$  since that would require some kind of singularity in some order of  $t$  derivative of  $x(t)$ . However, a proof is needed verify that the intuition is true. Of course, the exponent  $1/k$  could cause a singularity in some order of  $t$  derivative of  $x(t)$ , but we are not going concern ourselves with what happens in those cases.
- Behold:

$$\Delta x^{(1)} = \Delta x^{\ell/k} f_\ell^{1/k} .$$

c) Behold:

$$\begin{aligned}\Delta x^{(1)} &\propto \Delta x^{\ell/k} \\ \Delta x^{(2)} &\propto \Delta x^{\ell/k-1} \Delta x^{(1)} = \Delta x^{\ell/k-1} \Delta x^{\ell/k} = \Delta x^{2\ell/k-1} \\ \Delta x^{(3)} &\propto \Delta x^{3\ell/k-2} \\ &\vdots \\ \Delta x^{(n)} &\propto \Delta x^{n\ell/k-(n-1)} = \Delta x^{(\ell/k-1)n+1},\end{aligned}$$

where the generalization to the  $n$  derivative with respect to  $t$  is by inspection. The exponent formula is

$$p(n) = n\ell/k - (n-1) = (\ell/k - 1)n + 1.$$

- d) If  $\ell/k \geq 1$ , then exponent of the  $t$  derivatives of  $\Delta x$  strictly increases with  $n$  (if  $\ell/k > 1$ ) or is constant  $p = 1$ , and so in either case the  $t$  derivatives of  $\Delta x$  and  $x(t)$  are zero for  $x = x_i$ . This implies that for  $\ell/k \geq 1$ , there is a constant solution  $x(t) = x_i$ . We intuit that any other solution of  $x_i$  will converge to  $x_i$  as  $t \rightarrow \infty$  or diverge from  $x_i$ . However, a proof is needed verify that the intuition is true.
- e) If  $\ell/k < 1$  and  $p(n)$  never equals zero, then exponent of the  $t$  derivatives of  $\Delta x$  strictly decreases with  $n$ , and so there must be derivative  $\Delta x^{(n)}(x = x_i) = \infty$ : i.e., a derivative with a singularity. A singularity may cause a solution  $x(t)$  to reach  $x_i$  in a finite time, but what happens probably depends on the detailed behavior of  $f(x)$  and goes beyond the scope of this question. Yours truly guesses that most real physical solutions are unlikely to have such a singularity, except in idealized cases. Note, the derivatives of  $\Delta x$  for  $n$  lower than the one that gives the singularity will all be zero for  $x = x_i$ .
- f) The formula for  $\ell/k$  and the values for set  $n_{\text{st}} = 1, 2, 3, \dots, \infty$  are given by

$$\frac{\ell}{k} = 1 - \frac{1}{n_{\text{st}}} = \frac{n_{\text{st}} - 1}{n_{\text{st}}} = 0, \frac{1}{2}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots, 1.$$

The  $n_{\text{st}} = 1$  case is ruled out since that gives  $\ell = 0$  which implies  $f(x_i) \neq 0$  (i.e.,  $f_0 \neq 0$ ) which we assumed is not the case. The  $n_{\text{st}} = \infty$  case just gives  $\ell/k = 1$  which is the case already described in the part (d) answer. The formula  $n_{\text{st}}$  as a function of  $\ell/k$  is

$$n_{\text{st}} = \frac{1}{1 - \ell/k}$$

which is useful to see if a particular  $\ell/k < 1$  gives an actual stopping  $n_{\text{st}}$  which must be an integer greater than 1. If there is no actual stopping  $n_{\text{st}}$ , then we have the case discussed in the part (e) answer.

- g) An actual stopping  $n_{\text{st}}$  implies the function has a stationary point not at infinity for  $x = x_i$ . The small  $\Delta x$  solution with initial condition  $\Delta x(t=0) = 0$  is

$$\Delta x = Ct^{n_{\text{st}}},$$

where  $C$  is a constant of integration and there can be no lower powers of  $t$  since they would give lower order derivatives than  $\Delta x^{(n_{\text{st}})}$  that are nonzero at  $t = 0$  where  $x = x_i$ . If  $n_{\text{st}}$  is even  $\Delta x(t=0)$  is a minimum/maximum for  $C$  positive/negative. If  $n_{\text{st}}$  is odd  $\Delta x(t=0)$  is a rising/falling inflection point (which is also a stationary point) for  $C$  positive/negative.

- h) There is no absolutely right answer to this part. For the two likeliest  $\ell$  values given that  $f(x_i) = 0$ , yours truly thinks they will be in order of decreasing likelihood 1 and 2 since that is about the order they seem to turn up for in expansions of physics formulae about a point. For the two likeliest  $k$  values, yours truly thinks they will be in order of decreasing likelihood 1 and 2. Most physics formulae have no overall root function (i.e.,  $k = 1$ ) and of those that do, it seems the square root (i.e.,  $k = 2$ ) is probably the most common.

i) Behold:

$$1) \quad y^{(1)} = y f_1^{1/k} \quad 2) \quad \frac{dy}{y} = f_1^{1/k} dt \quad 2) \quad \ln\left(\frac{y}{y_0}\right) = f_1^{1/k}(t - t_0) \quad 3) \quad y = y_0 e^{f_1^{1/k}(t-t_0)}$$

$$4) \quad \Delta x = \Delta x_0 e^{f_1^{1/k}(t-t_0)} .$$

If  $f_1^{1/k} < 0$ , we have a convergent solution. If  $f_1^{1/k} > 0$ , we have a divergent solution. By the nature of the exponential function, it is clear that the convergent solution has a SPI.

j) Behold:

$$1) \quad y^{(1)} = y^{\ell/k} (\pm 1)^{\ell/k-1} f_\ell^{1/k} \quad 2) \quad y^{(1)} = y^\gamma C \quad \text{where we have simplified the notation,}$$

$$3) \quad \frac{dy}{y^\gamma} = C dt \quad 4) \quad \frac{y^{-\gamma+1} - y_0^{-\gamma+1}}{-\gamma+1} = C(t-t_0)$$

$$5) \quad y^{-\gamma+1} = (-\gamma+1)C(t-t_0) + y_0^{-\gamma+1}$$

$$6) \quad y = \frac{1}{\left[(-\gamma+1)C(t-t_0) + y_0^{-\gamma+1}\right]^{1/(\gamma-1)}}$$

$$7) \quad \Delta x = \pm \frac{1}{\left[(-\ell/k+1)(\pm 1)^{\ell/k-1} f_\ell^{1/k}(t-t_0) + |\Delta x_0|^{-\ell/k+1}\right]^{1/(\ell/k-1)}} .$$

If  $(-\ell/k+1)(\pm 1)^{\ell/k-1} f_1^{1/k} > 0$ , we have a convergent solution. If  $(-\ell/k+1)(\pm 1)^{\ell/k-1} f_1^{1/k} < 0$ , we have a divergent solution. In fact, the  $(-\ell/k+1)(\pm 1)^{\ell/k-1} f_1^{1/k} < 0$  case leads to divergence to an infinity in finite time. By the nature of the function, it is clear that the convergent solution has a SPI.

Note, convergence/divergence depends on the sign of  $\Delta x_0$  if  $\ell/k$  is even. So if  $\ell/k$  is even and perturbations are not somehow restricted in sign, the constant solution will be unstable in general since some perturbations will always put the system into a divergent solution.

k) Continuation of the part (h) answer: Going beyond, the required answer, the Friedmann equation is a FAODE with  $k = 2$ . Many Friedmann equation cases have no  $x_i$  points, and so have no stationary points, including no SPIs. There are, however, Friedmann equation cases with one  $x_i$  point and  $\ell = 1$ . These have  $n_{st} = 2$  and stationary points at finite time  $t$ . The ones you truly know of have cosh-like solutions with minima (bounce universes with two inverse-power density components, the lowest inverse-power one being positive and the highest being negative) and sine-like solutions with maxima (those that expand and then contract with two inverse power density components, the lowest inverse-power one being negative and the highest being positive). The latter case includes the matter-positive-curvature universe. There are Friedmann equation cases with one  $x_i$  point and  $\ell = 2$ . These include matter-positive-curvature-Lambda universes with density component constants adjusted to set the  $\ell = 1$  expansion constant set to zero. The constant solution for this kind of matter-positive-curvature- $\Lambda$  universe is the Einstein universe (i.e., static universe). If the positive curvature density component constant is made larger in absolute value (recall it is a negative quantity) than the Einstein universe case, then you have a forbidden zone in  $x$  (i.e., cosmic scale factor). Above the forbidden zone is a cosh-like solution and below is a sine-like solution. If the positive curvature density component constant is made smaller in absolute value (recall it is a negative quantity) than the Einstein universe case, then you have a Lemaitre universe which has an inflection point centering a low slope region that when adjusted to have a very low slope gives the Lemaitre universe an Einstein universe phase. For these solutions or related solutions, see Jeffery (2026, Exact Two-Density Component Solutions for the Cosmic Scale Factor From a General Approach Including a Simplified Exact Solution Formula for the Radiation-Matter Universe).

**Redaction:** Jeffery, 2018jan01

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002 qfull 00650 1 3 0 easy math: a FAODE with a stationary point that is not a SPI

19. In this problem, we will get some more insight into first order autonomous ordinary differential equations (FAODEs) with stationary points that are not stationary points at infinity (SPIs) by examining a solution beyond solution to lowest (nonzero) order around the stationary points. Consider the FAODE

$$x^{(1)} = f(x) ,$$

where  $f(x_i) = 0$  (i.e.,  $x = x_i$  gives a stationary point of some kind) and the independent variable is  $t$  (not necessarily time). However,

$$x^{(2)} = \frac{df}{dx}x^{(1)} = \frac{df}{dx}f(x) \neq 0$$

for  $x = x_i$ . This means the stationary point is not a SPI.

**NOTE:** There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c.

a) Let

$$g(x) = \frac{df}{dx}f(x)$$

and determine a formal solution for  $f(x)$ .

b) Assume  $x(t)$  has maximum and minimum at, respectively,  $x_i$  and  $-x_i$ . Now invent the simplest  $f(x)$  you can starting from the part (a) answer, except it has a general constant coefficient so as to give a general scale to the derivative  $x^{(1)}$ .

c) Now solve for  $x(t)$  given the part (b) answer. **HINT:** You could do this by integrating  $x(t)$ , but differentiating  $x(t)$  lead to solution by inspection.

d) Say a FAODE is given by

$$x^{(1)} = [f(x)]^{1/k} ,$$

where  $t$  is the independent variable (not necessarily time),  $k > 0$ ,  $f(x)$  is infinitely differentiable, and  $f(x) = \Delta x^\ell f_\ell + \dots$  is the expansion of  $f(x)$  around the stationary point  $x_i$  with  $\Delta x = x - x_i$  starting with the lowest nonzero order. Then the lowest order FAODE is

$$\Delta x^{(1)} = x^{\ell/k} f_\ell^{1/k} ,$$

In order for a solution of the FAODE to have stationary point that is not a SPI, there must be a stopping (derivative order)  $n_{st}$  given the formula

$$n_{st} = \frac{1}{1 - \ell/k}$$

where an actual stopping  $n_{st}$  must be an integer. If the formula gives a non-integer value, then there is a singularity in the behavior of some order of derivative of  $x(t)$  at  $x = x_i$  and that behavior takes some analysis to determine. An actual stopping  $n_{st}$  gives the only nonzero derivative order of  $x(t)$  at  $x = x_i$ . What are the  $\ell$  and  $k$  values for the FAODE used in the part (c) and are they consistent with a nonzero derivative order  $n = 2$  which is what we imposed in the preamble?

### SUGGESTED ANSWER:

a) Behold:

$$1) \quad g(x) = \frac{df}{dx}f(x) \qquad 2) \quad \frac{1}{2} \frac{d(f^2)}{dx} = g(x) \qquad 3) \quad f(x) = \pm \sqrt{2 \int g(x) + C} ,$$

where the integral is an indefinite integral (i.e., an antiderivative) and  $C$  is a constant of integration.

b) There is no absolutely right answer, but yours truly thinks the simplest  $f(x)$  that can be invented is

$$f(x) = \pm A \sqrt{x_i^2 - x^2} .$$

Note, the maximum and minimum of  $x(t)$  must be at, respectively,  $x_i$  and  $-x_i$ .

c) Behold:

$$1) \quad x^{(1)} = \pm A \sqrt{x_i^2 - x^2} \qquad 2) \quad x^{(2)} = (\pm A) \frac{(-x)}{\sqrt{x_i^2 - x^2}} x^{(1)} = -A^2 x = -\omega^2 x$$

$$3) \quad x = x_i \cos[\omega(t - t_0)] ,$$

where we recognized the simple harmonic oscillator differential equation and rewrote  $A$  as  $\omega$  for consistency with usual symbol usage. So the nonlinear FAODE is actually equivalent to a 2nd order linear differential equation.

d) Behold:

$$x^{(1)} = \pm A \sqrt{x_i^2 - x^2} = \pm A \sqrt{x_i^2 - (x_i + \Delta x)^2} = \pm A \sqrt{-2\Delta x + \dots},$$

where we require  $\Delta x \leq 0$ . Clearly,  $\ell = 1$  and  $k = 2$ , and so the stopping  $n_{\text{st}} = 2$  which is consistent with the imposed nonzero derivative order  $n = 2$ .

**Redaction:** Jeffery, 2018jan01