The RAMSES code and related techniques

I. Hydro solvers

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Outline

- The Euler equations
- Systems of conservation laws
- The Riemann problem
- The Godunov Method
- Riemann solvers
- 2D Godunov schemes
- Second-order scheme with MUSCL
- Slope limiters and TVD schemes
- 2D slope limiter.

The Euler equations in conservative form

A system of 3 conservation laws

$$\partial_t \rho + \nabla \cdot \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \nabla \cdot (\rho \mathbf{u} \times \mathbf{u}) + \partial_x P = 0$$

$$\partial_t E + \nabla \cdot \mathbf{u}(E + P) = 0$$

The vector of ***conservative variables*** $(\rho, \mathbf{m}, E)$

The Euler equations in primitive form

A non-linear system of PDE (quasi-linear form)

$$\partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0$$

$$\partial_t u + u \partial_x u + \frac{1}{\rho} \partial_x P = 0$$

$$\partial_t P + u \partial_x P + \gamma P \partial_x u = 0$$

The vector of ***primitive variables*** $(\rho, \mathbf{u}, P)$

We restrict our analysis to perfect gases $P = (\gamma - 1)\rho\epsilon$

The isothermal Euler equations

Conservative form with conservative variables $\mathbf{U} = (\rho, m)$

$$\partial_t \rho + \partial_x m = 0$$

$$\partial_t m + \partial_x \left(\rho u^2 + \rho a^2 \right) = 0$$

Primitive form with primitive variables $\mathbf{W} = (\rho, u)$

$$\partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0$$

$$\partial_t u + u \partial_x u + \frac{a^2}{\rho} \partial_x \rho = 0$$

a is the isothermal sound speed

Systems of conservation laws

General system of conservation laws with ***F*** flux vector.

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$$

Examples:

1- Isothermal Euler equations

$$\mathbf{U} = (\rho, m)$$

$$\mathbf{F} = \left(u\rho, um + \rho a^2\right)$$

2- Euler equation

$$\mathbf{U} = (\rho, m, E)$$

$$\mathbf{F} = (u\rho, um + P, u(E + P))$$

3- Ideal MHD equations

$$\mathbf{U} = \left(\rho, m_x, m_y, m_z, E, B_x, B_y, B_z\right)$$

$$\mathbf{F} = \left(v_x\rho, v_x m_x + P_{tot} - B_x^2, v_x m_y - B_x B_y, v_x m_z - B_x B_z, \right.$$

$$\left. 0, v_x B_y - v_y B_x, v_x B_z - v_z B_x\right)$$

Primitive variables and quasi-linear form

We define the Jacobian of the flux function as: $\mathbf{J}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$

The system writes in the quasi-linear (non-conservative) form

$$\partial_t \mathbf{U} + \mathbf{J} \partial_x \mathbf{U} = 0$$

We define the primitive variables $\mathbf{W}(\mathbf{U})$

and the Jacobian of the transformation $\mathbf{P} = \frac{\partial \mathbf{W}}{\partial \mathbf{U}}$

The system writes in the primitive (non-conservative) form

$$\partial_t \mathbf{W} + \mathbf{A} \partial_x \mathbf{W} = 0$$

The matrix A is obtained by $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$

The system is *hyperbolic* if ***A*** or ***J*** have real eigenvalues.

The advection equation

Scalar (one variable) linear (u=constant) partial differential equation (PDE)

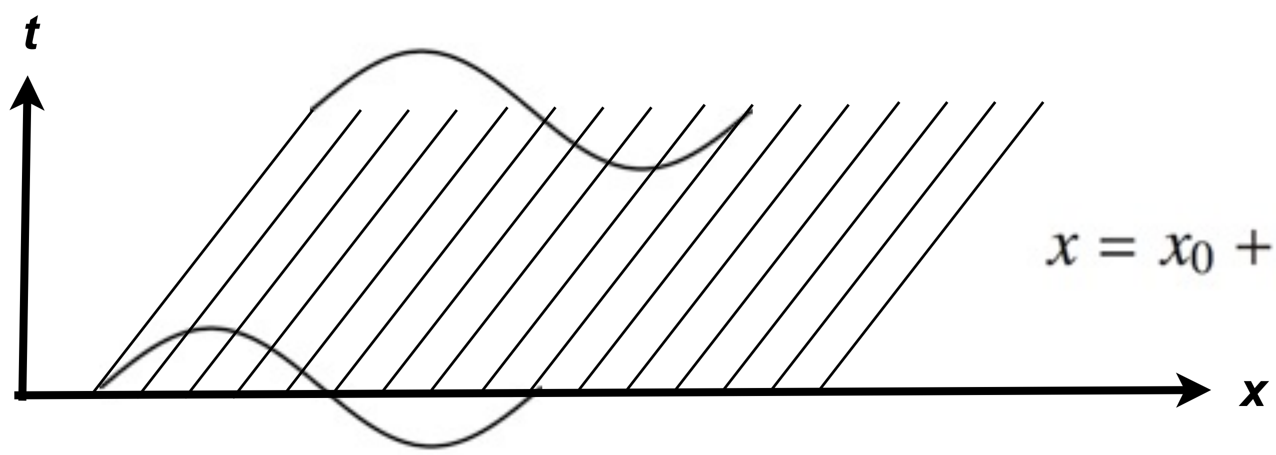
$$\partial_t \rho + u \partial_x \rho = 0$$

Initial conditions: $\rho(x, t=0) = \rho_0(x)$

Define the function: $\mathcal{I}(t) = \rho(x_0 + ut, t)$

Using the *chain rule*, we have: $\partial_t \mathcal{I} = u \partial_x \rho + \partial_t \rho = 0$

ρ is a ***Riemann Invariant*** along the ***characteristic curves*** defined by u



$$x = x_0 + ut$$

The isothermal wave equation

We linearize the isothermal Euler equation around some equilibrium state.

$$\mathbf{W} = \mathbf{W}_0 + \Delta\mathbf{W}$$

Using the system in primitive form, we get the ***linear*** system:

$$\partial_t \Delta\mathbf{W} + \mathbf{A}_0 \partial_x \Delta\mathbf{W} = 0$$

where the constant matrix has 2 real eigenvalues and 2 eigenvectors

$$\mathbf{A}_0 = \left\{ \begin{array}{cc} u & \rho \\ \dfrac{a^2}{\rho} & u \end{array} \right\} \qquad \begin{array}{l} \lambda^+ = u + a \\ \lambda^- = u - a \end{array} \qquad \begin{array}{l} \Delta\alpha^+ = \dfrac{1}{2}\left(\Delta\rho + \rho\dfrac{\Delta u}{a}\right) \\ \Delta\alpha^- = \dfrac{1}{2}\left(\Delta\rho - \rho\dfrac{\Delta u}{a}\right) \end{array}$$

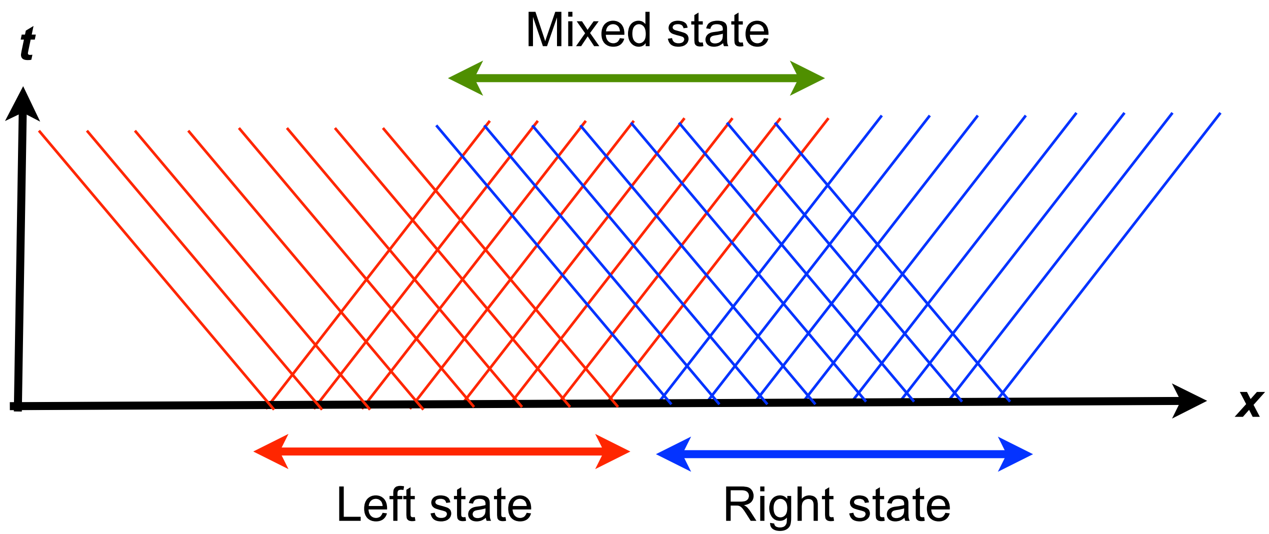
The previous system is equivalent to 2 independent *scalar linear* PDEs.

$$\partial_t \Delta\alpha^+ + (u + a)\partial_x \Delta\alpha^+ = 0$$
$$\partial_t \Delta\alpha^- + (u - a)\partial_x \Delta\alpha^- = 0$$

$\Delta\alpha^+$ ($\Delta\alpha^-$) is a Riemann invariant along characteristic curves moving with velocity $u + a$ $(u - a)$

Linear Riemann problem for isothermal waves

Initial conditions are defined by 2 semi-infinite regions with pieceweise constant initial states $(\Delta\rho_R, \Delta u_R)$ and $(\Delta\rho_L, \Delta u_L)$



"Star" state is obtained using the 2 Riemann invariants.

$$u - a < \frac{x}{t} < u + a$$

$$\Delta\rho^* = \Delta\alpha_L^+ + \Delta\alpha_R^-$$

$$\Delta u^* = \frac{a}{\rho}(\Delta\alpha_L^+ - \Delta\alpha_R^-)$$

The adiabatic wave equation

$$\mathbf{A}_0 = \left\{ \begin{array}{ccc} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma P & u \end{array} \right\}$$

$$\lambda^+ = u + a$$
$$\lambda^0 = u$$
$$\lambda^- = u - a$$

$$\Delta\alpha^+ = \frac{1}{2}\left(\frac{\Delta P}{a^2} + \rho\frac{\Delta u}{a}\right)$$
$$\Delta\alpha^0 = \Delta\rho - \frac{\Delta P}{a^2}$$
$$\Delta\alpha^- = \frac{1}{2}\left(\frac{\Delta P}{a^2} - \rho\frac{\Delta u}{a}\right)$$

We define the adiabatic sound speed: $a^2 = \gamma\frac{P}{\rho}$

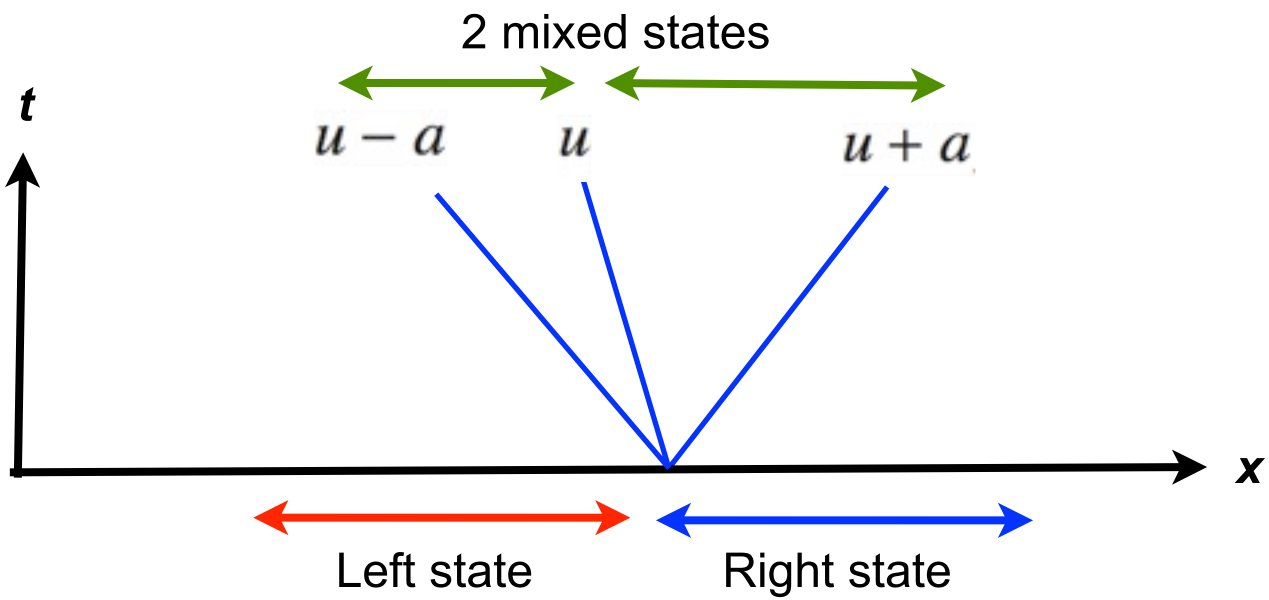
The system is equivalent to the 3 independent scalar PDEs:

$$\partial_t \Delta\alpha^+ + (u+a)\partial_x \Delta\alpha^+ = 0$$
$$\partial_t \Delta\alpha^0 + u\partial_x \Delta\alpha^0 = 0$$
$$\partial_t \Delta\alpha^- + (u-a)\partial_x \Delta\alpha^- = 0$$

$\Delta\alpha^+$, $\Delta\alpha^-$ and $\Delta\alpha^0$ are 3 Riemann invariants along characteristic curves moving with velocity $u+a$, $u-a$ and u.

Linear Riemann problem for adiabatic waves

Initial conditions are defined by 2 semi-infinite regions with pieceweise constant initial states $(\Delta\rho_R, \Delta u_R, \Delta P_R)$ and $(\Delta\rho_L, \Delta u_L, \Delta P_L)$.

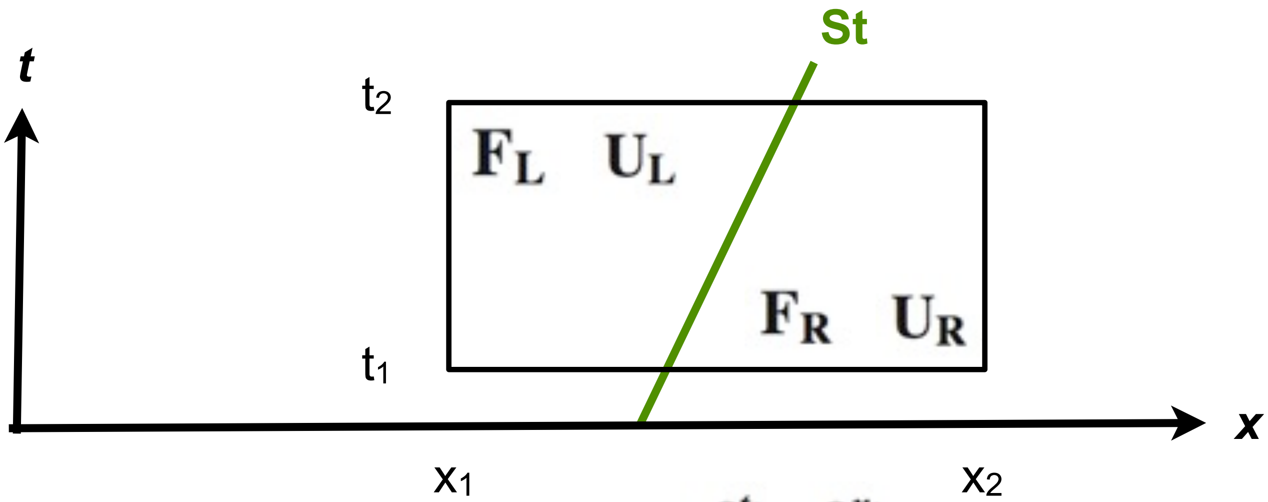


Left "star" state: (-,0,+)=(R,L,L) and right "star" state: (-,0,+)=(R,R,L).

$$\Delta u^*_{L,R} = \frac{a}{\rho}(\Delta\alpha^+_L - \Delta\alpha^-_R) \qquad \Delta\rho^*_R = \Delta\alpha^+_L + \Delta\alpha^0_R + \Delta\alpha^-_R$$

$$\Delta P^*_{L,R} = \frac{a}{\rho}(\Delta\alpha^+_L + \Delta\alpha^-_R) \qquad \Delta\rho^*_L = \Delta\alpha^+_L + \Delta\alpha^0_L + \Delta\alpha^-_R$$

Non-linear case: shocks and the RH relations



Integral form of the conservation law $$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left(\partial_t \mathbf{U} + \partial_x \mathbf{F}\right) dx dt = 0$$

$$\int_{x_1}^{x_2}\mathbf{U}(t_2)dx - \int_{x_1}^{x_2}\mathbf{U}(t_1)dx + \int_{t_1}^{t_2}\mathbf{F}(x_2)dt - \int_{t_1}^{t_2}\mathbf{F}(x_1)dt = 0$$

Rankine-Hugoniot relations: $\mathbf{F_R} - \mathbf{F_L} = S\ (\mathbf{U_R} - \mathbf{U_L})$

Bürger's equation: $\frac{u_R^2}{2} - \frac{u_L^2}{2} = S\ (u_R - u_L)$ gives $S = \frac{u_R + u_L}{2}$

Riemann invariants for non-linear waves

Define the 3 differential forms:

$$\mathrm{d}\mathcal{I}^{+} = \frac{1}{2}\left(\frac{\mathrm{d}P}{a^2} + \rho\frac{\mathrm{d}u}{a}\right) \quad \mathrm{d}\mathcal{I}^{-} = \frac{1}{2}\left(\frac{\mathrm{d}P}{a^2} - \rho\frac{\mathrm{d}u}{a}\right) \quad \mathrm{d}\mathcal{I}^{0} = \mathrm{d}\rho - \frac{\mathrm{d}P}{a^2}$$

These are Riemann invariants along the characteristic curves (u+a, u-a, u)

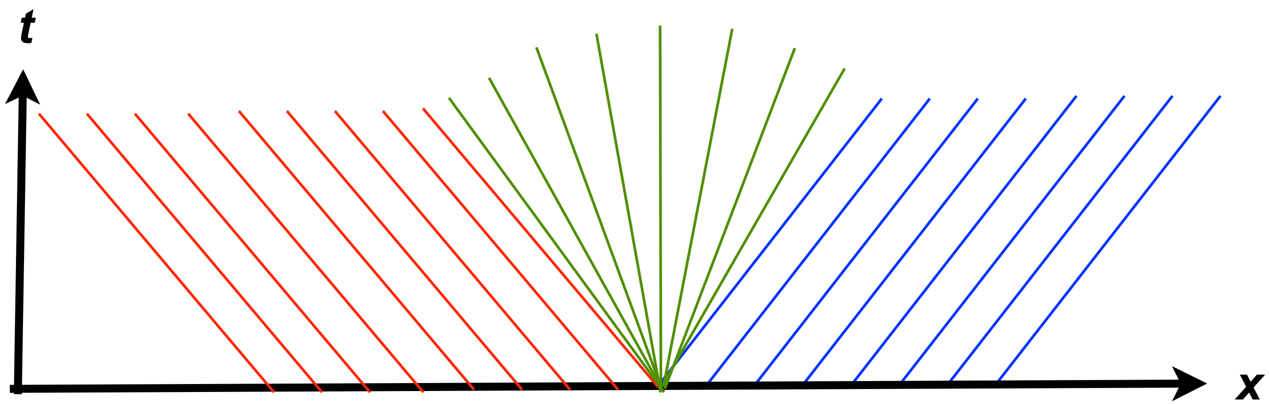
Exercise: using $\mathrm{d}P = \partial_t P + (u+a)\partial_x P$ and the Euler system in primitive form, show that the previous forms are invariants along their characteristic curve.

Right-going waves satisfy $\mathrm{d}\mathcal{I}^{-} = \mathrm{d}\mathcal{I}^{0} = 0$

Left-going waves satisfy $\mathrm{d}\mathcal{I}^{+} = \mathrm{d}\mathcal{I}^{0} = 0$

Entropy waves satisfy $\mathrm{d}\mathcal{I}^{+} = \mathrm{d}\mathcal{I}^{-} = 0$

Non-linear case: rarefaction waves



The entropy is conserved across the fan

$$P(x,t) = P_L \left(\frac{\rho}{\rho_L}\right)^{\gamma} \qquad a(x,t) = a_L \left(\frac{\rho}{\rho_L}\right)^{\frac{\gamma-1}{2}}$$

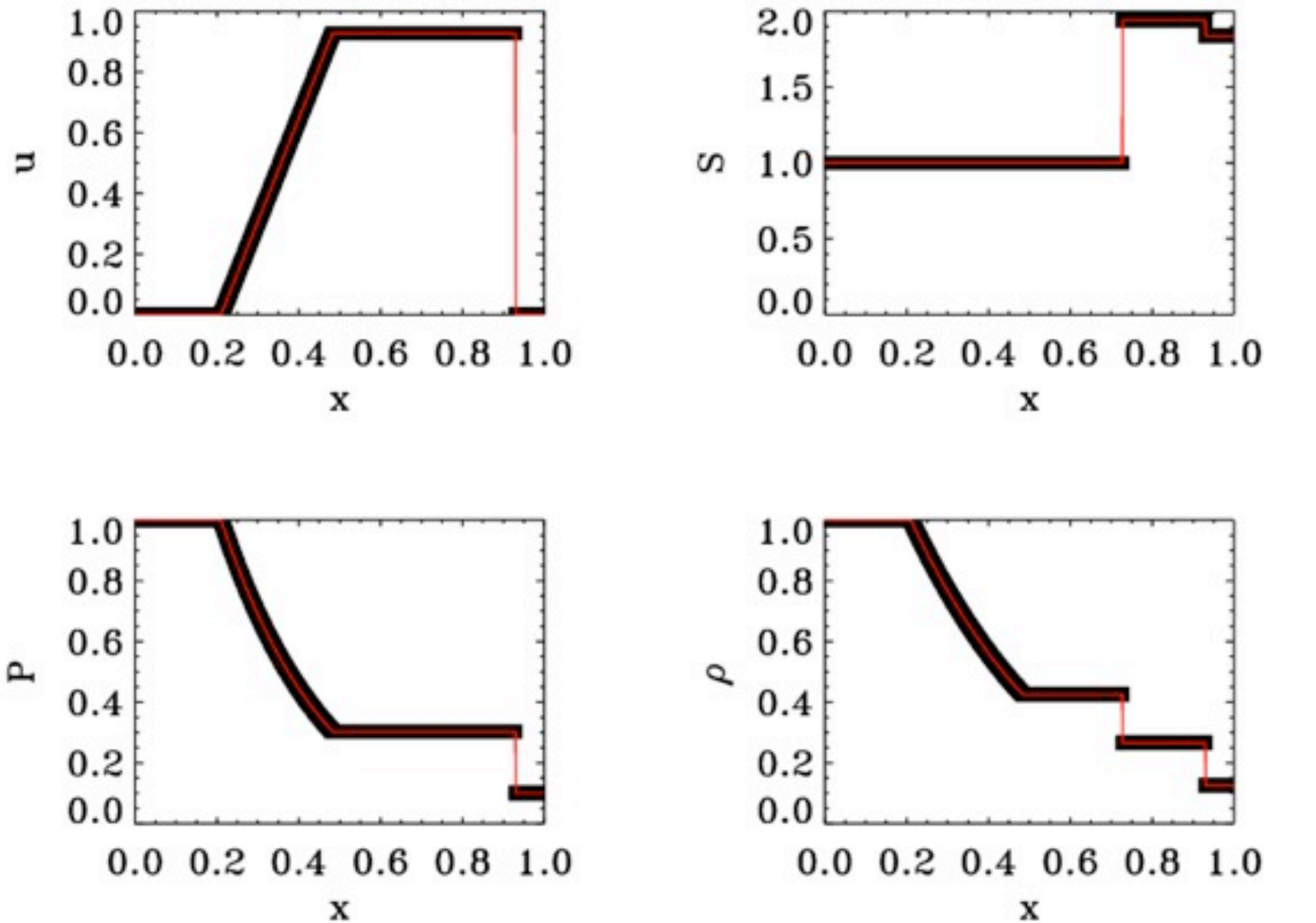
$\mathrm{d}\mathcal{I}^+ = 0$ across the fan, which gives $u + \dfrac{2a}{\gamma - 1} = \text{constant}$

Writing $x = (u-a)t$ we get $u(x,t) = \dfrac{2}{\gamma+1}\left(\dfrac{x}{t} + \dfrac{\gamma-1}{2}u_L + a_L\right)$

The Sod shock tube

Analytical solution: we match the pressure and the velocity at the tip of the rarefaction wave with the pressure and velocity after the shock.

$$(P_*, u_*)$$



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The Godunov method

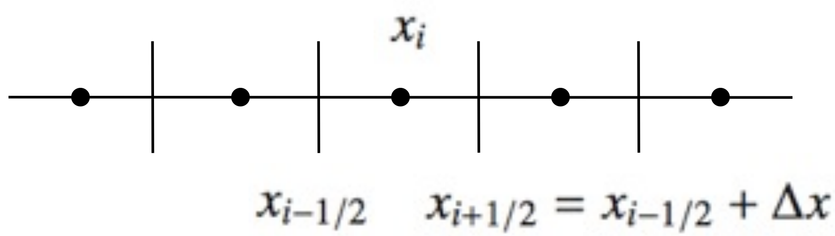
Sergei Konstantinovich Godunov

Sergei Konstantinovich Godunov

Born 17th July, 1929

Moscow

Finite volume scheme



Finite volume approximation of the advection equation:

$$u_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t^n)\mathrm{dx}$$

Use integral form of the conservation law:

$$\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{t^n}^{t^{n+1}} \mathrm{dxdt}\,(\partial_t u + a\partial_x u) = 0$$

Exact evolution of volume averaged quantities:

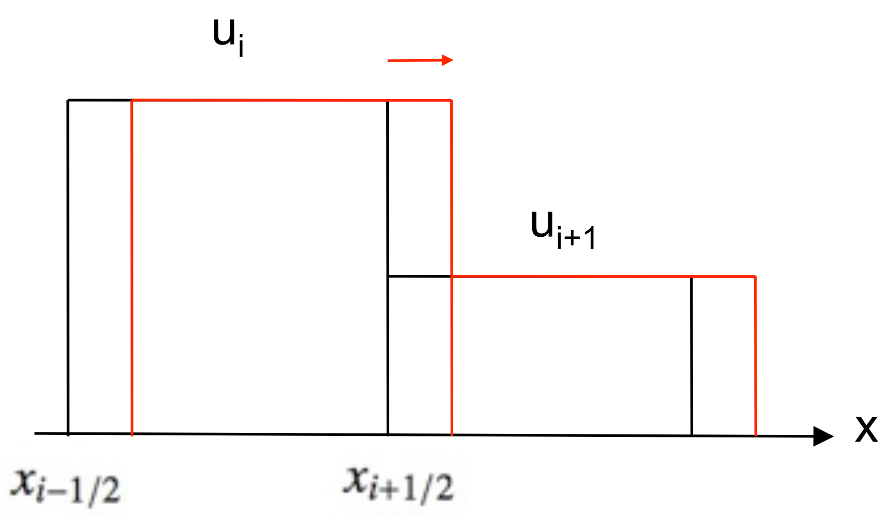
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

Time averaged flux function: $$u_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i+1/2},t)\mathrm{dt}$$

Godunov scheme for the advection equation

The time averaged flux function: $$u_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i+1/2}, t)\mathrm{dt}$$

is computed using the solution of the Riemann problem defined at cell interfaces with piecewise constant initial data.



For all t>0:
$$u(x_{i+1/2}, t) = u_i^n \quad \text{if} \quad a > 0$$
$$u(x_{i+1/2}, t) = u_{i+1}^n \quad \text{if} \quad a < 0$$

The Godunov scheme for the advection equation is identical to the upwind finite difference scheme.

Godunov scheme for hyperbolic systems

The system of conservation laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = 0$$

is discretized using the following integral form:

$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} + \frac{\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

The time average flux function is computed using the self-similar solution of the inter-cell Riemann problem:

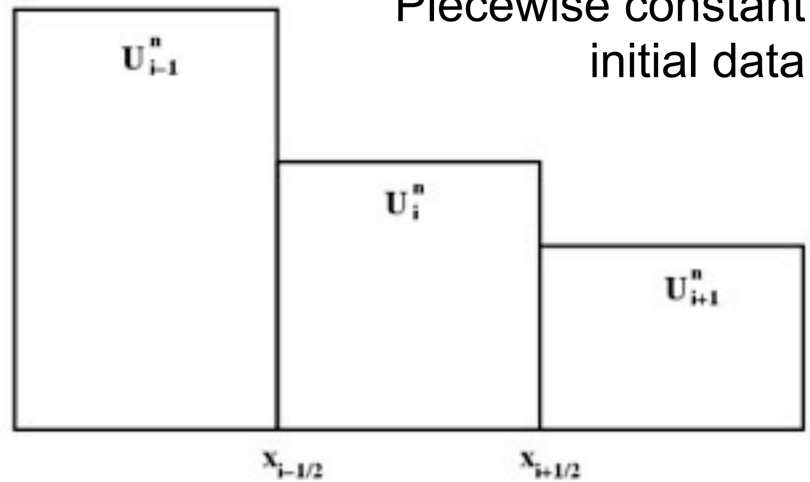
$$\mathbf{U}_{i+1/2}^*(x/t) = \mathcal{RP}\left[\mathbf{U}_i^n, \mathbf{U}_{i+1}^n\right]$$

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}(\mathbf{U}_{i+1/2}^*(0))$$

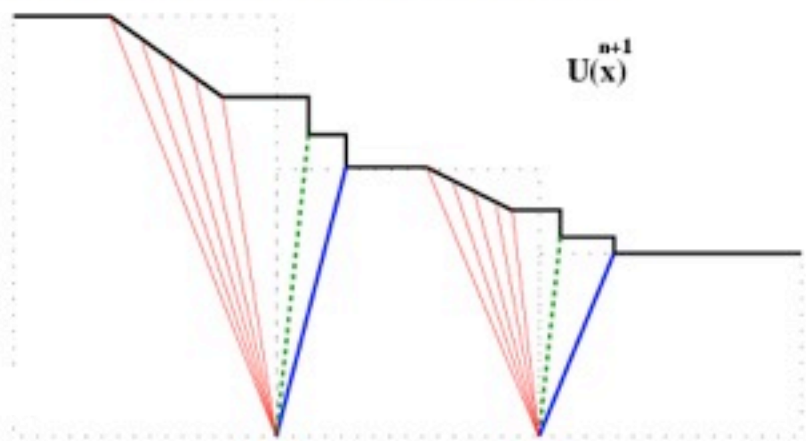
This defines the Godunov flux:

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}^*(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$$

Piecewise constant initial data



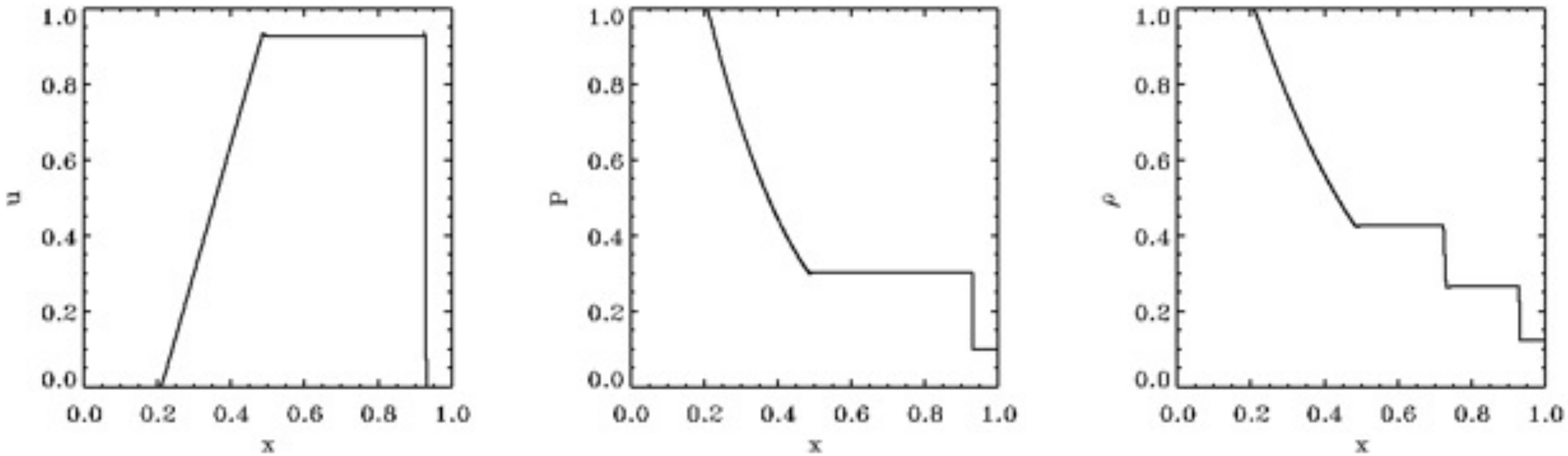
- **Godunov, S. K.** (1959), A Difference Scheme for Numerical Solution of Discontinuos Solution of Hydrodynamic Equations, *Math. Sbornik*, ***47***, *271-306*, translated US Joint Publ. Res. Service, JPRS 7226, 1969.



Advection: 1 wave, Euler: 3 waves, MHD: 7 waves

Riemann solvers

Exact Riemann solution is costly: involves Raphson-Newton iterations and complex non-linear functions.



Approximate Riemann solvers are more useful.

Two broad classes:

- Linear solvers

- HLL solvers

- **Toro, E. F.** (1999), *Riemann Solvers and Numerical Methods for Fluid Dynamics*, Springer-Verlag.

Linear Riemann solvers

Define a reference state as the arithmetic average or the Roe average

$$\mathbf{U}_{ref} = \frac{\mathbf{U}_L + \mathbf{U}_R}{2} \qquad \mathbf{U}_{ref} = \mathrm{Roe}\left[\mathbf{U}_L, \mathbf{U}_R\right]$$

Evaluate the Jacobian matrix at this reference state.

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}\left(\mathbf{U}_{ref}\right)$$

Compute eigenvalues and (left and right) eigenvectors

$$\mathbf{A} = \mathbf{L}^T \Lambda \mathbf{R}$$

The interface state is obtained by combining all upwind waves

$$\mathbf{A}\mathbf{U}_* = \mathbf{A}\frac{\mathbf{U}_L + \mathbf{U}_R}{2} - \mathbf{L}^T|\Lambda|\mathbf{R}\frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$

where $|\Lambda| = (|\lambda_1|, |\lambda_2|, \ldots)$

Non-linear flux function with a linear diffusive term.

$$\mathbf{F}^*(U_L, U_R) = \frac{\mathbf{F}_L + \mathbf{F}_R}{2} - \mathbf{L}^T|\Lambda|\mathbf{R}\frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$

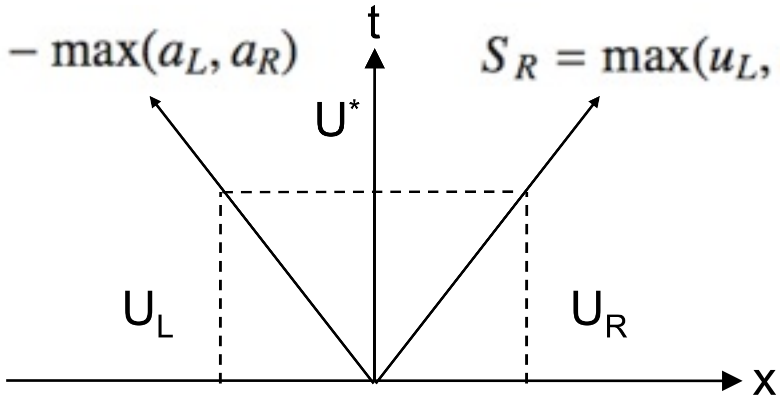
A simple example, the *upwind* Riemann solver:

$$\mathbf{F}^*(U_L, U_R) = a\frac{\mathbf{U}_L + \mathbf{U}_R}{2} - |a|\frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$

HLL Riemann solver

Approximate the true Riemann fan by 2 waves and 1 intermediate state:

$$S_L = \min(u_L, u_R) - \max(a_L, a_R) \qquad S_R = \max(u_L, u_R) + \max(a_L, a_R)$$



Compute U* using the integral form between $S_L t$ and $S_R t$

$$\mathbf{U}^*(\mathbf{U}_L, \mathbf{U}_R) = \frac{S_R \mathbf{U}_R - S_L \mathbf{U}_L - (\mathbf{F}_R - \mathbf{F}_L)}{S_R - S_L}$$

Compute F* using the integral form between $S_L t$ and 0.

$$S_L > 0 \quad \mathbf{F}^*(U_L, U_R) = \mathbf{F}_L$$

$$S_R < 0 \quad \mathbf{F}^*(U_L, U_R) = \mathbf{F}_R$$

$$S_L < 0 \quad \text{and} \quad S_R > 0 \quad \mathbf{F}^*(U_L, U_R) = \frac{S_R \mathbf{F}_L - S_L \mathbf{F}_R + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L}$$

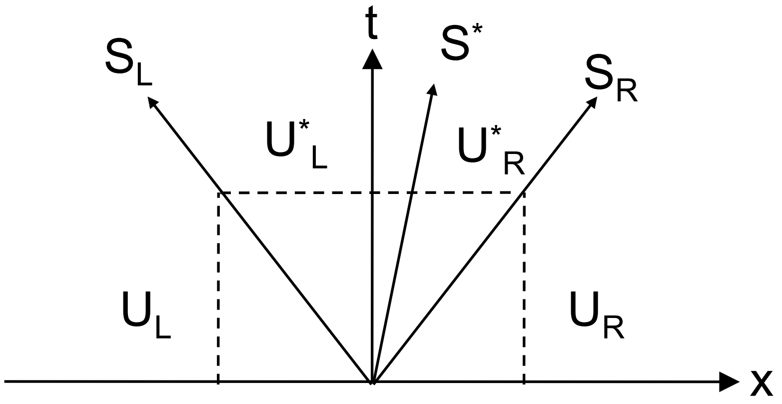
Other HLL-type Riemann solvers

Lax-Friedrich Riemann solver: $S_* = S_R = -S_L = \max(|u_L| + a_L, |u_R| + a_R)$

$$\mathbf{F}^*(U_L, U_R) = \frac{\mathbf{F}_L + \mathbf{F}_R}{2} - S_* \frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$

HLLC Riemann solver: add a third wave for the contact (entropy) wave.

$$S_* = \frac{(\rho u)^*_{HLL}}{\rho^*_{HLL}}$$

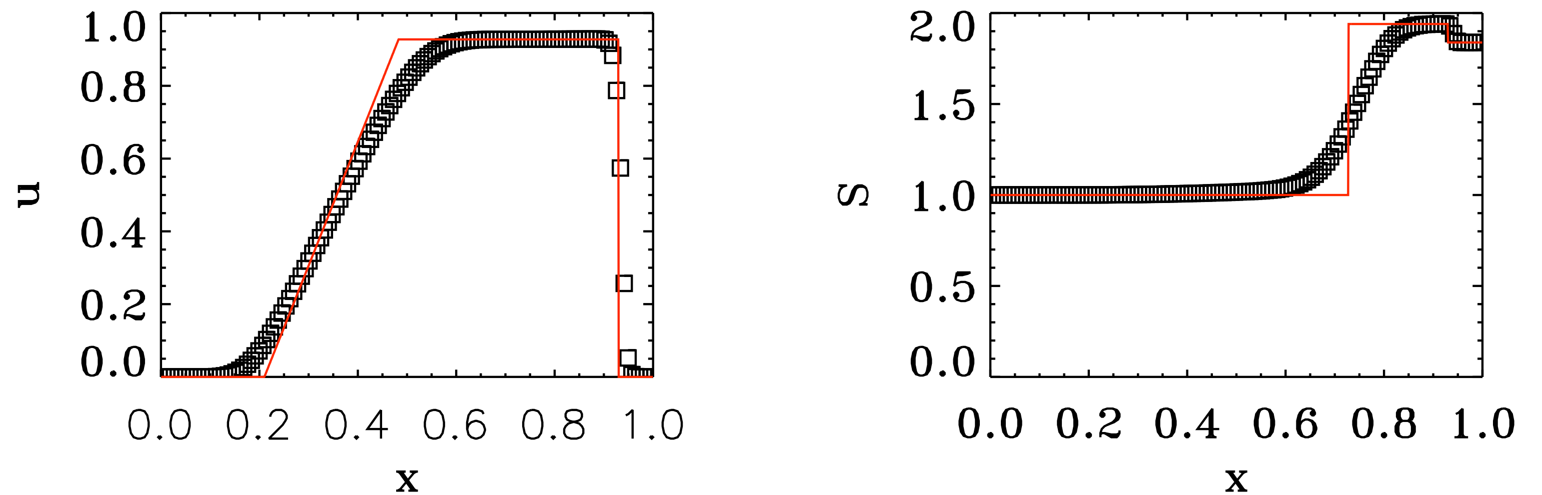


See Toro (1997) for details.

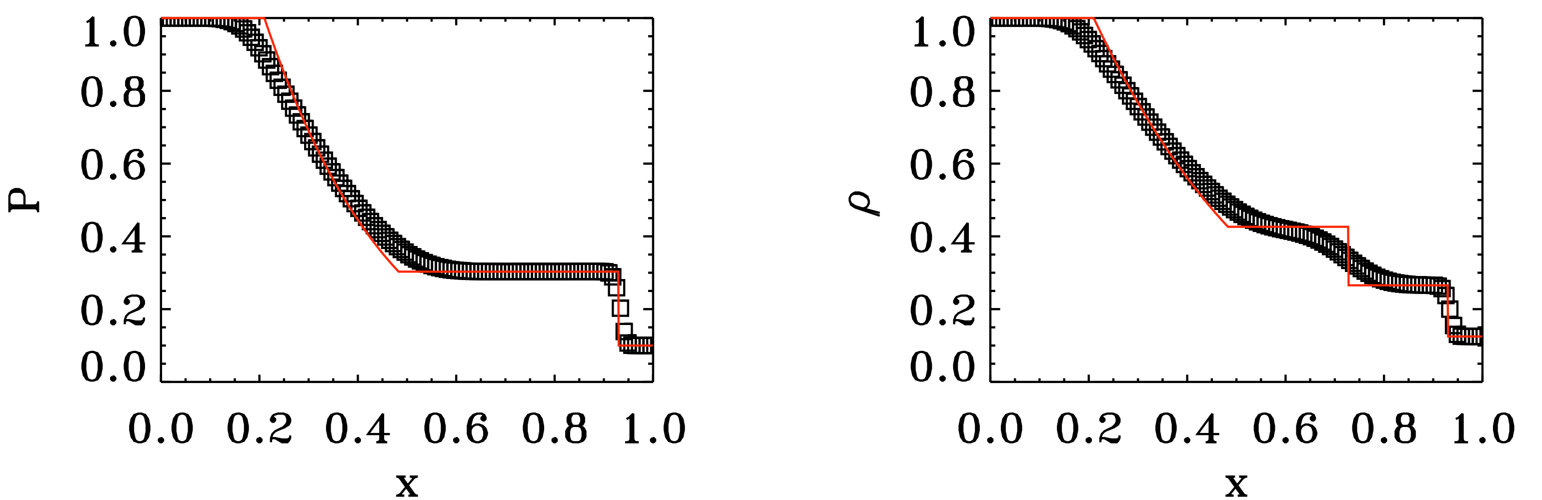
Sod test with the Godunov scheme

`riemann='llf'`

Lax-Friedrich Riemann solver



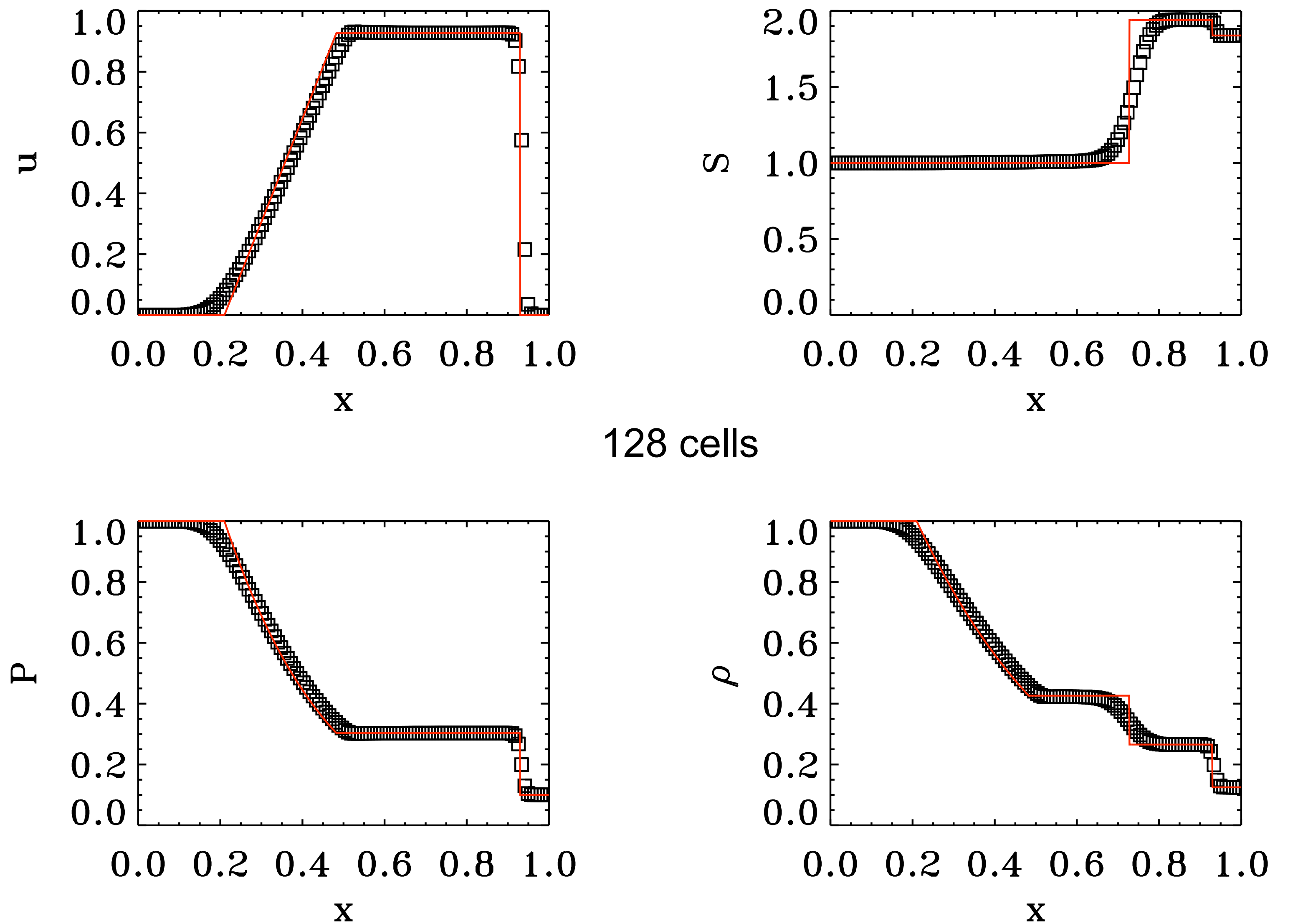
128 cells



Sod test with the Godunov scheme

`riemann='hllc'`

HLLC Riemann solver

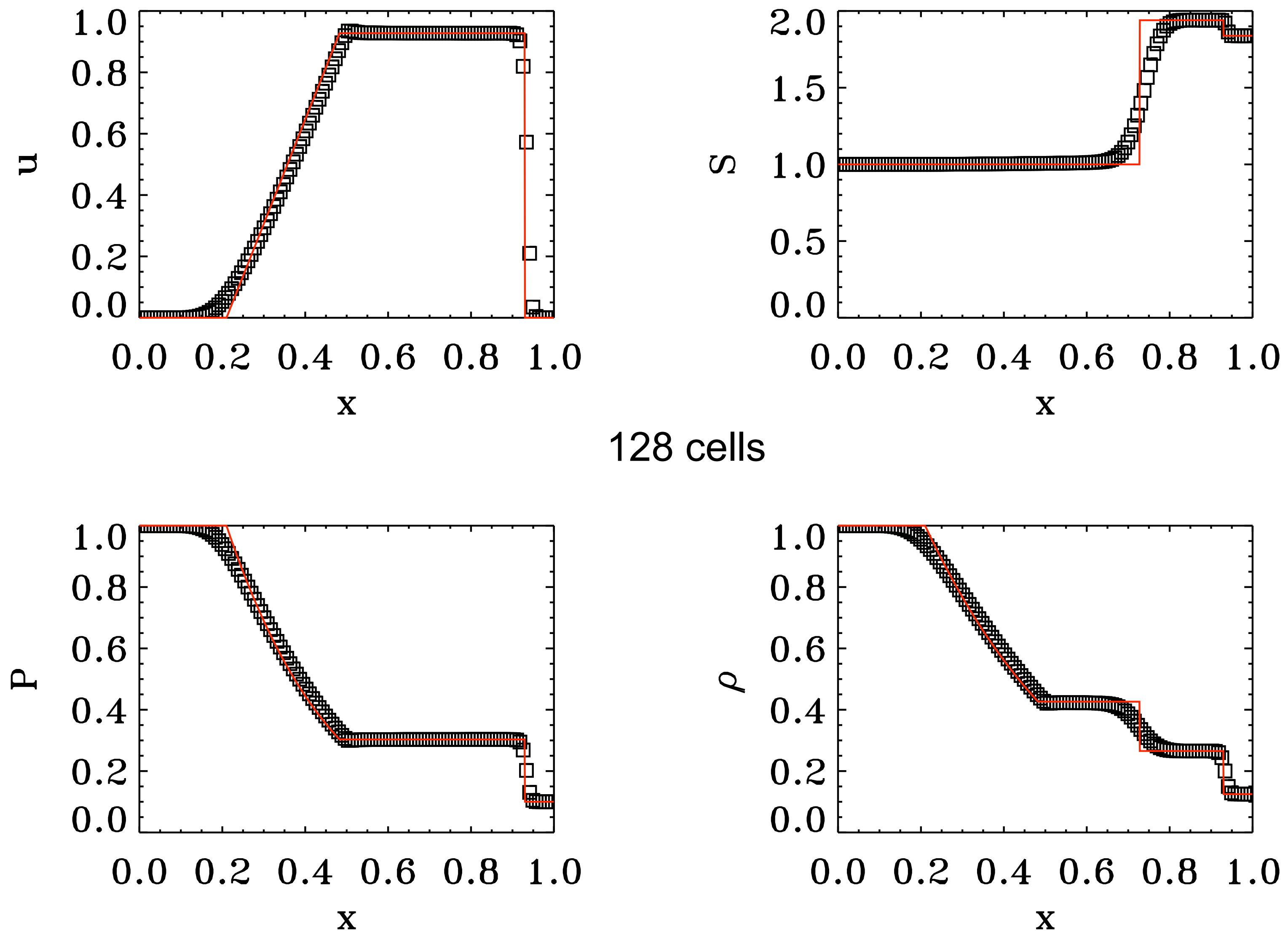


128 cells

Sod test with the Godunov scheme

`riemann='exact'`

Exact Riemann solver



Multidimensional Godunov schemes

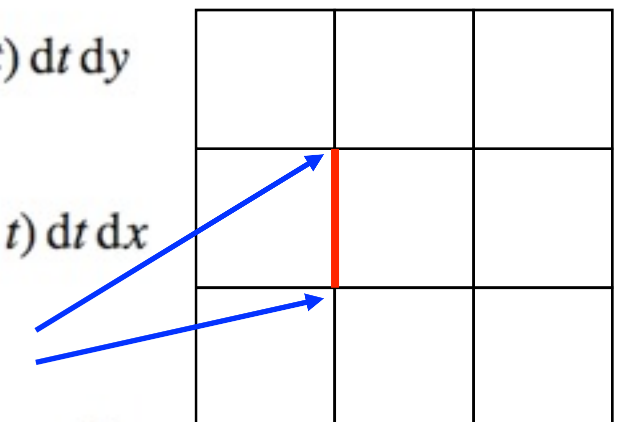
2D Euler equations in integral (conservative) form

$$\mathbf{U}^{n+1}_{i,j} - \mathbf{U}^{n}_{i,j} + \frac{\Delta t}{\Delta x}\left(\mathbf{F}^{n+1/2}_{i+1/2,j} - \mathbf{F}^{n+1/2}_{i-1/2,j}\right) + \frac{\Delta t}{\Delta y}\left(\mathbf{G}^{n+1/2}_{i,j+1/2} - \mathbf{G}^{n+1/2}_{i,j-1/2}\right) = 0$$

Flux functions are now time and space average.

$$\mathbf{F}^{n+1/2}_{i+1/2,j} = \frac{1}{\Delta t}\frac{1}{\Delta y}\int_{t^n}^{t^{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}} \mathbf{F}(x_{i+1/2}, y, t)\,\mathrm{d}t\,\mathrm{d}y$$

$$\mathbf{G}^{n+1/2}_{i,j+1/2} = \frac{1}{\Delta t}\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{G}(x, y_{j+1/2}, t)\,\mathrm{d}t\,\mathrm{d}x$$



2D Riemann problems interact along cell edges:

$$\mathbf{U}^{*}_{i+1/2,j+1/2}(x/t, y/t) = \mathcal{RP}\left[\langle\mathbf{U}\rangle^{n}_{i,j}, \langle\mathbf{U}\rangle^{n}_{i+1,j}\, \langle\mathbf{U}\rangle^{n}_{i,j+1}\, \langle\mathbf{U}\rangle^{n}_{i+1,j+1}\right]$$

Even at first order, self-similarity does not apply to the flux functions anymore.

Predictor-corrector schemes ?

A simple alternative: directional splitting

Perform 1D Godunov scheme along each direction in sequence.

X step: $$\mathbf{U}_{i,j}^{n+1} - \mathbf{U}_{i,j}^{n} + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2,j}^{n+1/2} - \mathbf{F}_{i-1/2,j}^{n+1/2}\right) = 0$$

Y step: $$\mathbf{U}_{i,j}^{n+2} - \mathbf{U}_{i,j}^{n+1} + \frac{\Delta t}{\Delta y}\left(\mathbf{G}_{i,j+1/2}^{n+3/2} - \mathbf{G}_{i,j-1/2}^{n+3/2}\right) = 0$$

Change direction at the next step using the same time step.

Compute Δt, X step, Y step, t=t+Δt Y step, X step t=t+Δt

Courant factor per direction: $C_x = (|u| + a)\frac{\Delta t}{\Delta x}$ $\quad C_y = (|v| + a)\frac{\Delta t}{\Delta y}$

Courant condition: $\max(C_x, C_y) < 1$

Cost: 2 Riemann solves per time step.

Second order based on corresponding 1D higher order method.

Used in the FLASH code: be careful with AMR !

Various 2D unsplit schemes

Godunov scheme

RAMSES

No predictor step.

Flux functions computed using 1D Riemann problem at time t^n in each normal direction.

2 Riemann solves per step.

Courant condition: $C_x + C_y < 1$

Runge-Kutta scheme

Predictor step using the Godunov scheme and $\Delta t/2$.

Flux functions computed using 1D Riemann problem at time $t^{n+1/2}$ in each normal direction.

4 Riemann solves per step.

Courant condition: $C_x + C_y < 1$

Corner Transport Upwind

Predictor step in transverse direction only using the 1D Godunov scheme.

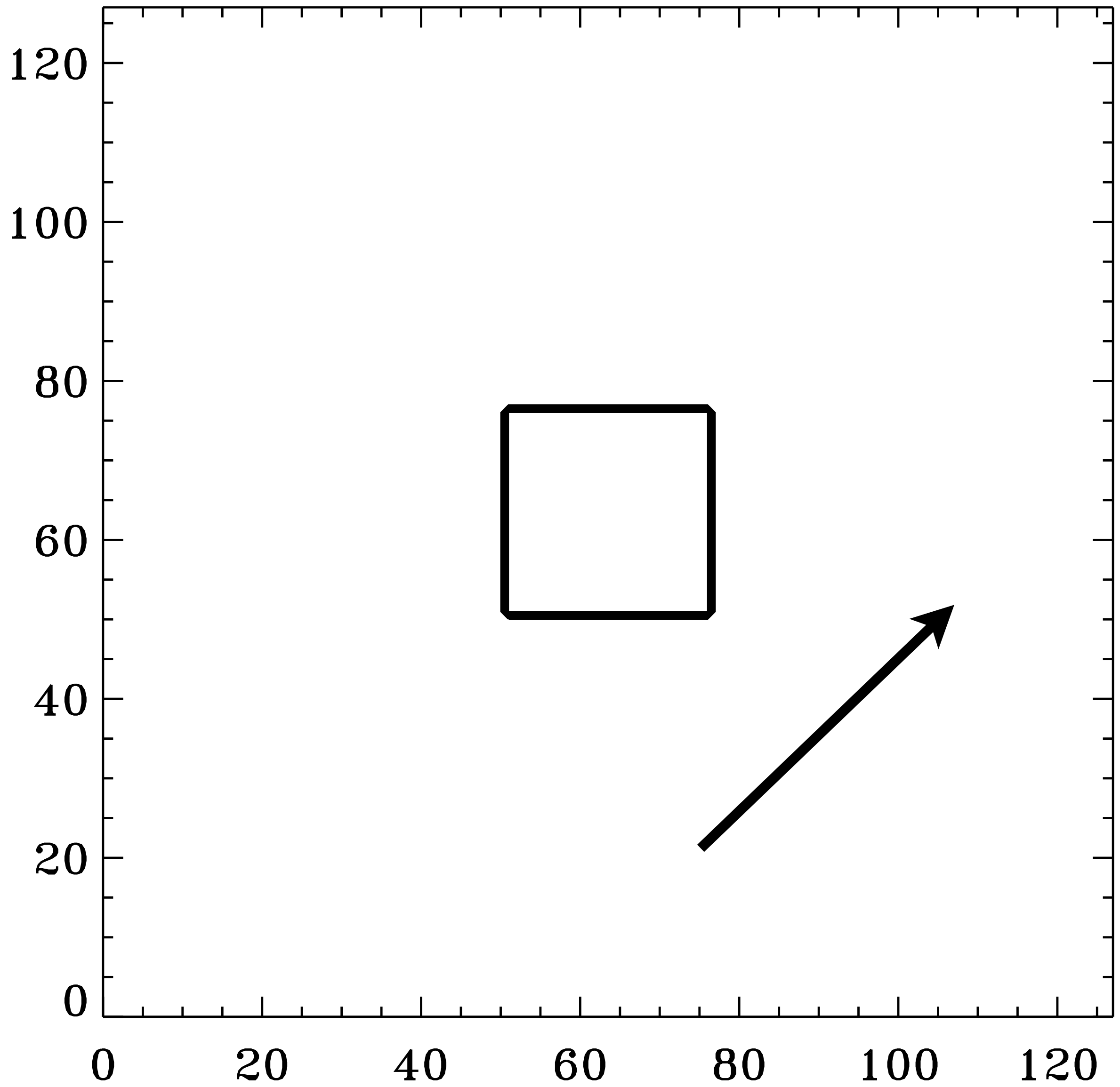
Flux functions computed using 1D Riemann problem at time $t^{n+1/2}$ in each normal direction.

4 Riemann solves per step.

Courant condition: $\max(C_x, C_y) < 1$

ATHENA

Advection of a square with Godunov scheme



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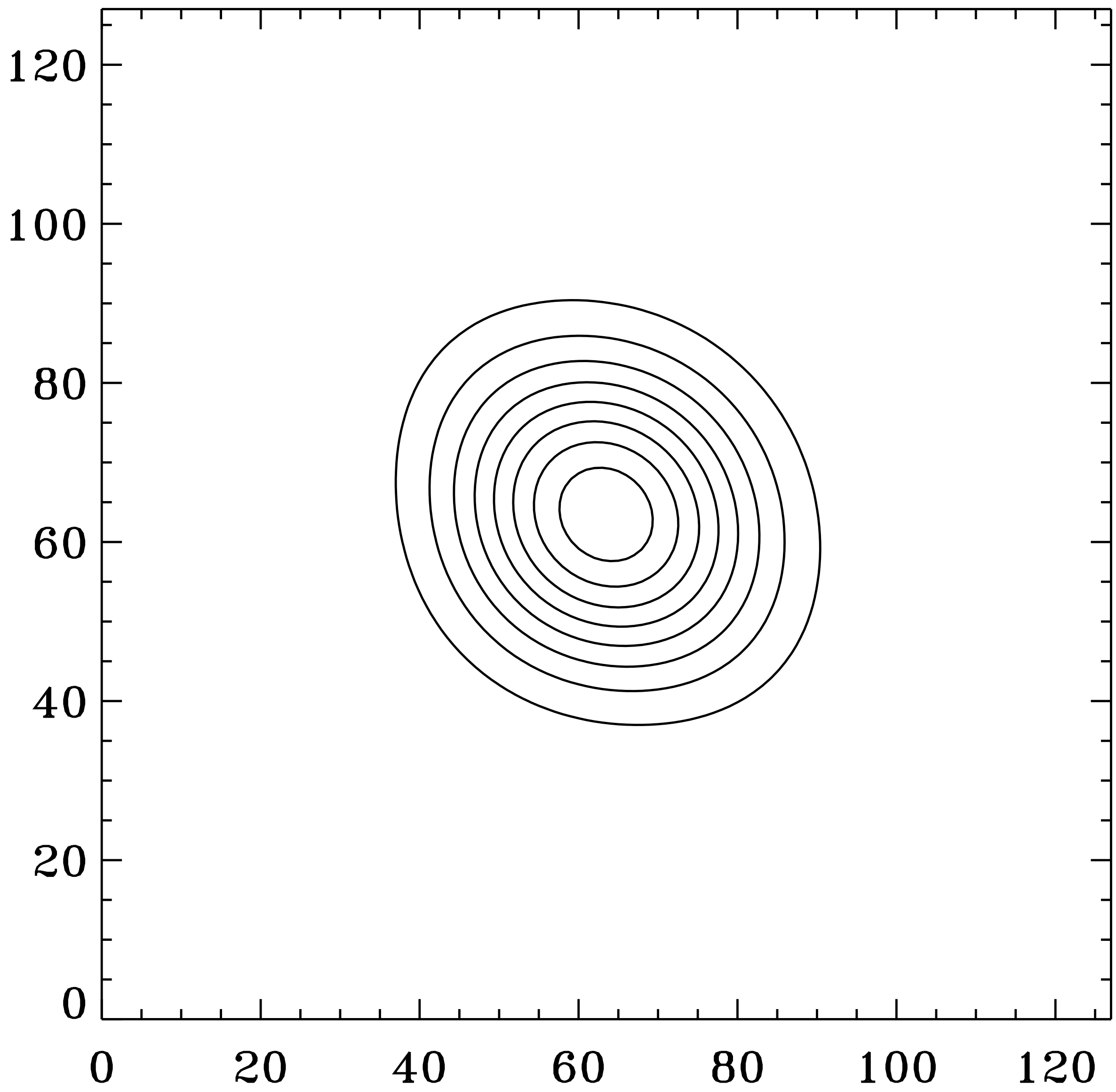
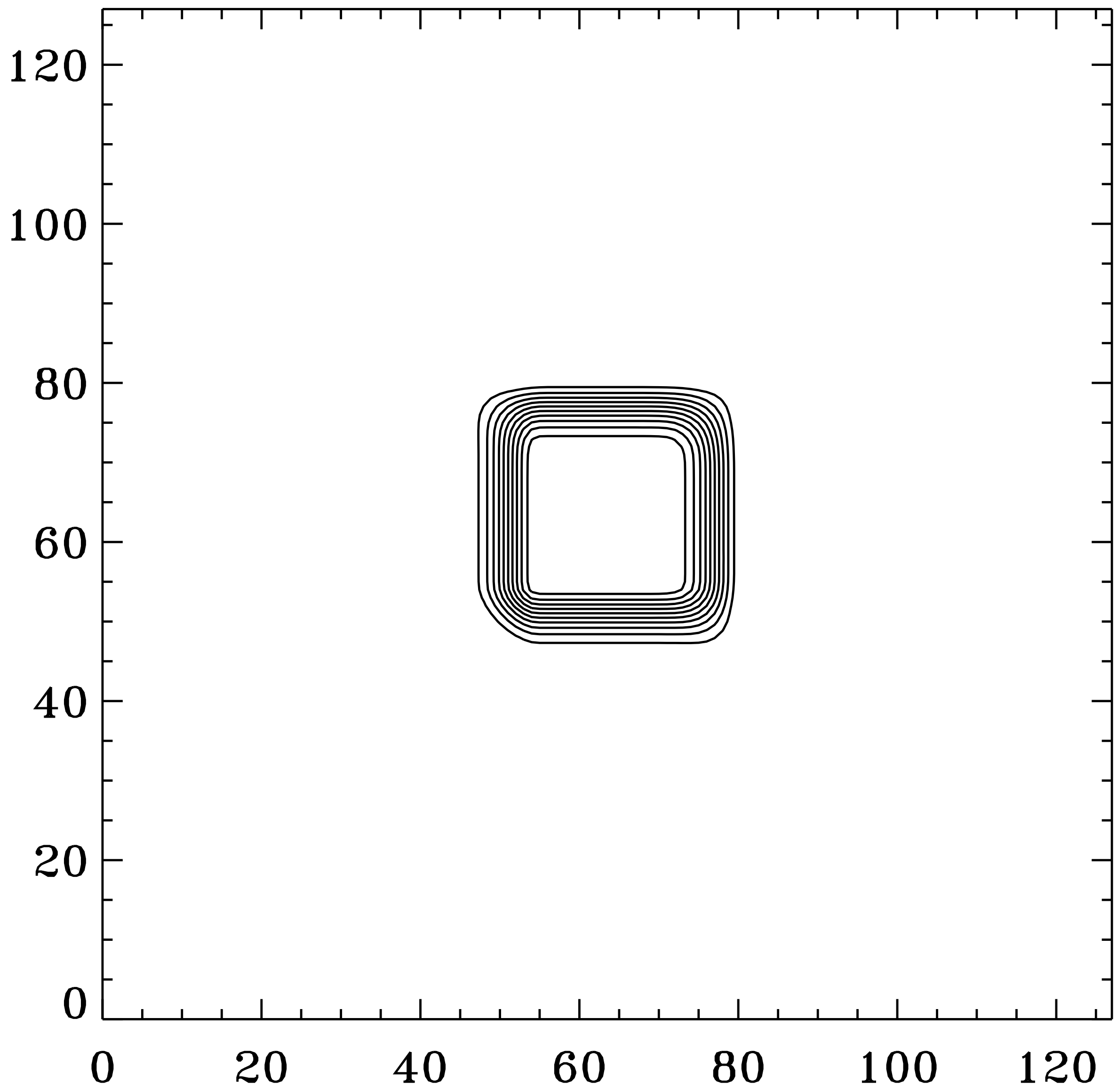


Second-order Godunov scheme



Outline

- The Euler equations
- Systems of conservation laws
- The Riemann problem
- The Godunov Method
- Riemann solvers
- 2D Godunov schemes
- Second-order scheme with MUSCL
- Slope limiters and TVD schemes
- Characteristics tracing and 2D slopes.

Godunov scheme for hyperbolic systems

The system of conservation laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = 0$$

is discretized using the following integral form:

$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} + \frac{\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

The time average flux function is computed using the self-similar solution of the inter-cell Riemann problem:

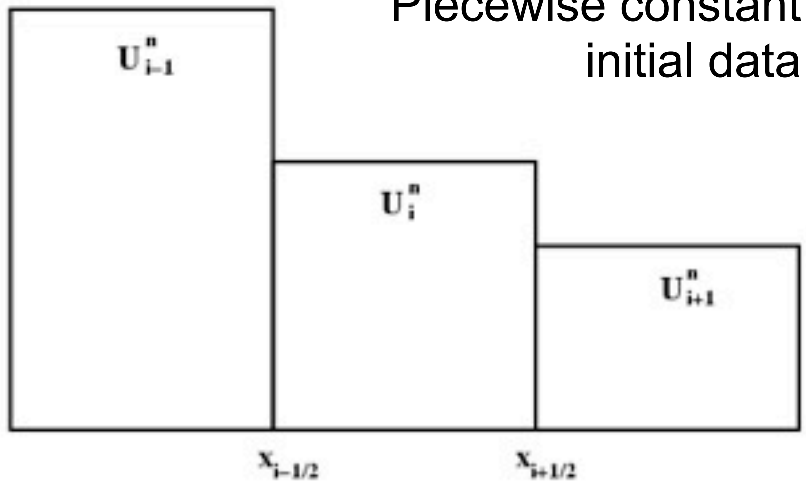
$$\mathbf{U}_{i+1/2}^*(x/t) = \mathcal{RP}\left[\mathbf{U}_i^n, \mathbf{U}_{i+1}^n\right]$$

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}(\mathbf{U}_{i+1/2}^*(0))$$

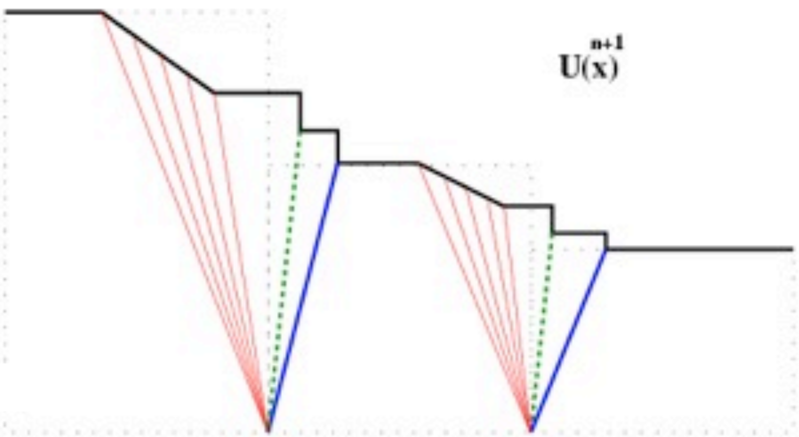
This defines the Godunov flux:

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}^*(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$$

Piecewise constant initial data



- **Godunov, S. K.** (1959), A Difference Scheme for Numerical Solution of Discontinuos Solution of Hydrodynamic Equations, *Math. Sbornik*, *47*, *271-306*, translated US Joint Publ. Res. Service, JPRS 7226, 1969.



Advection: 1 wave, Euler: 3 waves, MHD: 7 waves

Higher Order Godunov schemes

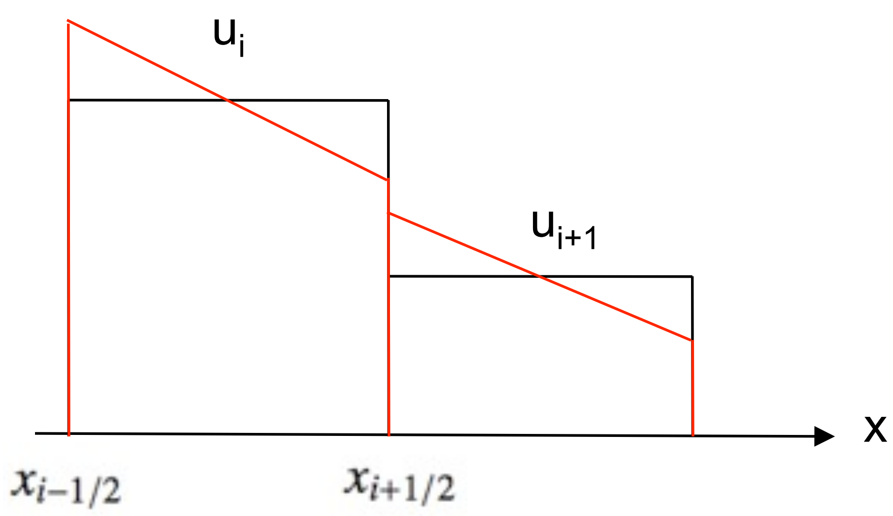
Godunov method is stable but very diffusive. It was abandoned for two decades, until…

Bram Van Leer

- **van Leer, B.** (1979), Towards the Ultimate Conservative Difference Scheme, V. A Second Order Sequel to Godunov's Method, *J. Com. Phys.*., *32*, 101–136.

Second Order Godunov scheme

Piecewise linear approximation of the solution:



The linear profile introduces a length scale: the Riemann solution is not self-similar anymore:

$$\mathbf{F}^{n+1/2}_{i+1/2} \neq \mathbf{F}(\mathbf{U}^*_{i+1/2}(0))$$

The flux function is approximated using a *predictor-corrector* scheme:

$$\mathbf{F}^{n+1/2}_{i+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}(x_{i+1/2}, t)\,\mathrm{d}t \quad \Longrightarrow \quad \mathbf{F}^{n+1/2}_{i+1/2} \simeq \mathbf{F}(\mathbf{U}^*_{i+1/2}(\frac{\Delta t}{2}))$$

The *corrected* Riemann solver has now *predicted* states as initial data:

$$\mathbf{U}^*_{i+1/2}(x/t) = \mathcal{RP}\left[\mathbf{U}^{n+1/2}_{i+1/2,L}, \mathbf{U}^{n+1/2}_{i+1,R}\right]$$

Summary: the MUSCL scheme for systems

Compute second order predicted states using a Taylor expansion:

$$\begin{cases} \mathbf{W}^{n+1/2}_{i+1/2,L} = \mathbf{W}^n_i + \dfrac{\Delta t}{2}\left(\dfrac{\partial \mathbf{W}}{\partial t}\right)_i + \dfrac{\Delta x}{2}\left(\dfrac{\partial \mathbf{W}}{\partial x}\right)_i \\ \mathbf{W}^{n+1/2}_{i+1/2,R} = \mathbf{W}^n_{i+1} + \dfrac{\Delta t}{2}\left(\dfrac{\partial \mathbf{W}}{\partial t}\right)_{i+1} - \dfrac{\Delta x}{2}\left(\dfrac{\partial \mathbf{W}}{\partial x}\right)_{i+1} \end{cases}$$

$$\begin{cases} \mathbf{W}^{n+1/2}_{i+1/2,L} = \mathbf{W}^n_i + (\mathbf{I} - \mathbf{A}\dfrac{\Delta t}{\Delta x})\dfrac{\Delta x}{2}\left(\dfrac{\partial \mathbf{W}}{\partial x}\right)_i \\ \mathbf{W}^{n+1/2}_{i+1/2,R} = \mathbf{W}^n_{i+1} - (\mathbf{I} + \mathbf{A}\dfrac{\Delta t}{\Delta x})\dfrac{\Delta x}{2}\left(\dfrac{\partial \mathbf{W}}{\partial x}\right)_{i+1} \end{cases}$$

Update conservative variables using corrected Godunov fluxes

$$\mathbf{F}^{n+1/2}_{i+1/2} = \mathbf{F}^*(\mathbf{W}^{n+1/2}_{i+1/2,L}, \mathbf{W}^{n+1/2}_{i+1/2,R}) \qquad \frac{\mathbf{U}^{n+1}_i - \mathbf{U}^n_i}{\Delta t} + \frac{\mathbf{F}^{n+1/2}_{i+1/2} - \mathbf{F}^{n+1/2}_{i-1/2}}{\Delta x} = 0$$

Monotonicity preserving schemes

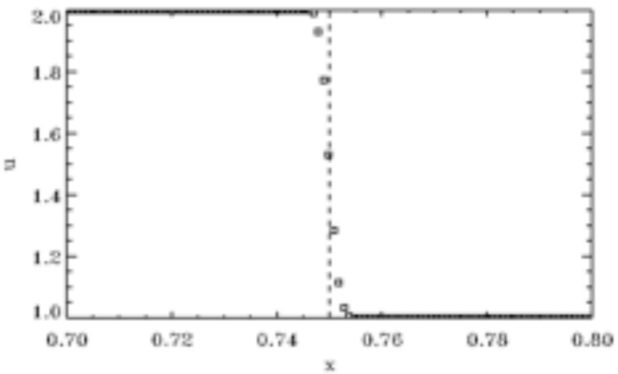
We use the central finite difference approximation for the slope:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{\Delta u_i}{\Delta x} = \frac{u_{i+1} - u_{i-1}}{2}$$

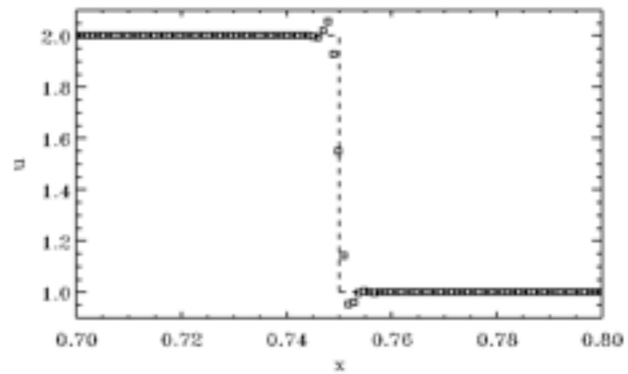
Second order linear scheme.

In this case, the solution is oscillatory, and therefore non physical.

first order



second order



Oscillations are due to the *non monotonicity* of the numerical scheme.

A scheme is monotonicity preserving if:

- No new local extrema are created in the solution

- Local minimum (maximum) non decreasing (increasing) function of time.

Godunov theorem: only first order linear schemes are monotonicity preserving !

Slope limiters

Harten introduced the Total Variation of the numerical solution:

$$TV^n = \sum_i^n |u_{i+1} - u_i|$$

Harten's theorem: a Total Variation Diminishing (TVD) scheme is monotonicity preserving.

$$TV^{n+1} \leq TV^n$$

Design non-linear TVD second order scheme using slope limiters:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{\Delta u_i}{\Delta x} = \lim(u_{i-1}, u_i, u_{i+1})\left(\frac{u_{i+1} - u_{i-1}}{2}\right)$$

where the slope limiter is a non-linear function satisfying:

$$0 \leq \lim(u_{i-1}, u_i, u_{i+1}) \leq 1$$

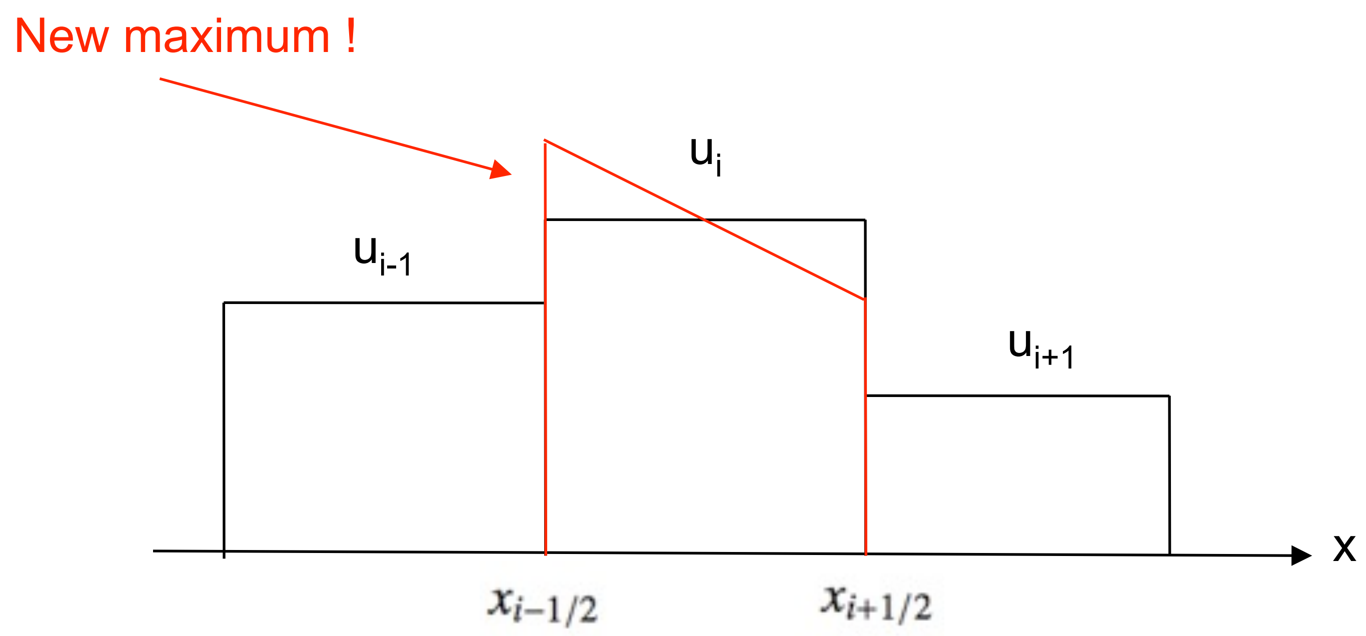
- Harten, Ami (1983), "High resolution schemes for hyperbolic conservation laws", *J. Comput. Phys* **49**: 357-393, doi:10.1006/jcph.1997.5713

No local extrema

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{\Delta u_i}{\Delta x} = \lim(u_{i-1}, u_i, u_{i+1})\left(\frac{u_{i+1} - u_{i-1}}{2}\right)$$

We define 3 local slopes: left, right and central slopes

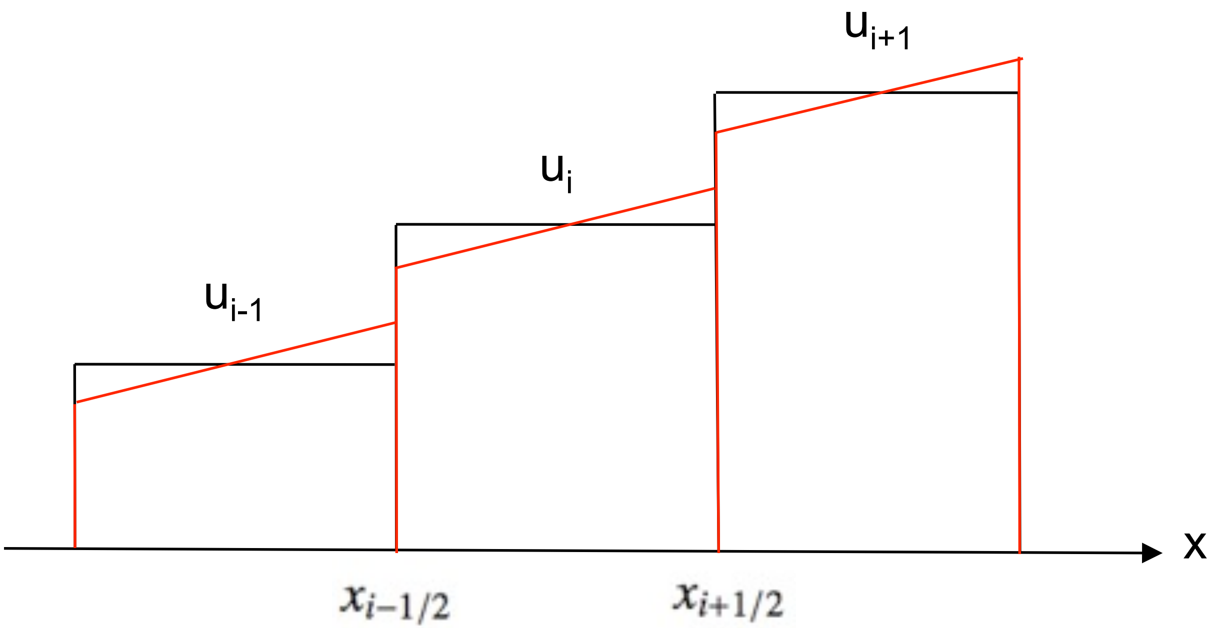
$\Delta u_L = u_i - u_{i-1}$ $\Delta u_R = u_{i+1} - u_i$ and $\Delta u_C = \dfrac{u_{i+1} - u_{i-1}}{2}$



For all slope limiters: $\Delta u_i = 0 \quad \text{if} \quad \Delta u_L \Delta u_R < 0$

The *minmod* slope

Initial reconstructed solution must be monotonous.



Linear reconstruction is monotone at time t^n

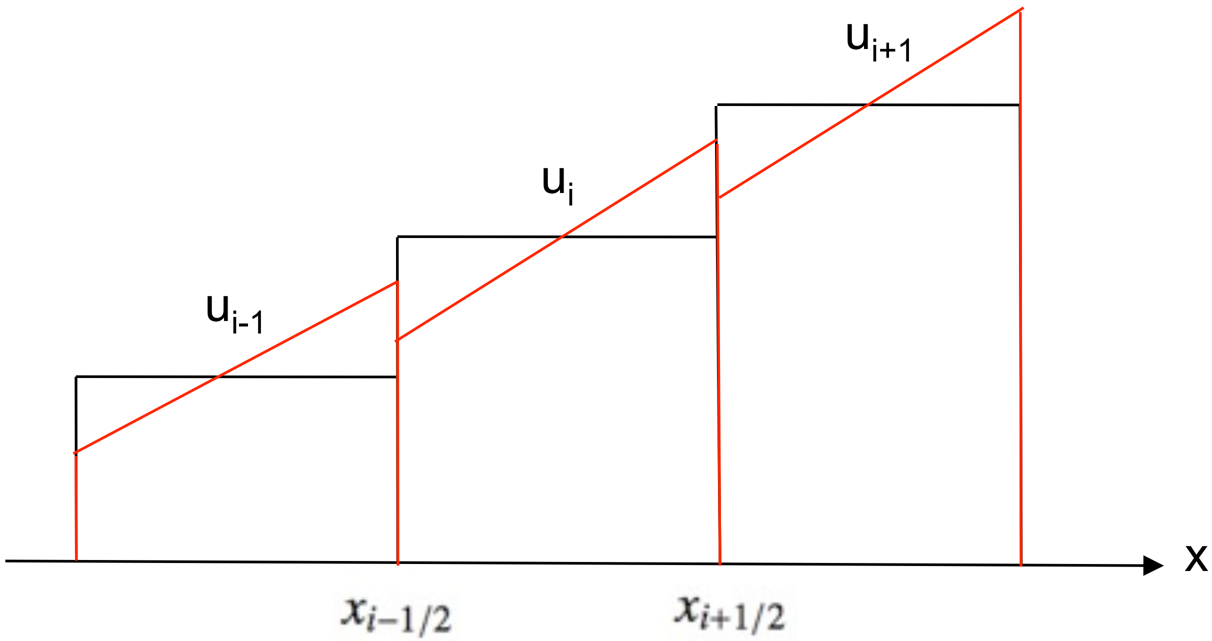
$$u^n_{i+1/2,L} = u^n_i + \frac{\Delta u_i}{2} \qquad u^n_{i-1/2,R} = u^n_i - \frac{\Delta u_i}{2}$$

Minmod slope limiting is never truly second order !

`slope_type=1` $\quad u^n_{i+1/2,L} \leq u^n_{i+1/2,R} \qquad \Delta u_i = \min(\Delta u_L, \Delta u_R)$

The *moncen* slope

Initial reconstructed states must be bounded by the initial average states.



Extreme values must be bounded by the *initial average* states.

$$u^n_{i-1/2,R} = u^n_i - \frac{\Delta u_i}{2} \qquad u^n_{i+1/2,L} = u^n_i + \frac{\Delta u_i}{2}$$

$$u^n_{i-1} \leq u^n_{i-1/2,R} \leq u^n_i \qquad u^n_i \leq u^n_{i+1/2,L} \leq u^n_{i+1}$$

`slope_type=2` $\Delta u_i = \min(2\Delta u_L, \Delta u_C, 2\Delta u_R)$

The *superbee* slope

Predicted states must be bounded by the initial average states.

$$u_{i+1/2,L}^{n+1/2} = u_i^n + (1-C)\frac{\Delta u_i}{2}$$

$$u_{i+1/2,R}^{n+1/2} = u_{i+1}^n - (1+C)\frac{\Delta u_{i+1}}{2}$$

TVD constraint is preserved by the Riemann solver.

$$u_i^n \le u_{i+1/2,L}^{n+1/2} \le u_{i+1}^n$$

$$u_{i-1}^n \le u_{i-1/2,R}^{n+1/2} \le u_i^n$$

The Courant factor now enters the slope definition.

$$\Delta u_i = \min\left(\frac{2}{1+C}\Delta u_L, \frac{2}{1-C}\Delta u_R\right)$$

The ultrabee slope

Final complete solution must be bounded by the initial average states.

Use the final state to compute the slope limiter.

$$u_i^{n+1} = u_i^n(1-C) + u_{i-1}^n C - \frac{C}{2}(1-C)\left(\Delta u_i - \Delta u_{i-1}\right) = 0$$

Upwind Total Variation constraint.

$$u_{i-1}^n \le u_i^{n+1} \le u_i^n$$

Strict Total Variation preserving limiter.

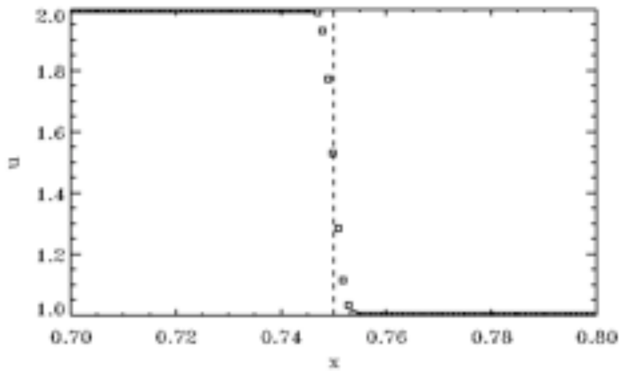
$$\text{if} \quad C > 0 \qquad \Delta u_i = \min\left(\frac{2}{C}\Delta u_L, \frac{2}{1-C}\Delta u_R\right)$$

$$\text{if} \quad C < 0 \qquad \Delta u_i = \min\left(\frac{2}{1+C}\Delta u_L, \frac{2}{-C}\Delta u_R\right)$$

Summary: slope limiters

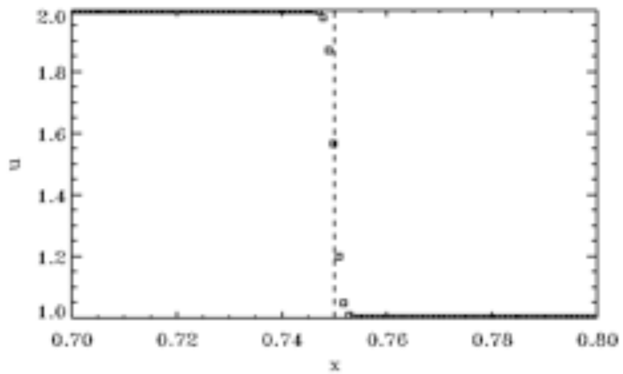
first order

slope_type=0



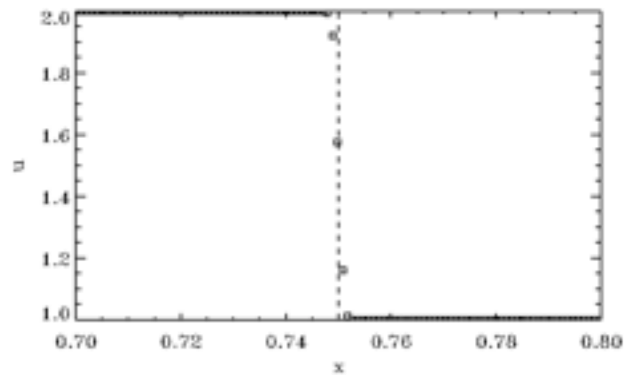
minmod

slope_type=1

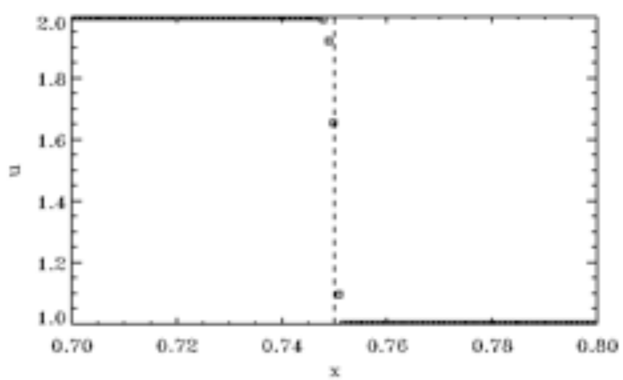


moncen

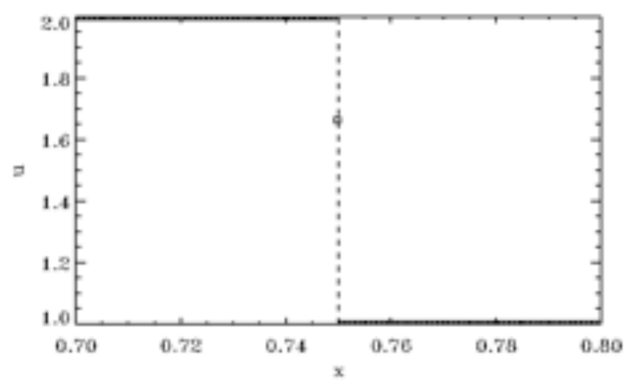
slope_type=2



superbee



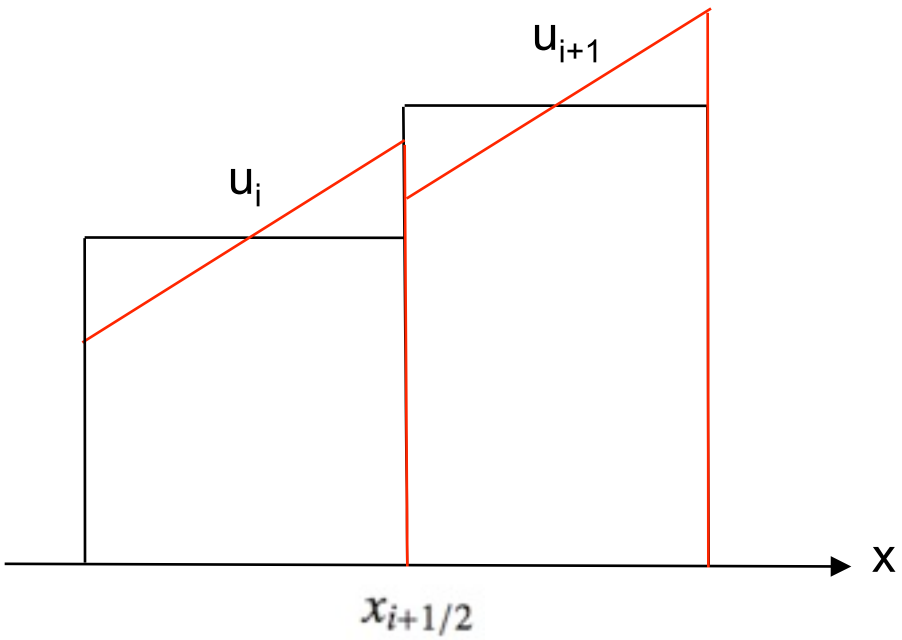
ultrabee



Summary: slope limiters

The previous analysis is valid only for the advection equation.

Non-linear systems: the wave speeds depend on the initial states (L and R).



MinMod is the only monotone slope limiter before the Riemann solver !

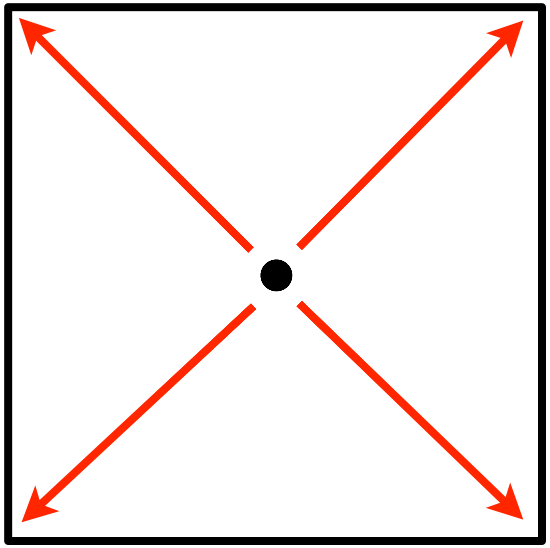
Superbee and Ultrabee must not be used for non-linear systems !

MonCen can be used, but with care: the characteristics tracing method.

2D slope limiter for unsplit MUSCL scheme

$$u^{n+1/2}_{i,j+1/2} = u^n_{i,j} - C_x \Delta_x u_{i,j} + (1 - C_y)\Delta_y u_{i,j}$$

$$u^{n+1/2}_{i+1/2,j} = u^n_{i,j} + (1 - C_x)\Delta_x u_{i,j} - C_y \Delta_y u_{i,j}$$



If 1D slope limiters are used, 2D schemes may become oscillatory.

Predicted states involve 2D neighboring cells.

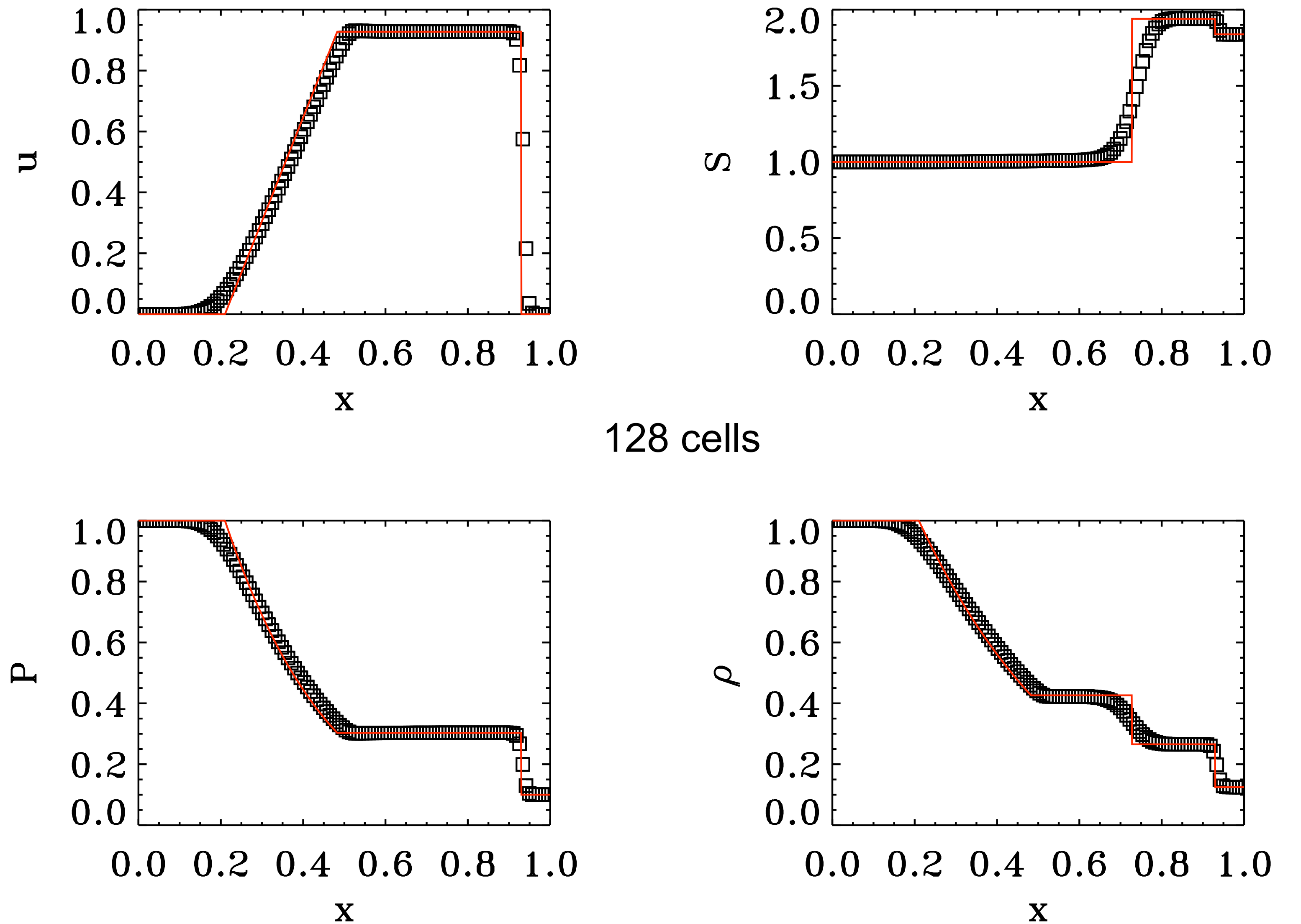
2D *moncen* slope: corner values must be bounded by the 8 neighboring initial values.

```
slope_type=3
```

Surech, Ambady, “Positivity Preserving Schemes in Multidimensions”, SIAM J. Sci. Comput., **22**, 1184-1198 (2000).

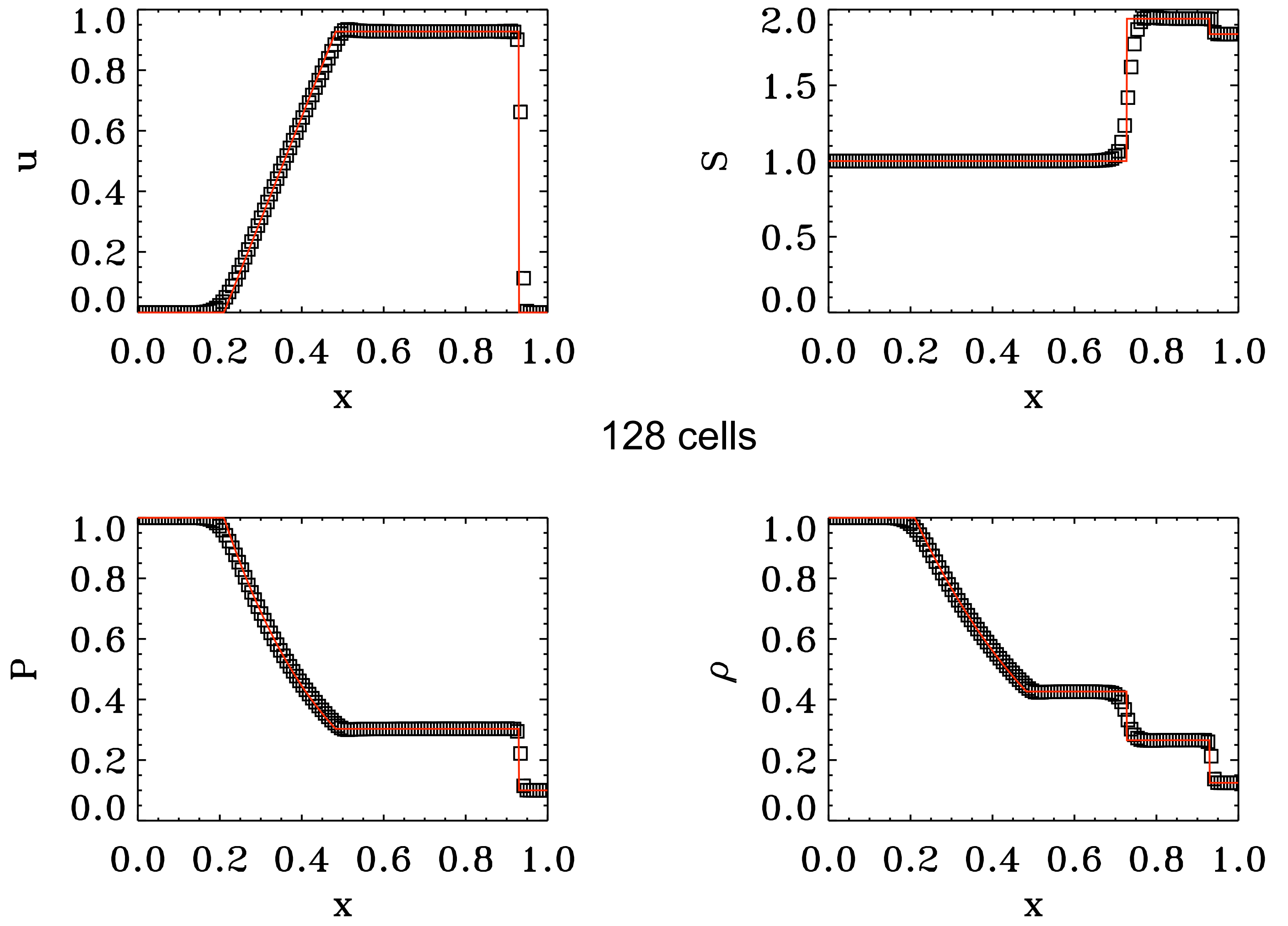
Sod test with HLLC first order

`slope_type=0`



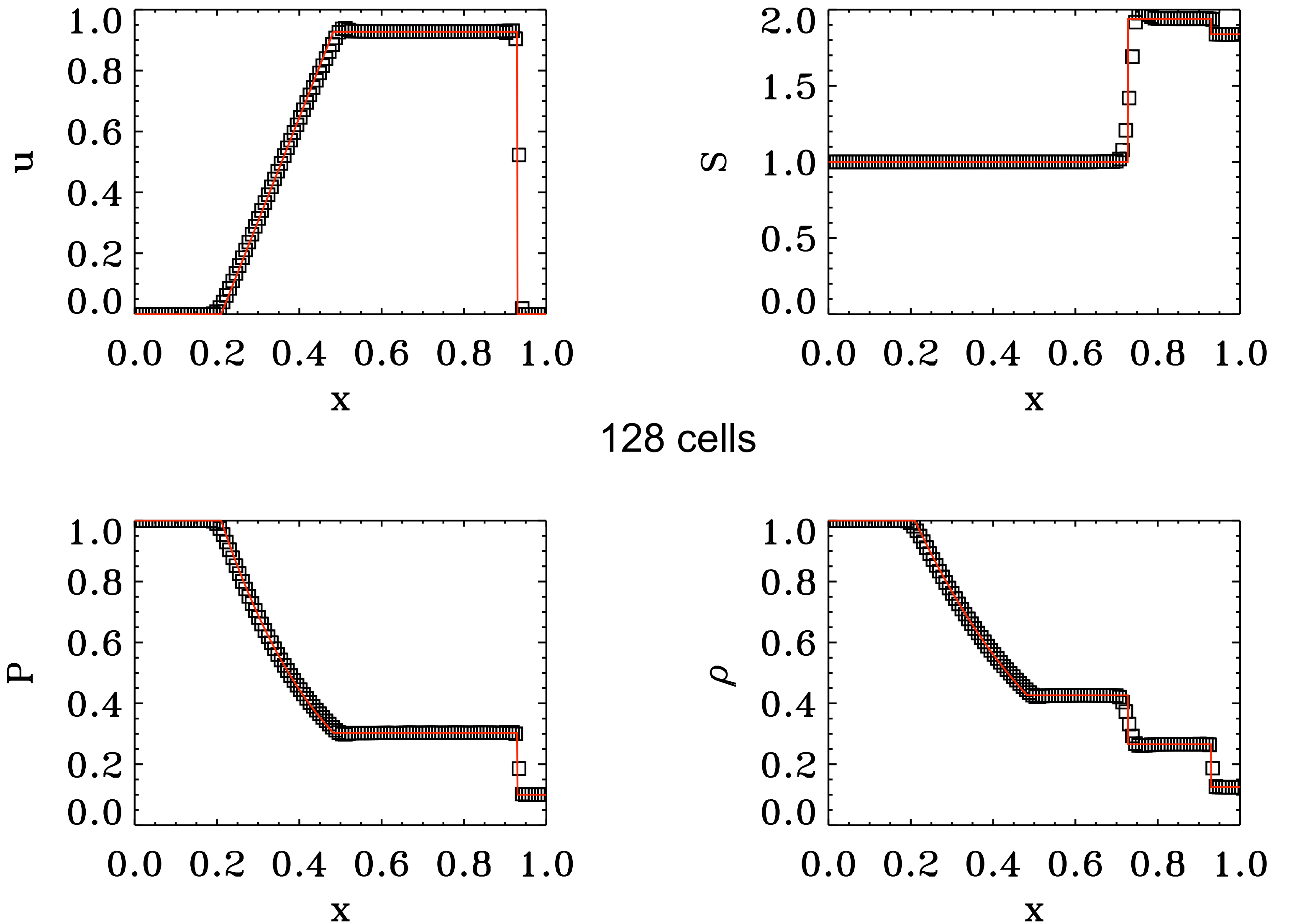
Sod test with HLLC and MinMod

slope_type=1



Sod test with HLLC and MonCen

slope_type=2



Beyond second order Godunov schemes ?

Smooth regions of the flow

More efficient to go to higher order.

Spectral methods can show *exponential convergence*.

More flexible approaches: use *ultra-high-order* shock-capturing schemes: 4th order scheme, ENO, WENO, discontinuous Galerkin and discontinuous element methods

Discontinuity in the flow

More efficient to refine the mesh, since higher order schemes drop to first order.

Adaptive Mesh Refinement is the most appealing approach.

What about the future ?

Combine the 2 approaches.

Usually referred to as "*h-p adaptivity*".

RAMSES Project 1
Supernova-driven MHD turbulence

Abstract: this project aims at reproducing a supernovae-driven turbulent flow, with properties close to the star forming interstellar medium in galaxies. You will first start with a simple 2D case, and then move to the full 3D case. You will study how an initially small seed magnetic field will be amplified by the turbulence.

RAMSES Project 2
Cooling halo and fragmenting disc

Abstract: this project aims at forming a galactic disc out of a cooling, collapsing halo. The properties of the resulting disc will be studied, as a function of the adopted thermodynamical model. Emphasis will be put on studying the properties of self-gravitating turbulence in the disc, as well as the amplification of the initial seed magnetic field.

RAMSES Project 3
Colliding flows and clumpy medium

Abstract: this project aims at producing a turbulent, multiphase, star-forming medium out of a colliding flow. Colliding flows occur in spiral galaxies, due to spiral shocks or large scale turbulence. Starting with a thermally unstable 2D flow, you will study the properties of the resulting turbulent medium. You will then move to a 3D, self-gravitating medium, with emphasis on studying the properties of the resulting protostellar clumps.