Physics 467/667: Thermal Physics **Spring 2019**

Lecture 6: The Second Law and Entropy

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6.1 Two Interacting Einstein Solids

In the previous section, we just learned how to count the Ω for an Einstein solid. Remember we are trying to understand how heats are transferred, which essentially at least two solids. Let's call the two solids A and B separately.

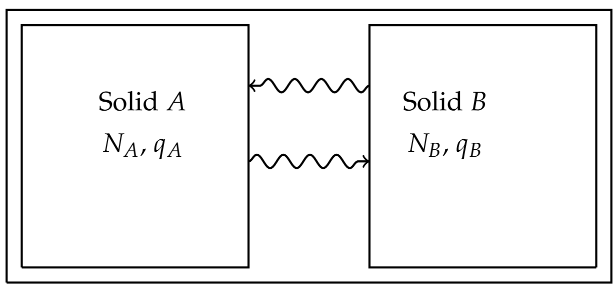

Figure 6.1: Two interacting Einstein solids isolated from the rest of the universe.

Assuming that A and B are weakly coupled (just like what we did on the ideal gas model), the individual energy units of the solids, q_A and q_B will change slowly. Under this assumption, the total number of energies q_{total} will be simple the sum of q_A and q_B.

To make life easier, let's fix q_{total}, what's the multiplicity for any arbitrary q_A? If we just count A,

$$\Omega(A) = \binom{q_A + N_A - 1}{q_A}, \tag{6.1}$$

In the meantime, we also needs to consider B,

$$\Omega(B) = \binom{q_B + N_B - 1}{q_B}, q_B = q_{\text{total}} - q_A. \tag{6.2}$$

Of course, the total number follows

$$\Omega(\text{total}) = \Omega(A)\Omega(B). \tag{6.3}$$

Exercises
Write a table of q_A, $\Omega(A)$, q_B, $\Omega(B)$, Ω(total), when q_A + q_B = 5, N_A=N_B=6.

$q(A)$	$\Omega(A)$	$q(B)$	$\Omega(B)$	Ω(total)
0				
1				
2				
3				
4				
5				

6.2 Stirling's Approximation

To apply these formulas to large systems, we need a trick for evaluating factorials of large numbers. Here is a trick called **Stirling's approximation**,

$$N! \approx N^N e^{-N} \sqrt{2\pi N} \tag{6.4}$$

This can be roughly understood that $N!$ is first approximated as N^N, then averaged by $(N/e)^N$,

$$N! \approx N^N e^{-N} \tag{6.5}$$

A more elegant way to express $N!$ is to use the so called **Gamma function**. Suppose you start with the integral,

$$\int_0^\infty e^{-ax} dx = 1/a \tag{6.6}$$

and differentiate repeatedly with respect to a, you will eventually get

$$\int_0^\infty x^n e^{-ax} dx = n! a^{-(n+1)} \tag{6.7}$$

Starting with this equation, you are able to prove eq 6.4. From the above, you can get the logarithm as follows

$$\ln N! \approx N \ln N - N - 1/2 \ln(2\pi N) \tag{6.8}$$

When N is very large, we can safely remove the last term,

$$\ln N! = N \ln N - N \qquad (\text{when } N \to \infty) \tag{6.9}$$

Alternatively, you can solve it in this way,

$$\begin{aligned} \ln N! &= \ln N + \ln(N-1) + \ln(N-2) + ... \\ &\approx \int_0^N \ln x dx \\ &= N \ln N - N - 1/2 \ln(2\pi N) \end{aligned} \tag{6.10}$$

6.3 Computer Programming

1. Write a code to calculate Ω as a function of q_A, when N_A=[300, 600, 3000, 6000], N_B=[200, 400, 2000, 4000], and q=100, plot them and try to find some tendency when N increases (hint: 4 plots).

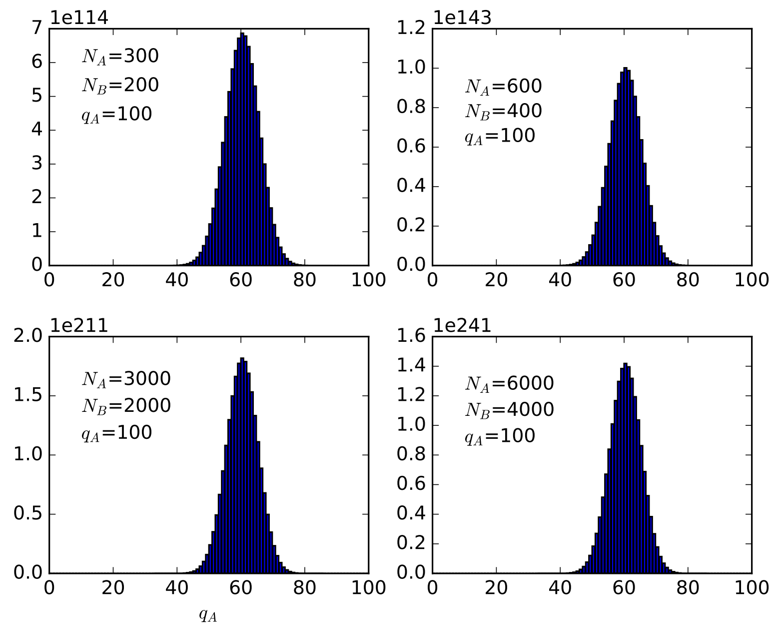

Figure 6.2: Ω as a function of N in two interacting Einstein solids.

2. Write a code to calculate the probability of $\Omega(q_A)$, when N_A=[300, 3000], N_B=[200, 2000], for q=[100, 1000], plot them and try to explain the differences. (hint: 2 plots)
3. Write a code to show the comparison of Stirling approximation in eq.6.10 and 6.9
4. The Gamma function is defined as

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx, \tag{6.11}$$

write a code to show the comparison of $\Gamma(n+1)$, $n!$, and $\sqrt{2\pi n}(n/e)^n$ in the range of [0,3.6]

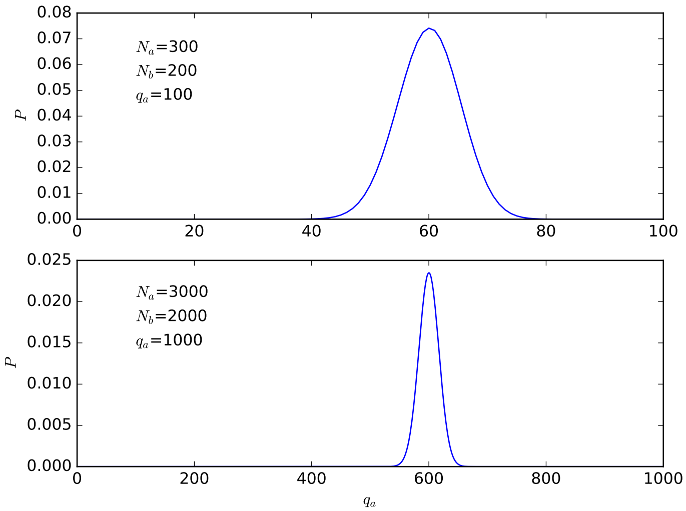

Figure 6.3: Probability distribution of $\Omega(N)$ in two interacting Einstein solids for different q values.

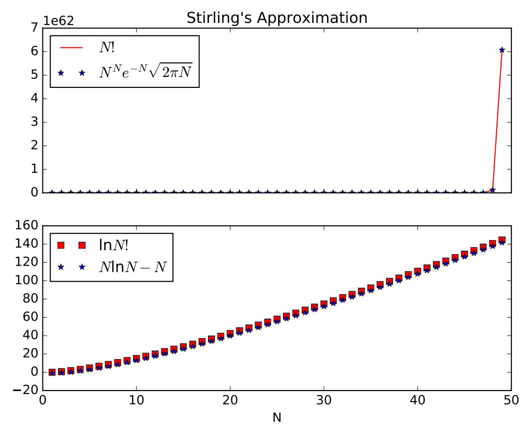

Figure 6.4: The accuracy of Stirling's approximation.

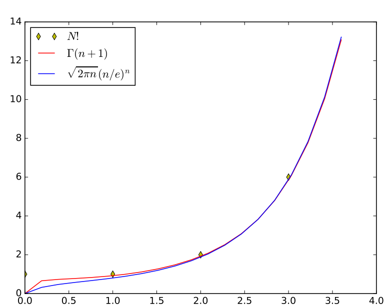

Figure 6.5: Comparison between the Gamma function and Stirling's approximation.