An accurate scheme to evaluate the linear dispersion relation for magnetized plasmas with arbitrary parallel distribution functions.

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Abstract

The collisionless plasmas that comprise the solar wind and occupy the earth’s magnetosphere have long been observed to have nonthermal components. Locally, it is a good approximation to model these plasmas under the assumptions that they are homogenous and embedded in a uniform magnetic field. There is a well known procedure to obtain the linear dispersion relation (LDR) for such plasmas, which allows for nonthermal velocity distributions. Unfortunately, due to the difficulty of solving this general LDR, most studies to date have instead assumed that the velocity distributions are Maxwellian or Maxwellian-like. Recently, a new method was developed to accurately evaluate a ‘generalized plasma dispersion function’ and it was shown to be effective at solving the Langmuir LDR for arbitrary distribution functions. Here we apply this method to build a scheme for evaluating the LDR for a warm magnetized plasma, allowing for arbitrary distribution functions parallel to the magnetic field. This scheme does not use the Padé approximation and is shown to be very accurate. We speculate that this scheme can be successfully combined with a novel root finding method that eliminates the need for a starting guess, and can hopefully lead to a powerful new dispersion solver.

Keywords: plasmas, dispersion relations

1. Introduction

The observed abundance and important dynamical role of plasma waves were central topics of this summer school. Perturbations in a finite temperature plasma can give rise to enhanced or damped fluctuations which can dynamically influence the plasma and its environment. Plasma instabilities can arise if one or more components of the plasma possesses free energy in the form of a nonthermal velocity distribution function (VDF). As an example, electrons in the solar wind are subject to an electron/electron beam instability, as they are observed to be comprised of separate populations, with ~ 10% consisting of an isotropic ‘halo’ component and an even higher energy anisotropic tail component known as the ‘strahl’ that are well modeled by so-called ‘Kappa’ distributions (e.g., Marsch 2006).

A basic starting point in investigating the dependence of the properties of the plasma (i.e., its density, temperature, and magnetic field strength and orientation) on the individual VDFs is the linear dispersion relation (LDR) that governs the types of waves or oscillations the collisionless plasma can support. Despite the fact that the procedure to obtain the LDR for warm magnetized plasmas is well laid out (e.g., Stix 1992), most studies of plasma waves treat each VDF as a linear combination of individual Maxwellian distributions. To better understand what modes can be excited in actual space environments, it is important to develop a dispersion solver capable of handling general VDFs.

The principal difficulty in treating a general VDF is the evaluation of the generalized plasma dispersion function

\[ Z(\zeta) = \int_{L} \frac{F(v)}{v - \zeta} dv, \]  

(1)
where $F(v)$ is an arbitrary input function that normally depends on both the VDF and its derivative, while $L$ is the Landau contour that must pass below the pole at $v = \zeta$, which physically represents the (Doppler-shifted) phase velocity of the wave. The presence of a pole leads to the famous effect of Landau damping for electrostatic plasma LDRs. The discovery of Landau damping first required viewing $F(v)$ as a function of a complex variable $v$ in order to employ the tools of residue calculus to properly handle the contour integration around the pole (Landau 1946). If $F(v)$ is further supposed to be an entire function (i.e. free of singularities for all finite values of $v$, which is almost always the case), then $Z(\zeta)$ will also be an entire function of $\zeta$ once its analytic continuation into the full complex plane is made explicit:

$$
Z(\zeta) = \begin{cases} 
\int_{-\infty}^{\infty} \frac{F(v)}{v - \zeta} dv, & \text{Im}(\zeta) > 0, \\
\text{PV} \int_{-\infty}^{\infty} \frac{F(v)}{v - \zeta} dv + \pi i f(\zeta), & \text{Im}(\zeta) = 0, \\
\int_{-\infty}^{\infty} \frac{F(v)}{v - \zeta} dv + 2\pi i f(\zeta), & \text{Im}(\zeta) < 0.
\end{cases}
$$

Here, $\text{PV}$ denotes the principal value of the integral, so the middle integral is by definition the Hilbert transform of $\pi F(v)$, and the domain of integration of each of these integrals is along the real axis. For the special case of a one-dimensional Maxwellian distribution, $F(v) = \exp(-v^2)/\sqrt{\pi}$, the well-known plasma dispersion function $Z_p(\zeta)$ is obtained. Its properties are well documented (Fried & Conte 1961), and efficient open-source algorithms exist to evaluate it for any value of $\zeta$ (e.g., S. Johnson’s Faddeeva Package\(^1\)). However, the singularity in these integrals have thwarted most attempts to reliably and efficiently evaluate $Z(\zeta)$ for arbitrary input functions $F(v)$.

A recent breakthrough was made by Xie (2013), who exploited methods developed by Weideman (1994, 1995) to avoid the singularity altogether. The basic idea is to expand the function $F(v)$ using an orthogonal basis set that is a manipulated form of a Fourier series, which allows the expansion coefficients to be easily and efficiently evaluated using a fast fourier transform (FFT). Each term in the series can be integrated analytically by again invoking residue calculus, i.e. by evaluating each integral in equation (2) over a closed contour in the complex plane consisting of the real axis plus a semi-circular arc extending to infinity. The power of this technique lies in the fact that the integral over the semi-circular arc can be shown to vanish, independently of the VDF, on account of the basis functions, allowing the integrals in equation (2) to be replaced by infinite analytic sums. In the case of the complex error function (which is simply related to $Z_p(\zeta)$), Weideman (1994) demonstrated that this scheme has excellent convergence properties when truncating the sums. For instance, he obtained 12 digits of accuracy with just 32 terms. Weideman (1995) extended this approach to compute the Hilbert transform of arbitrary functions. Xie (2013) adopted the algorithm from Weideman (1995) to compute the Hilbert transform in equation (2), and he further generalized the approach taken by Weideman (1994) to compute $Z(\zeta)$ for the cases $\text{Im}(\zeta) > 0$ and $\text{Im}(\zeta) < 0$.

Xie (2013) applied this scheme to solve the LDR for Langmuir waves, which describe the electrostatic longitudinal oscillations of a finite temperature, homogeneous, non-relativistic plasma. The present work is aimed at applying this scheme to such a plasma that is also threaded by a uniform magnetic field, in which case both electrostatic and electromagnetic waves arise from the LDR. Adding a uniform magnetic field immensely complicates the LDR, due to both the anisotropy it causes (so that susceptibility tensors must be calculated) and because the principle and higher harmonic resonances of particles gyrating about the magnetic field results in an infinite sum of Bessel functions. Nevertheless, it will be shown that extending Xie’s scheme to warm plasmas is quite straightforward and does not require use of the Padé approximation, which is known to sometimes give rise to spurious instabilities (Rönntmark 1982; see also Tjulin et al. 2000).

This report is organized as follows. In the §2, we write down the LDR for warm plasmas. In §3, we describe how to extend Xie’s scheme to evaluate this LDR. In §4, we summarize our results, concentrating on a simple measure to assess the accuracy and efficiency of this scheme. Finally, in §5, we present our conclusions and plans for future work.

\(^1\)[http://ab-initio.mit.edu/wiki/index.php/Faddeeva_Package]
2. Warm plasma linear dispersion relation

Here we provide the LDR from §10-7 of Stix (1992) in a dimensionless form suitable for numerical implementation. We adhere to Stix’s notation as much as possible, so our dimensionless variables, defined in Table 1, typically have the same symbols as the dimensional quantities from Stix (1992). This LDR, which we denote as \( D(\omega) \), assumes that the VDF perpendicular to the magnetic field is a non-relativistic Maxwellian, but it allows for an arbitrary (albeit still non-relativistic) parallel VDF. It was derived assuming field quantities that oscillate as \( \exp(ik \cdot r - i\omega t) \).

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>wave frequency</td>
<td>( \omega )</td>
<td>( \omega/(kv_1) )</td>
</tr>
<tr>
<td>plasma frequency</td>
<td>( \omega_p )</td>
<td>( \omega_p/(kv_1) )</td>
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<tr>
<td>cyclotron frequency</td>
<td>( \Omega )</td>
<td>( \Omega/(kv_1) )</td>
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<td>thermal velocity (( \perp ) to B)</td>
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<td>parallel velocity</td>
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</tr>
<tr>
<td>parallel VDF</td>
<td>( f(z) )</td>
<td>( v_1 H(v_0) )</td>
</tr>
<tr>
<td>input function of ( A_n )'s</td>
<td>( F(z) )</td>
<td>( v_1 H(v_0) )</td>
</tr>
<tr>
<td>dimensionless ( A_n )</td>
<td>( A_n )</td>
<td>( k\nu_1 A_n )</td>
</tr>
<tr>
<td>dimensionless ( B_n )</td>
<td>( B_n )</td>
<td>( kB_n )</td>
</tr>
<tr>
<td>dimensionless ( Y_n )</td>
<td>( Y_n )</td>
<td>( k\nu_1 Y_n )</td>
</tr>
</tbody>
</table>

Table 1: Definition of dimensionless quantities

The quantities in the middle column are the dimensionless ones used here. They are defined in terms of the dimensional quantities in the right column (often involving the same symbol) from Stix (1992).

Here, \( \mathbf{B} = B_0 \hat{\mathbf{b}} \) refers to the uniform background magnetic field, \( v_1 \) is an unspecified characteristic unit for velocity, and \( k \) is the magnitude of the wave vector \( \mathbf{k} \).

Explicitly, \( D(\omega) \) is given by the following determinant involving components of the dielectric tensor \( \epsilon \), the angle \( \theta \) that the wave vector makes with respect to the uniform background magnetic field (which is aligned with the \( z \)-axis), and the magnitude of the index of refraction \( n_\nu = (c/v_1)/\omega \), where \( c \) is the speed of light:

\[
D(\omega) \equiv \begin{vmatrix}
\epsilon_{xx} - n_\nu^2 \cos^2 \theta & \epsilon_{xy} & \epsilon_{xz} + n_\nu^2 \sin \theta \cos \theta \\
\epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\
\epsilon_{zx} + n_\nu^2 \sin \theta \cos \theta & \epsilon_{zy} & \epsilon_{zz} - n_\nu^2 \sin^2 \theta
\end{vmatrix} = 0. \tag{3}
\]

This form for \( D(\omega) \) assumes that the coordinate system is chosen so that \( \mathbf{k} \) lies in the \( x - z \) plane, which can be done without loss of generality. The dielectric tensor is composed of the individual susceptibilities of the constituent plasma species, denoted by \( \chi_s \):

\[
\epsilon = 1 + \sum_s \chi_s. \tag{4}
\]

Each susceptibility tensor is given by

\[
\chi_s = \left( \frac{\nu_0^2 \omega_p^2}{\omega} \right) \left\{ \frac{\langle z \rangle}{v_1^2 \cos \theta} \hat{\mathbf{e}}_\parallel + e^{-1} \sum_{n=-\infty}^{\infty} Y_n(\lambda) \right\}, \tag{5}
\]

with

\[
Y_n(\lambda) = \begin{bmatrix}
n^2 I_n A_n / \lambda & -i(n I_n - I_n' A_n) & n I_n B_n \sin \theta / (i \lambda) \\
-i(n I_n - I_n' A_n) & [n^2 I_n^2 / \lambda + 2I_n (I_n - I_n')] A_n & i(n I_n - I_n') B_n \sin \theta / \Omega \\
n I_n B_n \sin \theta / (i \lambda) & -i(n I_n - I_n') B_n \sin \theta / \Omega & I_n (\omega - n\Omega) B_n / v_1^2
\end{bmatrix}. \tag{6}
\]

Here \( I_n = I_n(\lambda) \equiv i^n J_n(i \lambda) \) are modified Bessel functions with argument \( \lambda = (v_\perp \sin \theta / \Omega)^2 \), and \( I_n' = dI_n/d\lambda \). The quantity \( B_n \) can be written in terms of \( A_n = A_n(\xi_n) \), which is an integral of the same type as \( Z(\zeta) \):

\[
A_n = -\frac{1}{\cos \theta} \int_{\xi_n}^{\lambda_n} \frac{F(z) \, dz}{z - \xi_n}, \tag{7}
\]
In these relations, \( \zeta_n = (\omega - n\Omega)/\cos\theta \); it is both frequency and species-dependent. The input function \( F(z) \) is related to the VDF \( f(z) \) and its derivative \( f'(z) = df(z)/dz \):

\[
F(z) = -\left(1 - \frac{z\cos\theta}{\omega}\right)f(z) + \frac{v^2\cos\theta}{\omega}f'(z).
\]

In equation (8), the quantity \( \langle z \rangle \) is the average velocity, \( \langle z \rangle = \int z f(z) dz \). Using this definition, it is straightforward to show that \( \epsilon_{zz} \) reduces to the Langmuir LDR for \( \theta = 0 \) and \( n = 0 \). The simplest benchmark of the warm plasma LDR is therefore the recovery of the solutions to the Langmuir LDR, which in our units is

\[
1 - \omega_p^2 \int_{-L}^{L} \frac{f'(z) dz}{z - \omega} = 0.
\]

The warm plasma LDR depends on \( 1 + (3 + N_\parallel)S \) parameters, where \( N_\parallel \) denotes the number of parameters associated with the parallel VDF \( f(z) \) and \( S \) is the total number of plasma species. The species-independent parameter is \( \theta \), while two of the species-dependent parameters are \( \omega_p \) and \( N_\parallel \Omega \). Since each species can have a different perpendicular Maxwellian VDF and a different arbitrary parallel VDF \( f(z) \), the remaining \( (1 + N_\parallel)S \) parameters are the perpendicular Maxwellian thermal velocity \( v_\perp \) and the parameters specifying the parallel VDF. The most common non-Maxwellian distributions, e.g. the Kappa distributions, have \( N_\parallel = 2 \).

3. Scheme to evaluate the dispersion relation

As described in the introduction, the scheme presented in Xie (2013) is an algorithm to evaluate the integral in equation (10). The main difference that arises when extending this scheme to evaluate equation (7) is that the input function \( F(z) \) given by equation (9) is frequency dependent, whereas \( f(z) \) is frequency independent. While this presents no complications in terms of the applicability of this scheme, it imposes a substantial increase in computational expense when actually solving \( D(\omega) \) for \( \omega \). The reason, of course, is because the scheme for evaluating \( D(\omega) \) has to be paired with a root finding algorithm that necessarily samples many different values of \( \omega \) in the process of converging to the actual root. Since \( F(z) \) will change with each of these samplings, \( S \) new FFTs will have to be computed with every evaluation of \( D(\omega) \), in contrast to the root finding process to solve equation (10), in which the same \( S \) FFTs can be reused for different \( \omega \).

Applied to the warm plasma LDR, Xie’s generalization of the schemes developed by Weideman (1994, 1995) can be summarized as the following 2-step procedure:

1. For each species, use an FFT to evaluate the expansion coefficients \( a_m \), based on the formal expansion

\[
F(z) = \lim_{N \to \infty} \sum_{m=-N}^{N} W(z)a_m(z)p_m(z),
\]

where \( W(z) \) is a weight function and \( p_m(z) \) are the basis functions. When the pole at \( \zeta_n \) lies on the real line, \( p_m(z) = (L + iz)^m/(L - iz)^{m+1} \) and \( W(z) = 1 \), while \( p_m(z) = [(L + iz)/(L - iz)]^m \) and \( W(z) = 1/(L^2 + z^2) \) when the pole is off the real line. Here the parameter \( L \) controls the convergence rate when truncating the sum from \( m = -N \) to \( m = N \). As discussed by Weideman (1995), \( L \) likely has an optimal value that in general depends on \( N \), but the truncated sum will still converge for other fixed values of \( L \). Weideman (1995) was able to determine an optimal value of \( L = 2^{-1/4}N^{1/4} \) for the case \( F(z) = e^{-z^2} \), which will likely suffice for arbitrary input functions. Note that the basis functions can be casted in Fourier form by letting \( z = L \tan(\phi/2) \) so that \( (L+iz)/(L-iz) = e^{i\phi} \); this is what makes it possible to use an FFT to determine the \( a_m \)’s.
2. Evaluate each $A_n$ using the truncated version of the following exact expansion

$$-A_n(\zeta) = \lim_{N \to \infty} \left\{ \frac{2\pi i}{L^2 + \zeta^2} \sum_{m=1}^{N} a_m \left( \frac{L + i\zeta}{L - i\zeta} \right)^m + \frac{\pi i a_0}{L(L - i\zeta)}, \quad \text{Im}(\zeta) > 0; \right. $$

(12a)

$$\left. \frac{2\pi i}{L - i\zeta} \sum_{m=0}^{N} a_m \left( \frac{L + i\zeta}{L - i\zeta} \right)^m, \quad \text{Im}(\zeta) = 0; \right. $$

(12b)

$$\left. \frac{2\pi i}{L^2 + \zeta^2} \sum_{m=1}^{N} a_m \left( \frac{L - i\zeta}{L + i\zeta} \right)^m + \frac{\pi i a_0}{L(L + i\zeta)} + 2\pi i F(\zeta), \quad \text{Im}(\zeta) < 0. \right. $$

(12c)

Here we have omitted the subscript ‘n’ on $\zeta_n = (\omega - n\Omega)/\cos \theta$ for simplicity of notation.

All that remains is to evaluate the infinite sums in equation (5) involving products of the modified Bessel functions with $A_n$ and $B_n$. The sums naturally separate into two distinct types: those involving $I_n(\lambda)$ and those involving $I'_n(\lambda)$. The former sums can be expressed as

$$\sum_{m=-\infty}^{\infty} a_m I_n = a_0 I_0 + \lim_{N \to \infty} \sum_{n=1}^{N} (a_n + a_{-n}) I_n, \tag{13}$$

where the second form, which follows from the fact that $I_n = I_{-n}$, is computationally advantageous. Our scheme is based on truncating this sum at a finite $N_B$. Using the properties $I'_n = (I_{n-1} + I_{n+1})/2$ for integer $n \neq 0$ and $I'_0 = I_1$, the second type of sum can be expressed as

$$\sum_{m=-\infty}^{\infty} a_m I'_n = \frac{1}{2} \lim_{N_B \to \infty} \left[ (a_{-1} + a_1) I_0 + (a_{-2} + a_2 - 1) I_1 + \sum_{n=1}^{N_B} (a_{-n} + a_{n+1} - 1) I_n + \sum_{n=1}^{N_B-1} (a_{-(n+1)} + a_{-(n-1)} + a_{n-1} + a_{n+1}) I_n \right]. \tag{14}$$

When truncating this sum at $N_B = 0$, it is to be understood that only the $I_{N_B+1}$ term is nonzero, so that this expression correctly gives $a_0 I_1$. Also, when $N_B = 1$ the final summation term is zero, correctly giving $a_0 I_1 + a_1 (I_0 + I_2)$. Finally, note from equation (6) that $a_0$ takes on the values $A_0, nA_0, n^2A_0, B_0, nB_0$ for sums involving $I_n$, and $A_0, nA_0, B_0, nB_0$ for sums involving $I'_n$.

This completes the presentation of our scheme to evaluate $D(\omega)$. Both the efficiency and accuracy of this scheme depends on how fast the truncated versions of the above sums converge to the limiting sums, which we now address.

4. Results

We have implemented this scheme to evaluate $D(\omega)$ in python using a wrapper to FFTW\(^2\), and we plan to make the code public once it is fully benchmarked. Thus far, we have reproduced the results of Xie (2013) by verifying that our own implementation to evaluate equation (10) produces identical output to the MATLAB program included with Xie’s paper. We also paired Xie’s evaluation scheme with a root finder to solve equation (10) and reproduce Figure 3.7 out of Gary (1993), which is an example of an electron/electron beam instability. Here we will use the “bump-on-tail” distribution function that gives rise to this instability to assess the speed of the full scheme, and its dependence on $N$ in equation (12) and $N_B$ in equations (13-14).

The bump-on-tail distribution can be implemented as one VDF or as separate drifting Maxwellian distributions for both the ‘core’ and the ‘beam’ electrons, as shown in Figure 1. When applying the trapezoid rule to calculate the FFT, it can be shown that the truncation of equation (12) at $N$ leads to a sampling of the input function $F(z)$ with $M = 4N$ values. This sampling is not linear in $z = L\tan(\phi/2)$, but rather it is linear in $\phi$. Hence, $F(z)$ will be sampled less densely at large velocities, which is generally advantageous. Figure 1 shows the excellent agreement between the sample points and the reconstructed input function based on the sum in equation (11) truncated at $N = 64$. The

\(^2\text{http://www.fftw.org/}\)
parameters needed to fully specify a Langmuir LDR model are given in the caption in Figure 1 and will be used in what follows.

While \( N = 64 \) is sufficient to accurately resolve \( F(z) \), it is important to show that Xie’s scheme to evaluate \( A_\theta(\zeta) = X + iY \) converges to a unique value as \( N \) increases. A simple quantitative measure of this convergence is a weighted percent difference:

\[
\text{weighted } \%\text{-difference} = \frac{|X\Delta X| + |Y\Delta Y|}{|X| + |Y|},
\]

where \( \Delta X \) and \( \Delta Y \) are ordinary percent differences, e.g. \( \Delta X = 100\% \frac{(X|_{N_1} - X|_{N_2})}{(X|_{N_1} + X|_{N_2})/2} \). In the bottom left panel of Figure 2 we plot the weighted %-difference of the Langmuir LDR (which recall is \( e_{\zeta} \) for \( \theta = 0 \) and \( n = 0 \)) in order to show the convergence behavior of \( A_\theta(\zeta) \) for each of the three sums given in equation (12). For example, the leftmost value shows that \( A_\theta(\zeta) \) evaluated at \( N_1 = 16 \) differs from \( A_\theta(\zeta) \) evaluated at \( N_2 = 32 \) by only 0.01%. We chose values of \( \zeta = 1.0 \) and \( 1.0 \pm 0.1i \) to demonstrate that the three sums have identical convergence properties as \( \text{Im}(\zeta) \to 0 \). Because \( |A_\theta(\zeta)| < 10^2 \) for these values, the weighted percent differences correspond to machine precision for \( N \geq 256 \), thereby confirming the accuracy of Xie’s scheme. Moreover, \( N = 256 \) should be the default value for this scheme since \( A_\theta(\zeta) \) becomes fully converged, and since the cost to evaluate \( N = 256 \) is not substantially greater than \( N = 64 \), as the top left panel shows. Similar results to these are obtained for \( A_\phi(\zeta) \) with \( n \neq 0 \).

The bottom right panel shows the same convergence measure applied to \( D(\omega) = X + iY \), this time varying the number of terms used in the sum of modified Bessel functions. To fully specify a model for the warm plasma LDR, values for \( \theta \) and \( \Omega \) are needed in addition to those given in the caption of Figure 1. The colors in Figure 2 represent different choices for \( \theta \), i.e. orientations between \( \mathbf{B} \) and \( \mathbf{k} \): nearly parallel (\( \theta = 5^\circ \), red line), nearly perpendicular (\( \theta = 85^\circ \), blue line), and oblique (\( \theta = 45^\circ \), black line). Meanwhile, the line styles distinguish different choices for \( \Omega \): dashed-dotted, solid, and dashed lines are for \( \Omega \) equal to 0.2, 1.0, and 10.0, respectively. It is clear that \( D(\omega) \) converges, implying that our scheme can reach any desired accuracy. However, the convergence is very sensitive to the value \( \lambda = (v_\perp/\Omega) \sin \theta \), so \( N_\theta \) will need to be chosen to meet the accuracy requirements.
Figure 2: **Top panels:** CPU time in seconds to evaluate the Langmuir LDR (left panel) and the warm plasma LDR (right panel) on a Macbook Air 1.8 GHz Intel i7 processor. **Bottom panels:** The weighted %-differences defined by equation (15). The left panel measures the convergence of the three sums in equation (12), while the right panel measures the convergence of the sums involving modified Bessel functions in equations (13-14). Here the dashed-dotted, solid, and dashed lines are for $\Omega$ equal to 0.2, 1.0, and 10.0, corresponding to $\lambda = 5 \sin \theta$, $\sin \theta$ and 0.1 $\sin \theta$, respectively.

That this $\lambda$-dependence is a generic feature of sums involving modified Bessel functions can be demonstrated quite simply using the identity $\sum_{n=-\infty}^{\infty} I_n(\lambda) = e^\lambda$. Figure 3 shows an ordinary percent difference of the quantity $\sum_{n=N_B}^{N_B} I_n(\lambda)$, i.e. an approximation whose limiting sum is $e^\lambda$. It demonstrates that the qualitative behavior of $D(w)$ shown in Figure 2 is independent of the choice of input function $F(z)$ and input parameters. It also implies that our scheme will be become very costly for large $\lambda$.

All of the computational expense is in the evaluation of the $Y_n(\lambda)$’s. Since our scheme amounts to computing a finite number of $Y_n(\lambda)$’s, we expect the execution time to increase linearly with the number of $Y_n(\lambda)$’s computed, i.e. to be linear in the quantity $N_B$. The top right panel in Figure 2 shows the run time for one evaluation of $D(\omega)$ as a function of $N_B$. The curves show a linear trend as expected; we terminate each one once a weighted percent difference of $10^{-12}$% has been reached. These evaluations were carried out for $N = 256$, and one evaluation of the Langmuir LDR takes $\sim 0.5$ ms according to the top left panel. Evaluation of the warm plasma LDR can exceed this amount by more than two orders of magnitude, depending on the desired accuracy and the magnitude of $\lambda$.

5. Conclusions and Future Work

We have presented a scheme to evaluate the warm plasma LDR that is based on finite truncations of two exact series expansions, one for $A_n$ and one for sums involving modified Bessel functions. The results presented here validated the truncation of these sums. While this scheme can in principle achieve arbitrary accuracies, the efficiency
Figure 3: Percent differences associated with the approximation $e^\lambda = \sum_{n=N_B}^{N_B} I_n(\lambda)$, which is an identity in the limit $N_B \to \infty$. This plot shows that the behavior of the weighted %-differences of $D(\omega)$ in the bottom right panel of Figure 2 is a generic feature of sums involving modified Bessel functions.

of this scheme can differ by orders of magnitude depending on the value of the argument to the modified Bessel functions, $\lambda$. Physically, a large $\lambda$ can be realized as a high frequency wave, and such small scale waves will be highly damped in general and therefore are not very interesting. Less commonly, however, for wave frequencies less than the cyclotron frequency, $\lambda$ represents the spread in wave phases that is seen by the gyrating particles. The cold plasma approximation demands that this spread is small, and hence a necessary condition for this approximation is $\lambda < 1$. In other words, very warm plasmas will have $\lambda > 1$, and we showed this to be the most computationally expensive regime.

The overall efficiency of this scheme to evaluate $D(\omega)$ can only be determined when it is coupled to a root finder to actually solve $D(\omega)$ for all of its roots. We can provide an estimate based on our experience solving the simpler Langmuir LDR, equation (10), which requires $\sim 3 \times 10^3$ evaluations to produce a curve of $\omega$ vs. $k_\parallel$ showing the electron-beam instability. If this instability is studied with the warm plasma LDR with $\lambda \approx 5$, Figure 2 shows that it takes $\sim 500$ms to evaluate $D(\omega)$ once. Hence, $3 \times 10^3$ evaluations requires 25 minutes of $D(\omega)$ evaluations when $N_B \sim 50$ terms are required. In the small $\lambda$ regime (i.e. for longitudinal oscillations or for small scale waves), the runtime will likely be under 1 minute.

Future work is required to fully benchmark this scheme against existing codes such as WHAMP (Rönnmark 1982). Ideally, the considerations involving $\lambda$ can be implemented automatically, so that the user can specify a desired accuracy and $N_B$ will be appropriately calculated.

The main focus of our future work is to automate the root finding process, which is notoriously tricky for two-dimensional problems such as ours (where both Re[D(\omega)] = 0 and Im[D(\omega)] = 0 must be solved). We note here that an alternative scheme to solve $D(\omega)$, one that uses an eigenvalue approach instead of a root finding algorithm, was recently put forth by Xie (2014). However, this approach uses the Padé approximation, which we want to avoid. A promising traditional root finding approach appears to be the public code cZero, based on the recent work of Johnson & Tucker (2009). The idea here is to exploit the fact that $D(\omega)$ is an analytic complex function, so that the winding theorem in complex function theory allows for the determination of the number of zeros inside any closed contour in the complex plane. By tiling the desired domain in the complex plane with triangles, cZero computes the winding number around each triangle to systematically close in on all of the isolated roots. It is completely analogous to one-dimensional bracketing and bisection (Press et al. 1992), which is guaranteed to find all roots for one-dimensional problems. This code, coupled with our scheme to evaluate $D(\omega)$, has the potential to automate the entire solution process, so that the properties (i.e. wave modes and corresponding growth rates) of waves supported by warm magnetized plasmas with arbitrary VDFs can be easily and reliably calculated.
References

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