28.1 The statistical behavior for Bosons

As stated in the introduction to Fermions and Bosons, quantum statistics starts to play a role in dense system and low temperatures. For an atom at room temperature, the quantum volume is

\[ \epsilon_0 = \frac{\hbar^2}{8mL^2} (1^2 + 1^2 + 1^2) = \frac{3\hbar^2}{8mL^2} \]  

(28.1)

\[ N_0 = \frac{1}{e^{(\epsilon_0-\mu)/kT} - 1} \]  

(28.2)

When \( T \) is very small, \( N_0 \) will be quite large. In this case, the denominator must be small,

\[ N_0 = \frac{1}{1 + (\epsilon_0-\mu)/kT - 1} = \frac{kT}{\epsilon_0 - \mu} \quad \text{(when } N_0 \gg 1) \]  

(28.3)

The chemical potential \( \mu \) must be equal to \( \epsilon_0 \) at \( T=0 \), and just a bit less than \( \epsilon_0 \) at small \( T \). Now the question is at which temperature we can observe that \( N_0 \) remains very large?

28.2 Computing the total number of Bosons

\[ N = \sum_s \frac{1}{e^{(\epsilon_s-\mu)/kT} - 1} \]  

(28.4)

In practice, we can turn it to an integral,

\[ N = \int_0^\infty g(\epsilon) \frac{1}{e^{(\epsilon_s-\mu)/kT} - 1} d\epsilon \]  

(28.5)

Where \( g(\epsilon) \) is the density of states, which has a similar form following the electron gas model.

\[ g(\epsilon) = \frac{2}{\sqrt{\pi}} \left( \frac{2\pi m}{\hbar^2} \right)^{3/2} V \sqrt{\epsilon} \]  

(28.6)
The trouble is that we cannot evaluate eq.(28.5) analytically. In order to work it out, we must guess some value for the $\mu$ term. A good starting point is let $\mu=0$. Changing the variable to $x = \epsilon / kT$

$$N = \frac{2}{\sqrt{\pi}} \left( \frac{2\pi m}{\hbar^2} \right)^{3/2} V \int_0^\infty \sqrt{\epsilon} d\epsilon \frac{e^{\epsilon/kT}}{e^{\epsilon/kT} - 1}$$

$$= \frac{2}{\sqrt{\pi}} \left( \frac{2\pi mkT}{\hbar^2} \right)^{3/2} V \int_0^\infty \sqrt{x} dx \frac{e^x}{e^x - 1} \tag{28.7}$$

The integral over $x$ gives 2.315, which leaves us with

$$N = 2.612 \left( \frac{2\pi mkT}{\hbar^2} \right)^{3/2} V \tag{28.8}$$

That result is wrong! It means that the number of particles purely depends on $T$. In fact, there can be only one $T$ corresponds to this value.

$$N = 2.612 \left( \frac{2\pi mkT_c}{\hbar^2} \right)^{3/2} V \tag{28.9}$$

When $T < T_c$, this will be no longer valid during the transformation from summation to integral. This is because the terms from $\epsilon = 0$ are missing. Therefore, it should be better expressed as

$$N_{\text{exited}} = 2.612 \left( \frac{2\pi mkT_c}{\hbar^2} \right)^{3/2} V \tag{28.10}$$

While the gap between $N$ and $N_{\text{exited}}$ is the number of atoms in the ground state.

$$N_0 = N - N_{\text{exited}} = \left[ 1 - \left( \frac{T}{T_c} \right)^{3/2} \right] N \tag{28.11}$$

The abrupt accumulation of atoms in the ground state below $T_c$ is called Bose–Einstein condensation.
Figure 28.2: Number of atoms in the ground state \(N_0\) and in excited states, for an ideal Bose gas in a three-dimensional box. Below \(T_c\) the number of atoms in excited states is proportional to \(T^{3/2}\). Copyright 2000, Addison-Wesley

### 28.3 Real World Examples

In 1995 BEC of a gas of weakly interacting atoms was first achieved using Rb-87. Later, BEC was achieved with dilute gases of atomic Li, Na, H, etc.

The superfluid phase of \(^4\text{He}\) is also considered to be an example of BEC.

### 28.4 Why does it happen?