Lens induced by stress in optical windows for high-pressure cells

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Calculations are described of the elastic stress, strain, and surface displacement for a high-pressure optical window consisting of a transparent cylinder which is supported on a flat seat with an axial hole (Poulter window). Under pressure the window is compressed, and its outside surface bulges through the hole in the support. The net optical effect is to create a weak positive lens and to increase the axial optical thickness of the window.

I. INTRODUCTION

Optical measurements of gases require windows to seal the optical measurement cell. Usually, transmission of the required light and operation without breakage are the only important criteria for the windows. However, laser, interferometric, and nonlinear optical measurements can impose additional, more stringent requirements on the windows. In particular, the results obtained for the nonlinear optical susceptibilities of gases by means of harmonic generation measurements are very sensitive to changes in focusing or optical power of the fundamental laser beam. In order to obtain accurate results, one must either eliminate or else quantitatively account for the changes in laser beam focusing and transmission that occur when the measurement cell is filled with gas. Since the calibration of the currently most accurate nonlinear-optical susceptibility measurements requires comparison of helium gas at 100 atm with other gases at much lower pressures, it is important to assess the possible systematic errors that may arise due to the effects of high gas pressure on the windows of the gas cell. An accurate assessment of the optical effects of pressure on windows of this type can be obtained from a theoretical calculation because of the simple geometry, and because it is possible to produce actual windows and seats which adequately conform to the geometrical ideal. Accordingly, we have carried out calculations of the linear elastic deformation, strain, and stress of a Poulter window. The optical effects of pressure on the window are calculated from the deformation and strain of the window.

II. METHOD OF CALCULATION

The calculation of the elastic response of the window to applied pressure follows closely the method presented previously by Alt and Kalus. The configuration of the window and its support are shown in Fig. 1. The elastic properties of a window composed of isotropic material are described by Young's modulus \( E \) and Poisson's ratio \( v \). The starting point for the calculation is the Navier equation for the displacement \( \mathbf{u} \):

\[
(1 - 2v) \Delta \mathbf{u} + \nabla (\nabla \cdot \mathbf{u}) = 0.
\]

The solution of this equation may be expressed in terms of an arbitrary biharmonic Galerkin vector \( \mathbf{G} \), where

\[
\mathbf{u} E / (1 + v) = 2(1 - v) \Delta \mathbf{G} - \nabla (\nabla \cdot \mathbf{G}).
\]

In the case of cylindrical symmetry, the Galerkin vector has the special form \( \mathbf{G} = Z(r,z) \hat{z} \), where \( Z \) may be written as a Fourier-Bessel expansion with the form

\[
Z = \sum_n \sin(az) [A_n I_0(\alpha r) + B_n a r I_1(\alpha r)] + \cos(az) \times [A_n^* I_0(\alpha r) + B_n^* a r J_1(\alpha r)] + \sum_r J_0(\gamma r) [C^* \sin(\gamma z) + D^* \gamma z \cosh(\gamma z)]
\]

where \( J_n \) and \( I_n \) are Bessel functions and modified Bessel functions of order \( n \), and \( Z_m \) are convenient, additional polynomial solutions of the biharmonic equation. One may choose \( \alpha_n \) and \( \gamma_k \) to be the \( n \)th and \( k \)th roots of \( \sin(x) \) and \( J_1(x) \), respectively. The problem is thus reduced to solving for numerical values of the coefficients of the Fourier–Bessel expansion of \( Z \) which will make \( \mathbf{u} \) satisfy the boundary conditions.

The boundary conditions for the problem are the imposed stresses or displacements of the surfaces of the window. The displacements \( u_i \) may be expressed in terms of \( Z \) by the following equations:

\[
u_r = (1 + v) \left[ -\frac{\partial^2}{\partial r \partial z} Z(r,z)/E, \right. \] (4a)

\[ u_o = 0, \] (4b)

\[ u_z = (1 + v) \left[ 2(1 - v) \nabla^2 - \frac{\partial^2}{\partial r^2} \right] Z(r,z)/E. \] (4c)

The strains \( \varepsilon_{ij} \) are given by \( \varepsilon_{rr} = \partial u_r / \partial r, \varepsilon_{00} = u_r, \varepsilon_{zz} = \partial u_z / \partial z, \) and \( \varepsilon_{rz} = (\partial u_r / \partial z + \partial u_z / \partial r)/2 \). The stresses \( \sigma_{ij} \) are found by applying Hooke's law, \( \sigma_{ij} = [\varepsilon_{ij} + \delta_{ij} \kappa_k \kappa_l / (1 - 2v)] E / (1 + v) \), to obtain the expressions given by Eqs. (7) of Ref. 1.

The boundary conditions at the hydrostatically loaded top surface (rim) of the window are \( v_{zz} = -p_0 (\sigma_{rr} = -p_0) \) and \( \sigma_{rr} = 0 \). In the case that the center hole shrinks to zero, these hydrostatic boundary conditions also apply along the bottom window surface. The polynomial

\( \sum_m Z_m \).
One only needs to consider the even terms for the coefficients of the truncated Fourier-Bessel expansion which solves for the elastic response of the window and the seat together. We have instead performed the simpler calculation which solves for the elastic response of the window alone, and have investigated a representative choice of forms for the support pressure function.

Uniform support pressure is the simplest possibility. The stress at the supported surface is then

$$\sigma_{zz} = -p_0 \left[ 1 - \left( \frac{w}{d} \right)^2 \right]^{-1},$$

where the stress is greater than the applied hydrostatic pressure because of the condition that the total force on top and bottom window surfaces must balance. This stress distribution would be obtained with a very stiff window resting on a very soft cylindrical seat of the same diameter. Uniform support pressure is obtained in this case because the deviation from flatness of the window and the surrounding support surface under load is negligible compared to the longitudinal compression of the support (a window supported on a rubber O ring might approximate this case). Clearly, the other limiting case is that of a soft window resting on a perfectly rigid support. The boundary stress in this case will be sharply peaked at the inner rim of the support because the rigid support forces the bottom window surface to be sharply bent at the edge of the aperture when pressure is applied. A notion of the form of the boundary stress may be obtained by considering the case of an infinite medium (with $v=1/3$) containing a spherical cavity of radius $R$. When a uniaxial stress $\sigma_{zz}$ is applied far from the cavity, the stress $\sigma_{zz}$ in the equatorial plane of the cavity will be magnified by the factor $[1 + (3/8) (R/r)^3 + (R/r)^5]$. One may expect the stress near the edge of the support for a Poulter window to be even more sharply peaked because the window will try to bulge out through the hole in the support and bend down the edges of the support. The function

$$\sigma_{zz} = -p_0 \left[ 1 + a \left( \sum_{m=1}^{6} \left( \frac{w}{2r} \right)^{2m-1} \right) \right]$$

with

$$a = \left[ 2 \left( \sum_{m=1}^{6} \left( 1 - \left( \frac{w}{d} \right)^{2m-1} \right) \right) (2m-1) \right]^{-1}$$

is found to produce an approximately flat supported surface within a distance $d/40$ from the edge. The displacements of the window surfaces are compared in Fig. 2 for soft and hard support. Contour plots of the stress distribution inside the window are qualitatively similar for the two cases (Fig. 2 of Ref. 1 shows a typical stress distribution). While we do not know the exact form for the support pressure function for a particular choice of window and support, our two choices for the function represent the limiting cases. Unless the final results are very sensitive to the form of the support function, it will be sufficient to interpolate between the results for the limiting cases. These and subsequent calculations have been done using $v = 0.168$ appropriate to fused silica glass.

The solution for a fused silica glass window resting on a stainless-steel sup-
The effects of strain on the optical properties of the window material are calculated by multiplying the strain-optical tensor by the nonzero strain components, adding this to the optical indicatrix of the unstrained isotropic glass, rotating to principal axis form, and inverting to obtain the dielectric tensor. The anisotropy induced by the strain is small, and the lowest order result for the refractive index change for light propagating in the $\hat{z}$ direction and polarized in the $\hat{r}$ direction is

$$\Delta n_r = -\left(n_0^2/2\right)\left[p_{11}\epsilon_{rr} + p_{12}\left(\epsilon_{\theta\theta} + \epsilon_{zz}\right)\right],$$

where $n_0$ is the refractive index of the unstrained glass, $p_{ij}$ are elements of the strain-optic tensor, and $\epsilon_{ij}$ are the strain components. For $\theta$ polarized light, $r$ and $\theta$ are interchanged in the above expression. Off the symmetry axis, the cylindrically symmetric strain field will result in an induced birefringence:

$$\Delta n_{r-\theta} = (n_0^2/2)\left(p_{11} - p_{12}\right)\left(\epsilon_{rr} - \epsilon_{\theta\theta}\right).$$

The net optical effect of the deformation and strain of the window is obtained by integrating the changes along the path of a ray traveling parallel to the $x$ axis. To lowest order, the expressions for the changes in optical thickness and optical path length are

$$\Delta OT = n_0\Delta h + h\Delta \bar{n}$$

and

$$\Delta OPL = (n_0 - 1)\Delta h + h\Delta \bar{n},$$

where $\Delta \bar{n}$ is the average change in refractive index of the window material along the path of the ray. The pressure-induced focal power $f^{-1}$ of the window is given by

$$f^{-1} = -\left(d^2/d\rho^2\right)\Delta OPL|_{\rho=0},$$

where the focal length $f$ is just the inverse of the focal power. Equation (11) ignores the effect of the increase in the refractive index $\Delta n_g$ of the gas under pressure. Accounting for the refractive index of the gas adds a term

$$f_g^{-1} = \Delta n_g R_{top}^{-1}$$

to the focal power given by Eqs. (11) and (12), where $R_{top}$ is the radius of curvature of the top surface of the window. Except for very thin windows supported near the rim, where the curvature of the top surface becomes large and the other contributions to the focal power are near zero, $f_g^{-1}$ is only a small correction to $f^{-1}$. The contribution given by Eq. (13) has not been included in the calculated results presented below. The results of our calculations are given in reduced units for a window of unit radius and unit stiffness, subjected to unit pressure; to obtain physical values one must multiply the reduced results by the appropriate factors of $(d/2)$ and $(p_0/E)$. 

Stress in optical windows
III. RESULTS AND DISCUSSION

Detailed results of the calculation are shown in Figs. 2 and 3 for a fused silica window with \( h/d = 1/2 \) and \( w/d = 1/3 \), for the limiting cases of soft [Eqs. (6)] and hard [Eq. (7)] support. With a soft support, the amplitude of the surface displacement is larger but the surface curvature at the edge of the support is reduced. On the axis, the main effect of a softer support is to increase the optical thickness of the window under pressure as a result of the more pronounced central bulge. The main qualitative features are essentially independent of the hardness of the support. Because the window bulges out into the hole in the support, the optical thickness of the window is increased on axis, even though it is decreased for the window directly above the supporting surface. The optical path length change is nearly constant across the face of the window because the variations in the \( (n_0 - 1) \Delta h \) contribution to \( \Delta OPL \) are 80% cancelled by the variations in the \( h \Delta n \) contribution. Table I summarizes the quantitative results for this fused silica window on a rigid support, in reduced units and also for the specific case of a window 12 mm in diameter and subjected to a pressure of 100 atm. The lens induced under pressure, with a focal length of 135 m, is very weak as a result of the fortuitous cancellation of terms. If a similar calculation is done for a sapphire window viewed along its \( c \) axis (ignoring the small mechanical anisotropy), one obtains a lens which is twice as strong even though sapphire is five times stiffer than silica.\(^7\,\,9,\,12\) The cancellation of terms is less complete in the calculation of the optical thickness, and in this case the OT change is three times smaller for sapphire, as one might expect due to its much higher stiffness. The effect of a weak pressure-induced lens on the results of gas-phase nonlinear-optical measurements\(^2\) is of order \( \gamma_0/f \), where \( \gamma_0 \) is the confocal parameter of the focused laser beam in the cell and \( f \) is the focal length of the pressure-induced lens. The experimental corrections are only of order 0.1% because of the fortuitously weak lensing for fused silica windows.

The variation of focal power and the optical thickness change as functions of the window thickness and the diameter of the central hole in the support, for a fused silica window, are shown in Figs. 4 and 5. In the limit \( w/d \to 1 \), the results depend only on the thickness of the window. A thin window supported near its edge will give near zero lensing because it is simply bent so that the stress distribu-

![FIG. 3. Changes in the refractive index, optical path length, birefringence, optical thickness, and geometrical thickness (top to bottom in the diagram) are shown for a fused silica Poulter window with \( d = 2 \), \( h = 1 \), \( w = d/3 \), \( \nu = 0.168 \), and \( \rho_0/E = 1 \), in the limiting cases of hard support (solid curves) and soft support (dashed curves). The window axis and the edge of the support are indicated by vertical dashed lines. Each quantity is averaged or integrated along the path of a light ray parallel to the window axis at radius \( r \). All parts of the window are compressed by the applied pressure, but the average compression is least along the axis. Thus, the refractive index has a maximum on axis, while the geometrical thickness has a maximum on axis. Their effects tend to cancel, resulting in a weak net positive lens (small curvature of \( \Delta OPL \), maximum at \( r = 0 \)) and a small net optical thickness change. The birefringence is maximum near the edge of the support and zero on axis.

![FIG. 4. The pressure-induced focal power \( f^{-1} \) of a fused silica Poulter window with \( d = 2 \) and \( \rho_0/E = 1 \) is shown for a range of thickness \( h \) and clear aperture \( w \), for the limiting cases of hard support (solid curves) and soft support (dashed curves). The focal power of the window is reduced by making it thinner or by increasing the diameter of the hole in the support, but this also lowers the breaking pressure of the window (Ref. 1). The focal power of the window increases as the support is made stiffer, the effect being more pronounced for small \( w/d \). As \( w/d \) decreases below 0.2, \( f^{-1} \) will reach a maximum, become independent of \( h \), and finally tend to zero at \( w = 0 \).]

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Reduced value</th>
<th>Physical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta n ), top</td>
<td>0.714</td>
<td>1.01x10^{-3}</td>
</tr>
<tr>
<td>( \Delta n ), bottom</td>
<td>0.115</td>
<td>1.62x10^{-5}</td>
</tr>
<tr>
<td>( \Delta h )</td>
<td>0.453</td>
<td>7.78x10^{-5}</td>
</tr>
<tr>
<td>( \Delta (n_0 - n_0) )</td>
<td>0.0091</td>
<td>1.28x10^{-6}</td>
</tr>
<tr>
<td>( \Delta OPL )</td>
<td>-0.301</td>
<td>-0.254 mm</td>
</tr>
<tr>
<td>( \Delta OT )</td>
<td>0.113</td>
<td>0.095 mm</td>
</tr>
<tr>
<td>( \Delta OPL )</td>
<td>0.414</td>
<td>0.350 mm</td>
</tr>
<tr>
<td>( f^{-1} )</td>
<td>0.316</td>
<td>0.741x10^{-2} m^{-1}</td>
</tr>
<tr>
<td>Curvature, top</td>
<td>0.355</td>
<td>0.833x10^{-3} m^{-1}</td>
</tr>
<tr>
<td>Curvature, bottom</td>
<td>3.796</td>
<td>8.904x10^{-2} m^{-1}</td>
</tr>
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</table>
Stress in optical windows

For light of wavelength 400 nm, the AOT given by microscopic focusing turns out to be relatively small, the pressure-induced change in window transmission can be large. One important application is in the Fresnel reflection from the top and bottom window surfaces (see Table I). This alters the Fresnel transmission changes are only of order 0.01%. Gross reflection coefficients of the surfaces, but the resulting transmission change in window transmission can be large as 15% when pressure is applied. Even with good antireflection coatings, the transmission change can be of order 1%. This very objectionable “etalon fringe” effect can be avoided by using windows with a small wedge angle between the top and bottom surfaces. For narrow beams, the wedge should be large enough to ensure that the reflected beams do not overlap.

The birefringence induced under pressure may significantly alter the polarization state of a laser beam transmitted through the window. The peak birefringence of the specific window presented in Table I will result in a retardation of 90 mrad and an extinction ratio of $2 \times 10^{-3}$. However, since the birefringence varies approximately quadratically with distance from the axis, the effect of the induced birefringence of the stressed window on the polarization state of a narrow beam can be minimized by centering the narrow beam on the window axis. For example, the extinction ratio for the window in Table I will remain below $10^{-6}$ for a centered, 0.5-mm-diam light beam.

In summary, the confocal parameter, power, and polarization of a laser beam transmitted through a Poulter window will be altered when pressure is applied to the window. These effects are not negligible for nonlinear-optical measurements of the highest accuracy because of the strong dependence of the measured signal on incident light intensity and polarization. The analysis of the results of high-pressure gas-phase nonlinear-optical experiments can be corrected for these effects using the quantitative estimates of the optical properties of stressed windows which have been presented above. These results may also be a useful guide in the design of experiments for other critical optical measurements of compressed gases.

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