Chapter 12 – Rotation of a Rigid Body

There is a lot of new material in this chapter about rotational dynamics. Although many of the problems can be understood and solved without using vectors to describe rotational motion, problems involving spinning tops and gyroscopes cannot. Therefore the summary starts with the definition of the vector cross product.

Earlier, the vector dot product was introduced, $\mathbf{A} \cdot \mathbf{B}$. It was a scalar and equal to the projection of vector $\mathbf{A}$ on the direction of vector $\mathbf{B}$ or analogously, the projection of vector $\mathbf{B}$ onto $\mathbf{A}$. In both cases the result is $AB \cos \theta$, where $\theta$ is the angle between the two vectors.

Vector Cross Product

The vector cross product, $\mathbf{A} \times \mathbf{B}$, produces a new vector with a direction perpendicular to the plane formed by the two vectors $\mathbf{A}$ and $\mathbf{B}$. The magnitude of the vector is $AB \sin \theta$, which is just the projection of vector $\mathbf{A}$ onto the direction perpendicular to $\mathbf{B}$ or analogously, the projection of $\mathbf{B}$ onto the direction perpendicular to $\mathbf{A}$.

Look at the two vectors below which lie in the plane of the paper. The vector $\mathbf{A} \times \mathbf{B}$ points in a direction perpendicular to the paper but that means it can either point up out of the sheet or down into the sheet.

\[
\begin{align*}
\mathbf{A} & \\
\mathbf{B} & \\
\end{align*}
\]

There are different rules for determining the correct direction. One is to imagine turning vector $\mathbf{A}$ toward vector $\mathbf{B}$, the new vector $\mathbf{A} \times \mathbf{B}$ points in the direction a right-hand screw would move if turned in the same direction. A right-hand screw would move out of the page so $\mathbf{A} \times \mathbf{B}$ points up out of the sheet. Note that the turning of $\mathbf{A}$ towards $\mathbf{B}$ begins with orienting the vectors tail-to-tail and then turning $\mathbf{A}$ through the smaller of the two possible angles towards $\mathbf{B}$.

Another rule for orienting $\mathbf{A} \times \mathbf{B}$ is called the right-hand rule and involves using your right hand! Notice that if you point your right hand with the thumb pointing up, your fingers curl towards the left. On the other hand, ha-ha, if you point your right hand with the thumb pointing down, your fingers curl to the right. This assumes you are not double jointed!
In the above example, take your right hand and point it in the direction of vector A in a manner that allows you to curl your fingers toward vector B. If you do that correctly, your thumb will be pointing up, the direction of the new vector \( \mathbf{A} \times \mathbf{B} \).

Notice that the vector formed by crossing \( \mathbf{B} \) into \( \mathbf{A} \), \( \mathbf{B} \times \mathbf{A} \), points in the opposite direction of \( \mathbf{A} \times \mathbf{B} \) but has the same magnitude, \( AB \sin \theta \).

For the unit vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \),

\[
\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j},
\]

and since \( \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \),

\[
\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}
\]

Center-of-Mass

Imagine a rigid object like the one shown below sitting on a frictionless table. The arrows represent two equal and opposite forces acting on the object. The forces act for a short time.

Since the forces are equal and opposite, the objects total momentum remains unchanged, that is it equals zero. But the forces will cause the object to rotate counter clockwise, CCW, about some point. That point is called the center of mass. Imagine drawing a grid on the object that divides into N pieces. The mass of the \( k \)th piece is \( m_k \), where “\( k \)” goes from 1 to N. When the object rotates about the CM, the sum of the momenta of all the pieces has to equal zero. We will use that fact to find the CM.
The diagram above shows the object with a superimposed coordinate system. Each of the \( k \) pieces of the object can be located by a vector, \( \mathbf{r}_k = x_k \mathbf{i} + y_k \mathbf{j} \), which goes from the origin to the \( k \)th box of the grid. The vector from the origin to the point near the left edge of the object is one such vector. The other vector goes from the origin to the center of mass, \( \mathbf{R}_{CM} = X_{CM} \mathbf{i} + Y_{CM} \mathbf{j} \), of the object. (We don’t know the actual values of \( X_{CM} \) and \( Y_{CM} \) but that does not stop us from drawing the picture.) The little vector, \( \mathbf{R}_k \), goes from the CM to the \( k \)th piece of the object and has components \( X_k = x_k - X_{CM} \) and \( Y_k = y_k - Y_{CM} \).

After the forces act, the object continues to rotate with a constant angular velocity, \( \omega \). The speed of the \( k \)th piece of the object is \( v_k = \omega R_k \), where
\[
R_k = \sqrt{(x_k - X_{CM})^2 + (y_k - Y_{CM})^2}
\]
is the distance from the CM to the \( k \)th piece of the object. The velocity vector of the \( k \)th piece is perpendicular to \( \mathbf{R}_k \). Draw a careful picture to convince yourself that the \( x \) and \( y \)-components of the velocity of the \( k \)th piece are given by,
\[
v_{k,x} = -v_k (y_k - Y_{CM})/R_k = -\omega (y_k - Y_{CM})
\]
\[
v_{k,y} = +v_k (x_k - X_{CM})/R_k = +\omega (x_k - X_{CM})
\]
where \( v_k \) was replaced by \( \omega R_k \) in both expressions allowing the \( R_k \)’s to cancel. The minus sign in \( v_{k,x} \) is necessary to guarantee that the \( \mathbf{v}_k \) points in the correct direction consistent with CCW rotation. Remember the total momentum has to equal zero,
\[
\mathbf{P} = \mathbf{P}_x + \mathbf{P}_y \mathbf{j} = 0
\]
where,
\[
P_x = \sum_{k=1}^{N} m_k v_{k,x} = -\omega \sum_{k=1}^{N} m_k (y_k - Y_{CM}) = 0
\]
\[
P_x = \sum_{k=1}^{N} m_k v_{k,y} = \omega \sum_{k=1}^{N} m_k (x_k - X_{CM}) = 0.
\]
The angular velocity is constant and can be brought outside the sum. Leaving
\[
0 = \sum_{k=1}^{N} m_k (y_k - Y_{CM}) = \sum_{k=1}^{N} m_k (x_k - X_{CM}).
\]
\( X_{CM} \) and \( Y_{CM} \) are constants and \( \sum_{k=1}^{N} m_k = M \), the total mass of the object. Solving for \( X_{CM} \) and \( Y_{CM} \) gives,
\[
X_{CM} = \frac{1}{M} \sum_{k=1}^{N} m_k x_k \quad \text{and} \quad Y_{CM} = \frac{1}{M} \sum_{k=1}^{N} m_k y_k.
\]
the coordinates of the center of mass of the object. The sums become integrals as \( N \) gets larger and larger making the pieces smaller and smaller,
Rotational Kinetic Energy

The rotating object in the CM example has a rotational kinetic energy because each of the N pieces is rotating about the center of mass. The total rotational kinetic energy is the sum of N individual rotational kinetic energies, \((KE)_k\),

\[
(KE)_k = \frac{1}{2} m_k v_k^2 = \frac{1}{2} m_k \omega^2 R_k^2,
\]

The total rotational kinetic energy is the sum of the pieces,

\[
(KE)_{\text{rot}} = \sum_k \frac{1}{2} m_k \omega^2 R_k^2 = \omega^2 \sum_k \frac{1}{2} m_k R_k^2 = \frac{1}{2} I \omega^2,
\]

where \(I\) is the moment of inertia of the object and is defined as,

\[
I = \sum_{k=1}^{N} m_k R_k^2 = \int R^2 dm.
\]

Notice the strong similarity between translational kinetic energy, \(\frac{1}{2} m v^2\) and rotational kinetic energy, \(\frac{1}{2} I \omega^2\). The moment of inertia, \(I\), is analogous to the mass and the angular velocity replaces the ordinary velocity. There is a table on page 347 that lists the moment of inertia, \(I\), for some simple shapes.

A moving rigid body typically has both rotational and translational kinetic energy plus whatever potential energy, usually gravitational, that applies to the given situation.

\[
\text{Total Energy} = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I \omega_{CM}^2 + \text{Potential Energy}.
\]

\(v_{CM}\) is the velocity of the CM and \(\omega_{CM}\) is the angular velocity about the CM.

**Torque**

Torque, typically represented by tau, \(\tau\), is the analog of force. Forces cause linear accelerations, movement along a line, and torques cause angular accelerations, rotations about an axis. The mass of an object acts as inertia (resistance) against a particular force changing its state of motion. Analogously, the moment of inertia moderates the amount of angular acceleration caused by a particular torque.

The quantitative relationship between force and acceleration is given by \(F = Ma\). Analogously, \(\tau = I a\), forces cause acceleration and torques cause angular accelerations.
Torque is a tad more complicated because the amount of rotation caused by a given force acting on an object depends on where the force acts. Consider the door below that lies in the plane of the page and can swing in either direction. We know from experience that pulling on the door or pushing on the door with forces parallel to the door will not work in opening it. Two such forces are shown. The best way to open a door is to push or pull with forces perpendicular to the door. But all pushes and pulls are not equal. Pushing or pulling on the side of the door near the hinge produces less opening “oomph” than pushing or pulling perpendicular to the door near the right-hand side away from the hinged edge. Although oomph is not a technical term, the oomph operating on the door is basically the thing called torque. Torque is defined as a cross product, \( r \times F \), where \( r \) is the vector from the axis of rotation to the point where the force \( F \) acts. Looking down from above at the door in the previous example, the picture below shows three different forces acting on the door and the vector \( r \) which goes from the hinged end of the door to the place where all three forces act.

To find the cross product for each of those three forces, the vectors \( r \) and \( F \) are redrawn below tail-to-tail.

In the first case, \( \theta = 180^\circ \) and the sine of \( 180^\circ \) is zero, so \( \tau = r \times F = rF \sin \theta = 0 \). In the middle example, \( \theta = 90^\circ \) and the sine of \( 90^\circ \) is one, so \( \tau = r \times F = rF \sin \theta = rF \) and by the right-hand rule the torque points out of the page producing a CCW rotation. In the last case, \( \theta \) is an intermediate angle so \( \tau = r \times F = rF \sin \theta \). Note that \( r \sin \theta \) is the
perpendicular distance between the axis of rotation and the applied force. (The arrow on the diagram points to \( r \sin \theta \).) Or analogously, \( F \sin \theta \) is the component of the force perpendicular to \( r \).

Note that a torque pointing out of the page produces a CCW angular acceleration and that a torque that points into the page produces a clockwise, CW, angular acceleration. For many problems the scalar equation,

\[
\tau = r F \sin \theta = I \alpha,
\]

will be all that is necessary as long as you remember to note whether the resulting angular acceleration, \( \alpha \), is CW or CCW.

**Pulleys and Ropes**

Although there are many different scenarios, ropes with or without mass, pulleys that are massless or have mass and can spin with or without friction, and ropes that do or don’t slip on the pulley, most problems will involve massless ropes that don’t slip on frictionless pulleys that do have mass. Pulleys with mass have rotational inertia, \( I \), that affects the dynamics of a problem. The key thing to remember in solving problems with ropes and pulleys is that the angular acceleration of the pulley, \( \alpha \), is caused by the net torque acting on the pulley. And the tangential acceleration at the rim of the pulley, \( a_T = \alpha R \), is the same as the linear acceleration of the rope as long as the rope is not slipping. Look at example problem 12.14 in the text to see how this works.

**Static Equilibrium**

A stationary object is one that is neither moving nor rotating. Such an object is in static equilibrium. An object in static equilibrium has no linear or angular acceleration, consequently \( \mathbf{F}_{\text{net}} = 0 \) and \( \mathbf{\tau}_{\text{net}} = 0 \). In most examples of static equilibrium these two vector equations lead to three scalar equations, \( F_x,_{\text{net}} = 0, F_y,_{\text{net}} = 0, \tau_{\text{net}} = 0 \) because the rotations are confined to the xy-plane, the plane of the page. The z-component of \( \tau_{\text{net}} = 0 \) is the only torque produced by forces in the xy-plane for an object that can only rotate about an axis perpendicular to the xy-plane.

**Rolling Motion**

When an object rolls along the ground without slipping, it travels a distance equal to its circumference each revolution. Therefore the linear velocity of the object is \( 2\pi R/T \),
where \( R \) is the radius and \( T \) is the time for one revolution, called the period. But the angular velocity of the spinning motion is just \( 2\pi/T = \omega \), so the linear velocity of a rolling object is related to its angular velocity by the simple equation \( v = \omega R \).

When an object is rolling without slipping, the part of the object in contact with the ground has zero velocity with respect to the ground. That is what rolling without slipping means! Furthermore, this means that at each instant, the object is rotating about that fixed point on the ground. See the picture below.

![Diagram](image)

The blue vector shows the “infinitesimal” rotation about the point of contact between the object and the ground. The CM of the object is at the midpoint of the shown vector and moves with velocity \( \omega R \). The tail end of the two vectors are in contact with the ground and are momentarily stationary, therefore the top of the object, where the two arrowheads are, is moving twice as fast as the center since it is twice as far from the stationary tail of the vectors, \( v_{\text{top}} = 2\omega R \).

In the text, they make a different but equally valid argument showing that the rolling motion can be considered as the superposition of two motions, translation and rotation. In both cases, the conclusions are the same. The part of the object in contact with the ground has \( v = 0 \) with respect to the ground. The center of the object, the rotational axis, has a velocity \( v = \omega R \) with respect to the ground. And lastly, a dot painted at the top of the rim of the object moves with a velocity \( v = 2\omega R \) with respect to the ground.

**Angular Momentum**

The angular momentum, \( L \), for a particle is defined as \( L = r \times p \), where \( p \) is the particle’s linear momentum. For a rigid body composed of lots of particles, \( \frac{dL}{dt} = \tau_{\text{net}} \), an equation analogous to \( \frac{dp}{dt} = F_{\text{net}} \). If the net torque is zero, \( \frac{dL}{dt} = 0 \), and the angular momentum is a conserved quantity just like the linear momentum.
For a rigid object rotating about an axis of symmetry, $L$ takes the particularly convenient form,

$$L = I \omega.$$

The vector sense of the angular velocity of an object with CCW rotation in the plane of the page is out of the page. For CW rotation, the angular velocity vector points into the page. The right-hand rule works perfectly here if you curl your fingers in the direction the object is rotating, your thumb points in the direction of $L$ and $\omega$.

**Tops and Gyroscopes**

The picture above corresponds to a top spinning with its angular momentum, $L$, pointing in the direction shown on the diagram. (The top of the top, denoted by a, is coming out of the page while the bottom of the top, b, is spinning into the page.) The normal and gravitational forces are shown with gravity acting at the CM. The $r$ vector from the point of contact of the top with the ground to the CM is also shown.

We are going to find the net torque about the point of contact of the top with the table. The normal force exerts no torque on the top since it goes through the point of contact. The torque caused by gravity, $r \times F_G$, points into the page. (Remember that the tails of $r$ and $F_G$ have to be connected before using the right-hand rule.) What does this torque cause? It causes the angular momentum to change, $\tau = \frac{dL}{dt}$, in the same direction as the torque produced by gravity, that is into the page. Since this change is perpendicular to the existing $L$, it does not cause the magnitude of the angular momentum vector to change but instead causes the direction of $L$ to change. The gravitational torque continually acts to change the direction of $L$ causing the top to precess CCW if viewed from above or into the page in the diagram above.

The non-spinning top just falls over because it has no angular momentum, $L$ at the start so that the $\frac{dL}{dt}$ caused by gravity results in a rotation toward the ground.
My suggestion is to start by doing some of the easier problems at the end of the chapter before tackling the assigned problems. Also try to keep in mind the analogies between linear and rotational motion summarized on page 356.