(2025apr15) 6001

Lecture 6: Topics

Leading ~~to~~ to the CMB and Very Simplified Recombination

6002

2025apr15

6003

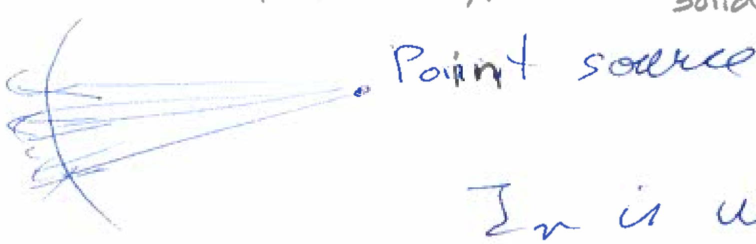
6.1) Representations of Specific Intensity

a) In frequency representation

$$I_\nu = \frac{dE}{d\nu \, dt \, dA \, d\Omega} \Rightarrow \frac{\text{ergs}}{\text{Hz s cm}^2 \text{ sr}}$$

per frequency, time, perpendicular area, solid angle

in cgs units



I_ν is used all the time in my own field of radiative transfer

We'll explicate it's use in a bit

But galaxy researchers tend to use surface brightness which annoyingly is $4\pi I_\nu$

Why, why? but they do

As for SED = Spectral Energy Distribution

I just use the word Flux which context defines

According to Google AI

= Luminosity per $\left\{ \begin{matrix} \text{frequency} \\ \text{wavelength} \end{matrix} \right\}$

6004

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b) Representations of specific Intensity + like Quantities

I_ν is per frequency

I_λ is per wavelengh

also I_E per energy
but this is just $h I_\nu$

so essentially frequency

h = Planck Constant
$= 6.626\,070\,15$ J.s (NIST)
exact by modern definition

Also wavenumber
$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\frac{c}{\lambda} = \frac{2\pi}{c}\nu$$
still used by some. Also essentially frequency.

There is also the logarithmic representation or fractional bandwith (Wiki, but mainly used for radio)

I will argue the log representation is the Natural One for plotting and sometimes ~~and~~ thinking about purposes. Coding? We probably just use I_ν probably just (~~though probably~~ not coding ~~because~~ of ~~all legacy~~) because all the formalism has been written in this formalism? Maybe the only sensible way given the way think about integrating energy.

The frequency rep. is better for that ~~energy purposes~~

2026 jan 24

Where does the logarithmic representation come from?

Recall the phase velocity relation,

$$c = \nu\lambda$$

$$\therefore \ln c = \ln\nu + \ln\lambda$$

$$d\ln c = d\ln\nu + d\ln\lambda$$

$$0 = \frac{d\nu}{\nu} + \frac{d\lambda}{\lambda}$$

$$\frac{d\nu}{\nu} = -\frac{d\lambda}{\lambda}$$

Note, if $d\nu > 0$, then $d\lambda < 0$, and so the -ve sign needed for the equality.

From p. 6003 for ν representation and analogous for λ representation

$$\frac{dE}{dt\,dA\,d\Omega} = I_\nu\,d\nu = I_\lambda(-d\lambda)$$

$$\therefore I_\nu\,d\nu = I_\lambda(-d\lambda)$$

$$\nu I_\nu \frac{d\nu}{\nu} = \lambda I_\lambda\left(-\frac{d\lambda}{\lambda}\right)$$

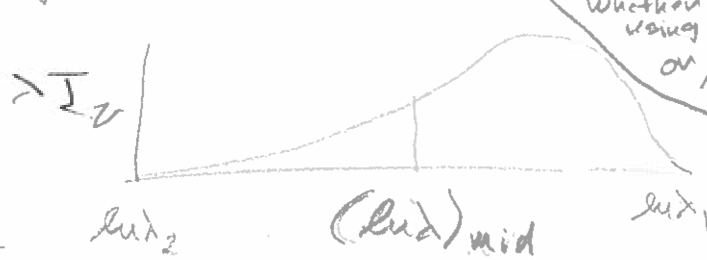
The log representation

$$\nu I_\nu = \lambda I_\lambda$$

c) It has the same value whether evaluated by ν or λ.

~~Seems~~ a good graphical simplification. You don't worry much about how different a spectrum would look whether using ν or λ.

Thus, plot of $\nu I_\nu = \lambda I_\lambda$ are the same aside from "mirror imaging"



6006

To show this definitively, note

$$\Delta \ln \nu = \ln \nu - (\ln \nu)_{mid}$$

$$= \ln c - \ln \lambda - [\ln \lambda - (\ln \lambda)_{mid}]$$

$$= -\Delta \ln \lambda$$

If $\ln \nu > 0$, $\Delta \ln \lambda < 0$ for the equality to hold

d) Another reason for using the log representation is that the width of spectral features tends to be proportional to their overall scale.

This is true for:

Doppler Shifting	Cosmological Redshifting
$\frac{\lambda_0}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}}$	$\frac{\lambda_0}{\lambda} = \frac{a_0}{a}$ Recall $z \equiv \frac{\lambda_0 - \lambda}{\lambda}$, $z + 1 = a_0/a$
$\Delta \ln \lambda = \ln \lambda_0 - \ln \lambda = \ln\left(\sqrt{\frac{1+\beta}{1-\beta}}\right)$	$\Delta \ln \lambda = \ln \lambda - \ln \lambda_0 = \ln(a_0/a)$
$\Delta \ln \lambda = \ln\left(\sqrt{\frac{1+\beta}{1-\beta}}\right)$ in general	$\Delta \ln \lambda = \ln(1+z)$ in general
$= \ln\left[(1+\frac{1}{2}\beta)(1+\frac{1}{2}\beta)\right]$	$= z$ to 1st order in small z
$= \ln(1+\beta) = \beta$ to 1st order in small β	
$\beta = \frac{v}{c}$	

In both cases, all shifts are by the same amounts in $\log\lambda$ (or $\log\nu$):

All the shifts have equal importance to the eye

This seems to be a good graphical simplification

Also makes all common line broadening (probably only Doppler broadening) the same size when plotted.

For Maxwell distribution of atom velocities along line of sight

$$\rho(v)\,dv = \frac{1}{\sqrt{\pi}\,v_0} e^{-(v/v_W)^2}\,dv$$

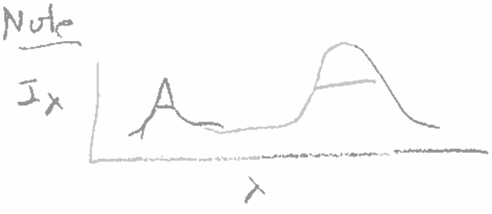
where $v_W = \left(\frac{2kT}{m}\right)^{\frac{1}{2}}$ or $\sigma_v = \frac{v_W}{\sqrt{2}}$ (Mihalus-279)

$= 12.85 \left(\frac{T}{10^4 K}\,\frac{1}{A}\right)^{1/2}$ km/s

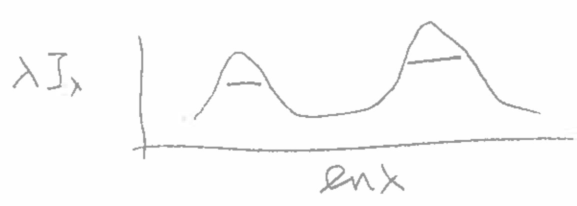
∴ The characteristic Doppler width is $\frac{\Delta\lambda}{\lambda} = \frac{v_W}{c} = \beta_W$ using the 1st order Doppler formula (p. 6006)

∴ $d\ln\lambda = \beta_W$

Note



but



So using the log representation makes all Doppler broaden lines have the same width.
Seems a good graphical simplification.
All widths have equal importance to the eye

e) And, of course, plotting versus $\log \nu$ or $\log \lambda$ allows compact plotting over bands that span many orders of magnitude

6.2 Blackbody Radiation: Representations and Other Details

a) $$\frac{dE}{dt\,dA\,d\Omega} = I_\nu d\nu = I_\lambda(-d\lambda) = \nu I_\nu \frac{d\nu}{\nu} = \lambda I_\lambda \left(-\frac{d\lambda}{\lambda}\right)$$

see p. 6003

see p. 6005

For blackbody radiation $I_\nu \to B_\nu$, $I_\lambda \to B_\lambda$

$$\nu I_\nu = \lambda I_\lambda \Rightarrow \nu B_\nu = \lambda B_\lambda$$

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^x - 1}$$ (Mihalas—7, Planck's law)

where $x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda}$

$$B_\lambda = B_\nu\left(-\frac{d\nu}{d\lambda}\right) = B_\nu\left[-\left(-\frac{c}{\lambda^2}\right)\right] = B_\nu\frac{c}{\lambda^2}$$

$$= \frac{2h}{c^2}\left(\frac{c}{\lambda}\right)^3(e^x-1)^{-1}\frac{c}{\lambda^2} = \frac{2hc^2}{\lambda^5}(e^x-1)^{-1}$$

$$\nu B_\nu = \frac{2h\nu^4}{c^2}\frac{1}{e^x-1} = \lambda B_\lambda = \frac{2hc^2}{\lambda^4}\frac{1}{e^x-1}$$

There also the photon representation ('photon intensity')

$$\underbrace{\frac{dE}{h\nu\, dt\, dA\, d\Omega}}_{\text{see p. 6003}} = N_\nu d\nu = \frac{2\nu^2}{c^2}\frac{1}{e^x - 1}\, d\nu$$

b) For analysis, it is convenient to make all these functions just functions of x

$$B_\nu d\nu = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \frac{x^3}{e^x-1}\,\underbrace{dx}_{d\nu = \left(\frac{kT}{h}\right)dx}$$

$$B_\lambda(-d\lambda) = 2hc^2\left(\frac{kT}{hc}\right)^5 x^5 (e^x-1)^{-1}\left(\frac{hc}{kT x^2}dx\right) = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \frac{x^3}{e^x-1}\,dx$$

$$\nu B_\nu \frac{d\nu}{\nu} = \lambda B_\lambda \frac{d\lambda}{\lambda} = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \frac{x^4}{e^x-1}\frac{dx}{x}$$
$$= \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \frac{x^3}{e^x-1}\,dx$$

These are all the same as functions of x which probably should have been obvious

$$N_\nu d\nu = \frac{2}{c^2}\left(\frac{kT}{h}\right)^3 \frac{x^2}{e^x-1}\,\underbrace{dx}_{d\nu = \left(\frac{kT}{h}\right)dx}$$

So, in fact, we have a universal function

$$f(x) = \frac{x^z}{e^x - 1}$$

to analyze where $z = 2, 3, 4, 5$ for physical interest.

I don't know if it has any special name, but

$$D_z(x) = \frac{z}{x^z}\int_0^x \frac{t^z}{e^t - 1}\,dt$$

is the Debye function (Wik)

Maybe $f(x)$ should just be called general Planck function.

6010

c) Integrating $f(x)$

When encounter a function, the first thing that should come to your mind is to integrate it.

$$F(z) = \int_0^\infty f(x)\,dx = \int_0^\infty \frac{x^z}{e^x - 1}\,dx$$

$$= \int_0^\infty \frac{x^z e^{-x}}{(1-e^{-x})}\,dx = \int_0^\infty x^z e^{-x} \sum_{l=0}^\infty e^{-lx}\,dx$$

using the geometric series (Arf-279)

Note $\frac{1}{1-y} = \sum_{l=0}^\infty y^l$ is NOT convergent for $y = 1$ (ie., $e^{-0} = 1$ or $x = 0$), but you can integrate to $x = 0$ since $x = 0$ has zero volume (Google AI) and the integration converges

$$F(z) = \sum_{l=0}^\infty \int_0^\infty x^z e^{-(l+1)x}\,dx$$

$$= \sum_{l=1}^\infty \int_0^\infty x^z e^{-lx}\,dx$$

with $l \to l-1$

$y = lz$
$dy = l\,dz$

$$= \sum_{l=1}^\infty l^{-(z+1)} \int_0^\infty y^z e^{-y}\,dy$$

$$= \sum_{l=1}^\infty l^{-(z+1)}\, z!$$

$$= z!\, \zeta(z+1)$$

The Riemann-zeta function

$$\zeta(s) = \sum_{l=1}^\infty l^{-s}$$

for $s > 0$ (Arf-332)

The factorial function Arf-543,
The generalization of factorial,
For integers $n \geq 0$, $z! = n!$,
$z!$ is undefined for negative integers,
It has infinities (Arf-543)

$$(z = -1/2)! = \sqrt{\pi}$$

2026jan24

6011

$s = z+1, \; z = s-1$

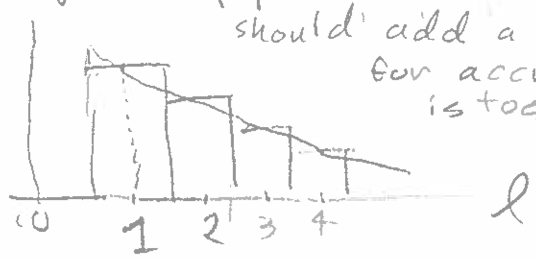
$$\zeta(s) = \begin{cases} \infty & s=1 \text{ which is the divergent harmonic series} & z=0 \\ \frac{\pi^2}{6} = 1.6449340668\ldots & s=2 & z=1 \\ 1.2020569032\ldots & s=3 & z=2 \\ \frac{\pi^4}{90} = 1.0823232337\ldots & s=4 & z=3 \\ 1.0369277551\ldots & s=5 & z=4 \\ \frac{\pi^6}{945} = 1.0173430620\ldots & s=6 & z=5 \end{cases}$$

From Wolfram Riemann-Zeta function and Arf-332

These are of physical interest to blackbody radiation

See p. 6009

Integral approximation, but you should add a few first terms for accuracy, Zero terms is too inaccurate. 1 term is enough for insight.



$$\zeta(s) \triangleq 1 + \int_{3/2}^{\infty} \ell^{-s}\, d\ell = 1 + \frac{\ell^{-s+1}}{-s+1}\Bigg|_{3/2}^{\infty} = 1 + \frac{(2/3)^{s-1}}{s-1} \text{ (in general) for } s>1$$

Diverges for $s=1$ as one should expect $\int_1^{\ell} \ell'^{-1} d\ell' = \ln(\ell')\Big|_1^{\ell}$

$= 1 + 2/3 = 1.666\ldots$ $\quad s=2$

$= 1 + \frac{4/9}{2} = 1 + 2/9 = 1.222\ldots$ $\quad s=3$

$= 1 + \frac{8/27}{3} = 1 + 8/81 \triangleq 1.1$ $\quad s=4$

$= 1 + \frac{16/81}{4} = 1 + 4/81 \triangleq 1.05$ $\quad s=5$

$= 1 + \frac{32/243}{5} \triangleq 1 + \frac{1}{40} = 1.025$ $\quad s=6$

$= 1$ $\quad s=\infty$

Not so good

6012

For $z = 3$, $s = 4$,

we recover the Blackbody radiation density law

$$F = \underbrace{\frac{4\pi}{c}}_{\substack{\text{Integrate over all solid angle (p. 6003)}\\ \text{and divide by } c \text{ to change}\\ \text{from flux to density}}} * \underbrace{\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4}_{\text{p. 6009}} * \underbrace{3!}_{\text{p. 6010}} \underbrace{\frac{\pi^4}{90}}_{\text{p. 6011}}$$

$$= \frac{4\pi}{c} \frac{2h}{c^2} \left(\frac{k}{h}\right)^4 \frac{\pi^4}{15} T^4 = aT^4$$

$$\sigma_{SB} = \frac{c}{4} a = \frac{\pi^5}{c^2} \frac{k^4}{h^3} \frac{2}{15} \quad \text{Stefan-Boltzmann constant}$$

$$= \frac{2\pi^5 k^4}{15 c^2 h^3} \quad \text{correct by Wiki}$$

$$= 5.670374419... \times 10^{-8} \frac{W}{m^2 K^4}$$

exact by modern definition, but since it includes π^5, it's irrational.

a is the just called the radiation constant

and no one seems to like its formula

$$a = \frac{8\pi^5 k^4}{15 c^3 h^3}$$

(Google AI: what is the blackbody radiation density

$$= 7.566... \times 10^{-16} \frac{J}{m^3 K^4}$$

also exact, but irrational, by modern definition.

(2021 nov 04)

(This page Duplicate 2020 apr [illegible] in 6009 notes)

(~~10~~ 6010a) 6013

c) Cases of interest

Energy density

$$\cancel{B = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\,dx}$$

$$n = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1}\,dx$$

(see p. 6009)

$$N = \frac{4\pi}{c} n = \frac{2}{c^2}\left(\frac{kT}{h}\right)^3 2!\,\zeta(3)$$ (see p. 6010)

$$B = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 3!\,\underbrace{\zeta(4)}_{\frac{\pi^4}{90}}$$

$$= \frac{\pi^4}{15}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4$$

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see p. 6011

No π factors from the odd Riemann-Zeta functions

$$\mathcal{E} = \frac{4\pi}{c} B$$

$$= \frac{4\pi}{c}\frac{\pi^4}{15}\frac{2h}{c^2}\left(\frac{k}{h}\right)^4 T^4$$

$$= \underbrace{a_r}_{\text{Radiation constant (Wiki Stefan-Boltzmann Law)}} T^4$$

$$\sigma = \frac{\pi^5}{15}\frac{2h}{c^2}\left(\frac{k}{h}\right)^4$$

Wiki Stefan-Boltzmann Law confirms

$a_r = 7.5657 \times 10^{-16}\ \frac{J}{m^3 K^4}$

$a = \frac{4\sigma}{c}$

10 U2
6U14 GOVS

B) Pressure of A Photon Gas in Thermodynamic equilibrium

d) Entropy & Pressure of Planck Photon gas

Energy not energy per unit volume

$\mathcal{E}$ is energy density

$$dE = T\,dS - P\,dV + \mu\,dN$$

$$\left(\frac{\partial E}{\partial V}\right)_S = -P\,,\quad \left(\frac{\partial E}{\partial S}\right)_V = T$$

Natural variables $E(S,V)$

$\mu = 0$ chemical potential zero for photon gas since number conservation not imposed in establishing thermodynamic equilibrium

But we have $E = aT^4 V = \mathcal{E}V$

a = radiation constant, $\mathcal{E} = aT^4$ (see p. 6U9)

$$= E(T,V)$$

$$dS = \left(\frac{dE}{T}\right)_V \quad \text{constant volume}$$

$$= \frac{dE}{\left(\frac{E}{aV}\right)^{1/4}}$$

$$S = (aV)^{1/4}\left(\tfrac{4}{3}\right)E^{3/4} + \text{Constant}$$

Constant = 0 set to zero. No reason not to

$$E = \left(\frac{\left(\frac{3}{4}\right)S}{(aV)^{1/4}}\right)^{4/3} = \frac{\left[\left(\frac{3}{4}\right)S\right]^{4/3}}{(aV)^{1/3}}$$

$$\left(\frac{\partial E}{\partial V}\right)_S = -\frac{1}{3}\frac{E}{V} = -\frac{1}{3}\mathcal{E}$$

and so $P = \frac{1}{3}aT^4 = \frac{1}{3}\mathcal{E}$

a famous formula

$$\mathcal{E} = aT^4$$

This result can also be derived from classical particles and quantum mechanical particles — a nice consistency.

~~Also 4th one for photon number~~

~~$dN = \frac{2\pi \nu^2}{c^2} \frac{d\nu}{...} = \frac{2}{c^2}\left(\frac{kT}{h}\right)^3 \frac{x^2\,dx}{e^x - 1}$~~

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d) ~~f)~~ Finding Planck law maximum (for use in Wien's law and other things)

$$f = \frac{x^z}{e^x - 1}$$

can be used to find maximum. ~~and its bound~~ and so all version of Wien's law

$$\frac{\partial f}{\partial x} = \frac{z x^{z-1}}{e^x - 1} - \frac{x^z e^x}{(e^x - 1)^2} \Rightarrow 0$$

$$z(e^x - 1) - x e^x = 0$$

$$\frac{x_0}{z} = 1 - e^{-x_0}$$

$$\frac{z}{x_0} = \frac{e^{x_0}}{e^{x_0} - 1}$$

$$x_0 = z(1 - e^{-x})$$ for iterative numerical solution. convergent iteration

$$x_0 = z \quad 0\text{th order}$$

$$x_0 = z(1 - e^{-z}) \quad 1^{st}\text{ order}$$

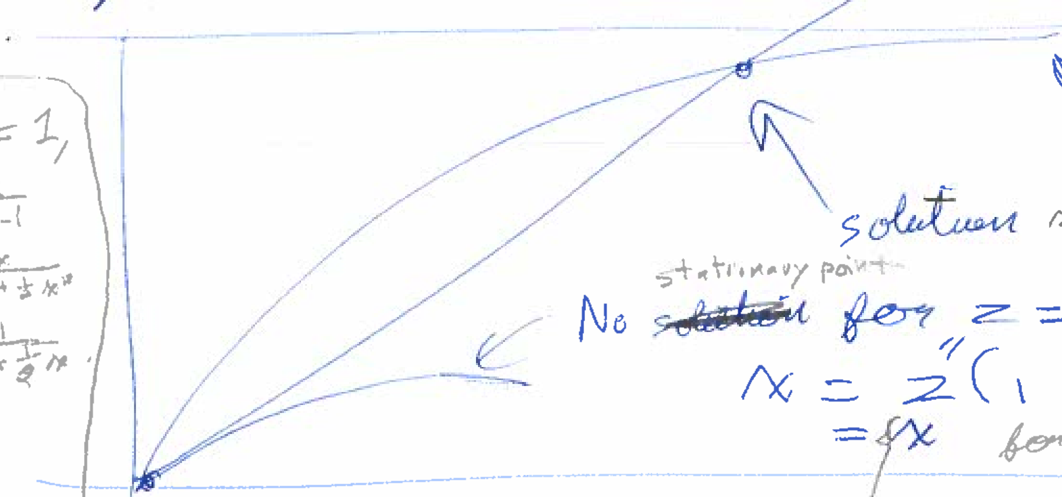
$$x_0 = z(1 - e^{-(z - \frac{1}{z})})$$ good approximation

(see radtran Planck function 2021aug10, aug14 notes)

Maximum relative error ~3% for $z = 1.5$

See math3.f in aalib.

Graphical Solution



$y = x$

$y = 1 - e^{-x}$

solution $x = z(1 - e^{-x})$

No ~~solution~~ stationary point for $z = 1$

$$x = z(1 - (1 - x)) = zx \quad \text{for } z = 1$$

0 solution

$zx > x$ for $z > 1$

For $z = 1$,

$$f = \frac{x}{e^x - 1} \approx \frac{x}{x + \frac{1}{2}x^2} = \frac{1}{1 + \frac{1}{2}x}$$

A maximum for $[0, \infty]$, but not a stationary point.

6016

$\angle$ 1e-18 precision

Result	z	$x_{maximum}$
photon number	2	1.593624...
τ rep	3	2.821439...
log rep	4	3.920690...
λ rep	5	4.965114...

See W&K Wien for 6 digit values? I think ~~this is~~ the good one for λ or ν of characteristic emission

g) 2nd Order Expansion Around Maxima

To get band to Δx about $x_{max} = x_0$

Expand to 2nd order in Δx

$$f = \frac{(x_0 + \Delta x)^z}{e^{x_0}\left(1 + \Delta x + \frac{\Delta x^2}{2}\right) - 1} = \frac{x_0^z\left(1 + z\frac{\Delta x}{x_0} + \frac{z(z-1)}{2}\left(\frac{\Delta x}{x_0}\right)^2\right)}{(e^{x_0}-1)\left[1 + \frac{\Delta x e^{x_0}}{(\ldots)} + \frac{1}{2}\frac{\Delta x^2 e^{x_0}}{(\ldots)}\right]}$$

$$\frac{e^{x_0}}{(\ldots)} = \frac{z}{x_0}$$ see p. 6013

$$\cancel{= f_0\left[1 + \frac{z\Delta x}{x_0} + \ldots\right]\left[1 - \frac{\Delta x e^{x_0}}{(\ldots)} - \frac{1}{2}\frac{\Delta x^2 e^{x_0}}{(\ldots)}\right]}$$

$$\cancel{= f_0\left[1 + \left(\frac{z}{x_0} - \frac{e^{x_0}}{(\ldots)}\right)\Delta x + \Delta x^2\left(\frac{z}{x_0}\frac{e^{x_0}}{(\ldots)} + \frac{1}{2}\ldots\right)\right]}$$

$$\cancel{= f_0\left[1 + \Delta x^2\left[\left(\frac{z}{x_0}\right)^2 \ldots \left(\frac{z}{x_0}\right) + \left(\frac{z}{x_0}\right)^2\right]\right]}$$

$$= f_0[1 + \Delta x^2$$

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$$f = f_0\left[1 + \left(\frac{Z}{x_0}\right)\Delta x + \frac{Z(Z-1)}{2x_0^2}\Delta x^2\right] \bigg/ \left[1 + \frac{Z}{x_0}\Delta x + \frac{1}{2}\frac{Z}{x_0}\Delta x^2\right]$$

$$= f_0[\ldots]\left[1 - \frac{Z}{x_0}\Delta x - \frac{1}{2}\frac{Z}{x_0}\Delta x^2 + \left(\frac{Z}{x_0}\right)^2\Delta x^2\right]$$

Avf-279 for geometric series

$$= f_0\Big[1$$

0 as it should since x_0 is a maximum. This is imposed by using $\frac{\partial^{x_0}}{(e^{x_0}-1)} = \frac{Z}{x_0}$ see p. 6013 + 6014

$$+ \Delta x\left(\frac{Z}{x_0} - \frac{Z}{x_0}\right)$$

$$+ \Delta x^2\left[-\left(\frac{Z}{x_0}\right)^2 + \frac{Z(Z-1)}{2x_0^2} - \frac{1}{2}\frac{Z}{x_0} + \left(\frac{Z}{x_0}\right)^2\right]$$

$\text{Coef} = \left(\frac{f - f_0}{f_0 \Delta x_0^2}\right)$

For Z=4
approx = -.125
exact = -.1197894 confirmed
Vam = -.1195941

cancel

$$f = f_0\Big[1 + \cancel{\ldots} + \frac{1}{2}\frac{Z}{x_0}\left(\frac{Z-1}{x_0} - 1\right)\Delta x^2\Big]$$ exact

~~For the~~ Now to zeroth or $x_0 \doteq Z$

$$\doteq f_0\left[1 + \frac{1}{2}\left(1 - \frac{1}{Z} - 1\right)\Delta x^2\right] = f_0\left[1 - \frac{1}{2Z}\Delta x^2\right]$$ approx

6018

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see p. 6007 and Riemann zeta

integrated specific intensity fraction $F = \int_0^{\infty} \frac{x^4}{e^x-1}\frac{dx}{x} = \int_0^{\infty}\frac{x^3}{e^x-1}dx = 3!\,\zeta(3+1)$

energy fraction

What is the integration around x_0?

$$F = f\,\frac{dx}{x} = \int_{-\Delta x}^{\Delta x} f_0\left[1 + f_2\Delta x^2\right]\left[1 - \frac{\Delta x}{x_0} + \left(\frac{\Delta x}{x_0}\right)^2\right]d\Delta x$$

$x_0 = x_{max,4}$

$x_0(1+\Delta x/x_0)$

Not even now

geometric series Ans −279

$$= \int_{-\Delta x}^{\Delta x}\frac{f_0}{x_0}\left[1 - \frac{\Delta x}{x_0} + \left(f_2 + \frac{1}{x_0^2}\right)\Delta x^2\right]d\Delta x$$

even interval

$$\left(-\frac{\Delta x^2}{2\Delta x_0}\right)\Bigg|_{-\Delta x}^{\Delta x}$$

But over even interval This term is zero.

$$= 2\frac{f_0}{x_0}\left[\Delta x + \frac{1}{3}\left(f_2 + \frac{1}{x_0^2}\right)\Delta x^3\right]$$

$$\underset{\text{exact}}{f_2} = \frac{1}{2}\frac{z}{x_0}\left(\frac{z-1}{x_0} - 1\right) \simeq \frac{1}{2}\left(1 - \frac{1}{z} - 1\right)$$

here $x_0 \simeq z$

$$= -\frac{1}{2z} = f_{2\,0}\ \text{approximated}$$

$$= 2\frac{f_0\Delta x}{x_0}\left[1 + \frac{1}{3}\left(f_2 + \frac{1}{x_0^2}\right)\Delta x^2\right]$$

F(int, −1.5, 1.5)
0.53738
0.54008
0.5378557

F(int −1, 1) =
- 0.367965 approx. coef
- 0.368617 exact coef.
- 0.3683219 numerical int., but not high accuracy

The exact coefficient is accurate (see p. 6013) and but to $\Delta x = 1$, it gives a better integral. Just a numerical accident that the approximate one is ~~better~~ for $\Delta x \geq 1$, is closer to the numerical integration which is NOT guaranteed high accuracy.

F(0, 2.821439..) ≃ 0.37
> rep peak

F(0, 3.920690..) = 0.59
4 log rep peak

F(0, 4.965114..) = 0.76
> rep peak

e) 2nd Order Expansion about x_0 point

$$\int_0^{\infty} \frac{x^z}{e^x - 1}\, dx \,, \quad \text{general Planck function } f(x) = \frac{x^z}{e^x - 1}$$

(see p. 6009)

But for blackbody radiation $z = 3$ is really the the correct choice for integrated flux (see p. 6009)

$$f(x) = \left(\frac{x_0^z}{e^{x_0} - 1}\right) + \Delta x$$

$$f' = \frac{z x^{z-1}}{e^x - 1} - \frac{x^z e^x}{(e^x-1)^2} = \frac{x^{z-1} e^x}{(e^x-1)^2}\underbrace{\left[z(e^x - 1) - x\right]}_{=0 \text{ for a stationary point}}$$

$$f'' = \frac{z(z-1)x^{z-2}}{(e^x - 1)} - \frac{z x^{z-1} e^x}{(e^x-1)^2} - \frac{x^z e^x}{(e^x-1)^2} + \frac{2 x^z e^{2x}}{(e^x-1)^3}$$

$$= \frac{z(z-1) x^{z-2}}{(e^x - 1)} \underbrace{- \frac{x^{z-1} e^x}{(e^x-1)^2}\left[z(e^x - 1) - x\right]}_{= -f'} + \frac{2 x^z e^{2x}}{(e^x-1)^3}$$

$$= \frac{x^{z-2}}{(e^x-1)^3}\left[z(z-1)(e^x-1)^2 + 2x^2 e^{2x}\right] - f'$$

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~~Lecture~~ 2025 apr 15 | 6019

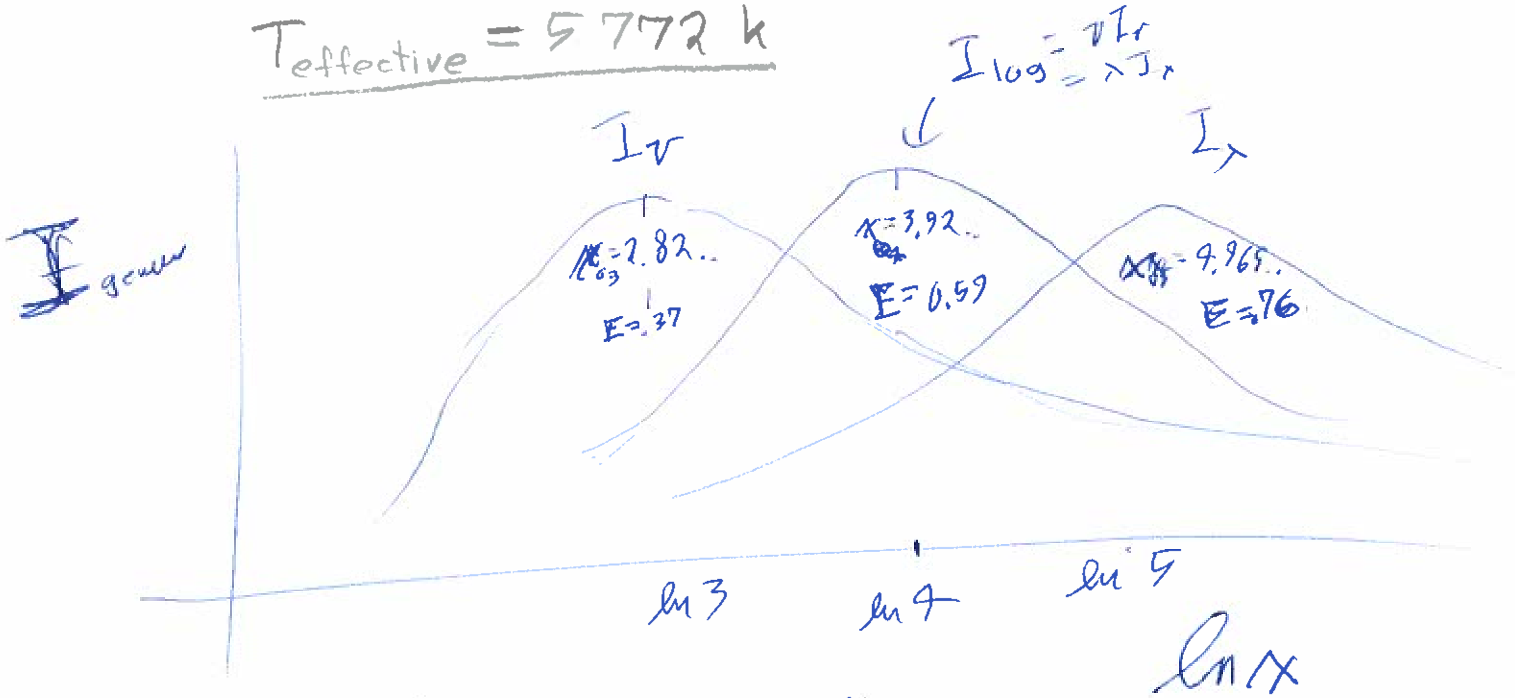
h) Why Log reprosentation

Maximum $x_{max} = x_{04}$ is the better representation of the region of maximum flux than x_{03} and x_{05}

$\underbrace{x_{03}}_{\text{max of } \nu \text{ representation}}$ and $\underbrace{x_{05}}_{\text{Max of } \lambda \text{ representation}}$

Plot for the solar

$T_{effective} = 5772\ k$



$$F = A\int_0^{x_{03}} \frac{x^3}{e^x - 1}\,dx = 0.37 \qquad F = A\int_0^{x_{04}} \frac{x^3}{e^x - 1}\,dx = 0.59 \qquad F = A\int_0^{x_{05}} \frac{x^3}{e^x - 1}\,dx = .76$$

So the x_{04} maximum is closer to the half way point to total integrated energy.

See p.6014

and $x_{03} \cong x_{04} - 1$ and $x_{04} + 1 \cong x_{05}$

and x_{04} is midway between the other two maximum

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$\boxed{2024\ \text{apr}\ 15}$

See p. 6016

and $F(x_{04} - 1, x_{04} + 1) \hat{=} 0.37$
$= 37\%$
of all the energy

$F(x_{04} - 1.5, x_{04} + 1.5) = 0.54$
$= 54\%$

So if you had to give an x, ν, or λ representative of where most energy is the x_{04} seems best

$$x_{04} = 3.920690\ldots$$

$$\nu_4 = \frac{kT}{h} x_{04}$$

$$\lambda_4 = \frac{hc}{kT} \frac{1}{x_{04}}$$

However, in numerical calculation I_λ and ν representation are best since?
See p. 6004
~~$\int I_\lambda d\lambda$ is integrated spectral energy, integrated intensity (total energy)~~

To conclude the log representation seems best for plotting spectra in general (see p. 6005–6007)

and for blackbody spectra the log rep. maximum actually is the natural x, ν, λ for characterizing where most energy is in a plot

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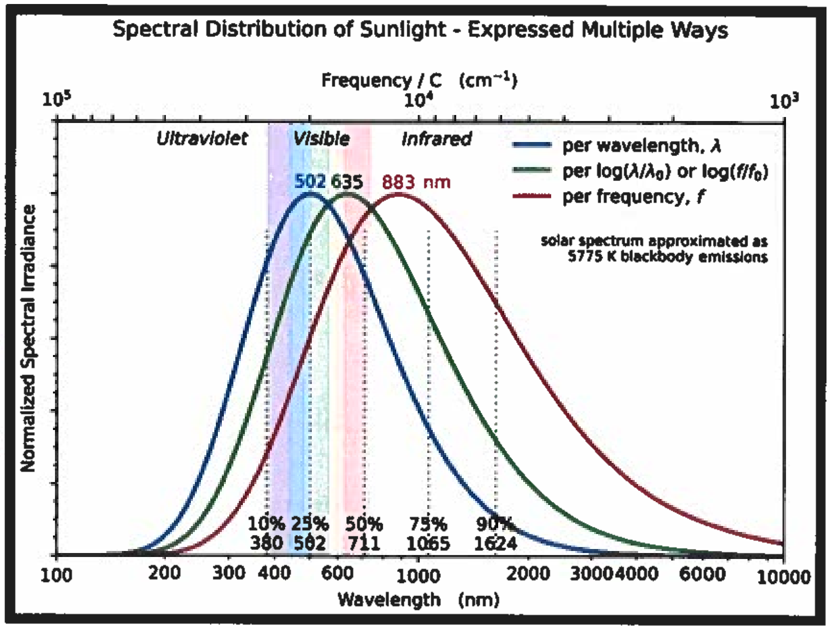
Caption: The normalized blackbody spectrum equivalent of the solar spectrum shown in 3 representations:

1. frequency representation ($x\_max = h\nu/(kT) = 2.821439372122078893...$, $F(x\_max) \cong 0.365$).
2. log representation ($x\_max = h\nu/(kT) = 3.920690394872886343...$, $F(x\_max) \cong 0.592$) (AKA fractional bandwidth).
3. wavelength representation ($x\_max = h\nu/(kT) = 4.965114231744276303...$, $F(x\_max) \cong 0.757$).



Note, $F(x\_max)$ is the fractional integrated specific intensity from $x = 0$ to $x = x\_max$.

The blackbody spectrum equivalent is the spectrum the Sun would have if it were a perfect blackbody radiator with a temperature exactly equal to the Sun's actual effective temperature (a characteristic photospheric temperature) which this plot sets to 5775 K which is slightly different from solar photosphere effective temperature = 5772 K (current best value).

Features:

1. Obviously, the shape of a spectrum depends on the representation, and so where a spectrum maximizes depends on the representation. The representation dependence of shape and maximum is clearly shown for the solar spectrum in the plot.

2. The 3 representations in the plot give maximizes at, respectively 502 nm = 0.502 μm (green band (fiducial range 0.495--0.570 μm), 635 nm = 0.635 μm (red (fiducial range 0.620--0.740 μm)), and 883 nm = 0.883 μm near-infrared (NIR, fiducial range 0.750--1.4 μm).

3. The energy distribution for the solar spectrum can be specified by percentiles for the wavelength representation: 10 % (< 0.380 μm), 25 % (< 0.502 μm), 50 % (< 0.711 μm), 75 % (< 1.065 μm), 90 % (< 1.624 μm), 90 % (< 1.624 μm),

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100 % (< ∞).

4. We suggest that the log representation is the natural representation for all spectra.

The argument:

1. The differential relationship among the representations is

$$dI = I\_\nu\ d\nu = \nu I\_\nu(d\nu/\nu) = -\lambda I\_\lambda(d\lambda/\lambda) = -I\_\lambda\ d\lambda \quad ,$$

where dI is differential integrated specific intensity, ν is frequency, I_ν is the specific intensity in the frequency representation, λ is wavelength, I_ν is specific intensity in the wavelength representation, and the minus sign is because increasing frequency dν > 0 causes corresponding decreasing wavelength dλ < 0.

We see that νI_ν = λI_λ and both are what is called the log representation. The version νI_ν is plotted versus d ln(ν) = dν/ν and the version λI_λ is plotted versus d ln(λ) =dλ/λ.

Since νI_ν = λI_λ, there is **NO** difference which of νI_ν and λI_λ you evaluate and plot, since plots of either are identical, except for mirror reflection about the either of the endpoints. These aspects of the log representation constitute the plotting convenience feature of the log representation.

2. Is there direct way to see that dν/ν = - dλ/λ for electromagnetic radiation (EMR)? Yes. We take the differential of the natural logarithm of the phase velocity relation νλ=c and then follow obvious steps: i.e.,

$$d\ln(c) = 0 = d[\ln(\nu\lambda)] = d[\ \ln(\nu) + \ln(\lambda)\ ] = d\ln(\nu)$$
$$d\ln(\nu) = -\ d\ln(\lambda)$$
$$d\nu/\nu = -\ d\lambda/\lambda \quad .$$

3. More important than the plotting convenience feature of the log representation is another feature.

The size of spectrum structures in frequency and wavelength tend to be proportional to their characteristic absolute size. For example, Doppler

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shift and cosmological redshift both shift frequency/wavelength by common factors C for all frequency/wavelength Thus, the logarithmic size of the shift is

$$\ln(\nu\_shifted/\nu) = \ln(C) = -\ln(\lambda\_shifted/\lambda)$$

throughout the spectrum.

The from the above example and other cases, it follows that representation tends to give spectrum structures of equal importance relative their band equal importance in plots. This spectrum structure feature seems yours truly a great boon.

UNDER CONSTRUCTION BELOW

5. In the log representation, blackbody spectrum specific intensity (i.e., Planck's law) is

$$\nu B\_\nu(T) = \lambda B\_\lambda(T) = 2c[(kT/(hc)]^{**}4 \frac{x^{**}4}{[\exp(x)-1]},$$

where the Planck constant h = 6.62607015*10**(-34) J·s (exactly), the vacuum light speed c = 2.99792458*10**8 m/s (exactly) ≅ 3*10**8 m/s = 3*10**5 km/s ≅ 1 ft/ns, and the x = hν/(kT) =hc/(kTλ) is dimensionless (i.e., unitless) variable incorporating frequency and wavelength information, where the Boltzmann contant k = 1.380649*10**(-23) J/K (exactly) and T is temperature. For reference for the fundamental physical constants, see also NIST: Fundamental Physical Constants.

Image link: Wikimedia Commons: File:Spectral Distribution of Sunlight.svg.
Local file: local link: representation.html.
File: Blackbody file: representation.html.

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6.3) Pressure of Photon Gas 3 Ways Derived

a) Entropy, Pressure, Energy

From Thermodynamic theory, we strongly believe the 1st law of thermodynamics in differential form

$$dE = TdS - PdV + \mu dN$$

change in number of particles

change in energy (internal energy) — Temperature — entropy — pressure — volume — chemical potential = zero for photons since photon number is NOT conserved in establishing thermo. Eq.

Analysis of photon gas Gives $E = aT^4$ energy density

a = a radiation constant See p. 6012

$E = aT^4 V$ energy in volume V

But this is $E = E(T, V)$ $\quad T = \left(\frac{E}{aV}\right)^{1/4}$

and use the 1st law we need $E(S, V)$

$$dS = \left(\frac{dE}{T}\right)_V$$ holding volume constant.

$$= \frac{dE}{\left(\frac{E}{aV}\right)^{1/4}}$$ Now we can integrate holding volume constant

$$S = (aV)^{\frac{1}{4}}\left(\frac{4}{3}\right) E^{\frac{3}{4}}\Big|_{E_0}^{E}$$

$$S = (aV)^{\frac{1}{4}}\left(\frac{4}{3}\right) E$$

we set $E_0 = 0$ since it seems reasonable to have zero entropy when $E = 0$

and $$E = \left[\frac{\left(\frac{3}{4}\right) S}{(aV)^{1/4}}\right]^{4/3} = \left[\frac{\left(\frac{3}{4}\right) S}{a^{1/4}}\right]^{4/3} V^{-1/3}$$

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Now we have $E = E(S, V)$

and $p \equiv -\left(\frac{\partial E}{\partial V}\right)_S$ ← constant entropy

$$= -\left(-\frac{1}{3}\right)\frac{E}{V}$$

$$p = \frac{1}{3} a T^4$$ a famous result for pressure of a photon gas

Radiation constant see p. 6012

But note especially

$$E \propto V^{-1/3}$$

$$\varepsilon \propto V^{-4/3} \propto (a^3)^{-4/3}$$

$$\varepsilon \propto a^{-4}$$ cosmic scale factor

For adiabatic expansion of a photon gas including in an expanding universe.

Historically, Wien found his approximation to the Planck Law by consider ~~an box of~~ radiation redshifting by the Doppler effect in an expanding reflecting box.

He found $T\lambda = \text{constant}$ must hold. We'll find that too for the expanding universe.

b) Classical Particle Derivation (gases) $\left\{ \frac{p}{m} \right.$ ~~classic~~ NR limit

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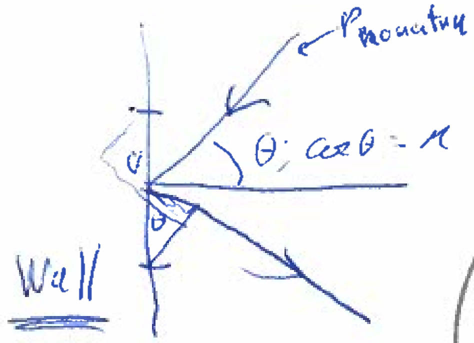


$$d\mathcal{P}\,dA = 2\mu P \,\frac{v}{4\pi}\, n(P)\, dP\, d\Omega\, \mu\, dA$$

Pressure

accounts for change in momentum of particle on bounce of rigid surface

component of momentum toward surface

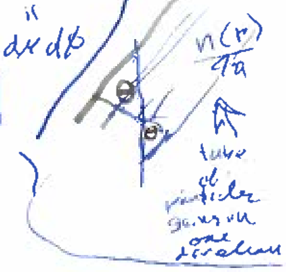
flow in one direction

number density of particles ~~per momentum~~ per momentum magnitude $|P|$

differential momentum magnitude

solid angle // $\sin\theta\, d\theta\, d\phi$ = $d\mu\, d\phi$

perpendicular surface area



$$\mathcal{P} = \frac{2}{4\pi}\underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^1 \mu^2 d\mu}_{\frac{1}{3}} \int_0^\infty vP\, n(P)\, dP$$

$$\mathcal{P} = \frac{1}{3}\int_0^\infty vP\, n(P)\, dP$$

Now NR limit $E_{kin} = \frac{P^2}{2m} = \frac{1}{2} vP$
non-relativistic limit
$vP = 2E(P)$

ER limit
extreme relativistic limit
$E_{kin} = Pc = vP$

$$\mathcal{P} = \begin{cases} \frac{2}{3}\varepsilon & \text{NR} \\ \frac{1}{3}\varepsilon & \text{ER} \end{cases}$$ where ε is energy per unit volume: i.e., energy density.

Note these results do not depend on $n(P)$: i.e., on how the particles are distributed in momentum magnitude, just on ~~the~~ instantaneous energy density. So the gas doesn't have to be in thermodynamic equilibrium

But if the gas is in thermodynamic equilibrium??

If the gas volume is changed adiabatically

No heat energy change ~~added~~ and no particle number change (or zero chemical potential)

Seems to be true whether there is thermody. equilibrium or not.

$$dE = -\mathcal{P}\, dV$$

$E = E(V,S)$ and $\mathcal{P} = \mathcal{P}(E)$

Note we have called the entities particles but they could also be compact wave packets in QM.

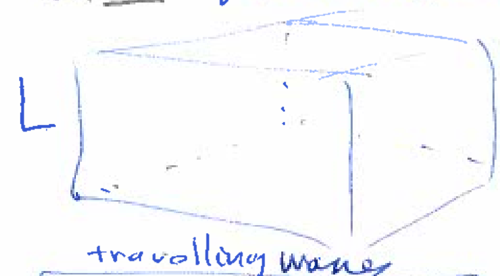
{ $\mathcal{P}$ just depends on instantaneous energy not on energy ~~density~~ distribution (even though it is one $\mathcal{P} = -\left(\frac{\partial E}{\partial V}\right)_S$)

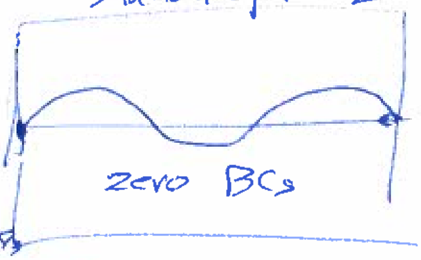
If thermodynamic equilibrium for photon gas $\mathcal{P} = \frac{1}{3}\varepsilon = \frac{1}{3}aT^4$ (see ...)

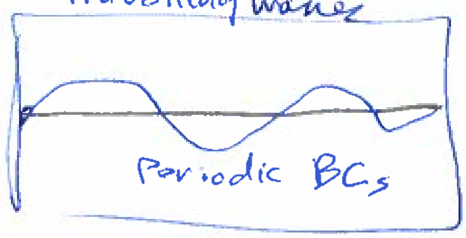
C) Quantum Derivation

Imagine an infinite square well that is a cube of side length L

Then $V = L^3$ or $L = V^{1/3}$







$L_x = n\frac{\lambda_x}{2}$

$k_x = \frac{2\pi}{\lambda_x} \propto \frac{1}{L_x} = \frac{1}{V^{1/3}}$

same for y and z

$L_x = n\lambda_x$

$k_x = 2\pi/\lambda_x$

$k_x \propto \frac{1}{L_x} = \frac{1}{V^{1/3}}$

Same for y, z

These give the same answer for density of states

Textbooks ~~argue~~ say the BCs shouldn't matter deep in interior, but since BCs matter in all other physics problems, textbooks seem to ~~state~~ be glib. I can understand why the shape of volume shouldn't matter for ~~..~~ high k (small λ) modes. Maybe that is all textbooks mean. ~~Periodic BCs are~~ What if you don't really have BCs like deep in a star or the universe as a whole. Textbooks pass over this crucial point in silence. Perhaps just a limiting argument. You get same answer asymptotically if you divide a volume up into cubes of any size, and so the answer must be the same even with no cubes — ~~..~~ or boundaries. Can't complete discussion here. May have something to do with assuming homogeneity and isotropy in deep interior of well.

Many case

$E \propto \sum_i k_i^q$ where sum is over all particles in the completely delocalized states i

$q = 1$ for ER, $E_{ur} = pc \propto \frac{1}{V^{1/3}}$

$q = 2$ for NR, $E_{nr} = \frac{p^2}{2m} \propto \frac{1}{V^{2/3}}$

Expanding adiabatically $dE = -P\,dV$

$$\therefore P = -\left(\frac{\partial E}{\partial V}\right)_S = \begin{cases} \frac{2}{3}\frac{E}{V} = \frac{2}{3}\varepsilon & \text{NR} \\ \frac{1}{3}\frac{E}{V} = \frac{1}{3}\varepsilon & \text{ER} \end{cases}$$

Adiabatic change.
- No heat added,
- particles stay in the states they are in if no particle-particle interaction

Now we think of the particles so far in completely delocalized states. But they can be in compact wave packets ~~and~~ just ~~like~~ as discussed in section (b) with classical particles and results don't change. P just depends on ε and not on the distribution, could be thermodynamic or anything.

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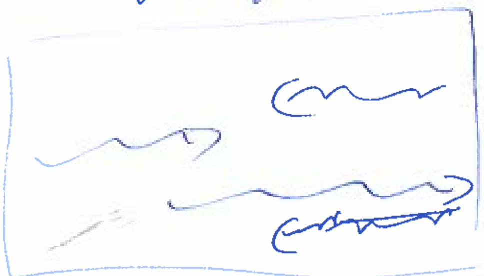
4) Proof that a Planckian Radiation Field (ie, a Blackbody Radiation Field) Stays Planckian under Universal Expansion (Adiabatic Expansion)

$r = r_0 a(t)$



instantaneous reflection

expanding reflecting box as Wien assumed in deriving his approximation to Blackbody radiation field

$\dot{r} = r_0 \dot{a}$

$\frac{\dot{r}}{r} = \frac{\dot{a}}{a}$

$v = \left(\frac{\dot{a}}{a}\right) r$

$v = H r$

$\therefore$ 1st Order Doppler formula

frequency instead of wavelength

$$\frac{d\nu}{\nu} = -\frac{dv}{c} = \frac{H dr}{c}$$

$$= -\frac{\frac{\dot{a}}{a} c\,dt}{c}$$

$dr = c\,dt$ for light

$$\frac{d\nu}{\nu} = -\frac{da}{a}$$

$$\ln \nu = \ln a^{-1}$$

$$\nu \propto a^{-1}$$

$$\nu = \nu_0 \left(\frac{a_0}{a}\right)$$

$$\lambda = \lambda_0 \left(\frac{a}{a_0}\right)$$

Same derivation all along.

$r = r_0 a(t)$

expanding universe

Repeat of Doppler shift derivation of cosmological Redshift, see p.4044. The GR derivation is on p.4040

$\dot{r} = r_0 \dot{a}$

$\frac{v}{r} = \frac{\dot{a}}{a}$

$v = \left(\frac{\dot{a}}{a}\right) r$

$v = H r$

1st order Doppler form

$$\frac{d\nu}{\nu} = -\frac{dv}{c} = -\frac{H dr}{c}$$

$$= -\frac{\frac{\dot{a}}{a} c\,dt}{c}$$

$$\frac{d\nu}{\nu} = -\frac{da}{a}$$

$$\ln \nu = \ln a^{-1}$$

$$\nu \propto a^{-1}$$

$$\nu = \nu_0 \left(\frac{a_0}{a}\right)$$

$d\nu = \left(\frac{a_0}{a}\right) d\nu$

$$\lambda = \lambda_0 \left(\frac{a}{a_0}\right)$$

~~6032~~ [cosmography]

6030 Total photon density $n \propto \frac{1}{a^3}$

and photon individual energies $E \propto \frac{1}{a}$

$\therefore$ Energy density $\propto nE \propto \frac{1}{a^4}$

This is independent of the distribution as a function of frequency (or energy)

The photons in the primordial photon gas (usually called CMB at all ~~times~~ despite not being in the microwave band in early times) to first order do not interact with matter or each or anything.

So photons in ~~band $d\nu_0$~~ bin $d\nu_0$ at fiducial time t_0 stay in that bin as it redshifts with time.

$$d\nu = \left(\frac{a_0}{a}\right) d\nu_0$$ (see p. 6029)

Similarly the ~~energy energy in~~ specific intensity (loosely the energy) in bin $d\nu_0$ stays in that bin

number density of photons in bin $d\nu$.

$$n_\nu d\nu = \left(\frac{a_0}{a}\right)^3 n_{\nu_0} d\nu_0$$

and

$$\underbrace{I_\nu d\nu}_{\text{general time}} = \underbrace{\left(\frac{a_0}{a}\right)^4 B_{\nu_0} d\nu_0}_{\text{at fiducial time which could be cosmic present}}$$

So these scale just like total density and energy

The $\left(\frac{a_0}{a}\right)^3$ is the scaling of the volume with time

We need another factor of $\left(\frac{a_0}{a}\right)$ to account for the scaling of individual photon energy

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$$\therefore I_\nu d\nu = \left(\frac{a_0}{a}\right)^4 \frac{2h\nu_0^3}{c^2} \frac{1}{e^{x_0}-1} d\nu_0$$

See p. 6008

$$= \frac{2h\nu^3}{c^2} \frac{1}{e^{x_0}-1} d\nu$$

Now define temperature parameter T such ~~that~~ that

$$\underbrace{\frac{h\nu}{kT} = x = x_0 = \frac{h\nu_0}{kT_0}}$$

See p. 6029

implying $\frac{\nu}{T} = \frac{\nu_0}{T_0}$

See p. 6029

$$\rightarrow T = \left(\frac{\nu}{\nu_0}\right) T_0 = \left(\frac{a_0}{a}\right) T_0$$

$$\therefore T = \left(\frac{a_0}{a}\right) T_0$$

Is T the actually photon gas ~~temperature at general time~~ t?

at a general time?

~~Well, it gives the actual photon gas spectrum at time t and that Planckian,~~

~~and so yes,~~

Well, $I_\nu d\nu = B_\nu(T) d\nu$

So this is the actually spectrum at time t and it is Planckian with T

and so yes T is the actual temperature of an actual Planck law distribution.

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So ~~photon density~~ and total energy scale ~~just as should~~ for adiabatic expansion and the distribution among frequency bins stays Planckian as long as we assert

$$T = T_0\left(\frac{a_0}{a}\right)$$

So why not just say this T is temperature? It should like a Planckian field at this temperature So it is a Planckian field at this temperature

I do wonder if this proof is actually complete or is it just overall consistency

~~And I guess it's complete~~

6.5) ~~Interlude~~ Digression on Specific intensity & Flux

a) Conservation of Specific Intensity for time-independent static system with no intervening opacity

$$dE = I_{\nu_1} d\nu_1 dt_1 dA_1 d\Omega_1 = I_\nu d\nu\, dt\, dA\, d\Omega$$

by energy conservation

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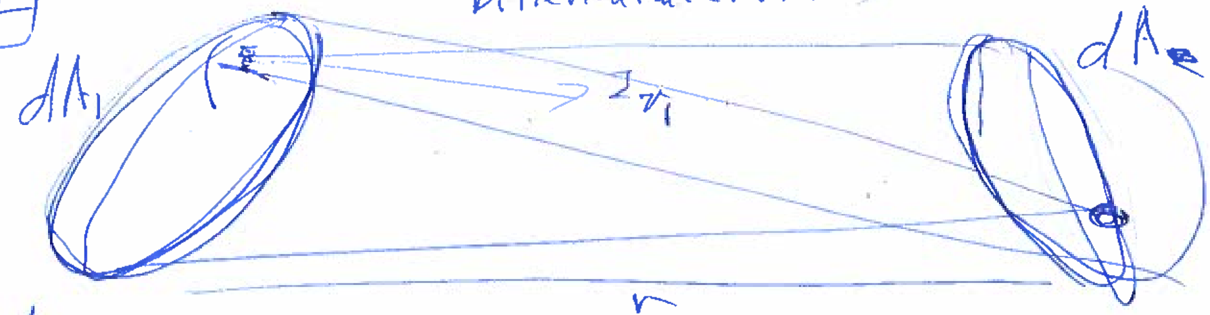
Differential emitters



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$$d\nu_1 = d\nu$$
$$dt_1 = dt$$
since time independent and static

$$\therefore \frac{dE}{d\nu\, dt} = I_{\nu_1} dA_1 d\Omega_1 = I_\nu dA d\Omega$$

Now
$$d\Omega_1 = \frac{dA}{r^2}, \quad d\Omega = \frac{dA_1}{r^2}$$

$dA_1 = r^2 d\Omega$

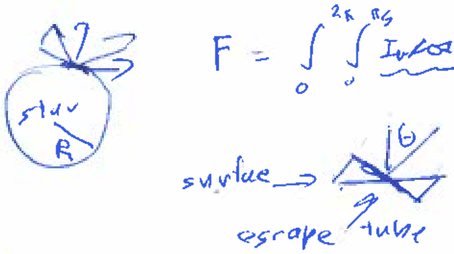
$I_{\nu_1} dA_1 d\Omega_1$
$= I_{\nu_1} r^2 d\Omega \frac{dA}{r^2}$
$= I_{\nu_1} d\Omega dA$

$$\cancel{I_{\nu_1} d\Omega_1 \frac{dA_1}{r^2} = I_\nu d\Omega \frac{dA}{r^2}}$$

$$I_{\nu_1} d\Omega\, d\Omega = I_\nu d\Omega\, d\Omega_1$$

$$I_{\nu_1} = I_\nu \quad \text{QED}$$

b) Specific intensity & Flux

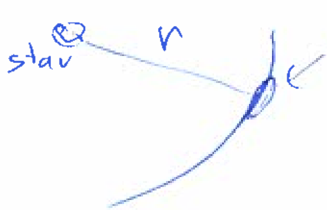


$$F = \int_0^{2\pi}\int_0^{\pi/2} I_\nu \cos\theta \sin\theta\, d\theta\, d\phi$$

~~Flux~~ for I_ν isotropic

Now $\mu = \cos\theta$, $d\mu = \sin\theta\, d\theta$ } radial cosine

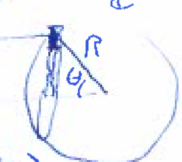
$$F = 2\pi \int_0^1 I_\nu \mu\, d\mu \quad \text{(assuming azimuthal symmetry)}$$



observer flux F_{obs}
$$L = 4\pi R^2 F$$
$$F_{obs} = \frac{L}{4\pi r^2} = F\left(\frac{R}{r}\right)^2$$

Note I_ν will depend on angle of emission θ in general.

Other perspective
should give same answer



~~$dF = I\, 2\pi R \cos\theta (\sin\theta)$~~

~~$I \cos\theta (2\pi R \sin\theta)\, d\theta$~~

~~The cone of transfer~~

$dA_\perp = \mu dA$

$\cos\theta (R^2 \sin\theta\, d\theta\, d\phi)$

$$I_{\nu,\,average} = \frac{2\pi \int I_\nu \mu\, d\mu\, R^2}{\pi R^2}$$

$$F_{obs} = d\Omega\, I_{\nu\,average}$$
$$= \frac{\pi R^2}{r^2} \frac{2\pi \int I_\nu \mu\, d\mu\, R^2}{\pi R^2} = F\left(\frac{R}{r}\right)^2$$

and so consistent