22.1 The critical exponents

A basic problem in the theory of phase transitions is to study the behavior of a given system in the neighborhood of its critical point. We know that this behavior is marked by the fact that the various physical quantities pertaining to the system possess singularities at the critical point. It is customary to express these singularities in terms of power laws characterized by a set of critical exponents that determine the qualitative nature of the critical behavior of the given system. To begin with, we identify an order parameter \( m \), and the corresponding ordering field \( h \), such that, in the limit \( h \to 0 \), \( m \) tends to a limiting value \( m_0 \), with the property that \( m_0 = 0 \) for \( T < T_c \) and \( \pm = 0 \) for \( T < T_c \). For a magnetic system, the natural candidate for \( m \) is the parameter \( L = \bar{\sigma} \) of Sections 12.5 and 12.6, while \( h \) is identified with the quantity \( \mu B/kT_c \); for a gasliquid system, one may adopt the density differential \( (\rho_l - \rho_c) \) or \( (\rho_g - \rho_c) \) for \( m \) and the pressure differential \( (P - P_c) \) for \( h \). The various critical exponents are then defined as follows.

The manner in which \( m_0 \to 0 \), as \( T \to T_c \) from below, defines the exponent \( \beta \):

\[
m_0 \sim (T_c - T)^\beta \quad (h \to 0, T < T_c)
\] (22.1)

at the same time, the coefficient \( q(t), r(t), s(t) \cdots \) maybe written as

\[
\chi_0 \sim \frac{\partial m}{\partial h} \sim \begin{cases} (T - T_c)^\gamma & (h \to 0, T \geq T_c) \\ (T_c - T)^\gamma' & (h \to 0, T < T_c) \end{cases}
\] (22.2)

The equilibrium value of the order parameter is then determined by minimizing \( \psi_0 \) with respect to \( m_0 \); retaining terms only up to the order displayed in (1), which for thermodynamic stability requires that \( s(t) > 0 \), we obtain

\[
m\big|_{T = T_c} \sim h^{1/\delta} \quad (T = T_c, h \to 0)
\] (22.3)

in the case of a gasliquid system, \( \delta \) is a measure of the “degree of flatness of the critical isotherm at the critical point, for then

\[
|P - P_c| \sim |\rho - \rho_c|^{\delta} \quad (T = T_c, P \to P_c)
\] (22.4)

Finally, we define exponents \( \alpha \) and \( \alpha' \) on the basis of the specific heat, \( C_V \), of the gas-liquid system:

\[
C_V \sim \begin{cases} (T - T_c)^{-\alpha} & (T \geq T_c) \\ (T_c - T)^{-\alpha'} & (T \leq T_c) \end{cases}
\] (22.5)
In connection with the foregoing relations, especially equations (5), we wish to emphasize that in certain cases the exponent in question is rather small in value; it is then more appropriate to write

\[ f(t) \sim |t|^{-\lambda - \frac{1}{\lambda}} \quad (|t| \ll 1) \]  

Table 2.1 Experimental Data on Critical Exponents

<table>
<thead>
<tr>
<th>Critical Exponents</th>
<th>Magnetic Systems(a)</th>
<th>Gas–liquid Systems(b)</th>
<th>Binary Fluid Mixtures(c)</th>
<th>Binary Alloys(d)</th>
<th>Ferroelectric Systems(e)</th>
<th>Superfluid He^4(f)</th>
<th>Mean Field Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha, \alpha' )</td>
<td>0.0–0.2</td>
<td>0.1–0.2</td>
<td>0.05–0.15</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-0.026</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.30–0.36</td>
<td>0.32–0.35</td>
<td>0.30–0.34</td>
<td>0.305 ± 0.005</td>
<td>0.33–0.34</td>
<td>---</td>
<td>1/2</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1.2–1.4</td>
<td>1.2–1.3</td>
<td>1.2–1.4</td>
<td>1.24 ± 0.015</td>
<td>1.0 ± 0.2</td>
<td>inaccessible</td>
<td>1</td>
</tr>
<tr>
<td>( \gamma' )</td>
<td>1.0–1.2</td>
<td>1.1–1.2</td>
<td>---</td>
<td>1.23 ± 0.025</td>
<td>1.23 ± 0.02</td>
<td>inaccessible</td>
<td>1</td>
</tr>
<tr>
<td>( \delta )</td>
<td>4.2–4.8</td>
<td>4.6–5.0</td>
<td>4.0–5.0</td>
<td>---</td>
<td>---</td>
<td>inaccessible</td>
<td>3</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.62–0.68</td>
<td>---</td>
<td>---</td>
<td>0.65 ± 0.02</td>
<td>0.5–0.8</td>
<td>0.675</td>
<td>1/2</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0.03–0.15</td>
<td>---</td>
<td>---</td>
<td>0.03–0.06</td>
<td>---</td>
<td>---</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Stierstadt et al. (1990).
(b) Voronel (1976); Rowlinson and Swinton (1982).
(c) Rowlinson and Swinton (1982).
(d) Als-Nielsen (1976); data pertain to beta-brass only.
(e) Kadanoff et al. (1967); Lines and Glass (1977).
(f) Ahlers (1980).

\[ f(t) \sim \ln(1/|t|) \quad (|t| \ll 1) \]  

For gas-liquid system obeying van der Waals equation, the various critical exponents are the same,

\[ \beta = \frac{1}{2}, \quad \gamma = \gamma' = 1, \quad \delta = 3, \quad \alpha = \alpha' = 0 \]  

In Table 2.1 we have compiled experimental data on critical exponents pertaining to a variety of systems including the ones mentioned above; for completeness, we have included here data on another two exponents, \( \nu \) and \( \eta \) which will be defined in Section 12.12. We find that, while the observed values of an exponent, say \( \beta \), differ very little as one goes from system to system within a given category (or even from category to category), these values are considerably different from the ones following from the mean field approximation. Clearly, we need a theory of phase transitions that is basically different from the mean field theory.