A Heuristic Two-Dark-Energy-Components Model for Cosmic Scale Factor Evolution and a Formalism for Exact Solutions of the Friedmann Equation with Three Density Components

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ABSTRACT

We present a heuristic two-dark-energy-components model for cosmic scale factor evolution. The first dark energy component is just the standard cosmological constant equivalent with pressure $P_\Lambda = w_\Lambda \rho_\Lambda c^2$, where Λ stands for cosmological constant and the equation of state parameter is constant $w_\Lambda = -1$. We will call the first dark energy the Λ dark energy and, for brevity, the model itself the $\Lambda\Gamma$ model where Γ is the symbol adopted for the second dark energy. The Γ dark energy component has pressure $P_\Gamma = w_\Gamma \rho_\Gamma c^2$ with $w_\Gamma = -1/2$. The secondary motivation for the $\Lambda\Gamma$ model is that the DESI DR2 results (Lodha et al. 2025; Abdul Karim et al. 2025) suggest a dynamical dark energy density that has been decreasing from cosmological redshift $z = 0.45$ (see, e.g., Lodha et al. 2025, Fig. 2) and the $\Lambda\Gamma$ model can give this effect with its two dark energy density components. (Note, assuming Λ-CDM evolution roughly applies, the DESI DR2 results suggest the dynamical dark energy density has been decreasing for lookback $\lesssim 4.5\,\mathrm{Gyr}$ or cosmic time $\gtrsim 9\,\mathrm{Gyr}$.) The $\Lambda\Gamma$ model also gives exact solutions for cosmic scale factor $a(t)$ and its inverse $t(a)$ which solutions give the $\Lambda\Gamma$ model physical elegance and makes it easy to test and use as a standard of comparison: these advantages are the primary motivation for introducing the $\Lambda\Gamma$ model rather than any other dynamical dark energy density model. However, we have no physical motivation for the Γ dark energy with $w_\Gamma = -1/2$. The $\Lambda\Gamma$ model is a special case of what we call the V model which in turn a special case of what we call the VU model. It may be that all exact solutions constructed using elementary operations and transcendental functions (e.g., root, trigomentric, hyperbolic, exponential, logarithmic functions, etc.) for the Friedmann equation with three density components all obeying (inverse) power laws

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are derivable from the VU model. In any case, the formalism for all exact solutions of the VU model is given in Appendix A. Of these solutions, those of any interest have probably been long known, but are apparently hard to to find collected, and so this paper is also a review of those exact solutions. (Note, some of the exact solutions are analogues to the non-exact standard solutions of the Friedmann equation reviewed by Bondi (1961, esp. p. 80–86).) Noteworthily, there is a V model solution with negative Λ dark energy that permits a sinusoidal cosmic scale factor evolution that never goes to zero: a kind of evolution not noted by Bondi (1961). However, the solution is unlikely to be ever physically real. Using universe age (i.e., the time from cosmic time zero to cosmic present) as a metric, we study the overall behavior of the $\Lambda\Gamma$ model with the variation of its parameters. A crude test of the $\Lambda\Gamma$ model with the $Om(z)$ diagnostic shows that $\Lambda\Gamma$ model may be crudely adequate to the fit the new observations with a cosmic present density parameters of order 0.53 for Λ dark energy, 0.18 for Γ dark energy, and 0.29 for matter. If a negative Γ dark energy is allowed, there may be other interesting cases to compare to observations. Given that $\Lambda\Gamma$ model is just a heuristic model with no physical reason for its equation of state parameter $w_\Gamma = -1/2$, any expectation that even a good fit of it to observations will be meaningful is very modest. In fact, the main value of this paper is the formalism for exact solutions of the Friedmann equation presented in § 3 and Appendix A, and the review of ancillary results presented in other appendices.

Unified Astronomy Thesaurus concepts: Cosmology (343); Accelerating universe (12); Cosmological constant (334); Cosmological evolution (336); Cosmological models (337); Cosmological parameters (339); Dark energy (351); Density parameter (372); Einstein universe (452); Expanding universe (502); Friedmann universe (551); Lambda density (898); Lemaître universe (914); Matter density (1014);

1. Introduction

Section 2 introduces $\Lambda\Gamma$ model and § 3 derives the Friedmann equation forms for what we call the VU model which specializes to what we call V Model which has the $\Lambda\Gamma$ model as a special case. Section 4 presents the exact solutions for the V model which specializes to the $\Lambda\Gamma$ model for the parameter $V = 2/3$. In § 5, we use the universe age (the time from cosmic time zero to cosmic present τ_0 in scaled time (see § 2)) as a metric to test the overall behavior of the $\Lambda\Gamma$ model solutions as their parameters are varied. We make a preliminary test of the $\Lambda\Gamma$ model using the $Om(z)$ diagnostic and data from (Lodha et al.

2025, Fig. 9) in § 6 and also present some ancillary formulae there. For future reference, we specialize the deceleration parameter diagnostic for the $\Lambda\Gamma$ model and present some ancillary formulae in § 7. A discussion is given in § 8. Appendix A investigates exact solutions for the VU model in terms of a generalized conformal time. Appendix B gives the exact solution for the radiation-matter-curvature universe, some exact special cases, and appropriate discussion. For reference, Appendix C gives the derivation of elementary one density component exact solutions of the Friedmann equation of significant interest and a table of these exact solutions with some of their details. Appendix D presents a formalism for approximate analytic solutions to the Friedmann equation by perturbation from exact (analytic) solutions $x_{\rm ex}(\tau)$. Appendix E shows that Friedmann equation exact solutions for $\tau(x)$ can in some cases be repurposed for solving for the comoving coordinate $\chi(z)$ that follows from the Robertson-Walker metric for a light signal reaching the observer at cosmological redshift $z = 0$ from cosmological redshift z in a universe model obeying the Friedmann equation.

2. The $\Lambda\Gamma$ Model

First, to be general, we assume for one ensity component (and in general there will be multiple such components) $\rho_p \propto x^{-p}$ (where p symbolizes power $p \geq 0$) and then obtain

$$\frac{\dot{\rho}_p}{\rho_p} = -p\frac{\dot{x}}{x} \ , \tag{1}$$

where x is cosmic scale factor and the time derivative is with respect to scaled cosmic time τ which for the $\Lambda\Gamma$ model is given by $\tau = H_0 t$ with H_0 being the Hubble constant and t being unscaled cosmic time. (Note we do not use a for the cosmic scale factor in this paper unless so noted.) We equate $\dot{\rho}_p/\rho_p$ to the usual perfect fluid equation of cosmology (e.g., Liddle 2015, p. 26) with perfect fluid pressure P_p parameterized by equation of state

$$P_p = w_p \rho_p c^2 \tag{2}$$

(with w_p being the equation of state parameter for power p) to obtain

$$-p\frac{\dot{x}}{x} = \frac{\dot{\rho}_p}{\rho_p} = -3\frac{\dot{x}}{x}\left(1 + \frac{P_p}{\rho_p c^2}\right) = -3\frac{\dot{x}}{x}\left(1 + w_p\right) \tag{3}$$

which we solve to obtain

$$p = 3\left(1 + w_p\right) \ , \quad w_p = \left(\frac{1}{3}\right)p - 1 \ , \quad \text{and} \quad p - 2 = 1 + 3w_p \ , \tag{4}$$

where the last expression is used in § 7. For Γ dark energy density component and as aforesaid in the abstract,

$$w_\Gamma = -\frac{1}{2} \ , \quad \text{and thus} \quad p_\Gamma = \frac{3}{2} \quad \text{and} \quad p_\Gamma - 2 = 1 + 3w_\Gamma = -\frac{1}{2} \ . \tag{5}$$

For the Λ dark energy density component, of course,

$$w_\Lambda = -1 \ , \quad \text{and thus} \quad p_\Gamma = 0 \quad \text{and} \quad p_\Gamma - 2 = 1 + 3w_\Gamma = -2 \ . \tag{6}$$

As well as the Γ dark energy, as aforesaid in the abstract, there is also the Λ dark energy (i.e., the ordinary constant dark energy which may, in fact, be just a cosmological constant) and matter. The brief radiation-dominated era (cosmic time from after inflation $\sim 10^{-35}\,$s to $\sim 50\,$kyr (e.g., Hergt & Scott 2024, p. 6)) of the observable universe is not being considered. Thus, the $\Lambda\Gamma$ model has three density components: Λ dark energy, Γ dark energy, and matter.

The secondary motivation for the $\Lambda\Gamma$ model is that the DESI DR2 results (Lodha et al. 2025; Abdul Karim et al. 2025) suggest a dynamical dark energy density that has been decreasing from cosmological redshift $z = 0.45$ (see, e.g., Lodha et al. 2025, Fig. 2) and the $\Lambda\Gamma$ model can give this effect with its two dark energy density components. (Note, assuming Λ-CDM evolution roughly applies, the DESI DR2 results suggest the dynamical dark energy density has been decreasing for lookback $\lesssim 4.5\,$Gyr or cosmic time $\gtrsim 9\,$Gyr.) The $\Lambda\Gamma$ model also gives exact solutions for cosmic scale factor $x(\tau)$ (conventionally written $a(t)$ where t is unscaled time as aforesaid) and its inverse $\tau(x)$ (conventionally written $t(a)$) which solutions give the $\Lambda\Gamma$ model physical elegance and makes it easy to test and use as a standard of comparison: these advantages are the primary motivation for introducing the $\Lambda\Gamma$ model rather than any other dynamical dark energy density model. However, we have no physical motivation for Γ dark energy with equation of state parameter $w_\Gamma = -1/2$ (or equivalently $p_\Gamma = 3/2$). Nevertheless, it is possible that $\Lambda\Gamma$ model could fit the cosmic scale factor evolution to some degree after the brief radiation-dominated era of the observable universe: i.e., after cosmic time $\sim 50\,$kyr (e.g., Hergt & Scott 2024, p. 6). In which case, the $\Lambda\Gamma$ model might become physically interesting.

3. The Friedmann Equation and its Forms for the VU Model, the V Model, and the $\Lambda\Gamma$ Model

3.1. The General Friedmann Equation

The Friedmann equation in general form, certain specializations, and certain defined quantities follow:

$$\left(\frac{dx/dt}{x}\right)^2 = \begin{cases} \dfrac{8\pi G}{3}\rho & \text{in general where } \rho \text{ can includes all density forms} \\ & \quad \text{including the cosmological constant form} \\ & \quad \text{(which may or may not be a constant dark energy),} \\ & \quad \text{and the curvature density which is actually not} \\ & \quad \text{a density and is negative/positive} \\ & \quad \text{for positive/negative curvature.} \\ & \quad \text{Note, } x \text{ (rather than the usual } a\text{) is used for} \\ & \quad \text{cosmic scale factor in this paper unless otherwise noted.} \\ \dfrac{8\pi G}{3}\rho_{\rm fid}\displaystyle\sum_k a_k x^{-p_k} & \text{where } \rho_{\rm fid} \text{ is a fiducial density,} \\ & \quad \text{the density components can be expressed scaling} \\ & \quad \text{as (inverse) powers } p_k \text{ of the cosmic scale factor,} \\ & \quad a_k = \rho_k/\rho_{\rm fid} \text{ are the density parameters, and} \\ & \quad \text{(i.e., the scaled densities), } a_k \text{ are} \\ & \quad \text{the parameter constants (usually symbolized} \\ & \quad \text{by } \Omega\text{'s with appropriate subscripts).} \end{cases}$$

$$H_{\rm fid} = \begin{cases} \sqrt{\dfrac{8\pi G}{3\rho_{\rm fid}}} & \text{is a Hubble-constant-like parameter.} \\ & \quad \text{If } \sum_k a_k \text{ is scaled to 1 (which is not always done),} \\ & \quad \text{then } \rho_{\rm fid} \text{ is the actual density when } x = 1 \\ & \quad \text{and is called the critical density, and} \\ & \quad H_{\rm fid} \text{ is the Hubble constant for time when } x = 1. \\ & \quad \text{An example of a case where } \sum_k a_k \\ & \quad \text{is not scaled to 1 is the static Einstein universe} \\ & \quad \text{where it is scale to 0.} \\ & \quad \text{In this paper, for generality we do not explicitly} \\ & \quad \text{scale } \sum_k a_k \text{ to 1 unless explicitly noted.} \end{cases}$$

$$d\tau = \begin{cases} H_{\rm fid}\,dt & \text{defines a scaled cosmic time } \tau. \\ & \text{Using scaled cosmic time effectively reduces} \\ & \text{number of free density parameter constants } a_k \\ & \text{by 1.} \end{cases}$$

$$\left(\frac{\dot{x}}{x}\right)^2 = \begin{cases} \sum_k a_k x^{-p_k} & \text{is the general scaled Friedmann equation.} \\ & \text{for (inverse) power density parameters.} \end{cases} \tag{7}$$

For a reference for most of the results in Equation (7), see, e.g., Liddle (2015, p. 51, 55–56).

Note, the Friedmann equation is a nonlinear 1st order differential equation. The fact that it is nonlinear means that sums of solutions are not solutions. In particular, note, the elementary (exact) one density component solutions (review Appendix C) cannot be summed to create exact solutions though they in some cases they approximate exact solutions to some degree. Note also, 1st order differential equations usually only have stationary points at infinity of the independent variable. However, those like the Friedmann equation where the right-hand-side is a square root can have stationary points at finite points of the independent variable. In fact, the Friedmann equation does have sinusoidal solutions has we show explicitly in § 4 and Appendix A. However, the cosmic scale factor $x(t)$ has no physical meaning when negative and and there are no oscillating Friedmann equation universe models for density components we consider, except with two physically very unlikely negative density components (see § 4.6).

The Friedmann equation is derived from general relativity with the assumption of universal homogeneity and isotropy (i.e., the assumption of the cosmological principle). There is a derivation from Newtonian physics (e.g., Liddle 2015, p. 22–24) also with the assumption of the cosmological principle plus hypotheses beyond Newtonian physics. In Newtonian physics, the Friedmann equation is an energy conservation equation rather than an dynamical equation (as many people would use this expression). In general relativity, it would probably also be considered a conservation equation also, but not exactly of energy since Newtonian gravitational potential energy has no simple meaning in general relativity (e.g., Penrose 2004, p.464–469). The Newtonian derivation does not, of course, give any information of space curvature. The curvature constant k that turns up in the Newtonian derivation is just constant integration.

There is no general exact solution of the general Friedmann equation given by Equation (7). Investigating special case exact solutions is, of course, one the points of this paper. Three density component exact solutions can be found in some cases and we take up which cases below in § 3.2. Interesting cases of exact solutions with more than three density components may not exist. A five density component of special case where one wishes there were an exact solution because all the components are of great interest is

$$\left(\frac{\dot{x}}{x}\right)^2 = \sum_{p=0}^{4} a_p x^{-p} \ , \tag{8}$$

where the general powers p_k have been specialized to integer powers p running 0 to 4. The integer powers have particular interesting meanings. The power $p = 0$ gives the cosmological constant or constant dark energy density parameter $p = 1$ gives some forms of quintessence density parameter, $p = 2$ gives the curvature density parameter (which as aforesaid is not actually a density form), $p = 3$ gives the (non-relativistic) matter density parameter, and $p = 4$ gives the radiation density parameter where radiation is a conventional shorthand for relativistic mass-energy which is usually, in fact radiation (e.g., Steiner 2008, p. 6–7; e.g., Melia 2014). Though there is no known general exact solution $\tau(x)$ nor $x(\tau)$ for Equation (8), there is a general exact solution for it in terms of conformal time $x(\eta)$ (where conformal time $d\eta = d\tau/x$) (Steiner 2008, p. 9) provided one regards the Weierstrass elliptic function as an analytic function (Wikipedia: Weierstrass elliptic function). But there is no known general exact solution for $\eta(\tau)$ nor $\tau(\eta)$ for Equation (8) (Steiner 2008, p. 11): one always has to do a numerical integration to get these functions. Hence, the general $x(\eta)$ solution cannot be regarded as an exact solution in a useful sense. We call exact solutions that cannot regarded as exact in a useful sense incomplete exact solutions. We do not consider incomplete exact solutions further in this paper.

Now the general scaled Friedmann equation form given by Equation (7) can be rearranged for easy numerical solution for $\tau(x)$:

$$d\tau = \pm \frac{dx}{x\sqrt{\sum_k a_k x^{-p_k}}} \ . \tag{9}$$

The solution can be by, e.g., the midpoint method (e.g., Wikipedia: Midpoint method) or Runge-Kutta methods (e.g., Wikipedia: Runge-Kutta methods). In this paper, we are interested primarily in exact (analytic) solutions and take little further interest in numerical solutions. Note, $\tau(x)$ exact solutions are the most directly solved for in all cases it seems and only in some cases can exact inverted $x(\tau)$ solutions be obtained from them. For the exact $\tau(x)$ solutions discussed in this paper, we investigate when inverted exact $x(\tau)$ solutions can be obtained in general in Appendix A.

Note, the inverted exact solutions $x(\tau)$ are, in fact, of great interest. For plotting the exact solutions $\tau(x)$ are completely adequate and by Newton-Raphson iteration of $\tau(x)$ or compiled tables of $\tau(x)$, one can always determine $x(\tau)$. However, cosmological modeling of the observable universe often requires fitting data to the cosmological redshift $z = (\lambda_0 - \lambda)/\lambda$, where here subscript 0 stands for cosmic present and unsubscripted means time of light signal emission. In fact, wavelength scales with with cosmic scale factor $x(\tau)$, and so cosmological redshift is related cosmic scale factor as follows:

$$z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{x_0}{x} - 1 \qquad \text{and} \qquad \frac{x}{x_0} = \frac{1}{1+z} \tag{10}$$

(e.g., Liddle 2015, p. 38–39). Now if one is only fitting cosmic quantities that depend on z, then one can always fit their cosmic evolution in terms z or x and afterward use $\tau(x) = \tau(z)$ to find the cosmic time evolution. However, if one is simultaneously fitting cosmic quantities and other processes that evolve with time (and which have no intrinsic dependence on z or x), then it can be convenient to fit the evolution of all the quantities with cosmic time τ in which case having an exact solution $x(\tau)$ is useful: the good alternative probably being compiled tables of $x(\tau)$. Whether or not having an exact solution $x(\tau)$ is advantageous depends on the particular case, of course. However, for understanding, it is elegant to have an exact solution $x(\tau)$ since everything evolves with time and not z or x. This point is why one generally see x plotted versus τ and not τ versus x even if the plot is generated by an exact or numerical solution $\tau(x)$.

3.2. The Three Density Component Friedmann Equation Form Yielding Exact Solutions

As noted above in § 3.1, exact solutions exist in some cases for the three density component Friedmann equation which we write

$$\frac{\dot{x}}{x} = \pm\sqrt{ax^{-r} + bx^{-q} + cx^{-p}} \ , \tag{11}$$

where scaled cosmic time τ is not necessarily scaled by the Hubble constant H_0, a, b, and c are density parameter constants scaled by some fiducial density that is not necessarily a critical density, and p, q, and r are distinct powers that are general other than that $p > q > r \geq 0$ and $q = (r + p)/2$ (which is proven necessary below). Note, the constraint $a + b + c = 1$ is not necessarily imposed. Also note for x very small, we have to leading order (LO) (choosing only the growing case from $x(\tau = 0)$ for clearest illustration) and assuming $c > 0$ with $p > 0$

$$\dot{x} = \sqrt{c}\, x^{1-p/2} \ , \qquad \text{and so} \qquad \tau_{\rm LO} = \frac{1}{\sqrt{c}}\left(\frac{x^{p/2}}{p/2}\right) \qquad \text{and} \qquad x_{\rm LO} = \left[\sqrt{c}\left(\frac{p}{2}\right)\tau\right]^{2/p} \ . \tag{12}$$

It follows that $x(\tau = 0)$ has zero slope, constant slope, and infinite slope for, respectively, $0 < p < 2$, $p = 2$, and $p > 2$. Thus, $x(\tau = 0)$ is a minimum only for $p < 2$. If density parameter constant c is the only non-zero density parameter constant and $p = 0$, then the solution is, of course, the exponential de Sitter universe solution

$$x = x_0 e^{\sqrt{c}\tau} , \tag{13}$$

where $x(\tau = 0) = x_0$ and which only goes to zero as $\tau \to -\infty$. Note, the solutions given by Equation (12) and (13) can also be obtained from the elementary one density component solutions given for reference in Appendix C.

What is needed for exact solutions to our knowledge is to transform the expression under the square root sign in Equation (11) into the quadratic equation form $ay^2 + by + c$ and then search integral tables to find indefinite integrals that allow transformed Friedmann equation to be solved exactly. To effect the transformation, we define a new cosmic scale factor

$$y = x^{1/V} \quad (\text{with } V \neq 0) \qquad \text{implying} \qquad x = y^V \qquad \text{and} \qquad \frac{dx}{x} = \frac{V\,dy}{y} . \tag{14}$$

Note, physically real x is a real number and positive and V as we define it below is real and positive, and so y must be real and positive. Now we transform Equation (11) for direct solution for τ as a function of y as follows

$$\begin{aligned} d\tau &= \pm\frac{dx}{x\sqrt{ax^{-r} + bx^{-q} + cx^{-p}}} = \pm\frac{V\,dy}{y\sqrt{ay^{-rV} + by^{-qV} + cy^{-pV}}} \\ &= \pm\frac{Vy^U\,dy}{\sqrt{ay^{-rV+2+2U} + by^{-qV+2+2U} + cy^{-pV+2+2U}}} , \end{aligned} \tag{15}$$

where we have introduced the parameter U which allows for a greater number of exact solutions. To get an exactly solvable Friedmann equation to our knowledge requires a quadratic equation form under the square root, and so we require

$$-rV + 2 + 2U = 2 , \qquad -qV + 2 + 2U = 1 , \qquad \text{and} \qquad -pV + 2 + 2U = 0 . \tag{16}$$

Note, if we had interchanged the 2 and 0 on the right-hand sides of the first and third expressions of Equation (16), we would not have made any new solutions possible as we show in Appendix A.1. Solving for p, q, and r gives

$$p = \frac{2 + 2U}{V} , \qquad q = \frac{1 + 2U}{V} , \qquad r = \frac{2U}{V} , \qquad \text{and also} \qquad U = \frac{rV}{2} . \tag{17}$$

Note,

$$q = \frac{r + p}{2} \tag{18}$$

necessarily to effect the transformation we have effected: i.e., q must be the average of p and r. In fact, it seems impossible to find another parameter in addition to V and U that allows p, q, and r to be independent and still gives a Friedmann equation form that has exact solutions. Thus only two of p, q, and r can be independent and three density component cases where the density parameter powers do not satisfy Equation (17) will not have exact solutions of the Friedmann equation to our knowledge.

Subtracting rV from pV and doing some other simple operations making use of Equation (17), we obtain

$$V = \frac{2}{p-r} \qquad \text{and} \qquad U = \frac{r}{p-r} \ . \tag{19}$$

From Equation (19), it is clear $V > 0$ and $U \geq 0$. As we show in Appendix A.2, to get an exactly solvable Friedmann equation requires integer $U \geq 0$. Thus, we require the ratio $W = r/p$ to satisfy

$$W = \frac{r}{p} = \frac{U}{U+1} = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1 \ , \tag{20}$$

where $W = 1$ corresponds to one density component solutions (which we, in fact, obtained above in Equations (12) and (13) for other reasons and are also given for reference in Appendix C)).

Note, requiring the equality $ax^{-r} = cx^{-p}$ gives the x and y equality cosmic scale factor values

$$x_{\mathrm{eq}\,pr} = \left(\frac{c}{a}\right)^{1/(p-r)} \qquad \text{and} \qquad y_{\mathrm{eq}\,pr} = \left(\frac{c}{a}\right)^{V/(p-r)} \tag{21}$$

These equality cosmic scale factor values are, of course, where the density components controlled by parameters a and c are equal.

Finally, the transformed Friedmann equation itself which allows for exact solutions for integer $U \geq 0$ (as we show in Appendix A.2) and which we call the VU Friedmann equation is

$$d\tau = \pm \frac{V y^U \, dy}{\sqrt{ay^2 + by + c}} \ . \tag{22}$$

Note, the solution $\tau(y)$ of Equation (22) will have at $y = 0$ a minimum for $U > 0$ (since $(d\tau/dx)_{(\tau=0)} = 0$) and a constant slope for $U = 0$ (since $(d\tau/dx)_{(\tau=0)} = \pm(V/\sqrt{c}) > 0$) which occurs only for $r = 0$. To give it a name, the model corresponding to Equation (22) we call the VU model which name we mentioned in § 1.

If Equation (22) has only two nonzero density parameter constants, we choose the nonzero density parameter constants to be a and c since this choice is most symmetrical, no disadvantage, and also easiest for understanding since these have, respectively, the smallest power parameter (i.e., r) and largest power parameter (i.e., p) and the zeroed density parameter constant b power parameter is always $q = (r + p)/2$ (see above Equation (18). In this case, Equation (22) reduces to

$$d\tau = \pm \frac{V y^U \, dy}{\sqrt{ay^2 + c}} \ . \tag{23}$$

Note, if one is specializing an already obtained three density component solution to a two density component solution, relabeling the density parameter constants is usually inconvenient and we do not do so in this paper unless otherwise noted.

If Equation (22) has only one density parameter, then V and U by Equation (19) are indeterminate since only one density parameter power is defined, and so to reduce Equation (22) to the one density parameter Friedmann equation consistently with the formalism in Appendix A.1, we choose the nonzero density parameter constant to be $a > 0$ and then the nature choice for V and U is

$$V = 1 \qquad \text{and} \qquad U = 0 \ , \tag{24}$$

With the above choices, Equation (22) reduces to

$$d\tau = \pm \frac{V y^U \, dy}{\sqrt{a}} = \pm \frac{x^{r/2-1}, dx}{\sqrt{a}} \tag{25}$$

which has the well known elementary one density component $x(\tau)$ solutions (which we, in fact, obtained above in Equations (12) and (13) (using c as the one nonzero density parameter constant) for other reasons and are also given for reference in Appendix C))

$$x = \begin{cases} \left[\sqrt{a} \left(\frac{r}{2} \right) \tau \right]^{2/r} & \text{for } r > 0; \\ x_0 e^{\sqrt{a}\,\tau} & \text{for } r = 0, \end{cases} \tag{26}$$

where we have chosen for clearest illustration growing solutions with $x(\tau = 0) = 0$ for the first solution (as for a Big Bang singularity at cosmic time zero) and $x(\tau = 0) = x_0$ for the second solution (which is actually the de Sitter universe solution).

We specialize the VU model to what we call the V model (which name we mentioned in § 1) for the case of $U = 0$. The V model Friedmann equation is

$$d\tau = \pm \frac{V \, dy}{\sqrt{ay^2 + by + c}} \ . \tag{27}$$

For the V model,

$$p = \frac{2}{V} \ , \qquad q = \frac{1}{V} = \frac{p}{2} \ , \qquad \text{and} \qquad r = 0 \ . \tag{28}$$

The Friedmann equation for the $\Lambda\Gamma$ model is the special case of the V model Equation (27) with $p = 3$ implying $q = 3/2$ and $V = 2/3$. Note, $r = 0$ implies $U = 0$ necessarily.

3.3. The VU Model, the V Model, and the $\Lambda\Gamma$ Model

In § 3.2 above, we introduced the VU model whose Friedmann equation form is Equation (22). If $U = 0$, the VU model specializes to the V model (which name we also introduced above in § 3.2) and if $V = 2/3$, there is a further specialization to the $\Lambda\Gamma$ model. The exact solutions for the $\Lambda\Gamma$ model and the more general V model are presented in § 4. Appendix A present from formalism (which includes a generalized conformal time $\tilde{\eta}$) from which all exact solutions VU model can be derived.

One aspect of the formalism for VU formalism that we make use of below in § 4 is the classification of exact solutions $\tau(x)$ and their inverse solutions $x(\tau)$ into classes the exp-like (for exponential behavior), sinh-like (for hyperbolic sine behavior), cosh-like (for hyperbolic cosine behavior), and sin-like (for sinusoidal behavior). The names exp-like, sinh-like, cosh-like, and sin-like denote the functional behavior of the doubly scaled cosmic scale factor $\tilde{y}$ as function of generalized conformal time $\tilde{\eta}$ (see Appendix A.2). However, we give another class power-like for those exact solutions where integer $U > 0$ since there behavior seems primarily power law and for odd $U > 0$ is exactly power law (see Appendix A.2).

Note, $x(\tau)$ is the inverse solution of $\tau(x)$, but vice versa is also true. The same can be said of the further scaled cases of $y(\tau)$ and $\tau(y)$, etc. that we use in this paper. Also, $x(\tau)$ and its inverse $\tau(x)$, etc. can also be considered collectively as a single solution. Therefore, we call whether a solution is called an inverse solution or not or whether $x(\tau)$ and its inverse $\tau(x)$, etc. are collectively called solution or not depends on context.

Another point of jargon in this paper is that historically or otherwise important cosmological models are usually referred to by their emphatic names which use "universe" instead of model: e.g., the Einstein universe (Wikipedia: Einstein's static universe: a static Friedmann equation model with an exact solution), the de Sitter universe (Wikipedia: De Sitter universe: an exponential expanding Friedmann model with exact solution), and the Lemaître universe (a matter-positive curvature-Λ universe) which has no exact solution (e.g., Bondi 1961, p. 82,84–85,120–122,165,168–170,175–176).

4. The Friedmann Equation Solutions for the V Model Which Specializes to the $\Lambda\Gamma$ Model for $V = 2/3$

In this section, we derive the interesting solutions (both $\tau(x)$ and $x(\tau)$ forms) for the V model. Recall, the $\Lambda\Gamma$ model is special case of the V model with $p = 3$ for matter, $q = 3/2$ for Γ dark energy component, $r = 0$ for Λ dark energy component, and $V = 2/3$. Another special case of the V model of interest has $p = 4$ for radiation, $q = 2$ for curvature, $r = 0$

for Λ dark energy component, and $V = 2/p = 1/2$. (Note, the curvature density parameter (conventional symbol Ω_k) is conventionally negative/positive for positive/negative curvature (e.g. Liddle 2015, p. 33,52).) This case is the radiation-curvature-Λ universe which is an analogue (when the q density parameter component is negative) to the Lemaître universe (a matter-positive curvature-Λ universe) which has no exact solution (e.g., Bondi 1961, p. 82,84–85,120–122,165,168–170,175–176).

In order for the $\tau(x)$ and $x(\tau)$ solutions for Equation (27) to be for the V model as well the $\Lambda\Gamma$ model, we leave V general in all our formulae for these solutions. Note, most, maybe all, of these solutions have probably been long known.

In Equation (27), a, b, and c are the density parameter constants. (Recall, the symbol a is not used for the cosmic scale factor in this paper, unless otherwise noted.) The values of density parameter constants are their weights. Note, we consider cases where a and b are positive, negative, or zero, but we assume $c \geq 0$ always and this holds always if the c density parameter is matter or radiation since they always have positive mass-energy.

For the solutions for the $\Lambda\Gamma$ model itself, density parameter constant c is the cosmic present matter density parameter constant, density parameter constant b is the cosmic present Γ dark energy density parameter constant, and density parameter constant a is the Λ dark energy density parameter constant (which is constant with cosmic time). We add descriptive terms Λ density parameter constant, Γ density parameter constant, and matter density parameter constant when we are referring to the $\Lambda\Gamma$ model specifically and there is a need for greater clarity.

???? However, for the $\Lambda\Gamma$ model, we restrict solutions to those that start from a Big Bang singularity at cosmic time zero (i.e., to those with $x(\tau = 0) = 0$), that strictly increase with time thereafter, and have $b > 0$ (except what we call below the V_1 solution requires $|b| < 2\sqrt{ac}$). Thus, we restrict what we call $\Lambda\Gamma$ model solutions to those V model solutions that at least minimally match the observable universe. There are 4 of these V model solutions which we call the V_i solutions (more explicitly $V_i(\tau)$ solutions) with index i running 1 to 4. We also find 4 solutions not conforming to our requirements for the $\Lambda\Gamma$ model solutions: we call these the V_{i-} solutions (more explicitly the $V_{i-}(\tau)$ solutions) with index i running from 2 to 5. The first 3 of the V_{i-} solutions are the same functions as like-numbered V_i solutions, but with different choices of density parameter constants and/or initial conditions. The inverse solutions (which are obtained first) are called the V_i^{-1} and V_{i-}^{-1} solutions (more explicitly the $V_i^{-1}(x)$ and $V_{i-}^{-1}(x)$ solutions).

For the $\Lambda\Gamma$ model solutions, we assume scaled cosmic time $\tau = H_0 t$ and choose $x = 1$ at cosmic present time τ_0 (which is the universe age defined in § 1) to yield the scaled

Hubble constant 1 (and unscaled Hubble constant H_0). By this choice, the density parameter constants obey the constraint

$$a + b + c = 1 \ . \tag{29}$$

The fiducial round-number Λ-CDM density parameter constant weights are $c = 0.3$ and $a = 0.7$ (e.g., Wikipedia: Lambda-CDM model: Parameters), and so $b = 0$. If we moved some density parameter constant weight from a to b, we would clearly increase the overall effect of dark energy and cause more rapid growth from the cosmic time zero, and so decrease the universe age τ_0. On the other hand, if we moved some model weight from c to b, we would weaken the initial growth in the matter dominated era, but would strengthen the later dark-energy dominated era, and so a priori it is not certain how τ_0 would change. In either case, if $b > 0$, the total dark energy would be decreasing as the universe age τ_0 is approached which is the effect found in the DESI DR2 results (Lodha et al. 2025; Abdul Karim et al. 2025) and, as aforesaid in the abstract and § 2, matching this effect with exact Friedmann solution is the primary motivation for introducing the ΛΓ model.

Note, we do not explicitly make scaled time $\tau = H_0 t$ nor apply the $a + b + c = 1$ constraint on the $V_i(\tau)$ and $V_i^{-1}(x)$ solutions in order to leave the formulae general, unless otherwise noted. and so the density parameter constants a, b, and c are independent for those solutions. This means that for the V model solutions, the implicit scale density is not necessarily a criticial density and the implicit inverse time scale is not necessarily a Hubble time: i.e., τ is not necessarily $H_0 t$.

4.1. The $\tau(x)$ V Model Solutions

For the cosmic time $\tau(x)$ V model solutions using a standard table of integrals (Wikipedia: List of integrals of irrational functions: Integrals involving $R = \sqrt{ax^2 + bx + c}$) and aided by Google AI for the arcosh case, we obtain

$$\tau_{\rm exp/sinh/cosh} = \begin{cases} \frac{V}{\sqrt{a}} \ln\left|\frac{2ay+b+2\sqrt{a}\sqrt{ay^2+by+c}}{b+2\sqrt{ac}}\right| & \text{Conditions: } a>0,\ c\geq 0, \\ & \quad \tau(y=x^{1/V}=0)=0. \\ & \text{Valid for solutions that increase} \\ & \quad \text{monotonically from } y=x^{1/V}=0. \\ & \text{Solution } V_{123}^{-1}(x)\text{: equivalent} \\ & \quad \text{to all of } V_1^{-1}(x),\ V_2^{-1}(x), \\ & \quad \text{and } V_3^{-1}(x) \text{ when their} \\ & \quad \text{conditions are applied.} \\ \frac{V}{\sqrt{a}} \ln\left|\frac{2a+b+2\sqrt{a}}{b+2\sqrt{ac}}\right| = \tau_0 & \text{Extra Conditions: } y=x^{1/V}=0 \text{ and} \\ & \quad a+b+c=1 \text{ is explicitly applied.} \\ \frac{V}{\sqrt{a}} \ln\left|\frac{1-c+a+2\sqrt{a}}{1-c-a+2\sqrt{ac}}\right| = \tau_0 & \text{Extra Condition: } b=1-(a+c) \\ & \quad \text{is explicitly applied.} \end{cases} \tag{30}$$

$$\tau_{\rm exp} = \begin{cases} \pm\frac{V}{\sqrt{a}} \ln|2ay+b| + C & \text{Conditions: } a>0,\ b^2-4ac=0. \\ & \text{Solution } V_2^{-1}(x)\text{: upper case only, } b>0, \\ & \quad C \text{ chosen to give } V_2(\tau=0)=0. \\ & \text{Solution } V_{2-}^{-1}(x)\text{: No constraint on } b \\ & \quad \text{and C chosen to make } V_{2-}(\tau) \\ & \quad \text{heuristically interesting.} \end{cases} \tag{31}$$

$$\tau_{\rm sinh} = \begin{cases} \frac{V}{\sqrt{a}} \text{arsinh}\left(\frac{2ay+b}{\sqrt{4ac-b^2}}\right) - \tau_{{\rm zero}\,x} & \text{Conditions: } a>0,\ c>0, \\ & \quad b^2-4ac<0,\ |b|<2\sqrt{ac}\,, \\ & \quad \text{zero time } \tau_{{\rm zero}\,x} \text{ chosen} \\ & \quad \text{to give } \tau(y=x^{1/V}=0)=0 \\ & \quad \text{so that } x(\tau=0)=0. \\ & \text{Solution } V_1^{-1}(x). \\ & \text{No solution } V_{1-}^{-1}(x). \\ \tau_{{\rm zero}\,x} = \frac{V}{\sqrt{a}} \text{arsinh}\left(\frac{b}{\sqrt{4ac-b^2}}\right) & \text{Zero time } \tau_{{\rm zero}\,x}. \end{cases} \tag{32}$$

$$
\tau_{\cosh} = \begin{cases}
\pm\dfrac{V}{\sqrt{a}}\operatorname{arcosh}\left(\left|\dfrac{2ay+b}{\sqrt{b^2-4ac}}\right|\right) & \text{Conditions: } a>0,\ b^2-4ac>0, \\
& \quad \text{upper/lower case time} \\
& \quad \text{increasing/decreasing with } y, \\
& \quad \text{sgn}(\ldots) \text{ is the sign function.} \\
& \text{Solution } V_3^{-1}(x)\text{: } b>0, \text{ upper case,} \\
& \quad \text{zero time } \tau_{\text{zero}\,x} \text{ exists,} \\
& \quad \text{and } \tau \geq \tau_{\text{zero}\,x}. \\
& \text{Solution } V_{3-}^{-1}(x) \text{ is for all other} \\
& \quad \text{physical cases.} \\
\tau_{\text{zero}\,x} = \dfrac{V}{\sqrt{a}}\operatorname{arcosh}\left(\left|\dfrac{b}{\sqrt{b^2-4ac}}\right| \geq 1\right) & \text{For } y = x^{1/V} = 0. \\
& \text{Condition: The zero time } \tau_{\text{zero}\,x} \\
& \quad \text{is chosen to give } x(\pm\tau_{\text{zero}\,x}) = 0. \\
& \text{Note, } \tau_{\text{zero}\,x} \text{ does not exist} \\
& \quad \text{if the arcosh function} \\
& \quad \text{argument} < 1 \\
& \quad \text{(e.g., when } b=0 \text{ and } c<0\text{).} \\
& \quad \text{In this case, there is no } x=0 \text{ line} \\
& \quad \text{intersection and the } y \text{ and} \\
& \quad x \text{ solutions that open upward} \\
& \quad \text{are physical and those that} \\
& \quad \text{open downward are not.} \\
& \text{Note, for } \tau_{\text{zero}\,x} = 0, \text{ one needs } c=0. \\
& \text{Note, if } \tau_{\text{zero}\,x} \text{ exists and } b>0, \\
& \quad \text{then the } y \text{ solution needs to} \\
& \quad \text{go negative (or at least to zero)} \\
& \quad \text{to reach the point where} \\
& \quad \text{the arcosh function has} \\
& \quad \text{argument 1 which gives } \tau = 0. \\
& \quad \text{Note, the cosh function solutions} \\
& \quad \text{for } y \text{ and } x \text{ are necessarily even} \\
& \quad \text{about } \tau = 0. \\
& \quad \text{The upshot is that } \tau = 0 \text{ is the} \\
& \quad \text{minimum value time for the } y \\
& \quad \text{and } x \text{ solutions, and therefore} \\
& \quad \text{they have positive coefficients} \\
& \quad \text{and open upward.} \\
& \text{By a corresponding argument,} \\
& \quad \text{if } \tau_{\text{zero}\,x} \text{ exists and } b<0, \\
& \quad \text{the } y \text{ and } x \text{ solutions} \\
& \quad \text{have negative coefficients} \\
& \quad \text{and open downward.}
\end{cases}
\tag{33}
$$

$$\tau_{\rm sin} = \begin{cases} -\left(\pm\frac{V}{\sqrt{|a|}}\right)\arcsin\left(\frac{2ay+b}{\sqrt{b^2-4ac}}\right) & \text{Conditions: indefinite integral,} \\ & \quad a<0,\ b^2-4ac>0, \\ & \quad |2ay+b| \le \sqrt{b^2-4ac}\,. \\ & \text{Solution } V_{5-}^{-1}(x). \\ & \text{No } V_5^{-1}(x) \text{ solution} \\ & \quad \text{nor } V_5(\tau) \text{ solution exists.} \end{cases} \tag{34}$$

$$\tau_{\rm power} = \begin{cases} \pm V\left(\frac{2}{b}\right)\sqrt{by+c} + \text{constant} & \text{General solution.} \\ \qquad \text{Conditions: } a=0,\ b\neq 0. & \\ \qquad \text{Actually this is } VU \text{ model ??? with } U=1 & \\ \qquad \text{as discussed in Appendix A.4.} & \\ \qquad \text{We include as a ``} V \text{ model'' for convenience.} & \\ V\left(\frac{2}{b}\right)\left(\sqrt{by+c}-\sqrt{c}\right) & \text{Solution } V_4^{-1}(x)\text{: } b>0, \\ & \quad \tau(y=x^{1/V}=0)=0 \text{ chosen.} \\ V\left(\frac{y}{\sqrt{c}}\right) & \text{Solution } V_4^{-1}(x)\text{: } b=0,\ c>0. \\ \pm V\left(\frac{2}{b}\right)\sqrt{by+c} & \text{Solution } V_{4-}^{-1}(x)\text{: } b<0, \\ & \quad \tau(y=c/|b|)=0, \\ & \quad \tau_{{\rm zero}\,x} = V(2/|b|)\sqrt{c} \\ & \quad \text{gives } x(\pm\tau_{{\rm zero}\,x})=0. \\ V\left(\frac{2}{b}\right)\left(\sqrt{b+c}-\sqrt{c}\right) & \text{Solution } V_4^{-1}(x)\text{: } b>0. \\ & \text{Extra Conditions: } y=x^{1/V}=1. \\ V\left[\frac{2(1-\sqrt{c}\,)}{1-c}\right] = V\left(\frac{2}{1+\sqrt{c}}\right) & \text{Extra Condition:} \\ & \quad b+c=1 \text{ applied.} \\ \quad = V\left(\frac{2}{1+\sqrt{1-b}}\right) & \\ \quad = V\left(\frac{2}{b}\right)\left(1-\sqrt{1-b}\right) & \end{cases} \tag{35}$$

4.2. The $x(\tau)$ V Model Solutions

The inverses of the cosmic time solutions $\tau(x)$ give the cosmic scale factor solutions $x(\tau)$ where only the region where $x(\tau)^{1/V} > 0$ are physical. The scale factor solutions are:

$$x_{\exp} = \begin{cases} \left(\dfrac{-b + b_0 e^{\pm\sqrt{a}V^{-1}\tau}}{2a}\right)^V & \text{Conditions: } a > 0,\ b^2 - 4ac = 0. \\ & \text{Extra Conditions: } \tau \geq 0,\ -b + b_0 \geq 0 \\ & \quad \text{for a physical solution at } \tau = 0. \\ & \text{Solution } V_2(\tau)\text{: upper case only, } b > 0 \text{ and } b_0 = b \\ & \quad \text{implying } x(\tau = 0) = 0. \\ & \text{Solution } V_{2-}(\tau)\text{: } b > 0,\ b_0 > b \text{ giving} \\ & \quad \text{increasing/decreasing exponential solutions} \\ & \quad \text{with the decreasing exponential solution} \\ & \quad \text{going to zero.} \\ & \text{Solution } V_{2-}(\tau)\text{: } b = 0,\ b_0 > 0 \text{ giving} \\ & \quad \text{increasing/decreasing exponential solutions.} \\ & \quad \text{An expanding/contracting de Sitter universe.} \\ & \text{Solution } V_{2-}(\tau)\text{: } b < 0,\ b_0 = 0 \text{ giving} \\ & \quad \text{an analogue to the (static) Einstein universe.} \\ & \text{Solution } V_{2-}(\tau)\text{: } b < 0,\ b_0 > 0 \text{ giving} \\ & \quad \text{increasing/decreasing exponential solutions.} \\ & \quad \text{An analogue to the Lemaître-Eddington universe:} \\ & \text{Solution } V_{2-}(\tau)\text{: } b < 0,\ b_0 < 0 \text{ giving} \\ & \quad \text{increasing/decreasing exponential solutions.} \\ & \quad \text{The negative coefficient } b_0 \text{ causes} \\ & \quad \text{the increasing exponential solution goes to zero.} \\ & \quad \text{An analogue to the Lemaître-Eddington universe.} \end{cases} \tag{36}$$

$$x_{\rm sinh} = \begin{cases} \left\{\dfrac{-b+\sqrt{4ac-b^2}\,\sinh[\sqrt{a}\,V^{-1}(\tau+\tau_{{\rm zero}\,x})]}{2a}\right\}^V & \begin{array}{l}\text{Conditions: } a>0,\ c>0,\\ \quad b^2-4ac<0,\ |b|<2\sqrt{ac},\\ \text{Solution } V_1(\tau)\text{: } b\geq 0.\\ \text{No solution } V_{1-}(\tau).\\ \text{The } \Lambda\text{-CDM model}\\ \quad \text{for } V=2/3 \text{ and } b=0.\\ \text{An analogue to the Lemaître}\\ \quad \text{universe for } b<0.\\ \text{An analogue to the (static) Einstein}\\ \quad \text{universe for } b<0 \text{ and } \tau_{{\rm zero}\,x}\to-\infty\end{array} \\ \tau_{{\rm zero}\,x} = \dfrac{V}{\sqrt{a}}{\rm arsinh}\left(\dfrac{b}{\sqrt{4ac-b^2}}\right) & \begin{array}{l}\text{Zero time } \tau_{{\rm zero}\,x} \text{ is chosen}\\ \quad \text{to give } x(\tau=0)=0.\end{array}\end{cases} \quad (37)$$

$$x_{\rm cosh} = \begin{cases} \left\{\dfrac{-b\pm\sqrt{b^2-4ac}\,\cosh[\sqrt{a}\,V^{-1}\tau]}{2a}\right\}^V & \begin{array}{l}\text{Conditions: } a>0,\ b^2-4ac>0.\\ \text{Solution } V_3(\tau)\text{: upper case,}\\ \quad \tau_{{\rm zero}\,x} \text{ exists, } b>0,\\ \quad \text{and } \tau\geq\tau_{{\rm zero}\,x}.\\ \text{Solution } V_{3-}(\tau)\text{: upper case,}\\ \quad \tau_{{\rm zero}\,x} \text{ exists, } b>0,\\ \quad \tau\leq-\tau_{{\rm zero}\,x}.\\ \text{Solution } V_{3-}(\tau)\text{: upper case,}\\ \quad \tau_{{\rm zero}\,x} \text{ does not exist}\\ \quad \text{and the solution is}\\ \quad \text{physical for all time.}\\ \text{Solution } V_{3-}(\tau)\text{: lower case,}\\ \quad \tau_{{\rm zero}\,x} \text{ exists, } b<0,\\ \quad \text{and the solution is physical for}\\ \quad \tau\in[-\tau_{{\rm zero}\,x},\tau_{{\rm zero}\,x}].\\ \text{No physical solution: lower case,}\\ \quad \tau_{{\rm zero}\,x} \text{ does not exist.}\end{array} \\ \tau_{{\rm zero}\,x} = \dfrac{V}{\sqrt{a}}{\rm arcosh}\left(\left|\dfrac{b}{\sqrt{b^2-4ac}}\right|\geq 1\right) & \text{Chosen to give } x(\tau=\tau_{{\rm zero}\,x})=0.\end{cases} \quad (38)$$

$$
x_{\sin} = \begin{cases}
\left\{ \dfrac{b \pm \sqrt{b^2 - 4ac}\,\sin[\sqrt{|a|}\,V^{-1}\tau]}{2|a|} \right\}^{V} & \text{Conditions: } a < 0,\ b^2 - 4ac \geq 0. \\
 & V_{5-}(x) \text{ solution: no } V_5(x) \text{ exists since} \\
 & \quad \text{the } V_{5-}(x) \text{ solution in all cases} \\
 & \quad \text{cannot grow to infinity.} \\
 & \text{The angular frequency } \omega = \sqrt{|a|}\,V^{-1} \\
 & \quad \text{and the period is } 2\pi/(\sqrt{|a|}\,V^{-1}). \\
\left\{ \dfrac{b + \sqrt{b^2 - 4ac}\,\cos[\sqrt{|a|}\,V^{-1}\tau]}{2|a|} \right\}^{V} & \text{Choosing a convenient phase.} \\
 & \text{For } b \geq \sqrt{b^2 - 4ac} > 0 \\
 & \quad \text{(implying } c \leq 0\text{), there is for the} \\
 & \quad \text{inequality a sinusoidal solution} \\
 & \quad \text{for all time that is greater than} \\
 & \quad \text{zero (and so is always a} \\
 & \quad \text{physical solution) and for the} \\
 & \quad \text{equality a physical sinusoidal} \\
 & \quad \text{solution for } \tau \in [-\tau_{\text{zero}\,x}, \tau_{\text{zero}\,x}] \\
 & \quad = (V/\sqrt{|a|})[\pi, \pi]. \\
 & \text{For } -\sqrt{b^2 - 4ac} < b < \sqrt{b^2 - 4ac} \\
 & \quad \text{(implying } c > 0\text{), there is a physical} \\
 & \quad \text{sinusoidal solution} \\
 & \quad \text{for } \tau \in [-\tau_{\text{zero}\,x}, \tau_{\text{zero}\,x}]. \\
 & \text{For } b \leq -\sqrt{b^2 - 4ac} \text{ (implying } c \leq 0\text{),} \\
 & \quad \text{there is no physical solution.} \\
 & \quad \text{If } c = 0,\ x = 0 \text{ only at} \\
 & \quad \sqrt{|a|}\,V^{-1}\tau = 0, \pm\pi, \pm 2\pi, \ldots \\
 & \text{For } b > 0 \text{ and } b^2 - 4ac = 0\text{, there is} \\
 & \quad \text{constant physical solution } b/(2|a|) \\
 & \quad \text{which is a neutral stable solution:} \\
 & \quad \text{no perturbations of } x \text{ are possible} \\
 & \quad \text{and positive perturbations of} \\
 & \quad \sqrt{b^2 - 4ac} \text{ just give proportionate} \\
 & \quad \text{amplitude oscillations.} \\
\tau_{\text{zero}\,x} = \dfrac{V}{\sqrt{|a|}} \arccos\left(\dfrac{-b}{\sqrt{b^2 - 4ac}} \right) & \text{The zero time if it exists for} \\
 & \quad \text{the cosine solution.}
\end{cases}
\tag{39}
$$

$$x_{\rm power} = \begin{cases} \left\{ \dfrac{[(b/2)V^{-1}\tau + \sqrt{c}\,]^2 - c}{b} \right\}^V & \text{Solution } V_4(\tau)\text{: } a=0,\ b>0,\ c>0. \\ & \quad \text{Note, } x \text{ grows strictly for } \tau \geq 0. \\ & \text{Actually this is } VU \text{ model ??? with } U=1 \\ & \text{as discussed in Appendix A.4.} \\ & \text{We include as a ``}V\text{ model'' for convenience.} \\ \left(\sqrt{c}\,V^{-1}\tau\right)^V & \text{Solution } V_4(\tau)\text{: } a=0,\ b=0,\ c>0. \\ & \text{It is the single (inverse) power law density} \\ & \quad \text{component solution.} \\ & \text{For } p=3 \text{ and } V=2/3, \\ & \quad \text{it is the Einstein-de Sitter universe.} \\ \left\{ \dfrac{c - [(b/2)V^{-1}\tau]^2}{|b|} \right\}^V & \text{Solution } V_{4-}(\tau)\text{: } a=0,\ b<0,\ c>0. \\ & \text{Note, } x^{1/V} \text{ is a parabola with maximum} \\ & \quad c/|b| \text{ at } \tau=0 \text{ and } x(\pm\tau_{{\rm zero}\,x})=0, \\ & \quad \text{where } \tau_{{\rm zero}\,x} = V(2/|b|)\sqrt{c}. \\ & \text{Note, substituting for } \tau \text{ with } \tilde{\tau} - \tau_{{\rm zero}\,x} \\ & \quad \text{gives the formula for } V_4(\tilde{\tau}), \\ & \quad \text{but now for } b<0. \end{cases} \tag{40}$$

The V_i solution index i increases with increasing ratio $b^2/(4ac)$ (assuming $ac \geq 0$): the ratios run $b^2/(4ac) < 1$, $b^2/(4ac) = 1$, $b^2/(4ac) > 1$, $b^2/(4ac) = \infty$ (since $a = 0$). Given that Λ-CDM model is such a good fit to the observable universe, we expect any viable $b^2/(4ac)$ value to be small, the V_1 solution (Equation (37)) is probably the only V_i solution of any viability. In fact, the V_1 solution with $b > 0$ is the solution we find in a crude fit to observations (see § 6). However, we are also interested in the overall behavior of the V model and the ΛΓ model as their parameters are varied, and thus on its flexibility to accommodate observations and not just on the ability of any particular V_i solution to do so. In § 5, we study the overall behavior of the ΛΓ model as a function of the density parameter constants (i.e., a, b, and c) in § 5.

We should note that the V model solutions (as partially noted above in Equations (37), (36), (38), (40), and (39)) include analogues to the non-exact standard solutions of the Friedmann equation reviewed by Bondi (1961, esp. p. 80–86). Of course, simple 1-density component solutions with parameters $a = b = 0$ and $c \neq 0$ and $p \geq 0$ follow as special cases of the V model solutions (see Equation (38)), including the Einstein-de Sitter universe with $p = 3$ (e.g., Bondi 1961, p. 82,166). The analogue solutions permit an analytic understanding of the corresponding non-exact solutions.

We explicate 4 noteworthy V model solutions in the following 4 subsections.

4.3. First Noteworthy V Model Solution

The V_1 solution (Equation (37)) for $V = 2/3$ (with $q = 3/2$ and $p = 3$) with $b = 0$ is just the matter-Λ universe solution used for the Λ-CDM model. For $V = 2/3$ (with $q = 3/2$ and $p = 3$) and $b > 0$, the solution is an analogue to a negative curvature version of the Λ-CDM model (which conventionally has has density parameter positive). A negative curvature Λ-CDM model has been explored in light of the DESI DR2 data (Lodha et al. 2025; Abdul Karim et al. 2025) by Chen & Zaldarriaga (2025), but only for small values of curvature density parameter constant Ω_k of order 0.002. But recall for $V = 2/3$ (with $q = 3/2$ and $p = 3$), the inverse power for the b parameter is $q = 3/2$, not 2 as for an actual curvature model. If $V = 1/2$ and $q = 2$ and $p = 4$, Equation (37) can be used to describe a radiation-curvature-Λ universe as mentioned in § 2.

4.4. Second Noteworthy V Model Solution

The V_1 solution (Equation (37)) for $b < 0$ and $V = 2/3$ (with $q = 3/2$ and $p = 3$) is an analogue solution to the Lemaître universe solution (a positive-curvature-matter-Λ universe solution which is not an exact solution) where the analogue to the density parameter constant $b < 0$ is the positive curvature (but negative-valued) density parameter at some fiducial time (e.g., Bondi 1961, p. 82,84–85,120–122,165,168–170,175–176). The Lemaître universe has a (quasi-static) Einstein universe phase. The analogue solution shows how an Einstein universe phase arises as we now show. From the V_1 solution (Equation (37)) for density parameter constant $b < 0$ (implying zero time $\tau_{\text{zero}\,x} < 0$), $\Delta\tau = \tau + \tau_{\text{zero}\,x}$, and $\Delta x = x - x(\tau = -\tau_{\text{zero}\,x}) = x - x(\Delta\tau = 0)$, we find

$$\frac{\Delta x}{x(\Delta\tau = 0)} = \begin{cases} \left[1 + \dfrac{\sqrt{4ac - b^2}}{-b}\sinh(\sqrt{a}\,V^{-1}\Delta\tau)\right]^V - 1 & \text{in general.} \\ \dfrac{V\sqrt{4ac - b^2}}{-b}\sinh(\sqrt{a}\,V^{-1}\Delta\tau) + \ldots & \text{expanding in small} \\ & [\sqrt{4ac - b^2}/(-b)] \\ & \times\sinh(\sqrt{a}\,V^{-1}\Delta\tau). \end{cases} \tag{41}$$

The expansion truncated to 1st order is valid for

$$\frac{\sqrt{4ac - b^2}}{-b}\sinh(\sqrt{a}\,V^{-1}\Delta\tau) << 1 \tag{42}$$

implying

$$\Delta\tau << -\frac{V}{\sqrt{a}}\text{arsinh}\left(\frac{b}{\sqrt{4ac-b^2}}\right) = -\tau_{\text{zero}\,x} = |\tau_{\text{zero}\,x}| \ . \tag{43}$$

Thus, we can make $\Delta x/x(\Delta\tau = 0)$ as small as we like for any $\Delta\tau$ by making $|\tau_{\text{zero}\,x}|$ sufficiently large. Thus, we can make an obvious Einstein universe phase by making $|\tau_{\text{zero}\,x}|$ sufficiently large. In making $|\tau_{\text{zero}\,x}|$ larger, we push the Big Bang time zero $\tau = 0$ further away from the $\Delta\tau = 0$ time. In fact, in the limit that $\tau_{\text{zero}\,x} \to -\infty$, the solution becomes an analogue Einstein universe where $x = -b/(2a)$ for all time. Note, if $\Delta\tau/(V/\sqrt{a}) << 1$, then the 1st order expansion simplifies to

$$\left.\frac{\Delta x}{x(\Delta\tau = 0)}\right|_{\text{1st}} = \frac{\sqrt{a}\sqrt{4ac-b^2}}{(-b)}\Delta\tau = \frac{\Delta\tau}{(-b)/\left(\sqrt{a}\sqrt{4ac-b^2}\right)} \ . \tag{44}$$

From the foregoing expressions, we see how parameters a, b, and c interact to create an Einstein universe phase. If one replaces density parameter constant b in the V_1 solution (Equation (37)) by positive curvature (but negative-valued) density parameter, one qualitatively approximates the Lemaître universe solution and sees how its parameters interact to create its Einstein universe phase.

4.5. Third Noteworthy V Model Solution

The parameter values of the V_{2-} solution (Equation (36)) with $V = 2/3$ (with $q = 3/2$ and $p = 3$) can be chosen to give an analogue Lemaître-Eddington universe (e.g., Bondi 1961, p. 84–85,117–121,159), an analogue to the (static) Einstein universe (e.g., Bondi 1961, p. 84,98–99,117–121,158–159,171), and the actual de Sitter universe with both exponentially expanding and contracting cases (e.g., Bondi 1961, p. 98–99,105,146–147,154,159,166). From the V_{2-} solution, we can see explicitly why the analogue Einstein universe is unstable to global perturbations b_0. Such global perturbations would put the universe model on either converging or diverging branches from the analogue Einstein universe. Thus, general global perturbations will always lead to divergence and the analogue Einstein universe is unstable just as is the actual Einstein universe. Of course, uniform global perturbations are not realistic. Dealing with more realistic local perturbations would take more hypotheses to explore.

4.6. Fourth Noteworthy V Model Solution

Fourth, the V_{5-} solution (Equation (39)) is remarkable since it permits an oscillating solution with $x > 0$ always (i.e., true oscillating Friedmann-equation universe model without extra hypotheses). That the Friedmann equation has oscillating solutions has probably been long known, but Bondi (1961) in his review of early cosmological models does not mention them. What he refers to as oscillating models (his Class V universe models) have Friedmann equation solutions that go into the unphysical negative x value range (Bondi 1961, p. 81–86,122). He does hypothesize that such universe models are cyclic: i.e., the positive range of the solution repeats itself after each solution zero (Bondi 1961, p. 82,86). However, the oscillating solution with $x > 0$ is unlikely to be ever physically real since it requires the parameters a and c both to be negative which necessarily requires b to be positive. Perhaps the most plausible case is where the b parameter is matter and thus has power $q = 3$ and the c parameter is some kind of negative energy stuff with necessarily $p = 6$. The negative energy stuff would formally have a negative pressure and equation of state parameter $w_p = 1$ (see Equation (4)).

Another point about the V_{5-} solution (which requires $a < 0$) is that it has an apparent forbidden zone for $b^2 - 4ac < 0$ and one may be concerned with what this means. To dismiss the concern, one should take a broader perspective. The Friedman equation has a forbidden zone when all the density parameters are negative: i.e., a region where there is no solution. The simplest way of dismissing the forbidden zone concern generally is to note that the curvature density parameter constant is actually an integral of motion and not a real mass-energy quantity. If somehow the density parameters that are real mass-energy quantities are all negative and a solution is demanded, then curvature density parameter constant must be positive and large enough to give a solution. Of course, as noted in § 4, a positive curvature density parameter constant gives a negative curvature. Returning to the V_{5-} solution, $b^2 - 4ac < 0$ means that in the integral for the scaled time V_{5-}^{-1} the square root is of a negative number always. So there is no real solution. So if the a, b, and c density parameter constants include the curvature density parameter constant, it must be increased to make a real solution. And if the a, b, and c density parameter constants do not include the curvature density parameter constant, it must be introduced and made large enough to make a real solution. In the latter case, the solution will not be a V model solution since it has 4 density parameters.

5. The Universe Age τ_0

Although the $\Lambda\Gamma$ model solutions (i.e., the V_i solutions with $V = 2/3$: Equations (37), (36) (38), and (40)) are elegant exact solutions, it is not obvious how general variations of parameters a, b, and c will affect their overall behavior. For example, in the V_1 solution (Equation (37)), the b parameter (i.e., the Γ density parameter constant) occurs four times (twice implicitly in $\tau_{\mathrm{zero}\,x}$), and so the effect of varying b on the V_1 solution is clearly not obvious.

What is needed is a single metric of overall solution behavior for the V_i solutions. The universe age (defined in § 1: i.e., the time from a cosmic time zero to cosmic present τ_0) seems a good choice: the faster overall growth of the solution, the smaller τ_0. Now with the $V = 2/3$ explicitly for numerical evaluation, the $V_{123}^{-1}(x = 1)$ solution (Equation (30) which gives the universe age appropriate for all of the $V_1(\tau)$, $V_2(\tau)$, and $V_3(\tau)$ solutions: i.e., Equations (37), (36), and (38)) is

$$\tau_0 = \frac{2}{3}\frac{1}{\sqrt{a}}\ln\left[\frac{1 - c + a + 2\sqrt{a}}{1 - c - a + 2\sqrt{ac}}\right] \tag{45}$$

and the $V_4^{-1}(x = 1)$ solution (Equation (35) which has $a = 0$ and $b \geq 0$) is

$$\tau_0 = \begin{cases} \dfrac{2}{3}\left(\dfrac{2}{1+\sqrt{1-b}}\right) = \dfrac{2}{3}\left(\dfrac{2}{b}\right)\left(1 - \sqrt{1-b}\right) & \text{in terms of } b. \\ \dfrac{2}{3}\left(\dfrac{2}{1+\sqrt{c}}\right) & \text{in terms of } c. \end{cases} \tag{46}$$

Unfortunately, Equation (45) is still too complex to just visualize the behavior of τ_0 as function of a and c. So we will consider a range of special case behaviors. Case 1 has $b = 0$, and so $c = 1 - a$. Case 1 is, in fact, the Λ-CDM model universe age case. The formula is

$$\tau_0 = \begin{cases} \frac{2}{3}\frac{1}{\sqrt{a}}\ln\left(\frac{1+\sqrt{a}}{\sqrt{1-a}}\right) & \text{in general: a well known formula (e.g., Liddle 2015, p. 63).} \\ \frac{2}{3} = 0.6666\ldots & \text{for } a = 0\text{: the Einstein-de Sitter universe age.} \\ \frac{2}{3}\left(\sum_{k=0}^{\infty}\frac{a^k}{2k+1}\right) = \frac{2}{3}\left(1 + \frac{a}{3} + \frac{a^2}{5} + \frac{a^3}{7} + \ldots\right) & \text{for } a \in (0,1). \\ 0.964099381639\ldots & \text{for } a = 0.7,\ c = 0.3\text{: the fiducial Λ-CDM density parameter constant weights giving the fiducial Λ-CDM model universe age.} \\ \infty & \text{for } a = 1\text{: the infinite-age de Sitter universe age.} \end{cases} \tag{47}$$

In fact, the Λ-CDM model universe age strictly increases with a (e.g., Liddle 2015, p. 63), and so increasing matter density parameter constant c increases the overall growth rate of the solution (i.e., decreases the Λ-CDM model universe age).

Case 2 has $c = 0$, and so $b = 1 - a$. In Case 2, the Γ dark energy is the analogue to matter in the Λ-CDM model. The formula is:

$$\tau_0 = \begin{cases} \frac{2}{3}\frac{1}{\sqrt{a}}\ln\left(\frac{1+a+2\sqrt{a}}{1-a}\right) & \text{in general. In fact, the Case 2 } \tau_0 \text{ formula} \\ \quad = \frac{4}{3}\frac{1}{\sqrt{a}}\ln\left(\frac{1+\sqrt{a}}{\sqrt{1-a}}\right) & \quad \text{is exactly 2 times the Case 1 } \tau_0 \text{ formula.} \\ \frac{4}{3} = 1.3333\ldots & \text{for } a = 0\text{: the pure Γ dark energy universe age.} \\ 1.9281987632789\ldots & \text{for } a = 0.7,\ b = 0.3\text{: the analogue to the fiducial Λ-CDM model universe age.} \\ \infty & \text{for } a = 1\text{: the infinite-age de Sitter universe age.} \end{cases} \tag{48}$$

Comparing Equations (47) and (48), we see that Γ dark energy gives a slower rate of growth than matter (i.e., gives larger universe age for comparable cases). In fact, all the Case 2 universe ages are exactly 2 times those of the corresponding Case 1 universe ages since, as noted above, the Case 2 is τ_0 formula is exactly 2 times the Case 1 τ_0 formula.

Case 3 has $a = 0$, and so $b = 1 - c$. In Case 3, the Γ dark energy is the analogue to Λ dark energy in the Λ-CDM model. Because $a = 0$, the $V_{123}^{-1}(x = 1)$ solution (Equation (45)) is inappropriate, and so instead, making use Equation (46), we find

$$\tau_0 = \begin{cases} \frac{2}{3}\left(\frac{2}{1+\sqrt{1-b}}\right) = \frac{2}{3}\left(\frac{2}{b}\right)\left(1-\sqrt{1-b}\right) & \text{general in terms of } b. \\ \frac{2}{3}\left(\frac{2}{1+\sqrt{c}}\right) & \text{general in terms of } c. \\ \frac{2}{3} = 0.6666\ldots & \text{for } b=0,\ c=1\text{: the Einstein-de Sitter universe age.} \\ \frac{2}{3}\left[\sum_{k=0}^{\infty}\frac{(2k-1)!!}{2^k(k+1)!}b^k\right] = \frac{2}{3}\left(1+\frac{b}{4}+\frac{b^2}{8}+\ldots\right) & \text{for } b \in [0,1]. \\ 0.861480842847\ldots & \text{for } b=0.7,\ c=0.3\text{: the analogue to the fiducial } \Lambda\text{-CDM model.} \\ \frac{4}{3}\sum_{k=0}^{\infty}(-1)^k c^{k/2} \\ \quad = \left(1-c^{1/2}+c-c^{3/2}+c^2-\ldots\right) & \text{for } c<1. \\ \frac{4}{3} = 1.3333\ldots & \text{for } b=1,\ c=0\text{: the pure } \Gamma \text{ universe age.} \end{cases} \tag{49}$$

Comparing Equations (47) and (49), we see that Γ dark energy gives faster growth than Λ dark energy (i.e., gives smaller universe age for comparable cases).

Case 4 has $a = c$, and so $b = 1 - 2a$. Case 4 is a constrained, and thus simplified, version of the $\Lambda\Gamma$ model useful for a general, but simplified, understanding of its behavior.

The formula is

$$
\tau_0 = \begin{cases}
\frac{2}{3}\frac{1}{\sqrt{a}}\ln\left(1+2\sqrt{a}\right) & \text{in general.} \\
\frac{2}{3}\sqrt{2}\ln\left(1+\sqrt{2}\right) = 0.83096698685\ldots & \text{for } a=c=1/2,\ b=0\text{: a very non-fiducial } \Lambda\text{-CDM universe age.} \\
\frac{2}{3}\sqrt{3}\ln\left(1+\frac{2}{\sqrt{3}}\right) = 0.886407892004\ldots & \text{for } a=b=c=1/3\text{: an equal-parameter-weight } \Gamma\Lambda \text{ model universe age.} \\
0.964070206453\ldots & \text{for } a=c=0.182,\ b=0.636\text{: this universe age is equal to 5 digits to the fiducial } \Lambda\text{-CDM model universe age.} \\
\frac{4}{3}\left[\sum_{k=0}^{\infty}(-1)^k\frac{(2\sqrt{a}\,)^k}{k+1}\right] & \text{for } 2\sqrt{a}=2\sqrt{c}\leq 1. \\
\quad = \frac{4}{3}\left[1-\frac{(2\sqrt{a})}{2}+\frac{(2\sqrt{a}\,)^2}{3}+\ldots\right] & \\
\frac{4}{3} = 1.3333\ldots & \text{for } a=c=0 \text{ and } b=1\text{: the pure } \Gamma \text{ dark energy } \Gamma\Lambda \text{ model universe age.}
\end{cases}
\tag{50}
$$

Cases 1–3 show that increasing the c parameter (i.e., the matter density parameter constant) increases the rate of solution growth relative growth given by the pure a parameter (i.e., the Λ density parameter constant) and increasing the b parameter (i.e., the Γ density parameter constant) does the same, but to a lesser degree. Thus, increasing the b parameter can be compensated for by decreasing the c parameter. But this is just an aspect of the fact that the $\Lambda\Gamma$ model cosmic scale factor solution is more flexible than the Λ-CDM model cosmic scale factor solution since it has two free density parameter constants (any two of a, b, and c) instead of just one like the Λ-CDM model cosmic scale factor solution (either of a or c). However, the $\Lambda\Gamma$ model is not at all extremely flexible, and so it can be falsified.

Case 4 shows that a constrained, and so simplified, $\Lambda\Gamma$ model can match the fiducial Λ-CDM model universe age which is very probably correct to within a few percent given that Λ-CDM model gives an extremely good fit to cosmic evolution in many respects. However, this matching is done with an implausibly large b parameter weight ($b = 0.636$) and implausibly small a and c parameter weights ($a = c = 0.182$) given the aforesaid overall goodness of the Λ-CDM model. However, it is likely that varying both a and c independently will give a range of relatively small b values that will make the ΛΓ model give a good fit to the fiducial Λ-CDM model universe age. But finding that range in itself is not an interesting project. In § 6, we undertake a more interesting project of trying to match the fiducial Λ-CDM model universe age and give a very limited fit to the DESI DR2 results (Lodha et al. 2025; Abdul Karim et al. 2025).

6. The $Om(z)$ Diagnostic

The $Om(z)$ diagnostic (Sahni et al. 2008) is essentially a way of rewriting the derivative of the cosmic scale factor $x(\tau) = 1/(1+z)$ (where z is cosmic redshift) in a way to emphasize features not otherwise obvious in direct presentations of $x(\tau)$ and its derivative. To slightly generalize from (Sahni et al. 2008) (but with a limitation to density components that evolve as inverse powers of the cosmic scale factor), we write the scaled Friedmann equation thusly

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_{n=0}^{N} a_n (1+z)^{p_n} \ , \tag{51}$$

where the a_n are cosmic present density parameters, $\sum_{n=0}^{N} a_n = 1$ (which give cosmic present $h = 1$), $p_0 = 0$, and $p_{n\geq 1} > 0$ are general powers that increase in size with index n implying p_N is the largest power. Using Equation (51), the $Om(z)$ diagnostic becomes

$$Om(z) = \begin{cases} \dfrac{h^2(z) - 1}{(1+z)^{p_N} - 1} = \dfrac{\left[\sum_{n=0}^{N} a_n (1+z)^{p_n}\right] - 1}{(1+z)^{p_N} - 1} & \text{general formula.} \\[2ex] \dfrac{\sum_{n=1}^{N} a_n p_n + (1/2)\left[\sum_{n=1}^{N} a_n p_n (p_n - 1)\right] z}{p_N + (1/2)[p_N(p_N - 1)]z} & \text{1st order in small } z \text{ formula.} \\[2ex] \dfrac{\sum_{n=1}^{N} a_n p_n}{p_N} & \text{for } z = 0. \\[2ex] a_N & \text{for } z \to \infty. \end{cases} \tag{52}$$

We now specialize the general formula in Equation (52) for the case of the V model, where $a = a_0$, $b = a_1$, and $c = a_2$, and $p_0 = 0$, $p_1 = q = p/2$, and $p_2 = p$:

$$Om(z) = \begin{cases} \dfrac{a + b(1+z)^q + c(1+z)^p - 1}{(1+z)^p - 1} & \text{the } V \text{ model general formula.} \\ \dfrac{bq + cp + (1/2)[bq(q-1) + cp(p-1)]z}{p + (1/2)p(p-1)z} & \text{the } V \text{ model 1st order formula.} \\ \dfrac{bq + cp}{p} = \left(\dfrac{1}{2}\right) b + c & \text{for } z = 0. \\ c & \text{for } z \to \infty = 0. \end{cases} \tag{53}$$

We now specialize the general formula in Equation (53) for the case of the $\Lambda\Gamma$ model (i.e., the V model with $p = 3$, $q = 3/2$, and $V = 2/3$):

$$Om(z) = \begin{cases} \dfrac{a + b(1+z)^{3/2} + c(1+z)^3 - 1}{(1+z)^3 - 1} & \text{the } \Lambda\Gamma \text{ model formula} \\ \quad = c + \dfrac{b[(1+z)^{3/2} - 1]}{3z[1 + z + (1/3)z^2]} & \\ c + \dfrac{(b/2)}{[1 + z + (1/3)z^2]} \left[1 + \dfrac{z}{4} - \dfrac{z^2}{24} \right. & \text{the } \Lambda\Gamma \text{ model} \\ \qquad \left. + \displaystyle\sum_{k=3}^{\infty} \dfrac{(-1)^{k-1}(2k-3)!!}{2^k (k+1)!} z^k \right] & \text{series expansion.} \\ & \text{Absolutely convergent} \\ & \text{for } z \in [-1, 1]. \\ c + \dfrac{(b/2)}{[1 + z + (1/3)z^2]} \left[1 + \dfrac{z}{4} \right. & \text{the } \Lambda\Gamma \text{ model series} \\ \qquad \left. - \dfrac{z^2}{24} + \dfrac{z^3}{64} + \ldots \right] & \text{expansion truncated} \\ & \text{to 3rd order.} \\ c + \dfrac{b}{2} & \text{for } z = 0. \\ c & \text{for } z \to \infty. \end{cases} \tag{54}$$

In determining Equation (54), we made use of the somewhat difficult to find general expansion formula for $(1+z)^{n/2}$ where n is an odd integer (i.e., $n = \pm 1, \pm 3, \pm 5 \ldots$), and so we give it here for general reference:

$$(1+z)^{n/2} = 1 + \sum_{k=1}^{\infty} (-1)^{\max[0, k-(n+1)/2]} \frac{(2k-n-2)!!}{2^k k!(-n-2)!!} \frac{n!!}{(n-2\ell)!!} z^k , \tag{55}$$

where (integer ≤ 0)!! $= 1$ in all cases. For $n > 0$, the series is absolutely convergent for $z \in [-1, 1]$ and for $n < 0$, for $z \in (-1, 1)$. For $z = 1$, there is conditional convergence for $n = -1$, but not for $n \leq -3$.

Lodha et al. (2025) applied $Om(z)$ diagnostic to several cosmological models fitted to various data combinations and to curves generated by Gaussian process regressions (i.e., a non-parametric way of generating curves) applied to various data combinations. Insofar as their models or Gaussian process regressions and data combinations are accurate, their $Om(z)$ curves are a true measure of the evolution of cosmic scale factor emphasizing as aforesaid features not otherwise obvious.

As a preliminary test of the $\Lambda\Gamma$ model, we have done a crude fit to the $Om(z)$ diagnostic curve of Lodha et al. (2025) (shown in their Figure 9) generated using a Gaussian process regression based on the DESI+CMB+Union3 data combination. The fit is crude since we just measured the Gaussian process regression $Om(z)$ curve off of the Figure 9 of Lodha et al. (2025). The criteria for the fit were that the $z = 0$ and $z = 3$ endpoints of Lodha et al. (2025) ($Om(z = 0) \approx 0.45$ and $Om(z = 3) \approx 0.315$) should fit within 2 standard deviations (respectively, ~ 0.07 and ~ 0.01), the matter density parameter constant $c \geq 0.29$ (since Abdul Karim et al. (2025, Fig. 16) strongly disfavors a lower value) and the universe age $t \geq 13.5\,\text{Gyr}$ (since the oldest globular clusters suggest $13.5\,\text{Gyr}$ is a lower limit on the age of the universe in round numbers (e.g., Valcin et al. 2026)).

For the Hubble constant value needed to convert scaled cosmic time τ into unscaled cosmic time t, we used $H_0 = 68.01\,(\text{km/s})/\text{Mpc}$ from Abdul Karim et al. (2025, Table V) since it was based on a model using DESI+CMB+Union3 data combination like the Gaussian process regression $Om(z)$ curve and the model otherwise seemed a reasonable proxy for the Gaussian process regression. We varied the $\Lambda\Gamma$ model density parameter constants a, b, c over all parameter space in steps of 0.01 consistent with the $a + b + c = 1$ constraint, $a > 0$ (since $a = 0$ just gives an approximate power-law dependence of x on τ (see the V_4 in Equation (40) and that is remote from the Λ-CDM behavior which is accepted as a good approximation in any case and $a < 0$ has solutions with maxima (see the Equation (39)), and $b \geq 0$ (since we need positive Γ dark energy to achieve the decreasing total dark energy implied by data of Lodha et al. 2025 and Abdul Karim et al. 2025).

Only two $\Lambda\Gamma$ model solutions matched all the constraints and they were nearly the same. We deem the slightly better one to have the Λ density parameter constant $a = 0.53$, the Γ density parameter constant $b = 0.18$, and the matter density parameter constant $c = 0.29$. The quantity $b^2 - 4ac = -0.5824$ which is less than zero, and so the V_1 solution (Equation (37)) is the fitted solution. We expected the V_1 solution to be best since it is closest the Λ-CDM solution. The universe age $t = 13.54\,\text{Gyr}$ which is significantly lower the

Λ-CDM universe age 13.797(23) Gyr favored by Planck Collaboration (Aghanim et al. 2021, p. 15). The relative deviation from the ΛΓ model $Om(z)$ from that of Lodha et al. (2025) (whose values were crudely measured) is $-15.6\,\%$ at $z = 0$ rising to $3.8\,\%$ at $z = 1.2$ and then declining to $-1.6\,\%$ at $z = 3$.

Given the crudeness of the fitting procedure, the fit solution achieved by the ΛΓ model is adequate for the observations so far. However, given that the ΛΓ model is just a heuristic model, the fit solution can only be considered as interesting.

7. The Deceleration Parameter

The deceleration parameter (symbolized here by $q_{\rm dec}$ to distinguish it for the power q) can also be used as a diagnostic. It is essentially a way of rewriting the 2nd derivative of the cosmic scale factor $x(\tau) = 1/(1+z)$ (where z is cosmological redshift) in a way to emphasize features not otherwise obvious in direct presentations of $x(\tau)$ and its 1st and 2nd derivatives. We do not make use of the deceleration parameter as diagnostic in this paper, but for future reference, we give general and special case formulae below.

Making use of the standard general formula for the deceleration parameter (e.g., Liddle 2015, p. 53), the Friedmann acceleration equation (e.g., Liddle 2015, p. 27), and Equation (4) and Equation (51) (both of which imply a limitation to density components that evolve as inverse powers of the cosmic scale factor), we obtain

$$q_{\rm dec} = \begin{cases} -\dfrac{\ddot{x}x}{\dot{x}^2} = -\dfrac{\ddot{x}}{x}\dfrac{1}{h(x)^2} = \dfrac{1}{2}\left[\dfrac{\sum_{n=0}^{N} a_n(p_n-2)(1+z)^{p_n}}{\sum_{n=0}^{N} a_n(1+z)^{p_n}}\right] & \text{general formula.} \\ -1 + \dfrac{1}{2}\left[\dfrac{\sum_{n=1}^{N} a_n p_n (1+z)^{p_n}}{\sum_{n=0}^{N} a_n(1+z)^{p_n}}\right] & \text{general formula recalling } \sum_{n=0}^{N} a_n = 1 \text{ and } p_0 = 0. \\ = -1 + \left(\dfrac{1}{2}\right)\displaystyle\sum_{n=1}^{N} a_n p_n & \text{for } z = 0. \\ -1 + \dfrac{p_N}{2} & \text{for } z \to \infty. \end{cases} \tag{56}$$

$$
q_{\rm dec} = \begin{cases}
-1 + \dfrac{1}{2}\left[\dfrac{bq(1+z)^q + cp(1+z)^p}{a + b(1+z)^q + c(1+z)^p}\right] & \text{for the } V \text{ model} \\
 & \quad\text{recalling } q = p/2. \\
-1 + \dfrac{1}{2}(bq + cp) = -1 + \dfrac{p}{2}\left(\dfrac{b}{2} + c\right) & \text{for } z = 0 \\
\quad = -1 + \dfrac{p}{4}(1 - a + c) & \text{and using } b = 1 - (a+c)/2. \\
-1 + \dfrac{p}{2} & \text{for } z \to \infty.
\end{cases} \tag{57}
$$

$$
q_{\rm dec} = \begin{cases}
-1 + \dfrac{1}{2}\left[\dfrac{(3/2)b(1+z)^{3/2} + 3c(1+z)^3}{a + b(1+z)^{3/2} + c(1+z)^3}\right] & \text{for the } \Lambda\Gamma \text{ model.} \\
1 + \dfrac{3}{2}\left(\dfrac{b}{2} + c\right) = -1 + \dfrac{3}{4}(1 - a + c) & \quad z = 0 \text{ and using} \\
 & \quad b = 1 - (a+c). \\
\dfrac{1}{2} & \text{for } z \to \infty \\
 & \quad\text{and also for the} \\
 & \quad \Lambda\text{-CDM model} \\
 & \quad\text{with } z \to \infty.
\end{cases} \tag{58}
$$

$$
q_{\rm dec} = \begin{cases}
-1 + \dfrac{3}{2}\left[\dfrac{c(1+z)^3}{a + c(1+z)^3}\right] & \text{for the } \Lambda\text{-CDM model.} \\
-1 + \dfrac{3}{2}c = \dfrac{1}{2} - \dfrac{3}{2}a & \text{for } z = 0 \text{ and} \\
 & \quad\text{using } c = 1 - a. \\
-0.55\left[\dfrac{0.5 - 1.05 \times (a/0.7)}{-0.55}\right] & \text{for } z = 0 \text{ and} \\
 & \quad\text{fiducial } a = 0.7.
\end{cases} \tag{59}
$$

8. Discussion

It would be an interesting project to see if the $\Lambda\Gamma$ model V_1 solution (Equation (37)) can give a good fit to all data cited by Lodha et al. (2025) and Abdul Karim et al. (2025) with a least-squares fit done by varying the a parameter (i.e., the Λ density parameter constant: conventionally the constant density parameter Ω_Λ) and the c (i.e., the matter density parameter constant and conventionally the cosmic present matter density parameter constant $\Omega_{\rm M,0}$). The fact that the Λ-CDM cosmic scale factor solution (i.e, the matter-Λ universe solution) fits all the aforesaid data to the eye very well (Abdul Karim et al. 2025, Fig. 1) suggests that the b parameter (i.e., Γ density parameter constant: the cosmic present Γ dark energy density with the constraint $b = 1 - (a + c)$) will be relatively small in a good fit. However, the crude fit to the $Om(z)$ curve Lodha et al. (2025) with Λ density parameter constant $a = 0.53$ (conventionally Ω_Λ) Γ density parameter constant $b = 0.18$, and matter density parameter constant $c = 0.29$ (conventionally $\Omega_{\rm matter}$) (§ 6) suggests the possibility that Γ dark energy density parameter constant b might larger than very small.

Note, the $\Lambda\Gamma$ model is somewhat related to the $\Omega_1\Omega_2$-ΛCDM model introduced by Kumar (2025) which in addition to the Λ and matter density components has density components that $\Omega_1 = \Omega_{1,0}x^{-1}$ and $\Omega_2 = \Omega_{2,0}x^{-2}$, where $\Omega_{1,0}$ and $\Omega_{2,0}$ are cosmic present values. For some values of $\Omega_{1,0}$ and $\Omega_{2,0}$, the $\Lambda\Gamma$ model (which has density component $\Omega_{3/2} = \Omega_{3/2,0}x^{-3/2}$ in the notation of Kumar (2025)) will approximate the $\Omega_1\Omega_2$-ΛCDM model. However, the $\Omega_1\Omega_2$-ΛCDM model has more flexibility and, in fact, Kumar (2025) find a best fit to cosmological observations for $\Omega_{1,0} = -0.112^{+0.063}_{-0.048}$, $\Omega_{2,0} = 0.010^{+0.002}_{-0.010}$ $\Omega_{\rm matter} = 0.2917 \pm 0.0060$, and $\Omega_\Lambda = 1 - \Omega_{1,0} - \Omega_{2,0} - \Omega_{\rm matter} = 0.810$. Note Kumar (2025) has a negative energy density component and they argue that that is theoretically plausible. In our, analysis we have not considered negative energy density components. However, the V_1 model by Equation (37) (which is an analogue to the Lemaître universe for $b < 0$) with $V = 2/3$, Λ density parameter constant $a = 0.81$, Γ density parameter constant $b = -0.10$, and matter density parameter constant $c = 0.29$ would probably approximate the best fit of Kumar (2025) to some degree. Note, in Equation (37), the square-root factor $\sqrt{4ac - b^2} = 0.964\ldots$ with the a, b, and c values just given.

At this point, we need to note that there have been many papers assessing the evidence for dynamical dark energy and finding it not convincing (e.g., Hergt et al. 2026; Ong et al. 2025), and so the Λ-CDM model may survive as the standard model of cosmology. We will not discuss this issue further.

To recapitulate from the abstract, given that $\Lambda\Gamma$ model is just a heuristic model with no physical reason for its equation of state parameter $w_\Gamma = -1/2$, any expectation that even a good fit of it to observations will be meaningful is very modest. Also recapitulating from the abstract, the main value of this paper is the formalism for exact solutions of the Friedmann equation presented in § 3 and Appendix A, and the review of ancillary results presented in other appendices.

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A. Exact Solutions of the VU Model

In this appendix, we investigate the exact solutions of the VU model specified in § 3 whose Friedmann equation form is (i.e., Equation (22)). Appendix A.1 shows the VU model model solutions with $U < 0$ are, in fact, the exact solutions to those for the VU model model with $U \geq 0$, and so the cases of $U < 0$ need not be considered further. In Appendix A.2, we show that there are exact solutions only for integer $U \geq 0$ and that these are only $\tau(x)$ solutions, except that exact inverse solutions $x(\tau)$ exist for $U = 0, 1, 3$ and no other cases useful cases. In Appendix A.3, we give a table of the model VU exact solutions for density parameters having power-law behavior with powers limited to $0, 1, 2, 3, 4$ and a discussion of the most interesting of these exact solutions. We illustrate an example VU model solution for $U = 1$ in Appendix A.4 and for an important example VU model solution for $U = 3$, the radiation-matter universe, in Appendix A.5. Note, radiation-matter universe is a two density component VU model with power parameters $r = 3$ and $p = 4$. The radiation-matter universe is extremely important since it applies to the radiation dominated phase and early matter dominated phase of the observable universe. The exact solution $x(\tau)$ formula given here is apparently novel and slightly simpler, but mathematically equivalent, to the exact solution $x(\tau)$ formula given by Galanti & Roncadelli (2021, Eq. (20)).

A.1. The VU Model with $U < 0$ Is Redundant to the VU Model with $U > 0$

As mentioned in § 3.2, we could have interchanged the 2 and 0 on the right-hand sides of the first and third expressions in Equation (16). Does this lead to more exact solutions than Equation (22) which has $U \geq 0$? To investigate, we do the interchange and obtain

$$r = \frac{2 + \tilde{2}U}{\tilde{V}} \ , \qquad q = \frac{1 + \tilde{2}U}{\tilde{V}} \ , \qquad \text{and} \qquad p = \frac{\tilde{2}U}{\tilde{V}} \ , \tag{A1}$$

where we have used the tildes to distinguish the interchanged case where necessary. Note, since $p > r \geq 0$,

$$r - p = \frac{2}{\tilde{V}} \qquad \text{implies} \qquad \tilde{V} < 0 \tag{A2}$$

and then

$$r = \frac{2 + \tilde{2}U}{\tilde{V}} \geq 0 \qquad \text{implies} \qquad \tilde{U} \leq -1 \ . \tag{A3}$$

Mathematical intuition causes us to suspect that the interchanged case leads to the same integrals as the original case with $\tilde{U} = -1 - U$, $\tilde{V} = -V$, $\tilde{y} = y^{-1}$, and $d\tilde{y} = -y^{-2}\,dy$. The proof that this is so is

$$\begin{aligned} d\tau(\tilde{V},\tilde{U}) &= \pm\frac{\tilde{V}\tilde{y}^{\tilde{U}}\,d\tilde{y}}{\sqrt{a+b\tilde{y}+c\tilde{y}^2}} = \pm\frac{Vy^{(1+U)}y^{-2}\,dy}{\sqrt{a+by^{-1}+cy^{-2}}} \\ &= \pm\frac{Vy^{U}\,dy}{y\sqrt{a+by^{-1}+cy^{-2}}} = \pm\frac{Vy^{U}\,dy}{\sqrt{ay^2+by+c}} \\ &= d\tau(V,U)\ : \end{aligned} \tag{A4}$$

i.e., the interchanged case is the same as the original case (i.e., Equation (22)) with the appropriate change of parameters. The upshot of the proof is all cases of $\tilde{U} \leq -1$ can be transformed into cases $U \geq 0$ and vice versa. In Appendix A.2, we show that there are exact solutions only for integer $U \geq 0$, and so the VU model with $U < 0$ gives no new exact solutions and is redundant to the VU model with $U \geq 0$.

A.2. Formalism for Exact Solutions of the VU Model

In this appendix subsection, we present the formalism for the exact solutions of the VU model. Recall, the VU model has the Friedmann equation form Equation (22) given in § 3.2. If $U = 0$, the VU model specializes to the V model (which we introduced above in § 3.2) and if $V = 2/3$, there is a further specialization to the $\Lambda\Gamma$ model. The exact solutions for the V model (which immediately specialize to those for the $\Lambda\Gamma$ model) are presented in § 4. Note, as we showed in Appendix A.1, we only need to consider VU model cases of $U > 0$.

Note, if only two of the density parameter constants a, b, and c are nonzero, we choose the nonzero density parameter constants to be a and c since this choice is most symmetrical, no disadvantage, and also easiest for understanding since these have, respectively, the smallest power parameter (i.e., r) and largest power parameter (i.e., p) and the zeroed density parameter constant b power parameter is always $q = (r + p)/2$ (see Equation (18) in § 3.2). If there is only one nonzero density parameter constant, we choose it to be $a > 0$ since all of the VU model formalism we develop below is based integrals which require $a \neq 0$ and in this case V and U by Equation (19) are indeterminate and their most natural choice is $V = 1$ and $U = 0$ as we discussed in § 3.2. The solutions are for the case of one nonzero density parameter constant are, of course, elementary one density component exact solutions discussed in Appendix C .

We now make the following transformation to Equation (22) (including completing a square):

$$\begin{aligned} d\tau &= \pm\frac{Vy^U\,dy}{\sqrt{ay^2+by+c}} = \pm\frac{Vy^U\,dy}{\sqrt{|a|}\sqrt{\mathrm{sgn}(a)[y^2+(b/a)y+(c/a)]}} \\ &= \pm\frac{Vy^U\,dy}{\sqrt{|a|}\sqrt{\mathrm{sgn}(a)\{[y+b/(2a)]^2+[(c/a)-b^2/(4a^2)]\}}} \\ &= \pm\frac{V[B\tilde{y}-b/(2a)]^U B\,d\tilde{y}}{B\sqrt{|a|}\sqrt{\mathrm{sgn}(a)\tilde{y}^2+\mathrm{sgn}(aD)}} = \pm\frac{V[B\tilde{y}-b/(2a)]^U\,d\tilde{y}}{\sqrt{|a|}\sqrt{\mathrm{sgn}(a)\tilde{y}^2+\mathrm{sgn}(aD)}} \\ &= \pm\frac{V[B\tilde{y}-b/(2a)]^U\,d\tilde{\eta}}{\sqrt{|a|}}\ , \end{aligned} \tag{A5}$$

where

$$\mathrm{sgn}(s) = \begin{cases} 1 & \text{for } s>0; \\ 0 & \text{for } s=0; \\ -1 & \text{for } s<0 \end{cases} \tag{A6}$$

is the sign function and we define

$$D = \frac{c}{a} - \frac{b^2}{4a^2} = \frac{4ac-b^2}{4a^2} = -\left(\frac{b^2-4ac}{4a^2}\right)\ , \qquad B = \begin{cases} \sqrt{|D|} & \text{for } D\neq 0; \\ 1 & \text{for } D=0, \end{cases} \tag{A7}$$

$$\tilde{y} = \frac{y+b/(2a)}{B} \qquad \text{giving} \qquad d\tilde{y} = \frac{dy}{B}\ , \tag{A8}$$

and generalized conformal time $\tilde{\eta}$ by

$$d\tilde{\eta} = \frac{d\tilde{y}}{\sqrt{\mathrm{sgn}(a)\tilde{y}^2+\mathrm{sgn}(aD)}} = \pm\frac{\sqrt{|a|}\,d\tau}{V[B\tilde{y}-b/(2a)]^U}\ . \tag{A9}$$

Note, there is no solution for the case of $a<0$ and $aD<0$ (i.e., $a<0$ and $D>0$). Note also, the conventional conformal time η (given our scaling factors for cosmic scaling factor and cosmic time) is defined by $d\eta = d\tau/x(\tau)$ (e.g., Steiner 2008, p. 3). Actually, the generalized conformal time $\tilde{\eta}$ is consistent with the conventionl conformal time η aside from constant factors only in the special case of $U/V=1$ implying $r=2$ as the following shows:

$$\begin{aligned} d\tilde{\eta} &= \pm\frac{\sqrt{|a|}\,d\tau}{V[B\tilde{y}-b/(2a)]^U} = \pm\frac{\sqrt{|a|}\,d\tau}{Vy^U} = \pm\frac{\sqrt{|a|}\,d\tau}{Vx^{U/V}} = \pm\frac{\sqrt{|a|}\,d\tau}{Vx^{r/2}} = \pm\frac{\sqrt{|a|}\,d\tau}{Vx} \quad \text{for } r=2 \\ &= \pm\frac{\sqrt{|a|}}{V}\,d\eta\ . \end{aligned} \tag{A10}$$

Of course, the conventionl conformal time η can be used for any case of the Friedmann equation, not just the case where it is consistent with our generalized conformal time.

The solutions for the generalized conformal time $\tilde{\eta}(\tilde{y})$ and their inverses $\tilde{y}(\tilde{\eta})$ are, respectively,

$$\tilde{\eta} = \begin{cases} \pm\ln(|\tilde{y}|) \\ \mathrm{arsinh}(\tilde{y}) \\ \mathrm{sgn}(\tilde{y})\mathrm{arcosh}(|\tilde{y}|) \\ \arcsin(-1 \geq \tilde{y} \leq 1) \end{cases} \quad \text{and} \quad \tilde{y} = \begin{cases} \pm e^{\pm\tilde{\eta}} & \text{for } (a > 0, aD = 0); \\ \sinh(\tilde{\eta}) & \text{for } (a > 0, aD > 0); \\ \pm\cosh(\tilde{\eta}) & \text{for } (a > 0, aD < 0); \\ \sin(\tilde{\eta}) & \text{for } (a < 0, aD > 0), \end{cases} \tag{A11}$$

where the plus/minus cases are the same for $\pm e^{\pm\tilde{\eta}}$ (i.e., the plus/minus cases are correlated), where we have neglected constants of integration for simplicity and since they can be imposed in the expressions for $\tau(x)$ and/or $x(\tau)$, and where the integrals are obtained from Wikipedia (Wikipedia: List of integrals of irrational algebraic functions). Note, the solution $\eta = \mathrm{sgn}(\tilde{y})\mathrm{arcosh}(|\tilde{y}|)$ just specifies the upper half-plane (plus $\tilde{y} = 0$) branches of the solution.

The generalized conformal time is not actually used for the case of $U = 0$ which is why it never appeared in the solutions for the V model in Section 4. As a compact reference, we recapitulate here the exp-like, sinh-like, cosh-like, and sin-like solutions from Section 4 in scaled form without for clarity introducing any explicit constants of integration which would be needed for particular solutions. Note all the V model solutions have exact $\tau(x)$ and $x(\tau)$ forms. The solutions for $\tau(\tilde{y})$ are

$$\Delta\tau = \pm\frac{V}{\sqrt{|a|}} \begin{cases} \tilde{\eta} = \int d\tilde{\eta} = \int \dfrac{d\tilde{y}}{\sqrt{\mathrm{sgn}(a)\tilde{y}^2 + \mathrm{sgn}(aD)}} & \text{in general.} \\ \ln(\tilde{y}) & \text{the exp-like solution with } (a > 0, aD = 0). \\ \mathrm{arsinh}(\tilde{y}) & \text{the sinh-like solution with } (a > 0, aD > 0). \\ \mathrm{sgn}(\tilde{y})\mathrm{arcosh}(\tilde{y}) & \text{the cosh-like solution with } (a > 0, aD < 0). \\ \arcsin(\tilde{y}) & \text{the sin-like solution with } (a > 0, aD > 0). \end{cases} \tag{A12}$$

and, using Equations (A7) and (A8), for

$$y(\tau) = -\frac{b}{2a} + B\tilde{y} = \begin{cases} \dfrac{-\mathrm{sgn}(a)b - \left(\sqrt{|b^2 - 4ac|}\right)\tilde{y}}{2|a|} & \text{(for } D \neq 0); \\ \dfrac{-b + 2a\tilde{y}}{2a} & \text{(for } D = 0) \end{cases} \tag{A13}$$

are

$$y = \begin{cases} \dfrac{-b + 2ae^{\pm\sqrt{|a|}V^{-1}\Delta\tau}}{2a} & \text{the exp-like solution with } (a > 0, aD = 0). \\ & \quad \text{An appropriate constant of integration} \\ & \quad \text{added to } \Delta\tau \text{ would change the } 2a \\ & \quad \text{numerator to } b_0 \text{ as seen in Equation (36).} \\ \dfrac{-b \pm \sqrt{4ac - b^2}\sinh\left(\sqrt{|a|}V^{-1}\Delta\tau\right)}{2a} & \text{the sinh-like solution with } (a > 0, aD > 0). \\ \dfrac{-b \pm \sqrt{b^2 - 4ac}\cosh\left(\sqrt{|a|}V^{-1}\Delta\tau\right)}{2a} & \text{the cosh-like solution with } (a > 0, aD < 0). \\ \dfrac{b \pm \sqrt{b^2 - 4ac}\sin\left(\sqrt{|a|}V^{-1}\Delta\tau\right)}{2|a|} & \text{the sin-like solution with } (a < 0, aD > 0). \end{cases} \tag{A14}$$

Note, the generalized conformal time solutions for $U = 0$ include no power-like solutions. Given our VU model formalism (where $a \neq 0$ is assumed), the power-like (exact) solutions only occur for odd integer $U > 0$ as we will show below.

From Equation (A9), we find the general solution for $U > 0$ for $\tau(\tilde{y})$ is

$$\Delta\tau = \pm\frac{V}{\sqrt{|a|}}\int\left[B\tilde{y}(\tilde{\eta}) - \frac{b}{2a}\right]^U d\tilde{\eta} \tag{A15}$$

Given the nature of the $\tilde{y}(\tilde{\eta})$ solutions from Equation (A11), there can be no exact solutions for non-integer $U > 0$. The case of exp-like solution with $b = 0$ is not an exception. The exp-like solution requires $D = 0$ and, with $b = 0$, this implies $c = 0$. Thus, the exp-like solution is for $a \neq 0$ only and must have must have $V = 1$ and $U = 0$ as mentioned above, and so there is no $U > 0$ case for the exp-like solution and the exp-like solution with $U = 0$ and $b = 0$ is, in fact, the exponential-expanding/contracting de Sitter universe: see Appendix C.

What can we say about exact inverse solutions $\tilde{\eta}(\tau)$ of Equation (A15) for integer $U > 0$? If $b \neq 0$, then there will be an integrated terms from the integral in Equation (A15) consisting of exponential, hyperbolic, or sinusoidal functions of $\tilde{\eta}$ and a term $[-b/(2a)]^U\tilde{\eta}$. There can be no exact inverse solutions $\tilde{\eta}(\tau)$ in this case, and thus no exact inverse solutions $\tilde{y}(\tau)$ (which would follow from Equation (A11)).

If $b = 0$ (i.e., there are only density components for density parameter constants a and c) and we exclude the already discussed exp-like solution case with $b = 0$, then the integrand of Equation (A15) can be expanded using the binomial theorem thusly

$$B^U \left(s \pm s^{-1}\right)^U = B^U \sum_{k=0}^{U} (\pm)^{U-k} \binom{U}{k} s^{2k-U} \ , \tag{A16}$$

where $s = e^{\tilde{\eta}}$ for the hyperbolic sinh and cosh functions, $s = s^{i\tilde{\eta}}$ for the sine function, and the upper case applies to the cosh function and the lower case to the sinh and sin functions. If U is even, then integration will give terms containing powers of exponentials of $\tilde{\eta}$ and the term for $k = U/2$ which will be linear in $\tilde{\eta}$. Clearly, this integrated expression has no exact inverse solution $\tilde{\eta}(\tau)$. On the other hand, if U is odd, there will be no term linear in $\tilde{\eta}$ and an exact inverse solution is possible. In fact, the integral $\int \sin^U(\tilde{\eta})\, d\tilde{\eta}$ for odd U consists of a sum of all odd powers k from 1 to U of variable $\cos(\tilde{\eta})$ plus a constant of integration (e.g., Suello 2015, p. 194). The integrals $\int \sinh^U(\tilde{\eta})\, d\tilde{\eta}$ and $\int \cosh^U(\tilde{\eta})\, d\tilde{\eta}$ will have the same behavior (aside factors of ± 1) with, respectively, variables $\cosh(\tilde{\eta})$ and $\sinh(\tilde{\eta})$ replacing variable $\cos(\tilde{\eta})$. Note, the coefficients of the variables $\cos(\tilde{\eta})$, $\cosh(\tilde{\eta})$, and $\sinh(\tilde{\eta})$ are rather complex. Given that general quintic equations (with a constant term which is true in the case of finding an exact inverse solution $\tilde{\eta}(\tau)$ since there is the term there is the τ term from the left-hand side of Equation (A15)) and higher degree equations cannot be solved exactly in terms of algebraic expressions: i.e., expressions built up from constants by the basic algebraic operations of addition, subraction, multiplication, division, whole number powers, and roots (i.e., fractional powers) (Wikipedia: Quintic function: Finding roots of a quintic polynomial; Wikipedia: Algebraic expression), we believe that quintic and higher degree equations for exact inverse solutions $\tilde{\eta}(\tau)$ for odd integer U (which are not completely general since they have only odd powers of the variable but with rather complex coefficients and a constant) in terms of algebraic expressions probably cannot be found for any case and even if there are some cases, they are not likely to be very useful. The conclusion of this paragraph is that exact inverse solutions $\tilde{\eta}(\tau)$ for integer $U > 3$ (and therefore for $y(\tau)$ and $x(\tau)$ for integer $U > 3$) in algebraic expressions are not very likely and even if some exist, they are not likely to very useful. We will pursue the issue of exact inverse solutions $\tilde{\eta}(\tau)$ for integer $U > 3$ no further.

Exact inverse solutions $x(\tau)$ for $U = 1$ and $U = 3$ with $b = 0$ do exist. The $U = 1$ case just requires solving a linear equation, of course. The $U = 3$ case requires solving a cubic equation, but since no even powers of the unknown variable (i.e., $\cosh(\tilde{\eta})$, $\sinh(\tilde{\eta})$, or $\cos(\tilde{\eta})$) appear, the cubic equations are depressed cubic equations which have simpler solutions than for general cubic equations (e.g., Wikipedia: Cubic equation: Depressed cubic). We give an example of determining exact solutions for $U = 1$ in Appendix A.4 and an important example for $U = 3$, the radiation-matter universe, in Appendix A.5. Note, the radiation-matter universe in the VU model formalism (which has $b = 0$) has density parameter powers

$r = 3$ and $p = 4$, and so $V = 2/(p - r) = 2$ and $U = r/(p - r) = 3$.

At this point, it is a useful reference to give the integral solutions for $\tau(\tilde{\eta})$ for the exp-like, sinh-like, cosh-like, and sin-like solution cases for density parameter constant $b = 0$: i.e., for VU model cases with only two density components with only density parameter constants a and c nonzero and $B = \sqrt{|c/a|}$, except the exp-like case has has only density parameter constant a nonzero and $B = 1$). We neglect redundant constants of integration and plus-minus cases. For the exp-like solution case (which has $a > 0$ and $D = 0$) with density parameter constant $b = 0$ (which requires $V = 1$ and $U = 0$: see § 3.2), we have

$$\Delta\tau_{\rm exp} = \pm\frac{VB^U}{\sqrt{a}}\tilde{\eta} = \pm\frac{1}{\sqrt{a}}\ln(|\tilde{x}|) \tag{A17}$$

where in this case we define

$$\tilde{x} = \tilde{y}^V = y^V = x \tag{A18}$$

since $b = 0$ and $B = 1$. For the exp-like solution case, there is a unique solution which is the de Sitter universe again: i.e.,

$$x = x_0 e^{\pm\sqrt{a}\Delta\tau} \ . \tag{A19}$$

The exp-like solution with $b \neq 0$ is given by Equations (A14) and (36).

For the sinh-like solution case, we have

$$\Delta\tau_{\rm sinh} = \pm\frac{VB^U}{\sqrt{a}}\begin{cases} \int \sinh^U(\tilde{\eta})\,d\tilde{\eta} & \text{in general and recall } (a > 0, aD > 0); \\ \tilde{\eta} = \operatorname{arsinh}(\tilde{y}) & \text{for } U = 0; \\ \cosh(\tilde{\eta}) = \sqrt{1+\tilde{y}^2} & \text{for } U = 1; \\ \frac{1}{4}\sinh(2\tilde{\eta}) - \frac{\tilde{\eta}}{2} & \text{for } U = 2; \\ \frac{1}{3}\cosh^3(\tilde{\eta}) - \cosh(\tilde{\eta}) = \frac{1}{3}\left(1+\tilde{y}^2\right)^{3/2} - \sqrt{1+\tilde{y}^2} & \text{for } U = 3; \\ \frac{1}{32}\sinh(4\tilde{\eta}) - \frac{1}{4}\sinh(2\tilde{\eta}) + \frac{3}{8}\tilde{\eta} & \text{for } U = 4, \end{cases} \tag{A20}$$

where the harder integrals were obtained from Wikipedia (Wikipedia: List of integrals of trigonometric functions) or Klerer & Grossman (1971, p. 167) and confirmed by Google AI and note in this case we define

$$\tilde{x} = \tilde{y}^V = \left(\frac{y}{\sqrt{|c/a|}}\right)^V = \frac{x}{\left(\sqrt{|c/a|}\right)^V} \ . \tag{A21}$$

For the cosh-like solution case, we have

$$\Delta\tau_{\rm cosh} = \pm\frac{VB^U}{\sqrt{a}}\begin{cases} \displaystyle\int \cosh^U(\tilde\eta)\,d\tilde\eta & \text{in general and recall } (a>0, aD<0); \\ \tilde\eta = {\rm sgn}(\tilde y){\rm arcosh}(\tilde y) & \text{for } U=0; \\ \sinh(\tilde\eta) = \sqrt{\tilde y^2-1} & \text{for } U=1; \\ \dfrac{1}{4}\sinh(2\tilde\eta) + \dfrac{\tilde\eta}{2} & \text{for } U=2; \\ \dfrac{1}{3}\sinh^3(\tilde\eta) + \sinh(\tilde\eta) = \dfrac{1}{3}\left(\tilde y^2-1\right)^{3/2} + \sqrt{\tilde y^2-1} & \text{for } U=3; \\ \dfrac{1}{32}\sinh(4\tilde\eta) + \dfrac{1}{4}\sinh(2\tilde\eta) + \dfrac{3}{8}\tilde\eta & \text{for } U=4, \end{cases} \tag{A22}$$

where the harder integrals were obtained from Wikipedia (Wikipedia: List of integrals of trigonometric functions) or Klerer & Grossman (1971, p. 168) and confirmed by Google AI and note in this case we again define

$$\tilde x = \tilde y^V = \left(\frac{y}{\sqrt{|c/a|}}\right)^V = \frac{x}{\left(\sqrt{|c/a|}\right)^V} \ . \tag{A23}$$

For the sin-like solution case, we have

$$\Delta\tau_{\rm sin} = \pm\frac{VB^U}{\sqrt{a}}\begin{cases} \displaystyle\int \sin^U(\tilde\eta)\,d\tilde\eta & \text{in general and recall } (a<0, aD>0); \\ \tilde\eta = \arcsin(\tilde y) & \text{for } U=0; \\ -\cos(\tilde\eta) = -\sqrt{1-\tilde y^2} & \text{for } U=1; \\ -\dfrac{1}{4}\sin(2\tilde\eta) + \dfrac{\tilde\eta}{2} & \text{for } U=2; \\ \dfrac{1}{3}\cos^3(\tilde\eta) - \cos(\tilde\eta) = \dfrac{1}{3}\left(1-\tilde y^2\right)^{3/2} - \sqrt{1-\tilde y^2} & \text{for } U=3; \\ \dfrac{1}{32}\sin(4\tilde\eta) - \dfrac{1}{4}\sin(2\tilde\eta) + \dfrac{3}{8}\tilde\eta & \text{for } U=4, \end{cases} \tag{A24}$$

where the harder integrals were obtained from Wikipedia (Wikipedia: List of integrals of trigonometric functions) or Klerer & Grossman (1971, p. 98–99) and confirmed by Google AI and note in this case we again define

$$\tilde x = \tilde y^V = \left(\frac{y}{\sqrt{|c/a|}}\right)^V = \frac{x}{\left(\sqrt{|c/a|}\right)^V} \ . \tag{A25}$$

Note from Equations (A20), (A22), and (A24) that the exact inverse solutions for the $U = 3$ cases are all obtained from depressed cubic equations in, respectively, the variables $\cosh(\tilde{\eta})$, $\sinh(\tilde{\eta})$, and $\cos(\tilde{\eta})$. The cubic equations would not be depressed if one replaced specified variables by $\tilde{y}$. To show this explicitly, consider the $U = 3$ case for Equation (A20):

$$\begin{aligned} \text{Constant} &= \frac{1}{3}\cosh^3(\tilde{\eta}) - \cosh(\tilde{\eta}) = \frac{1}{3}\left[\left(1+\tilde{y}^2\right)^{3/2} - 3\sqrt{1+\tilde{y}^2}\right] = \frac{1}{3}\left(\tilde{y}^2 - 2\right)\sqrt{1+\tilde{y}^2} \\ \text{Constant}^2 &= \frac{1}{9}\left(\tilde{y}^4 - 4\tilde{y}^2 + 4\right)\left(\tilde{y}^2 + 1\right) \\ \text{Constant}^2 &= \frac{1}{9}\left(\tilde{y}^6 - 3\tilde{y}^4 + 4\right) \ , \end{aligned} \tag{A26}$$

where the last expression is not a depressed cubic in powers of $\tilde{y}^2$ since an even power of $\tilde{y}^2$ appears: i.e., $\tilde{y}^4 = (\tilde{y}^2)^2$.

Analyzing the VU model solutions using generalized conformal time is useful in understanding the solution behaviors, in determining when exact solutions $\tau(x)$ exist, and in determining when the exact inverse solutions $x(\tau)$ exist. However, one does not actually need the generalized conformal time to determine solutions for $\tau(x)$ for integer $U \geq 0$. Table integrals exist for the first expression in Equation (A5): for $U = 0, 1, 2$ and $a > 0$ and $a < 0$ from Wikipedia (Wikipedia: List of integrals of irrational functions: Integrals involving $R = \sqrt{ax^2 + bx + c}$) and for $U = 3$ for $a > 0$ from Klerer & Grossman (1971, p. 80). Further searching in more extensive integral tables may lead to other cases of U leading to exact solutions. Of course, there may also be ways to solve the VU model Friedmann equation for exact solutions other than the way discussed in this paper and also ways to solve more general three density component Friedmann equation forms for exact solutions than those involving density parameters obeying (inverse) powers of the cosmic scale factor.

To summarize this appendix subsection, we have presented the VU model formalism from which we believe all exact solutions of the VU model can be derived. The set of exact solutions is infinite, but those any interest are finite and we discuss those we recognize as such in Appendix A.3. The VU model formalism gives exact solutions for the three density component Friedmann solution where the density components all obey power laws. The three density parameter constants are labeled a, b, and c and there corresponding powers cosmic scale factor x are labeled p, q, and r and these powers obey $p > q > r \geq 0$. Further restrictions on the powers and on the auxiliary parameters V, U and W for exact solutions are specified by Equations (17), (19), (20) and (24). If $r = 0$ which implies $U = 0$, the power $q = p/2$, but $p > 0$ is completely general. If $r = 0$ and there are only two nonzero density components, let them have density parameter constants a and c. If there is only one nonzero density component, choose it have density parameter constant a and in this case $V = 1$ and $U = 0$ by special specification (i.e., Equation (24)) and the solution is just the de Sitter universe (i.e., an exponetially expanding/contracting universe). If $U > 0$ (and note, $U < 0$ gives no extra solutions as proven in Appendix A.1), there are exact solutions $\tau(x)$ only for integer U. Exact inverse solutions $x(\tau)$ (at least of any use) for integer $U > 0$, exist only for density parameter constant $b = 0$, and $U = 1$ and $U = 3$.

A.3. Interesting VU Model Exact Solutions

In Appendix A.2, we present for formalism for exact solutions of the VU model. The question then arises what are the interesting VU model exact solutions out the infinite set of VU model exact solutions. One condition for interesting solutions is that they are for cases where the density parameters have power-law behavior with the generally considered interesting powers $0, 1, 2, 3, 4$. The physical reasons for these powers being interesting, see, e.g., Steiner (e.g., 2008, p. 6–7) and Melia (2014). The reasons are also briefly described in Appendix C.

The VU models with powers limited to $0, 1, 2, 3, 4$ specified by their characteristics (i.e., the set powers (r, q, p) and parameters $[V, U, W]$) are tabulated in Table 1.

Table 1. The Sets of Powers (r, q, p) Over the Parameters $[V, U, W]$ for Three Density Component Exact Solutions for the VU Model†

$r \setminus p$	1	2	3	4	$\left(r, q = \dfrac{r+p}{2}, p\right)^{\ddagger}$ $[V, U, W]$
0	$(0, 1/2, 1)$ $[2, 0, 0]$	$(0, 1, 2)$ $[1, 0, 0]$	$(0, 3/2, 3)^1$ $[2/3, 0, 0]$	$(0, 2, 4)^2$ $[1/2, 0, 0]$	$\left(0, q = \dfrac{p}{2}, p\right)$ $\left[\dfrac{2}{p}, 0, 0\right]$
1	...	$(1, 3/2, 2)$ $[2, 1, 1/2]$	...	...	$\left(r, q = \dfrac{r+p}{2}, p\right)$ $\left[\dfrac{2}{p-r}, \dfrac{r}{p-r}, \dfrac{r}{p}\right]$
2	...	...	$(2, 5/2, 3)^3$ $[2, 2, 2/3]$	$(2, 3, 4)^4$ $[1, 1, 1/2]$	$\left(r, q = \dfrac{r+p}{2}, p\right)$ $\left[\dfrac{2}{p-r}, \dfrac{r}{p-r}, \dfrac{r}{p}\right]$
3	...	...	...	$(3, 7/2, 4)^5$ $[2, 3, 3/4]$	$\left(r, q = \dfrac{r+p}{2}, p\right)$ $\left[\dfrac{2}{p-r}, \dfrac{r}{p-r}, \dfrac{r}{p}\right]$

†The physically, educationally, and/or heuristically interesting VU model solutions (specified by their sets of parameters) with exact solutions. Recall, for an exact solution $\tau(x)$, integer $U > 0$ is needed and, for an inverted exact solution $x(\tau)$, ratio $W = r/p = U/(U+1) = 0, 1/2, 2/3, 3/4, \ldots < 1$ is needed.

‡For convenient reference, this column recapitulates the formulae relating power parameters r and p to the other parameters.

[1]The $\Lambda\Gamma$ model which is heuristically interesting as a dynamical dark energy model, but there is very modest expectation that it could be physically real since there is no physical explanation for a dark energy density component that varies as $x^{-3/2}$, where x is the cosmic scale factor. If the model $b = 0$, then the $\Lambda\Gamma$ model reduces to the the matter-λ universe which is just the Λ-CDM model solution of the Friedmann equation which is, of course, a physically interesting model of great importance.

[2]The radiation-curvature-Λ universe is physically interesting as it could hold in some other pocket universe if those exist. It is also educationally interesting since it is an analogue to the matter-curvature-Λ universe which is a viable model for the observable universe if the observable universe has some curvature (i.e., curvature density parameter constant $\Omega_k \neq 0$.

In fact, density components for powers greater than 4 (which is the power for a radiation universe: i.e., for an extreme relativistic gas, e.g., and most obviously a photon gas) and non-integer powers do not seem to have been much discussed and may not have any physical motivation unlike powers 0, 1, 2, 3, 4 which do (e.g., Steiner 2008, p. 6–7). Even for heuristic reasons, powers greater than 4 or non-integer powers are probably mostly uninteresting, except for the $\Lambda\Gamma$ model's Γ dark energy component power 3/2 since the $\Lambda\Gamma$ model is heurisitically interesting for reasons presented in § 2 and which reasons are the motivation for this paper. The upshot is that most determinable exact solutions implied Table 1 will probably not be interesting.

However, besides the $\Lambda\Gamma$ model, there are at least two determinable VU model exact solutions which are interesting. The first is the $V = 1/2$ and $U = 0$ case which implies an exact solution (which has probably been long known) for the radiation-curvature-Λ universe which we discuss in § 4. The second is for the $V = 1$ and $U = 2$ case which gives an exact solution (which has probably been long known) for the radiation-matter-curvature universe which universe is heuristically interesting since the observable universe may have some small curvature (e.g., Chen & Zaldarriaga 2025). The exact solution (which has probably long been known) is given and discussed in Appendix B.

A.4. The Exact Solutions $\tau(x)$ and $x(\tau)$ for the VU Model with $r = p/2$, $b = 0$, and $U = 1$

Exact solutions $x(\tau)$ for $U = 1$ and $U = 3$ with $b = 0$ do exist. We illustrate this with an example for $U = 1$ in Appendix A.4

A.5. The Exact Solutions $\tau(x)$ and $x(\tau)$ for the Radiation-Matter Universe

For $Q = 3$, we have only one interesting case: the radiation-matter universe which has $p = 4$, $q = 3$, $R = 1$, and $g = h = 1$. The radiation-matter universe is, of course, Λ-CDM model in the radiation-matter era which is prior to the matter era which is prior to the matter-Λ era where cosmic present is located.

From Equation (???refeq-friedmann-equation-scaled-form-general-derived) in § ???refThe Two Density Component Friedmann Equation Equation (???refeq-z-solution-g-h-1-point-origin) in § ???refThe Solution for z for g=h=1, and Equation (???eq-F-w-tau-eta-scaling) in § ???refComplete Two Density Component Solutions and Equations (???refeq-generalized-conformal-time-Q-parameter-2) and (???eq-explicit-implicit-F-factor) in § ???refComplete Two Density Component Solutions, we find that

$$\tilde{x} = \tilde{y}^2 = \sinh^2(\tilde{\eta}) \ , \tag{A27}$$

$$\frac{1}{2}\, d\tilde{\tau} = \sinh^3(\tilde{\eta})\, d\tilde{\eta} \ , \tag{A28}$$

and

$$\tilde{\tau} = \frac{\tau}{\tau_{\rm scale}} = \frac{\tau}{(c/a)^{3/2}/\sqrt{a}} = \frac{\tau}{c^{3/2}/a^2} \ , \tag{A29}$$

where we have set $F = R/2 = 1/2$ in Equation (???refeq-explicit-implicit-F-factor) from § ???refComplete Two Density Component Solutions. We now solve for $w(\tilde{\eta})$ requiring $w(\tilde{\eta} = 0) = 0$ so that the zero point of scaled cosmic time and generalized conformal time agree which implies that the cosmic time zero gives the generalized scale factor zero (since $z = \sinh(\tilde{\eta})$), and so the Big Bang singularity occurs at the fiducial cosmic time zero as one would like. The solution is

$$\begin{aligned}
\frac{1}{2}\tilde{\tau} &= \int_0^{\tilde{\eta}} \sinh^3(\tilde{\eta}')\, d\tilde{\eta}' \\
&= \int_0^{\tilde{\eta}} \sinh(\tilde{\eta}')[\cosh^2(\tilde{\eta}') - 1]\, d\tilde{\eta}' \\
&= \left[\frac{1}{3}\cosh^3(\tilde{\eta}') - \cosh(\tilde{\eta}')\right]\Bigg|_0^{\tilde{\eta}}
\end{aligned}$$

$$= \frac{1}{3}\cosh^3(\tilde{\eta}) - \cosh(\tilde{\eta}) + \frac{2}{3} \tag{A30}$$

which could have been done by a table integral (see, e.g., Wikipedia: List of integrals of hyperbolic functions: Integrals involving only hyperbolic sine functions). We can now find $\tilde{\tau}(\tilde{x})$ explicitly:

$$\begin{aligned}\tilde{\tau} &= \frac{2}{3}[1+\sinh^2(\tilde{\eta})]^{3/2} - 2[1+\sinh^2(\tilde{\eta})]^{1/2} + \frac{4}{3} = \frac{2}{3}(1+\tilde{x})^{3/2} - 2(1+\tilde{x})^{1/2} + \frac{4}{3} \\ &= \begin{cases} \frac{2}{3}(\tilde{x}-2)\sqrt{1+\tilde{x}} + \frac{4}{3} & \text{in general.} \\ 0 & \text{for } \tilde{x} = 0. \\ \frac{4}{3}\left(1 - \frac{1}{\sqrt{2}}\right) = 0.39052429\ldots & \text{for } \tilde{x} = 1. \\ \frac{4}{3} = 1.333\ldots & \text{for } \tilde{x} = 2. \\ \frac{8}{3} = 1.666\ldots & \text{for } \tilde{x} = 3. \\ \frac{2}{3}\tilde{x}^{3/2} & \text{for } \tilde{x} >> 1. \end{cases}\end{aligned} \tag{A31}$$

We will explicate the solution $w(y)$ in § ???refThe Radiation-Matter Universe t(a) Solution. Here we are interested in finding the inverse solution $y(w)$.

For inverse solution $y(w)$, we first find $\cosh(\tilde{\eta})$ as a function of w. Defining

$$u = \cosh(\tilde{\eta}) \ , \tag{A32}$$

we can rearrange Equation (A30) as a cubic equation

$$0 = u^3 - 3u + \left(2 - \frac{3}{2}w\right) \tag{A33}$$

Equation (A33) is, in fact, a depressed cubic equation in that it lacks a u^2 term (e.g., Wikipedia: Cubic equation: Depressed cubic). Depressed cubic equations have simpler solutions than general cubic equations.

The solution of Equation (A33) follows from a standard procedure (e.g., Press et al. 1992, p. 179). First we define parameters

$$\tilde{Q} = 1 \qquad \text{and} \qquad \tilde{R} = 1 - \frac{3}{4}w \ , \tag{A34}$$

where the tildes are needed to distinguish parameters from the totally different parameters Q and R that we use for other purposes. For

$$\tilde{R}^2 = \left(1 - \frac{3}{4}w\right)^2 \leq 1 = \tilde{Q}^3 \tag{A35}$$

implying $w \in [0, 8/3]$, there are three real roots written in terms of parameter

$$\theta = \arccos\left(\frac{\tilde{R}}{\sqrt{\tilde{Q}^3}}\right) = \arccos\left(1 - \frac{3}{4}w\right) \tag{A36}$$

In fact, the 1st and 3rd of the three real root formulae specified by the procedure (e.g., Press et al. 1992, p. 179) are ruled out since they give some values for $u < 1$ and since $u = \cosh(\tilde{\eta}) \geq 1$ always. That leaves the 2nd root formula which we rearrange thusly

$$\begin{aligned} u &= -2\cos\left\{\frac{\arccos[1-(3/4)w] + 2\pi}{3}\right\} = -2\cos\left\{\frac{\pi - \arccos[(3/4)w - 1] + 2\pi}{3}\right\} \\ &= -2\cos\left\{\pi - \frac{\arccos[(3/4)w-1]}{3}\right\} = 2\cos\left\{\frac{\arccos[(3/4)w-1]}{3}\right\} \ , \end{aligned} \tag{A37}$$

where we used the inverse trigonometric identity $\arccos(x) = \pi - \arccos(-x)$ (e.g., Wikipedia: Inverse trigonometric functions: Relationships among the inverse trigonometric functions) and the trigonometric identity $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ (e.g., Wikipedia: Trigonometric functions: Sum and difference formulas). So our formula for allowed root u is

$$u = \begin{cases} 2\cos\left\{\frac{\arccos[(3/4)w-1]}{3}\right\} & \text{for } w \in [0, 8/3]; \\ 1 & \text{for } w = 0; \\ \sqrt{3} & \text{for } w = 4/3; \\ 2 & \text{for } w = 8/3. \end{cases} \tag{A38}$$

For

$$\tilde{R}^2 = \left(1 - \frac{3}{4}w\right)^2 \geq 1 = \tilde{Q}^3 \tag{A39}$$

implying $w \geq 8/3$, there is only one real root

$$u = \begin{cases} A + \frac{1}{A} & \text{for } w \geq 8/3; \\ 2 & \text{for } w = 8/3, \end{cases} \tag{A40}$$

where

$$A = \left\{\left[\left(\frac{3}{4}\right)w - 1\right] + \sqrt{\left[\left(\frac{3}{4}\right)w - 1\right]^2 - 1}\right\}^{1/3} \ . \tag{A41}$$

From Equation (A27) and Equation (A32) above and the first line of Equation (???refeq-hyperbolic-function-identities) in § ???refSummary of Hyperbolic Function Identities, we find the solution for the scaled cosmic scale factor

$$\tilde{x}(w) = \sinh^2(\tilde{\eta}) = \cosh^2(\tilde{\eta}) - 1 = u^2 - 1 = \begin{cases} \left[2\cos\left\{\dfrac{\arccos[(3/4)w - 1]}{3}\right\}\right]^2 - 1 \\ \qquad\qquad \text{for } w \in [0, 8/3]. \\ \left(A + \dfrac{1}{A}\right)^2 - 1 \qquad \text{for } w \geq 8/3. \\ \left(\dfrac{3}{2}w\right)^{2/3} \qquad \text{for } w >> 1 \\ \qquad\qquad \text{(i.e., the large } w \\ \qquad\qquad \text{asymptotic solution).} \end{cases} \tag{A42}$$

We call Equation (A42) the 2nd exact formula for $y(w)$. What we call 1st exact formula (Galanti & Roncadelli 2021, p. 3) is given by Equation (???refeq-radiation-matter-y(w)-GR) in § ???refCOMPARISON OF FORMULAE FOR THE EXACT SOLUTION OF THE RADIATION-MATTER UNIVERSE COSMIC SCALE FACTOR.

The functional behavior of $y(w)$ is not obvious (except for the large w asymptotic solution), but it can be easily investigated using the inverse $w(y)$ which we have already found above in Equation (A31).

B. The Radiation-Matter-Curvature Universe

As mentioned in § 3, an interesting case of the VU model is the radiation-matter-curvature universe for which the powers are for radiation $p = 4$, matter $q = 3$, and curvature $r = 2$ which imply $V = 2/(p - r) = 1$, $U = 2$, and $y = x$ in Friedmann equation form Equation (22). The radiation-matter-curvature universe is heuristically interesting since hypothetical other universes than the observable universe may conform to it and the observable universe may have some small curvature (e.g., Chen & Zaldarriaga 2025). The radiation-matter universe describes, of course, the observable universe before dark energy of some kind became important: in the Λ-CDM model before cosmic time $\sim 8\,\text{Gyr}$ (e.g., Hergt & Scott 2024, p. 6). For the radiation-matter universe, there is a well known rather simple exact solution $\tau(x)$ (which we give below) and a rather complex exact solution $x(\tau)$ (Galanti & Roncadelli 2021, Eq. (20)) for which there is a mathematically equivalent slightly simpler formula (Jeffery 2026b).

For the radiation-matter-curvature universe exact solution, the curvature density parameter constant is a (conventionally $\Omega_{k,0}$) the matter density parameter constant is b (conventionally $\Omega_{\text{matter},0}$), and the radiation density parameter constant is c (conventionally $\Omega_{\text{radiation},0}$). We only consider expanding-universe cases, except for the $a < 0$ case, and so drop the plus/minus coefficient for the solution, except in that case. Making use of Wikipedia (Wikipedia: List of integrals of irrational functions: Integrals involving $R = \sqrt{ax^2 + bx + c}$;

Integrals involving $S = \sqrt{ax+b}$), the solution obtained from Equation (22) is

$$\tau = \begin{cases} \dfrac{x^2}{2\sqrt{c}} - \dfrac{1}{6}\dfrac{b}{c\sqrt{c}}x^3 & \text{Series solution for } c > 0, \\ \quad + \dfrac{1}{4\sqrt{c}}\left[-\dfrac{1}{2}\left(\dfrac{a}{c}\right) + \dfrac{3}{8}\left(\dfrac{b}{c}\right)^2\right]x^4 + \ldots & \quad \text{in small } x. \\ & \quad \text{Truncating to 4th order} \\ & \quad \text{in small } x \\ & \quad \text{in a decimal precision 18} \\ & \quad \text{calculation, use the series} \\ & \quad \text{solution for } x \lesssim 2.5 \times 10^{-4} \\ & \quad \text{to keep relative error in } \tau \\ & \quad \text{less than} \sim 2 \times 10^{-12}. \\ \dfrac{\sqrt{ax^2+bx+c} - \sqrt{c}}{a} & \text{General solution for} \\ \quad - \dfrac{b}{2a\sqrt{a}} \ln \left| \dfrac{2ax + b + 2\sqrt{a}\sqrt{ax^2+bx+c}}{b + 2\sqrt{ac}} \right| & \quad a > 0,\ b \geq 0,\ c \geq 0, \\ & \quad \text{not } b = c = 0,\ \tau(x=0) = 0. \\ & \quad \text{The solution increases} \\ & \quad \text{strictly from } x = 0, \text{ except} \\ & \quad \text{for the minimum at } x = 0. \\ & \quad \text{This solution is for} \\ & \quad \text{negative curvature.} \\ \dfrac{x}{\sqrt{a}} & \text{Solution for } a > 0, \\ & \quad \text{valid for } x \\ & \quad \text{asymptotically large.} \\ \dfrac{c^{3/2}}{b^2}\left(\dfrac{2}{3}\right)\left[(Z-2)\sqrt{1+Z} + 2\right] & \text{Solution for } a = 0,\ b > 0, \\ & \quad c > 0,\ Z = x/x_{\rm eq}, \text{ and} \\ & \quad x_{\rm eq} = c/b \text{ is the radiation-} \\ & \quad \text{matter equality} \\ & \quad \text{cosmic scale factor.} \\ & \quad \text{The solution increases} \\ & \quad \text{strictly from } Z = 0 \\ & \quad \text{as proven for } Z = 0 \\ & \quad \text{by the series solution above.} \\ & \quad \text{This solution is for} \\ & \quad \text{zero curvature.} \end{cases} \tag{B1}$$

$$\tau = \begin{cases} \pm\left\{-\frac{\sqrt{ax^2+bx+c}}{|a|} + \frac{b}{2|a|\sqrt{|a|}}\left[\arcsin\left(\frac{2|a|x-b}{\sqrt{b^2+4|a|c}}\right)-\frac{\pi}{2}\right]\right\} & \text{Solution for } a<0 \text{ here and, below, } b^2-4ac>0, \text{ upper/lower case is for } x \text{ in/decreasing with } \tau. \text{ This solution is for positive curvature.} \\ 0 & \text{For } x_{\rm max} = [b+\sqrt{b^2+4|a|c}/(2|a|)]. \\ \tau_{\pm\rm endpoint} = \pm\left\{-\frac{\sqrt{c}}{|a|} + \frac{b}{2|a|\sqrt{|a|}}\left[\arcsin\left(\frac{-b}{\sqrt{b^2+4|a|c}}\right)-\frac{\pi}{2}\right]\right\} & \text{For } x=0, \text{ physical solution only for } \tau\in(\tau_{-\rm endpoint},\tau_{+\rm endpoint}). \\ \pm\left(-\frac{\sqrt{c-|a|x^2}}{|a|}\right) & \text{For case of } b=0. \\ \pm\left\{-\frac{\sqrt{ax^2+bx}}{|a|} + \frac{b}{2|a|\sqrt{|a|}}\left[\arcsin\left(2\frac{x}{b/|a|}-1\right)-\frac{\pi}{2}\right]\right\} & \text{For case of } c=0. \end{cases} \quad \text{(B2)}$$

In the subsections below, we will explicate the series solution (Appendix B.1), the radiation-matter-negative-curvature universe (Appendix B.2), the radiation-matter-positive-curvature universe (Appendix B.3, and matter-positive-curvature universe (Appendix B.4. We will not explicate here the behavior of the case with $a = 0$ since that is done extensively in Jeffery (2026b).

B.1. The Radiation-Matter-Curvature Universe Series Solution

As noted in Equation (B1), the series solution truncated to 4th order in small x (with $c > 0$) should be used for decimal precision 18 for $x \lesssim 2.5 \times 10^{-4}$ to keep relative error in τ less than $\sim 2 \times 10^{-12}$. For reference, we present here a general explication of when to switch from a formally exact solution $y(x)$ (here y being a generic dependent variable) to a series solution in small x (here x being a generic independent variable) as x becomes small. Note, the explication applies for a series solution in small Δx around a general point x, *mutatis mutandis*.

A formally exact solution $y(x)$ can become numerically inexact due to round-off error as x becomes small. In these cases, switching to a series expansion $y(x) = \sum_{\ell=L}^{\infty} c_\ell x^\ell$ will become more exact at some point as x becomes small. We explore what we call the expansion error EE by the following cases:

$$EE = \begin{cases} \dfrac{y + \Delta y - y_{n+1}}{y_L} & \text{The definition of } EE \text{ where } y \text{ is the} \\ & \quad \text{exact solution, } \Delta y \text{ is the round-off error} \\ & \quad \text{(which in general is not known a priori),} \\ & \quad y_{n+1} \text{ is the expansion series for terms} \\ & \quad n+1 \text{ and greater} \\ & \quad \text{(which is formally not known),} \\ & \quad y_L \text{ is the (complete) expansion series} \\ & \quad \text{which starts with} \\ & \quad \text{the term } L \text{ where } L \neq 0 \text{ in general.} \\ \dfrac{y - y_{n+1}}{y_L} & \text{The case where the round-off error } \Delta y = 0. \\ & \quad \text{In this case, } EE \to 0 \text{ as } x \to 0 \\ & \quad \text{for } x \text{ sufficiently small.} \\ \dfrac{y - y_{n+1}}{y_L} \mathrel{\underset{\sim}{\propto}} x^{n+1-L} & \text{For } \Delta y = 0 \text{ and } x \text{ sufficiently small.} \\ & \quad \text{The quantity } x^{n+1-L} \text{ is the} \\ & \quad \text{(fiducial) estimated relative error in the} \\ & \quad \text{series truncated at term } n. \\ \dfrac{\Delta y + c_{n+1} x^{n+1}}{c_L x^L} \mathrel{\underset{\sim}{\propto}} \dfrac{\Delta y + x^{n+1}}{x^L} & \text{For } \Delta y \neq 0 \text{ and } x \text{ sufficiently small.} \end{cases} \tag{B3}$$

From the above formulae for EE, we argue for x sufficiently small, but $\Delta y << x^{n+1}$, that EE should decrease with decreasing x. The point where EE begins to increase indicates that Δy is no longer negligible and the exact solution $y(x)$ is no longer numerically as accurate as the series solution. The point x just before EE begins to increase is $x_{\rm min}$ and for $x \leq x_{\rm min}$, the series solution is more accurate and should be used. In general, there is no way to predict $x_{\rm min}$ since $y(x)$ may be rather complex, and so $x_{\rm min}$ is itself determined numerically. Usually, $x_{\rm min}$ only needs to be determined roughly.

Note, the coefficient c_{n+1} can be determined approximately for sufficiently small x from

$$c_{n+1} \approx \frac{y + \Delta y - y_{n+1}}{x^{n+1}} \ , \tag{B4}$$

provided $\Delta y << y_{n+1}$. There will often be a region as x becomes small, but $x > x_{\rm min}$, where the above formula gives a relatively constant value and this can be recognized as the approximate value of c_{n+1}.

If we are aiming at having an estimated relative error in y series evaluation of 10^{-m}, the maximum satisfactory x value obeys

$$x_{\rm max,sat}^{n+1-L} = 10^{-m} \quad \text{implying} \quad x_{\rm max,sat} = 10^{-m/(n+1-L)} \quad \text{and} \quad n = L-1+\frac{(-m)}{\log(x_{\rm max,sat})} \ . \tag{B5}$$

Clearly, to increase $x_{\rm max,sat}$ (but with $x_{\rm max,sat} < 1$), we must increase n, the number of terms in the truncated series. To make the combination of series and exact solution satisfy estimated relative error 10^{-m}, we simply increase n until $x_{\rm min} \lesssim x_{\rm max,sat}$: recall, the series and the exact solution are still converging as x decreases until $x \approx x_{\rm min}$. Note, the estimated relative error x^{n+1-L} decreases for all x as n increases, but this means the x values where $\Delta y/x^L$ is comparable to x^{n+1-L} will increase as n increases. So $x_{\rm min}$ will increase with increasing n, and so $x_{\rm min}$ is a moving target (i.e., an increasing target) for satisfying $x_{\rm min} \lesssim x_{\rm max,sat}$.

For the series solution $\tau(x)$ for $a > 0$ given by Equation (B1), if we wanted estimated relative error 10^{-18} (a standard Fortran 95 double precision value) given $L = 2$ and $n+1-L = 4+1-2 = 3$, then $x_{\rm max,sat} = 10^{-6}$ which is much smaller than the $x_{\rm min} \approx 2.5\times10^{-4}$ we found for the switch to the series. If $x_{\rm min}$ stayed fixed as n increases (though it should increase at least somewhat as argued above), then for $x_{\rm max,sat} \approx x_{\rm min} \approx 2.5 \times 10^{-4}$ to give estimated relative error 10^{-18} for all values of x, we find we need $n = 6$ from Equation (B5). Clearly, one may need many terms in an expansion around small x to evaluate an exact solution with large round-off error as x becames small if one wants to have double-precision accuracy for all values of x for a combination of series and exact solution.

B.2. The Radiation-Matter-Negative-Curvature Universe

The case with $a > 0$ (i.e., negative curvature) in Equation (B1) obviously does not allow an exact inverse solution (i.e., an exact solution $x(\tau)$) because of the combination of square root and logarithmic terms. However, an interpolation inverse solution $x_{\rm IP}(\tau)$ (necessarily approximate) can be found. Alhough not needed for calculational purposes, $x_{\rm IP}(\tau)$ is useful in understanding scale factor evolution as a function of cosmic time since after all the universe in general respects evolves with cosmic time, not with the cosmic scale factor. First, from small x series for $\tau(x)$ for $a > 0$, we find

$$x_{\rm 1st} = \left(2\sqrt{c}\,\tau\right)^{1/2} \quad \text{implies} \tag{B6}$$

$$\begin{aligned} x_{\rm 2nd} &= \left[2\sqrt{c}\left(\tau + \frac{1}{6}\frac{b}{c\sqrt{c}}x_{\rm 1st}^3\right)\right]^{1/2} = \left[2\sqrt{c}\,\tau\left(1 + \frac{\sqrt{2}}{3}\frac{b}{c^{3/4}}\tau^{1/2}\right)\right]^{1/2} \\ &= \sqrt{2\sqrt{c}\,\tau}\left(+\frac{1}{3\sqrt{2}}\frac{b}{c^{3/4}}\tau^{1/2}\right) \ , \end{aligned} \tag{B7}$$

where the order count is in powers of $\tau^{1/2}$. Second, the large x asymptotic formula

$$x_{\rm 1st\,asy} = \sqrt{a}\,\tau \quad \text{implies a 2nd order asymptotic solution} \tag{B8}$$

$$x_{\rm 2nd\,asy} = \sqrt{a}\,\tau + \frac{b}{2a}\ln\left|\frac{2ax_{\rm 1st\,asy} + b + 2\sqrt{a}\sqrt{ax_{\rm 1st\,asy}^2 + bx_{\rm 1st\,asy} + c}}{b + 2\sqrt{ac}}\right| \ . \tag{B9}$$

Actually, there is some freedom about what terms to include in the formula for $x_{\rm 2nd\,asy}$ and the choice we arrived at gave better results in the interpolation formula than other obvious choices. The interpolation formula we arrived at after some experimentation is

$$x_{\rm IP} = \left\{\frac{2}{3}\left(\frac{c}{b\pi/2}\right)\arctan\left[\frac{x_{\rm 2nd}}{c/(b\pi/2)}\right] + \frac{1}{3}\left(\frac{c}{b}\right)\tanh\left[\frac{x_{\rm 2nd}}{c/b}\right]\right\}(1 + x_{\rm 2nd\,asy}) \ . \tag{B10}$$

Although Equation (B10) looks a bit mysterious, it is arrived at by natural process to create an interpolation formula between using $x_{\rm 2nd}$ and $x_{\rm 2nd\,asy}$. The two functions arctangent arctan(...) and hyperbolic tangent tanh(...) give $x_{\rm 2nd}$ for small τ and then they slow the growth of the $x_{\rm 2nd}$ contribution and saturate at $x_{\rm eq} = c/b$ (the equality x for radiation and matter). In arriving at Equation (B10), a somewhat fitted average of arctangent and hyperbolic tangent worked best. The $x_{\rm 2nd\,asy}$ contribution is relatively unimportant until of order $x_{\rm eq} = c/b$ is reached and becomes the major source of growth becoming linear growth asymptotically.

The interpolation formula $x_{\rm IP}(\tau)$ was actually optimized for $a = b = c = 1$ (or with τ scaled, a, b, and c equal). The accuracy of $x_{\rm IP}(\tau)$ was determined by computing $\tau(x)$ using the exact formuala for $a > 0$ (or the series for small x) for a logarithmically spaced grid of x values, recreating x from $x_{\rm IP}(\tau)$ and using using relative error $[x_{\rm IP}(\tau) - x]/x$. The absolute value of relative error was everywhere less than 0.86 %. The relative error, of course, goes to 0 as x goes small, it is -2.4×10^{-12} at $x = 10^{-5}$, decreases to -8.0×10^{-3} at $x \approx 0.63096$, increases to 8.6×10^{-3} at $x \approx= 2.5119$, decreases to -9.5×10^{-4} at $x = 10$, and increases (but decreases in absolute value) to -4.4×10^{-5} at 10^5 by which point the absolute value is asymptotically going to zero. We expect the interpolation formula $x_{\rm IP}(\tau)$ to be relatively good for the curvature density parameter constant a, the matter density parameter constant b, and the radiation density parameter constant c comparable in size and worsen as they depart from being comparable in size. The interpolation formula $x_{\rm IP}(\tau)$ fails if any of a, b, and c are zero. Some more investigation and refinement of interpolation formulae can be done if that is ever of interest.

B.3. The Radiation-Matter-Positive-Curvature Universe

The radiation-matter-positive-curvature universe solution Equation (B2 (i.e., the radiation-matter-curvature universe solution with $a < 0$) has a maximum x value formula which we identified in Equation (B2). It is easy to prove this a maximum value formula starting from an anticipated formula for the stationary points

$$x_{\rm max\,/\,min} = \frac{b \pm \sqrt{b^2 + 4|a|c}}{2|a|} \ , \tag{B11}$$

where the upper/lower case is form maximum/minimum value. Note, the minimum is negative, and therefore is not part of the physical solution for the cosmic scale factor. Now the zeros of the denominator of the Equation (22) version of the Friedmann equation give inifinities for $d\tau/dx$ and therefore zeros for $dx/d\tau$ which locate the stationary points. Substituting Equation (B11) into the denominator of Equation (22) (with special case x substituted for the general case y) gives

$$\begin{aligned} \sqrt{ax^2+bx+c}\,\Big|_{x_{\rm max\,/\,min}} &= \sqrt{a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)}\,\Bigg|_{x_{\rm max\,/\,min}} \\ &= \sqrt{a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]}\,\Bigg|_{x_{\rm max\,/\,min}} \\ &= \sqrt{a\left(\frac{b^2-4ac}{4a^2}+\frac{4ax-b^2}{4a^2}\right)} = 0 \end{aligned} \tag{B12}$$

as anticipated. Note, the $c = 0$ case is special. From Equation (22) with $V = 1$ $U = 2$, $y = x$, and $c = 0$, we obtain

$$d\tau = \pm\frac{x\,dx}{\sqrt{ax^2+bx}} = \pm\frac{\sqrt{x}\,dx}{\sqrt{-|a|x+b}} \tag{B13}$$

from which we see that there is no minimum and only a maximum at $x_{\rm max} = b/|a|$.

Equation (B2) obviously does not allow an exact inverse solution (i.e., an exact solution $x(\tau)$) because of the combination of square root and inverse arcsine terms. However, a partial inverse solution is useful understanding the behavior of $x(\tau)$. This formula is

$$x = \begin{cases} \Big\{ b + \sqrt{b^2 + 4|a|c} & \text{In general.} \\ \quad \times \sin\Big[(2|a|\sqrt{|a|}/b) & \\ \qquad \times \Big(\pm\tau + \sqrt{ax^2 + bx + c}/|a| \Big) + \pi/2 \Big] \Big\} (2|a|)^{-1} & \\ \dfrac{b + \sqrt{b^2 + 4|a|c}}{2|a|} = x_{\max} & \text{For } \tau = 0 \text{ and } x = 0. \\ 0 & \text{For } \tau = \tau_{\pm\text{endpoint}}. \end{cases} \quad \text{(B14)}$$

We could investigate interpolation formulae for $x(\tau)$ starting from Equation (B14), but we leave this to future work.

An inverse formula $x(\tau)$ is easily obtained for the case of $b = 0$ (i.e., the radiation-positive-curvature universe) from Equation (B2):

$$\begin{aligned} \tau &= \pm\left(-\frac{\sqrt{c - |a|x^2}}{|a|} \right) \\ x &= \begin{cases} \sqrt{\dfrac{c - (|a|\tau)^2}{|a|}} & \text{In general and where we have suppressed the} \\ \quad = \sqrt{\dfrac{c}{|a|} - |a|\tau^2} & \quad \text{negative case solution as not physically real.} \\ \sqrt{\dfrac{c}{|a|}} & \text{For } \tau = 0 \text{ which is the maximum.} \\ 0 & \text{For } \tau = \tau_{\pm\text{endpoint}} = \pm\sqrt{c}/|a|. \end{cases} \end{aligned} \quad \text{(B15)}$$

There is no exact inverse formula $x(\tau)$ of $c = 0$ (i.e., the matter-positive-curvature universe) which recall from Equation (B2) is

$$\tau = \pm\left\{ -\frac{\sqrt{ax^2 + bx}}{|a|} + \frac{b}{2|a|\sqrt{|a|}} \left[\arcsin\left(2\frac{x}{b/|a|} - 1 \right) - \frac{\pi}{2} \right] \right\} . \quad \text{(B16)}$$

However, it is useful in further explicating this solution below in Appendix B.4 to rewrite Equation (B16) in terms of scaled variables in order to make the constants disappear. The rescalings are

$$dZ = \frac{dx}{b/|a|} \qquad \text{and} \qquad dw = \frac{d\tau}{b/(|a|\sqrt{|a|})} , \quad \text{(B17)}$$

and these lead to the rescaled matter-positive-curvature universe solution

$$\begin{aligned} w &= \pm\left\{-\sqrt{Z-Z^2}+\frac{1}{2}\left[\arcsin(2Z-1)-\frac{\pi}{2}\right]\right\} \\ &= \pm\left[-\sqrt{Z-Z^2}-\frac{1}{2}\arccos(2Z-1)\right] . \end{aligned} \tag{B18}$$

B.4. The Matter-Positive-Curvature Universe

The matter-positive-curvature universe as discussed above in Appendix B.3 is a special case of the radiation-matter-positive-curvature universe (i.e., the case with $c = 0$). However, the formalism in Appendix B.3 does not resemble the standard formalism for this case which makes use of (standard) conformal time. In fact, other than starting from Equation (22) and obtaining the standard formalism directly there is no obvious path to show that the formalism in Appendix B.3 is equivalent to the standard formalism. As we show below, there is an obvious path going in the other direction. For reference, we derive the standard formalism here.

With $c = 0$ (and recall $V = 1$, $U = 2$, and $y = x$) and the scalings of Equation (B17), Equation (22) reduces to

$$dw = \pm\frac{Z\,dZ}{\sqrt{Z-Z^2}} = \pm\frac{\sqrt{Z}\,dZ}{\sqrt{1-Z}} \tag{B19}$$

which inverses to

$$\frac{dZ}{dw} = \begin{cases} \pm\dfrac{\sqrt{1-Z}}{\sqrt{Z}} & \text{In general.} \\ 0 & \text{For } Z=1 \text{ which is the maximum of } Z(w). \\ \pm\infty & \text{For } Z=0 \text{ which are obviously the zeros of } Z(w). \end{cases} \tag{B20}$$

We now introduce conformal time specified by $d\eta = d\tau/Z$ and obtain

$$d\eta = \pm\frac{dZ}{\sqrt{Z-Z^2}} . \tag{B21}$$

From Wikipedia (Wikipedia: List of integrals of irrational functions: Integrals involving $R = \sqrt{ax^2+bx+c}$), the following solution is obtained

$$\eta - \eta_0 = -\arcsin(-2Z+1) \tag{B22}$$

which rearranges to

$$Z = \frac{1}{2}[1+\sin(\eta-\eta_0)] = \frac{1}{2}[1-\cos(\eta)] , \tag{B23}$$

where we have chosen integration constant $\eta_0 = \pi/2$ for consistency familiar versions of the solution. The solution for cosmic time w is

$$w = \int_0^\eta Z\,d\eta' = \int_0^\eta \frac{1}{2}[1 - \cos(\eta')]\,d\eta' = \frac{1}{2}[\eta - \sin(\eta)] \ . \tag{B24}$$

The $Z(\eta)$ and $w(\eta)$ solutions are for $\eta \in [0, 2\pi]$ where the endpoints are a big bang and a big crunch. The stationary points for $Z(\eta)$ occur at $\eta = 0$ and $\eta = 2\pi$ for minima $Z = 0$ and $\eta = \pi$ for a maximum $Z = 1$ (or $x = b/|a|$) as we noted above.

There is no exact solution $Z(w)$, but approximate ones can be derived for educational reasons. Note,

$$w|_{\rm 1st} = \frac{1}{2}[\eta - \sin(\eta')]\Big|_{\rm 1st} = \frac{1}{2}\left[\eta - \eta + \frac{1}{6}\eta^3\right]\Big|_{\rm 1st} = \frac{1}{12}\eta^3 \tag{B25}$$

and

$$Z\Big|_{\rm 1st} = \frac{1}{2}[1 - \cos(\eta)]\Big|_{\rm 1st} = \frac{1}{2}\left[1 - 1 + \frac{1}{2}\eta^2\right]\Big|_{\rm 1st} = \frac{1}{4}\eta^2 = \frac{1}{4}(12w)^{2/3} = \left(\frac{3}{2}\right)^{2/3} w^{2/3} \ . \tag{B26}$$

The last expression suggests the following interpolation formulae:

$$Z_{\rm approx} = \begin{cases} \sin^{2/3}(w) & \text{Always an underestimate (except for being exactly correct by design at } w = 0, \pi, 2\pi\text{), maximum relative error} \sim 0.24\text{, root-mean-square relative error} \sim 0.15. \\ \left\{ \left(\frac{3}{2}\right)^{2/3} \left[\frac{\cos(2w) + 1}{2}\right]^1 + \sin^3(w) \right\} \sin^{2/3}(w) & \text{Overestimate for } 0 < w \lesssim 0.17\pi \text{ and } \pi > w \gtrsim 0.83\pi\text{, otherwise an underestimate (except for being exactly correct by design at } w = 0, \pi, 2\pi\text{), maximum relative error} \sim 0.061\text{, root-mean-square relative error} \sim 0.036. \end{cases} \tag{B27}$$

Note, the basic form for second approximation seems natural. However, the powers for the two coefficient terms (i.e., 1 and 3) were found by a rough fitting procedure that sought natural-good-fit values, not absolute-best-fit values. Better fits to $Z(w)$ can be found, but they probably have no educational value.

To summarize the exact solutions for the matter-positive-curvature universe, we present the forms given above (in terms of η and w) and the symmetric-about-the-origin forms (in terms of $\eta' = \eta - \pi$ and $w' = w - \pi/2$):

$$\begin{cases} Z = \frac{1}{2}[1-\cos(\eta)] & \text{For } \eta \in [0, 2\pi],\ w \in [0, \pi]. \\ w = \frac{1}{2}[\eta - \sin(\eta)] & \\ Z = \frac{1}{2}[1+\cos(\eta')] & \text{For } \eta' \in [-\pi, \pi],\ w' \in [-\pi/2, \pi/2]. \\ w' = \frac{1}{2}[\eta' + \sin(\eta')] \ . & \end{cases} \tag{B28}$$

To show the consistency of the standard formalism with solution $w(Z)$ given by Equation (B18), we note that

$$\tilde{\eta} = \mp \arccos(2Z-1) \tag{B29}$$

(where the choice of cases anticipates what is needed for consistency) implying

$$\begin{aligned} \tilde{w} &= \frac{1}{2}[\eta' + \sin(\eta')] = \frac{1}{2}\{\mp \arccos(2Z-1) + \sin[\mp \arccos(2Z-1)]\} \\ &= \mp\frac{1}{2}\left[\arccos(2Z-1) + \sqrt{1-(2Z-1)^2}\right] \\ &= \pm\left[-\sqrt{Z-Z^2} - \frac{1}{2}\arccos(2Z-1)\right] \end{aligned} \tag{B30}$$

which is indeed Equation (B18) recovered, and so shows consistency.

C. Elementary One Density Component Exact Solutions of the Friedmann Equation of Significant Interest

In this appendix, we derive the elementary one density component exact solutions of the Friedmann equation of significant interest and give a table of these exact solutions. The solutions are of significant interest because they have historically been considered possible model universes, may apply in alternate pocket universes of a multiverse (if that exists), or apply to the early universe, and, of course, are just educationally interesting in understanding the Friedmann equation including for this paper.

The primary scaled Friedmann equation for one density component governed by general power p is

$$\frac{\dot{x}}{x} = \pm\sqrt{\Omega_{p,0}x^{-p}} \ , \tag{C1}$$

where 0 indicates the fiducial cosmic scale time t_0, the primary scaled time coordinate is $\tau = H_0 t$ with H_0 being the Hubble constant, $\dot{x} = dx/d\tau$, and x is the cosmic scale factor. (Recall we do not use a for the cosmic scale factor in this paper unless so noted.) If there were, in fact, just one density component, $\Omega_{p,0}$ would be 1. However, we leave $\Omega_{p,0}$ general to allow for the case that there are other nonzero density components that are negligible until later times than the one we are considering. We will only consider cases of p of significant interest. Therefore, we will only consider the growing solutions (i.e., solutions to the upper case of Equation (C1)) and solutions with with simple $\tau = 0$ behavior. For $p = 0$ (which leads to an exponential solution), we set $x(\tau = 0) = x_0$. For $p > 0$ (which leads to the power-law solutions), we set $x(\tau = 0) = 0$.

Solving Equation (C1) for $p = 0$, gives

$$1)\quad d\tau = \frac{x^{-1}\,dx}{\sqrt{\Omega_{p,0}}} \qquad 2)\quad \tau = \frac{1}{\sqrt{\Omega_{p,0}}}\ln\left(\frac{x}{x_0}\right) \qquad 3)\quad x = x_0 \exp\left(\sqrt{\Omega_{p,0}}\,\tau\right) \tag{C2}$$

and for $p \neq 0$,

$$\begin{aligned} &1)\quad d\tau = \frac{x^{p/2-1}\,dx}{\sqrt{\Omega_{p,0}}} \qquad 2)\quad \tau = \frac{1}{\sqrt{\Omega_{p,0}}}\left(\frac{2}{p}\right)x^{p/2} \\ &3)\quad x = \left[\sqrt{\Omega_{p,0}}\left(\frac{p}{2}\right)\tau\right]^{2/p} = \left[\sqrt{\Omega_{p,0}}\left(\frac{p}{2}\right)\tau\right]^{\gamma}\ , \end{aligned} \tag{C3}$$

where we have defined $\gamma = 2/p$. The elementary one density component solutions of significant interest are:

$$x = \begin{cases} x_0 \exp\left(\sqrt{\Omega_{p,0}}\,\tau\right) & \text{for } p=0\text{: a pure cosmological constant universe} \\ & \quad\text{or de Sitter universe (Wikipedia: de Sitter universe);} \\ \left[\sqrt{\Omega_{p,0}}\left(\frac{p}{2}\right)\tau\right]^{2/p} = \left[\sqrt{\Omega_{p,0}}\left(\frac{p}{2}\right)\tau\right]^{\gamma} & \text{in general for } p>0 \\ & \quad\text{with } \gamma = 2/p; \\ \left[\sqrt{\Omega_{1,0}}\left(\frac{1}{2}\right)\tau\right]^{2} & \text{for } p=1 \text{ and } \gamma=2\text{:} \\ & \quad\text{some quintessence universes;} \\ \left(\sqrt{\Omega_{2,0}}\,\tau\right) & \text{for } p=2 \text{ and } \gamma=1\text{:} \\ & \quad\text{a pure negative curvature universe} \\ & \quad\text{(for which } \Omega_{2,0}>0\text{),} \\ & \quad\text{some cosmic-string universes,} \\ & \quad\text{and the } R_h = ct \text{ universe;} \\ \left[\sqrt{\Omega_{3,0}}\left(\frac{3}{2}\right)\tau\right]^{2/3} & \text{for } p=3 \text{ and } \gamma=2/3\text{; the matter universe or} \\ & \quad\text{Einstein-de Sitter universe} \\ & \quad\text{(Wikipedia: Einstein-de Sitter universe);} \\ \left[\sqrt{\Omega_{4,0}}\,(2)\,\tau\right]^{1/2} & \text{for } p=4 \text{ and } \gamma=1/2\text{:} \\ & \quad\text{a radiation universe;} \\ \left[\sqrt{\Omega_{5,0}}\left(\frac{5}{2}\right)\tau\right]^{2/5} & \text{for } p=5 \text{ and } \gamma=2/5\text{: an artificial universe} \\ & \quad\text{where all mass-energy is classical kinetic energy} \end{cases} \tag{C4}$$

(e.g., Steiner 2008, p. 6–7; e.g., Melia 2014 for the $R_h = ct$ universe).

The density evolutions of the one-component solutions are

$$\Omega_p = \begin{cases} \Omega_{p,0} & \text{for } p=0; \\ \Omega_{p,0}x^{-p} = \Omega_{p,0}\left[\sqrt{\Omega_{p,0}}\left(\frac{p}{2}\right)\tau\right]^{-2} = \left(\frac{2}{p}\right)^{2}\tau^{-2} & \text{for } p>0. \end{cases} \tag{C5}$$

Note that for $P = 0$, the density is constant and for $P > 0$, it scales as $1/\tau^2$ in all cases which is a remarkable fact.

Recall from § 2, that it is usual to parameterize the perfect fluid of cosmology pressure P_p by the equation of state

$$P_p = w_p \rho c^2 \ , \tag{C6}$$

where the constant w_p is the equation of state parameter for power p. Substituting Equation (C6) into Equation (3) and solving for ρ gives

$$\rho = \rho_0 \left(\frac{x}{x_0}\right)^{-3(1+w)} \tag{C7}$$

and the following results:

$$1)\quad p = \frac{2}{\gamma} = 3(1+w_p) \qquad 2)\quad \gamma = \frac{2}{p} = \frac{2}{3(1+w_p)} \qquad 3)\quad w_p = \frac{p}{3} - 1 = \frac{2}{3\gamma} - 1\ . \tag{C8}$$

For reference, we present the elementary one density component solutions of significant interest in terms of unscaled cosmic time t and some of their details in Table 2 with $\Omega_{p,0} = 1$. Note, for $x_0 = x(t_0) = 1$, we must have cosmic present

$$t_0 = \begin{cases} 0 & \text{for } p = 0 \text{ since we define } x(t_0 = 0) = 1 \\ & \quad \text{in this case;} \\ \dfrac{2/p}{H_0} = \dfrac{\gamma}{H_0} = \gamma\left(\dfrac{13.968\,\text{Gyr}}{h_{70}}\right) & \text{for } p > 0 \end{cases} \tag{C9}$$

where 13.968 Gyr is the Hubble time for Hubble constant $H_0 = 70\,(\text{km/s})/\text{Mpc}$ and $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$. Note also, the deceleration parameter for elementary one density component solutions is

$$q_0 = -\frac{x_0 \ddot{x}_0}{\dot{x}_0^2} = -\frac{(\gamma-1)\gamma}{\gamma^2} = \frac{1}{\gamma} - 1 = \frac{p}{2} - 1 = \frac{1}{2}(1+3w) \tag{C10}$$

(e.g., Liddle 2015, p. 53).

Table 2. Elementary One Density Component Solutions of the Friedmann Equation of Significant Interest

w_p	$p = \frac{2}{\gamma}$	$\gamma = \frac{2}{p}$	$x(t)$	t_0	$q_0 = \frac{1}{\gamma} - 1$	ρ
$w_p = -1$	$p = 0$	$\gamma = \infty$	$x_0 e^{H_0 t}$	0	-1	ρ_0
$w_p \neq -1$	$\left\{ \begin{array}{c} p = \\ 3(1+w_p) \end{array} \right\}$	$\frac{2}{3(1+w_p)}$	$x_0 \left(\frac{t}{t_0}\right)^{\gamma}$	$\left\{ \begin{array}{c} \frac{\gamma}{H_0} = \gamma \times \\ \left(\frac{13.968\,\mathrm{Gyr}}{h_{70}}\right) \end{array} \right\}$	$\left\{ \begin{array}{c} \frac{P}{2} - 1 = \\ \frac{1}{2}(1+3w_p) \end{array} \right\}$	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w_p = -\frac{2}{3}$	$p = 1$	$\gamma = 2$	$x_0 \left(\frac{t}{t_0}\right)^2$	$\frac{2}{H_0}$	$-\frac{1}{2}$	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w_p = -\frac{1}{3}$	$p = 2$	$\gamma = 1$	$x_0 \left(\frac{t}{t_0}\right)$	$\frac{1}{H_0}$	0	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w_p = 0$	$p = 3$	$\gamma = \frac{2}{3}$	$x_0 \left(\frac{t}{t_0}\right)^{2/3}$	$\frac{2}{3}\frac{1}{H_0}$	$\frac{1}{2}$	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w_p = \frac{1}{3}$	$p = 4$	$\gamma = \frac{1}{2}$	$x_0 \left(\frac{t}{t_0}\right)^{1/2}$	$\frac{1}{2}\frac{1}{H_0}$	1	$\rho_0 \left(\frac{t_0}{t}\right)^2$
$w_p = \frac{2}{3}$	$p = 5$	$\gamma = \frac{2}{5}$	$x_0 \left(\frac{t}{t_0}\right)^{2/5}$	$\frac{2}{5}\frac{1}{H_0}$	$\frac{3}{2}$	$\rho_0 \left(\frac{t_0}{t}\right)^2$

D. A Formalism for Approximate Analytic Solutions to the Friedmann Equation by Perturbation from Exact Solutions

In this Appendix, we present a formalism for approximate analytic solutions to the Friedmann equation by perturbation from exact (analytic) solutions $x_{\rm ex}(\tau)$. The perturbation formalism is presented for educational reasons: to show what can be done. The perturbation formalism is probably of no practical use: i.e., it cannot replace high-accuracy numerical solutions when no exact solution is available.

To scope our presentation, we make several restrictions. We consider only cases where the cosmic scale factor $x(\tau)$ starts from a Big Bang singularity at cosmic time zero. The perturbations do not include a constant density component: i.e., a Λ density parameter that could be either of true cosmological constant or a constant dark energy. Any included constant density component must appear in the exact solutions $x_{\rm ex}(\tau)$. General exp-like, sinh-like, cosh-like, and sin-like exact solutions $x_{\rm ex}(\tau)$ for the V model (which all have a constant density component) are presented in § 4. These general exact solutions may be all that exist for cases where the density parameters can be written as (inverse) power laws of the cosmic scale factor x. In any case, we restrict the perturbation formalism to these cases if there is a constant density component.

We posit that an exact Friedmann equation cosmic scale factor $x(\tau)$ (exact, but not an exact solution that can be written as formula in general) can be written as the sum of an exact (analytic) solution $x_{\rm ex}(\tau)$ plus a perturbation solution $\delta x(\tau)$:

$$x = x_{\rm ex} + \delta x(\tau) \quad \text{with Friedmann equation} \quad \frac{dx}{d\tau} = x\sqrt{\sum_k a_k x^{-p_k} + \sum_\ell a_\ell x^{-p_\ell}} \ , \qquad \text{(D1)}$$

where the density parameter constants a_k and the corresponding powers p_k are for the exact solution $x_{\rm ex}(\tau)$ and the density parameter constants a_ℓ and the corresponding powers p_ℓ are needed for $x(\tau)$: i.e., they are the perturbations from the Friedmann equation for $x_{\rm ex}(\tau)$. We are assuming the scaled time $d\tau = H_0 dt$ where H_0 is a Hubble constant, and thus for $x(\tau) = 1$, the density parameter constants sum to 1: i.e.,

$$\sum_k a_k + \sum_\ell a_\ell = 1 \ . \qquad \text{(D2)}$$

This normalization means that varying the perturbation density parameters a_ℓ requires adjusting the exact solution density parameters a_k to maintain the summation to 1.

We now expand in small $\delta x(\tau)$ and $\sum_\ell a_\ell x^{-p_\ell}$ and work towards a qualitative approximate 1st order differential equation for $\delta x(\tau)$:

$$
\begin{aligned}
\frac{dx}{d\tau} &= x\sqrt{\sum_k a_k x^{-p_k} + \sum_\ell a_\ell x^{-p_\ell}} \\
\frac{dx_{\rm ex}}{d\tau} + \frac{d(\delta x)}{d\tau} &= (x_{\rm ex}+\delta x)\sqrt{\sum_k \frac{a_k x_{\rm ex}^{-p_k}}{(1+\delta x/x_{\rm ex})^{p_k}} + \sum_\ell \frac{a_\ell x_{\rm ex}^{-p_\ell}}{(1+\delta x/x_{\rm ex})^{p_\ell}}} \\
\left[\frac{dx_{\rm ex}}{d\tau} + \frac{d(\delta x)}{d\tau}\right]\bigg|_{\rm 1st} &= (x_{\rm ex}+\delta x)\sqrt{\sum_k a_k x_{\rm ex}^{-p_k} + \sum_\ell a_\ell x_{\rm ex}^{-p_\ell} - \frac{\delta x}{x_{\rm ex}}\left(\sum_k a_k x_{\rm ex}^{-p_k} p_k + \sum_\ell a_\ell x_{\rm ex}^{-p_\ell} p_\ell\right)} \\
&= (x_{\rm ex}+\delta x)\sqrt{\sum_k a_k x_{\rm ex}^{-p_k} + \sum_\ell a_\ell x_{\rm ex}^{-p_\ell} - \frac{\delta x}{x_{\rm ex}}\left(\sum_k a_k x_{\rm ex}^{-p_k} p_k\right)} \\
&= (x_{\rm ex}+\delta x)\sqrt{\sum_k a_k x_{\rm ex}^{-p_k}}\left(1 + \frac{1}{2}\frac{\sum_\ell a_\ell x_{\rm ex}^{-p_\ell}}{\sum_k a_k x_{\rm ex}^{-p_k}} - \frac{1}{2}\frac{\sum_k a_k x_{\rm ex}^{-p_k} p_k}{\sum_k a_k x_{\rm ex}^{-p_k}}\frac{\delta x}{x_{\rm ex}}\right) \\
&= x_{\rm ex}\sqrt{\sum_k a_k x_{\rm ex}^{-p_k}}\left(1+\frac{\delta x}{x_{\rm ex}}\right)\left(1 + \frac{1}{2}\frac{\sum_\ell a_\ell x_{\rm ex}^{-p_\ell}}{\sum_k a_k x_{\rm ex}^{-p_k}} - \frac{1}{2}\frac{\sum_k a_k x_{\rm ex}^{-p_k} p_k}{\sum_k a_k x_{\rm ex}^{-p_k}}\frac{\delta x}{x_{\rm ex}}\right) \\
\frac{d(\delta x)}{d\tau}\bigg|_{\rm 1st} &= \frac{dx_{\rm ex}}{d\tau}\left[\frac{1}{2}\frac{\sum_\ell a_\ell x_{\rm ex}^{-p_\ell}}{\sum_k a_k x_{\rm ex}^{-p_k}} + \left(1 - \frac{\sum_k a_k x_{\rm ex}^{-p_k} p_k/2}{\sum_k a_k x_{\rm ex}^{-p_k}}\right)\frac{\delta x}{x_{\rm ex}}\right] \\
\frac{d(\delta x)}{d\tau}\bigg|_{\rm 1st,ap} &= \frac{dx_{\rm ex}}{d\tau}\left(\frac{1}{2}\frac{\sum_\ell a_\ell x_{\rm ex}^{-p_\ell}}{\sum_k a_k x_{\rm ex}^{-p_k}}\right) = \frac{dx_{\rm ex}}{d\tau} f(x_{\rm ex}) \ ,
\end{aligned}
\tag{D3}
$$

where for the last expression, we have dropped a 1st order term that might be small in many cases and where we define the modulation function

$$
f(x_{\rm ex}) = \frac{1}{2}\frac{\sum_\ell a_\ell x_{\rm ex}^{-p_\ell}}{\sum_k a_k x_{\rm ex}^{-p_k}} \ . \tag{D4}
$$

Because we have dropped the aforesaid 1st order term, the differential equation for we have found for $d(\delta x)/d\tau$ can only be judged a qualitative approximate 1st order differential equation.

Unfortunately, despite the significant approximation of dropping the twice aforesaid 1st order term, the differential equation Equation (D3) is still intractable to solve. However, we expect

$$
\delta x_{\rm 1st,ap} = x_{\rm ex} f(x_{\rm ex}) \tag{D5}
$$

will be a qualitative approximate 1st order solution to Equation (D3) provided the modulation function $f(x_{\rm ex}) \lesssim 1$. Equation (D5) will be best if all the p_ℓ satisfy $\min(p_k) < p_\ell <$

max(p_k) which implies that $f(x_{\rm ex})$ vanishes asymptotically as $x_{\rm ex} \to 0$ and $x_{\rm ex} \to \infty$. In between the two limiting cases, there will be one or more stationary points for $f(x_{\rm ex})$. In fact, there should be a global maximum (or mininum for the dominant density parameter constant $a_\ell < 0$) where $\delta x_{\rm 1st,ap} = x_{\rm ex} f(x_{\rm ex})$ approaches being exactly true and this is where the perturbation is strongest.

Can we do better than Equation (D5) with a perturbation solution that explicates the perturbation behavior for all sizes of the perturbation? We believe a good suggestion is the approximate interpolation perturbation solution

$$\delta x_{\rm interp} = \begin{cases} [x_{\rm ex} + \sum_\ell x_{p_\ell}(\tau)] \tanh\left[f(x_{\rm ex})\right] \\ \quad = \left\{ x_{\rm ex} + \sum_\ell \left[\sqrt{a_\ell}\left(\frac{p_\ell}{2}\right)\tau\right]^{2/p_\ell} \right\} \tanh\left[f(x_{\rm ex})\right] \quad \text{in general.} \\ 0 & \text{in the limit where all } a_\ell = 0. \\ x_{\rm ex} f(x_{\rm ex}) & \text{in the limit where } \sum_\ell x_{p_\ell}(\tau) << x_{\rm ex} \text{ and } f(x_{\rm ex}) << 1, \\ & \quad \text{and note, these conditions are effectively.} \\ & \quad \text{the same condition.} \\ \sum_\ell x_{p_\ell}(\tau) & \text{in the limit where } \sum_\ell x_{p_\ell}(\tau) >> x_{\rm ex} \text{ and } f(x_{\rm ex}) >> 1, \\ & \quad \text{and note, these conditions are effectively} \\ & \quad \text{the same condition.} \\ & \quad \text{Note, this is very crude approximate solution,} \\ & \quad \text{unless one of the density parameter constants } a_\ell \text{ is} \\ & \quad \text{overwhelmingly dominant.} \\ & \quad \text{Also note, in this case } x_{\rm ex} \text{ is the perturbation.} \\ x_{p_\ell}(\tau) & \text{exact in the limit that } x_{\rm ex} = 0 \text{ everywhere and there} \\ & \quad \text{is a single nonzero density parameter constant } a_\ell. \end{cases} \tag{D6}$$

Note, the $x_{p_\ell}(\tau)$ are the elementary one density component solutions for $p_\ell > 0$ given by Equation (C4) in Appendix C. Note also, the hyperbolic tangent function tanh is introduced just as a plausible and not optimized way to smoothly bound any growth of the modulation function $f(x_{\rm ex})$ to 1.

The approximate interpolation (full) solution and its derivative are, respectively,

$$x_{\rm interp} = x_{\rm ex} + \delta x_{\rm interp} = x_{\rm ex} + \left[x_{\rm ex} + \sum_\ell x_{p_\ell}(\tau)\right] \tanh\left[f(x_{\rm ex})\right]$$

and

$$\frac{dx_{\rm interp}}{d\tau} = \frac{dx_{\rm ex}}{d\tau} + \frac{d(\delta x_{\rm interp})}{d\tau}$$

$$= \frac{dx_{\rm ex}}{d\tau} + \left\{ \frac{dx_{\rm ex}}{d\tau} + \sum_\ell \left(\frac{2}{p_\ell}\right) \left[\frac{x_{p_\ell}(\tau)}{\tau}\right] \right\} \tanh\left[f(x_{\rm ex})\right] + \left[x_{\rm ex} + \sum_\ell x_{p_\ell}(\tau)\right] \left\{\frac{1}{\cosh^2[f(x_{\rm ex})]}\right\} \frac{df}{d\tau} \ , \tag{D7}$$

where

$$\frac{df}{d\tau} = f(x_{\rm ex}) \sqrt{\sum_k a_k x_{\rm ex}^{p_k}} \left(\frac{\sum_k a_k x_{\rm ex}^{p_k} p_k}{\sum_k a_k x_{\rm ex}^{p_k}} - \frac{\sum_\ell a_\ell x_{\rm ex}^{p_\ell} p_\ell}{\sum_\ell a_k x_{\rm ex}^{p_\ell}} \right) \ . \tag{D8}$$

To test the approximate interpolation solution formula Equation (D7), one could see if it can reproduce a known exact solution $x(\tau)$ treating one of the density parameter constants of the exact solution as a perturbation. In cases, where no exact solution exists, one can use Equation (9) to obtain a high accuracy numerical solution $\tau(x_i)$ from a finely spaced grid of x_i points by, e.g., the midpoint method (e.g., Wikipedia: Midpoint method) or Runge-Kutta methods (e.g., Wikipedia: Runge-Kutta methods). The high accuracy solution can be fed into to Equation (D7) to obtain values $x_{\rm interp}[\tau(x_i)]$ which can then be compared to original grid of x_i points: e.g., by computing the relative error $\{x_{\rm interp}[\tau(x_i] - x_i\}/x_i$. An alternative that avoids the need for a high accuracy solution is to find a crude solution set of times τ_i and increments $\Delta\tau_i$ from a crude grid of x_i points by the midpoint method. Then solve for $x_{\rm interp,i}$ from $dx_{\rm interp}/d\tau$ and the increments $\Delta\tau_i$ by the midpoint method. Both numerical solutions τ_i and $x_{\rm interp,i}$ will very approximate, but their level of approximation is the same, and so the closer the agreement of $x_{\rm interp}[\tau(x_i)]$ and x_i, the better the approximate interpolation formula for the case being studied.

One test case of interest is the best fitting $\Lambda\Gamma$ solution $x(\tau)$ (i.e., Γ_1 discussed in § 6 and given by the exact Equation (37) with $V = 2/3$) treating the Γ density parameter constant b as a perturbation. A second test case of interest is the radiation-matter-Λ universe which has no exact solution and is the Λ-CDM model extended to include radiation era (cosmic time from after inflation $\sim 10^{-35}$ s to ~ 50 kyr (e.g., Hergt & Scott 2024, p. 6)). One can use the radiation density parameter constant as the perturbation to the exact solution matter-Λ universe (see the exact solution Equation (37) with $V = 2/3$ and $b = 0$ which is just the Λ-CDM model, in fact). Or one can treat the matter density parameter constant as the perturbation to the radiation-Λ universe (see the exact solution Equation (37) with $V = 1/2$ and $b = 0$). A third test case of interest is one where all five of the density parameter of general interest (i.e., those with powers $p = 0, 1, 2, 3, 4$). In this case, one would treat the constants of the density parameter with powers $p = 1, 2, 3$ as perturbations to the radiation-Λ universe (see the exact solution Equation (37) with $V = 1/2$ and $b = 0$). We leave tests of the approximate interpolation solution formula Equation (D7) to future work.

E. Exact Solutions for Comoving Coordinate $\chi(z)$

The Friedmann equation exact solutions for $\tau(x)$ can in some cases be repurposed for solving for the comoving coordinate $\chi(z)$ that follows from the Robertson-Walker metric for a light signal reaching the observer at cosmological redshift $z = 0$ from cosmological redshift z in a universe model obeying the Friedmann equation as we show below in this appendix

The formula for $\chi(z)$ is

$$\chi(z) = \int_t^{t_0} \frac{c\,dt}{a_{\rm Ga} x} = \frac{c/H_0}{a_{\rm Ga}} \int_\tau^{\tau_0} \frac{d\tau}{x} = \left(\frac{c/H_0}{a_{\rm Ga}}\right) [\eta(z=0) - \eta(z)] \ , \tag{E1}$$

where here the fiducial time t_0 is set to cosmic present, H_0 is the Hubble constant, $d\tau = H_0\,dt$ as throughout this paper, $a_{\rm Ga}$ is the absolute value of the Gaussian curvature radius for curved space and is an arbitrary length otherwise (since it cancels from all cosmological distance measure formulae in this case as we show below), x is cosmic scale factor set to 1 for cosmic present, and η is the (conventional) conformal time (defined $d\eta = d\tau/x$ here written as a function of z by means shown below) (e.g., Coles & Lucchin 2002, p. 11–13). The factor $a_{\rm Ga}$ can be derived from the curvature density parameter for cosmic present:

$$\Omega_{k,0} = -\frac{k_{\rm unscaled}}{H_0^2 x_0^2} = -\frac{kc^2}{H_0^2 a_{\rm Ga}^2} \ , \tag{E2}$$

where $k_{\rm unscaled}$ is the unscaled dimensioned curvature, x_0 is the dimensionless cosmic scale factor for cosmic present, and k is the (scaled dimensionless) curvature which is scaled to have only values 1 (for positive curvature: i.e., for a hyperspherical space), 0 (for zero curvature: i.e., for a Euclidean or flat space), and -1 (for negative curvature: i.e., for a hyperbolic space), and 1 (for positive curvature: i.e., for a hyperspherical space) (e.g., Liddle 2015, p. 56). The formula for $a_{\rm Ga}$ for $k \neq 0$ is

$$a_{\rm Ga} = \frac{c/H_0}{\sqrt{-k\Omega_{k,0}}} = \frac{c/H_0}{\sqrt{|\Omega_{k,0}|}} = \frac{(4.2827\ldots\ {\rm Gpc})h_{70}}{\sqrt{|\Omega_{k,0}|}} = \frac{(13.968\ldots\ {\rm Gly})h_{70}}{\sqrt{|\Omega_{k,0}|}} \ , \tag{E3}$$

where $h_{70} = H_0/[70\,({\rm km/s})/{\rm Mpc}]$ is the reduced Hubble contant in terms of the fiducial value 70 (km/s)/Mpc and as aforesaid $a_{\rm Ga}$ is arbitrary length if $k = 0$ (e.g., Coles & Lucchin 2002, p. 11).

The comoving coordinate $chi(z)$ is used for the theoretical formulae for proper distance $D_{\rm P}$, luminosity distance $D_{\rm L}$, and angular-diameter distance $D_{\rm A}$:

$$\begin{aligned} D_{\rm P} &= a_{\rm Ga}\chi(z) \\ D_{\rm L} &= a_{\rm Ga} r[\chi(z)](1+z) \end{aligned}$$

$$D_{\rm A} = a_{\rm Ga} r[\chi(z)](1+z)^{-1} \ , \tag{E4}$$

where

$$r = \begin{cases} \sin(\chi) & \text{for } k=1 \text{ (positive curvature space: i.e., hyperspherical space);} \\ \chi & \text{for } k=0 \text{ (Euclidean or flat space);} \\ \sinh(\chi) & \text{for } k=-1 \text{ (negative curvature space: i.e., hyperbolic space)} \end{cases} \tag{E5}$$

(e.g., Coles & Lucchin 2002, p. 11,13,19). Equation (E4) gives the theoretical formulae to be fitted to observed proper distance (only an observable to 1st order in small z), luminosity distance, and angular-diameter distance. These distance are all cosmological distance measures and are important in determining the observational Friedmann solution of the observable universe. Given Equations (E1) and (E5), we note that $a_{\rm Ga}$ cancels out of the cosmological distance measure formulae as noted above.

The repurposing of the $\tau(x)$ solutions follows from the formulae for η

$$\eta(\{p_k\},\{a_k\}) = \int \frac{d\tau}{x} = \int \frac{dx}{x^2\sqrt{\sum_k a_k x^{-p_k}}} = \int \frac{dx}{x\sqrt{\sum_p a_p x^{-(p_k-2)}}} = \tau(\{p_k-2\},\{a_k\}) \ , \tag{E6}$$

where we have made use of the Friedmann equation form Equation (9), $\{p_k\}$ means set of power parameters, $\{a_k\}$ means set of density parameter constants, $\tau(\{p_k-2\},\{a_k\})$ is the solution of the Friedmann equation for set of power parameters $\{p_k-2\}$ and density parameter constants $\{a_k\}$, and $\eta(\{p_k\},\{a_k\})$ is the solution for conformal time for set of power parameters $\{p_k\}$ and density parameter constants $\{a_k\}$. The upshot is if there is an exact solution for $\tau(\{p_k-2\},\{a_k\})$, there is an exact solution for $\eta(\{p_k\},\{a_k\})$.

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