Geometric methods for orbit integration
spacecraft trajectories

Cassini-Huygens trajectory around Saturn, 2004-2008
Planetary orbits

lines = current orbits of the four inner planets

dots = orbits of the inner planets over 50,000 years, 4.5 Gyr in the future

Ito & Tanikawa (2002)
Cosmological simulations
Springel et al. (2005)
Galactic dynamics

box and tube orbits in a galactic potential

orbits of stars near the Galactic center

Eisenhauer et al. (2005)
Large Hadron Collider
~100 orbits

~100-1000 orbits

~10^9 orbits

~10^6 orbits

~10^10 orbits
Consider following a particle in the force field of a point mass. Set $G=M=1$ for simplicity. Equations of motion read

\[ \dot{r} = v \quad ; \quad \dot{v} = F(r) = -\frac{\hat{r}}{r^2} \]

Examine three integration methods with timestep $h$:

1. Euler’s method
   \[ r_{n+1} = r_n + hv_n \quad ; \quad v_{n+1} = v_n + hF(r_n) \]
2. modified Euler’s
   \[ r_{n+1} = r_n + hv_n \quad ; \quad v_{n+1} = v_n + hF(r_{n+1}) \]
3. leapfrog
   \[ r' = r_n + \frac{1}{2}hv_n \quad ; \quad v_{n+1} = v_n + hF(r') \quad ; \quad r_{n+1} = r' + \frac{1}{2}hv_{n+1} \]
4. Runge-Kutta method

Euler methods are first-order; leapfrog is second-order; Runge-Kutta is fourth order

To keep the playing field level, use equal number of force evaluations per orbit for each method (rather than equal timesteps)
eccentricity = 0.2
200 force evaluations per orbit
plot shows fractional energy error $|\Delta E/E|$
Liouville’s theorem

The flow in phase space generated by a dynamical system governed by a Hamiltonian conserves volume
A **geometric integration algorithm** is a numerical integration algorithm that exactly preserves some geometric property of the original set of differential equations.

**Volume-conserving algorithms:**

- conserve phase-space volume, i.e. satisfy Liouville’s theorem
- appropriate for Hamiltonian systems
- e.g. modified Euler, leapfrog but *not* Runge-Kutta
Energy-conserving algorithms:
• conserve energy, i.e. restrict the system to a surface of constant energy in phase space
• appropriate for systems with time-independent Hamiltonians, e.g. motion in a fixed potential
• does not include modified Euler, leapfrog, Runge-Kutta

Time-reversible algorithms:
• integrate forward in time for N steps, reverse all velocities, integrate backward in time for N steps, reverse velocities, and the system is back where it started
• appropriate for time-reversible systems, e.g. gravitational N-body problem
• includes leapfrog but not modified Euler or Runge-Kutta

\[ r' = r_n + \frac{1}{2}hv_n; \quad v_{n+1} = v_n + hF(r') \quad \text{and} \quad r_{n+1} = r' + \frac{1}{2}hv_{n+1} \]
Symplectic algorithms:

- if the dynamical system is described by a Hamiltonian $H(q,p)$ then

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} ; \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

- if $y(t)=[q(t),p(t)]$ then the flow from $y(t_0)$ to $y(t_1)$ generated by a Hamiltonian is a symplectic or canonical map

- an integration method is symplectic if the formula for advancing by one timestep

$$y_{n+1}=y_n+g(t_n,y_n,h)$$

is also a symplectic map, i.e. if it can be generated by a Hamiltonian

- for one-dimensional systems symplectic = volume-conserving (actually area-conserving)

- for systems of more than one dimension symplectic is more general

- modified Euler and leapfrog are symplectic
The motivation for geometric integration algorithms is that preserving the phase-space geometry of the flow determined by the real dynamical system is more important than minimizing the one-step error.
Geometric integrators for cosmology

As Volker showed, the Hamiltonian in comoving coordinates is

\[ H(q, p, t) = \sum p_i \cdot q_i - L = H_A(q, p, t) + H_B(q, p, t) \]

with

\[ H_A = \sum_i \frac{p_i^2}{2m_i a^2(t)}, \quad H_B = - \sum_{i > j} \frac{G m_i m_j}{a(t)|x_i - x_j|^3}. \]

Drift and kick operators correspond to motion under \( H_A \) and \( H_B \):

\[ x'_i = x_i + \frac{p_i \Delta t}{m_i a^2(t)}, \quad p'_i = p_i - \frac{G m_i m_j (x_i - x_j)}{a(t)|x_i - x_j|^3} \Delta t \]
Geometric integrators for planetary systems

To follow motion in the general potential $\Phi(r,t)$ we may use the Hamiltonian splitting

$$H(q,p,t) = H_A(q,p,t) + H_B(q,p,t)$$

with

$$H_A = \frac{1}{2}p^2, \quad H_B = \Phi(q,t)$$

Motion of a test particle in a planetary system is described by

$$\Phi(r,t) = -\frac{GM_*}{r} - \sum_j \frac{Gm_j}{|r - r_j|}$$

In this case a much better split is

$$H_A = \frac{1}{2}p^2 - \frac{GM_*}{r}, \quad H_B = \sum_j \frac{Gm_j}{|r_i - r_j|}$$
The workhorse for long orbit integrations in planetary systems is the mixed-variable symplectic integrator (Wisdom & Holman 1991)

\[ H(r, p) = H_A + H_B, \]

with

\[ H_A = \frac{1}{2}p^2 - \frac{GM_*}{r}, \quad H_B = \sum_j \frac{Gm_j}{|r_i - r_j|} \]

- integrate \( H_A \) and \( H_B \) using leapfrog
- motion under \( H_A \) is analytic (Keplerian motion) and motion under \( H_B \) is also analytic (impulsive kicks from the planets)
- this is a geometric integrator (symplectic and time-reversible)
- errors smaller than leapfrog by of order \( m_{\text{planet}}/M_* \sim 10^{-4} \)
- long-term errors reduced to \( O(m_{\text{planet}}/M_*)^2 \) by techniques such as warmup (start with small timesteps and adiabatically change them)
The workhorse for long orbit integrations in planetary systems is the mixed-variable symplectic (MVS) integrator (Wisdom & Holman 1991)

- what it does well: long (up to Gyr) integrations of planets on orbits that are not too far from circular and don’t come too close
- what it doesn’t do well: close encounters and highly eccentric orbits

The most popular public software packages for solar-system and other planetary integrations are MERCURY (John Chambers) and SWIFT (Hal Levison, Martin Duncan) - URLs are on the wiki

- include several integrators: MVS, Bulirsch-Stoer, Forest-Ruth, etc.
- can handle close encounters + test particles
- can include most important relativistic corrections

Following 9 planets for $10^6$ yr takes about 30 minutes
eccentricity of Mercury over 5 Gyr from 2,500 integrations differing by < 1 mm in semi-major axis of Mercury

(Laskar & Gastineau 2009)
Leapfrog with variable timestep (1)

- we want to allow a variable timestep that depends on phase-space position, $h = \tau(r,v)$
- time-reversible integrators have almost all the good properties of symplectic integrators
- define a symmetric function $s(h,h')$, e.g. $s(h,h') = (h+h')/2$

\[
\begin{align*}
r' &= r_n + \frac{1}{2}hv_n \; ; \; \; v' = v_n + \frac{1}{2}hF(r') \\
s(h,h') &= \tau(r',v') \\
v_{n+1} &= v' + \frac{1}{2}h'F(r') \; ; \; \; r_{n+1} = r' + \frac{1}{2}h'v_{n+1}
\end{align*}
\]

This is time-reversible but not symplectic
$e=0.5$

200 steps per orbit
Leapfrog with variable timestep (2)

**Time transformation:**

- we want to allow a variable timestep that depends on phase-space position $h = \tau(q,p)$
- introduce a new time variable $t'$ by $dt = \tau(q,p) \, dt'$; then unit timestep in $t'$ corresponds to desired timestep in $t$
- introduce extended phase space $Q=(q_0,q)$ with $q_0=t$ and $P=(p_0,p)$ with $p_0=-H$. Then set
  $$H'(Q,P) = \tau(q,p)[H(q,p)+p_0]$$

If $(q,p)$ satisfy Hamilton’s equations with Hamiltonian $H$ and time $t$, then $(Q,P)$ satisfy Hamilton’s equations with Hamiltonian $H'$ and time $t'$

- works very well on eccentric orbits but only for one particle (can't synchronize timesteps of different particles)
Leapfrog with variable timestep (3)

• we have a general differential equation \( \frac{dy}{dt} = f(t, y) \) that is known to be time-reversible

• we want an integration scheme that is time-symmetric with a variable timestep that depends on \( y \), \( h = \tau(y) \)

• define a symmetric function \( s(h, h') \), e.g. \( s(h, h') = (h + h')/2 \)

• pick your favorite one-step integrator, \( y_{n+1} = y_n + g(y_n, h) \) (e.g. Runge-Kutta)

• introduce a dummy variable \( z \) and set \( z_n = y_n \) at step \( n \)

\[
\begin{align*}
    y' &= y_n + g(z_n, h/2) \quad ; \quad z' = z_n - g(y', -h/2) \\
    s(h, h') &= \tau(y') \\
    z_{n+1} &= z' + g(y', h/2) \quad ; \quad y_{n+1} = y' - g(z_{n+1}, -h/2)
\end{align*}
\]

This is time-reversible (Mikkola & Merritt 2006)
What has been left out

• individual timesteps
• regularization (Burdet, Kustaanheimo-Stiefel, etc.)
• non-geometric methods for N-body integration (e.g. Hermite methods, multistep and multivalue methods)
• roundoff error
• homework