Brachistochrone inside the Earth: The Gravity Train

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1 Goldstein 2.6 Solution

Find the Euler-Lagrange equation describing the brachistochrone curve for a particle moving inside a spherical Earth of uniform mass density. Obtain a first integral for this differential equation by analogy to the Jacobi integral h. With the help of this integral, show that the desired curve is a hypocycloid (the curve described by a point on a circle rolling on the inside of a larger circle). Obtain an expression for the time of travel along the brachistochrone between two points on Earth's surface. How long would it take to go from New York to Los Angeles (assumed to be 4800 km apart on the surface) along a brachistochrone tunnel (assuming no friction) and how far below the surface would the deepest point of the tunnel be [1]?

2 Potential Energy inside the Earth

First we need an expression for potential energy as a function of r inside of the Earth. The integrated mass at a distance r from the center of the Earth is just

$$M = \int_0^r 4\pi r^2 \rho dr = \frac{4}{3}\pi r^3 \rho \tag{1}$$

If the mass density, ρ, is constant throughout the Earth then

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \tag{2}$$

where R is the radius of the Earth and M is the total mass of the Earth. Expressing acceleration due to gravity in terms of (1) and (2) gives

$$F(r) = \frac{GMm}{r^2} = \frac{GM}{R^3} r \tag{3}$$

if m is the mass of our train.

Potential is simply the integral of force and so becomes

$$U(r) = \int_0^r F(r) dr = \frac{GMm}{2R^3} r^2 \tag{4}$$

In order to find the velocity, we will write and expression for the total energy (which is a constant):

$$E = T + U = \frac{1}{2} m v^2 + \frac{GMm}{2R^3} r^2 \tag{5}$$

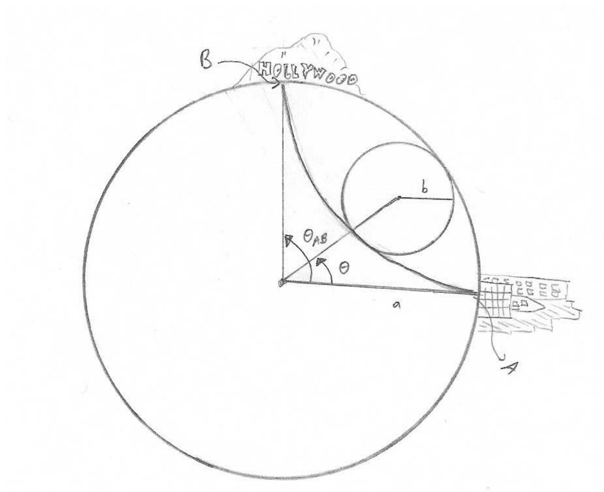

Figure 1: Hypocyloid curve.

letting $v = 0$ at $r = R$ gives the total energy and then solving for v gives:

$$v = \sqrt{\frac{g(R^2 - r^2)}{R}} \tag{6}$$

where I have used the constant

$$g \equiv \frac{GM}{R^2} \tag{7}$$

3 Minimization

I want to minimize the time traveled by the train in the tunnel. The time required for the train to slide through the tunnel from point A to point B is

$$t = \int_A^B \frac{ds}{v} \tag{8}$$

where ds is the distance along the path.

3.1 Solution in Polar Coordinates

In polar coordinates, taking θ to be the angle measured from point A as shown in figure 1, polar coordinate transforms apply:

$$x = r\cos\theta \tag{9}$$

$$y = r\sin\theta \tag{10}$$

$$r^2 = x^2 + y^2 \tag{11}$$

The distance element becomes (with a little bit of differentiation and algebra):

$$ds = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} = \sqrt{r^2 + r'^2} \tag{12}$$

where $r' = dr/d\theta$.

We then wish to minimize the time, (8),

$$t = \int_A^B \frac{ds}{v} = \int_0^{\theta_{AB}} \sqrt{\frac{(r^2 + r'^2)R}{g(R^2 - r^2)}} \tag{13}$$

subject to the constraints

$$r(0) = R \tag{14}$$

$$r(\theta_{AB}) = R \tag{15}$$

Since the integral is independent of the variable θ, we may use the "second form" of the Euler-Lagrange equation or the Beltrami identity:

$$f - r'\frac{\partial f}{\partial r'} = C \tag{16}$$

With some simplification from Mathematica,

$$\frac{r^2\sqrt{\frac{R(r'^2 - r^2)}{g(r^2 - R^2)}}}{r^2 - r'^2} = C \tag{17}$$

and solving for r' (again using the help of Mathematica) gives two roots:

$$r' = \pm\frac{r\sqrt{2C^2 + \frac{r^2}{g(r-R)} - \frac{r^2}{g(r+R)}}}{\sqrt{2}C} \tag{18}$$

This equation can then be integrated to show a hypocycloid, but not easily. See [2] for a full solution in polar coordinates.

3.2 Solution in Cartesian Coordinates

It may have been easier to do this in cartesian coordinates. Going back to (8), in cartesian coordinates we have:

$$t = \int_A^B \frac{ds}{v} = \int_A^B \sqrt{\frac{(x'^2 + y'^2)R}{g(R^2 - (x^2 + y^2))}} \tag{19}$$

and since this equation doesn't depend on time, using the Beltrami identities again gives two coupled equations:

$$f - x'\frac{\partial f}{\partial x'} = C_1 = \frac{y'^2\sqrt{\frac{R(x'^2 - y'^2)}{g(R^2 - x^2 - y^2)}}}{y'^2 - x'^2} \tag{20}$$

$$f - y'\frac{\partial f}{\partial y'} = C_2 = \frac{x'^2\sqrt{\frac{R(x'^2 - y'^2)}{g(R^2 - x^2 - y^2)}}}{x'^2 - y'^2} \tag{21}$$

Adding (20) and (21) and squaring gives:

$$R(x'^2 + y'^2) = gC^2(R^2 - x^2 - y^2) \tag{22}$$

where

$$C = C_1 + C_2 \tag{23}$$

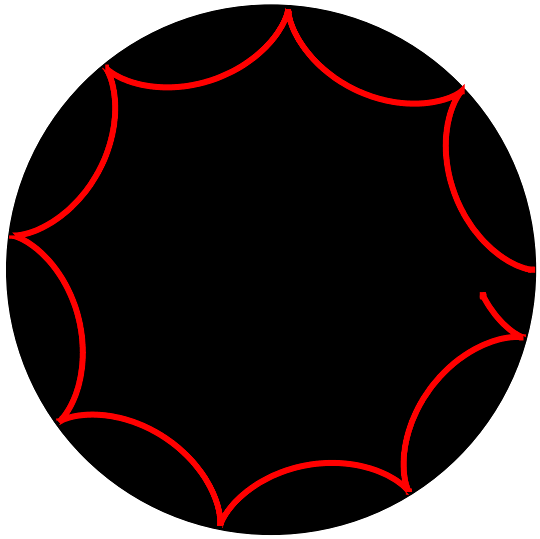

Figure 2: Solution for the train traveling from New York to L.A. and then around the world.

The solution to (22) is a hypocycloid parametrically described by:

$$x(t) = (R-b)\cos t + b\cos(\frac{R-b}{b}t) \tag{24}$$

$$y(t) = (R-b)\sin t - b\sin(\frac{R-b}{b}t) \tag{25}$$

with R being the radius of the outside circle and b the radius of the inside circle. Differentiating (24) and (25) and plugging back into (22) allows us to find the constant C in terms of b:

$$C = \sqrt{\frac{R(R-b)}{gb}} \tag{26}$$

4 Time Traveled, Depth of Tunnel and Length of Tunnel

To find the time traveled, we must simply re-integrate the time integral.

$$t = \int_A^B \frac{ds}{v} = \int_A^B \sqrt{\frac{(x'^2 + y'^2)R}{g(R^2 - (x^2+y^2))}} = \int_A^B C dt = \int_A^B \sqrt{\frac{R(R-b)}{bg}} dt \tag{27}$$

The time interval dt can be found in terms of b by noting the following:

$$r^2 = x^2 + y^2 = 2b^2 - 2bR + R^2 + 2b(R-b)\cos(\frac{Rt}{b}) \tag{28}$$

Knowing that each turn will bring the train back to the surface, so that $r = R$ gives (28) (with a little algebra) as:

$$1 = \cos(\frac{Rt}{b}) \tag{29}$$

or in other words,

$$\frac{Rt}{b} = 2\pi n \tag{30}$$

where n is an integer. Just looking at the first turn ($n = 1$) and solving (30) gives the time interval

$$dt = \frac{2\pi b}{R} \tag{31}$$

So our time integral becomes:

$$t = \sqrt{\frac{R(R-b)}{bg}} \frac{2\pi b}{R} \tag{32}$$

Left then is to find an expression for b in terms of the variables we know. From the section on hypocycloids, (41) gives the number of cusps, for one cusp, this gives

$$\theta_{AB} = \frac{2\pi b}{R} \tag{33}$$

Also from the arc length of a circle,

$$\theta_{AB} = \frac{s}{R} \tag{34}$$

where s is the distance between the two cities along the surface of the Earth. Therefore, combining (33) and (34),

$$b = \frac{s}{2\pi} \tag{35}$$

The time traveled is therefore:

$$t = \sqrt{\frac{R(R-b)}{bg}} \frac{2\pi b}{R} = \sqrt{\frac{s}{R} \frac{(2\pi R - s)}{g}} \approx 27.43 minutes \tag{36}$$

The maximum depth is just $2b \approx 1528km$ (see figure 1). The arc length, or the distance the train will travel while in the tunnel is 5380 km (see (40)).

The solution for a train traveling from New York to L.A. is plotted in figure 2. A similar solution in Cartesian coordinates can be found in [3].

Interestingly, a hypocycloid with $a/b = 2$ forms a straight line shortest time path solution through the center of the Earth taking

$$t = \pi \sqrt{\frac{R}{g}} \approx 42.24 minutes \tag{37}$$

It can be shown that any straight line solution will take this long, including the straight line connecting New York to L.A [4]. So our train gets passengers there about 15 minutes earlier, but the tunnel has to be much deeper.

5 Hypocycloids

A hypocycloid is the curve produced by following a small circle with radius a as it rolls around in a larger circle of radius b [5]. Parametrically, the hypocycloid can be described by the equations

$$x(t) = (a-b)\cos\theta + b\cos(\frac{a-b}{b}\theta) \tag{38}$$

$$y(t) = (a-b)\sin\theta - b\sin(\frac{a-b}{b}\theta) \tag{39}$$

where θ describes the angle traveled around the large circle. The path length [5], in terms of the angle traveled around the large circle, θ is

$$s(\theta) = \frac{8(a-b)b}{a} \sin^2(\frac{a}{4b}\theta) \tag{40}$$

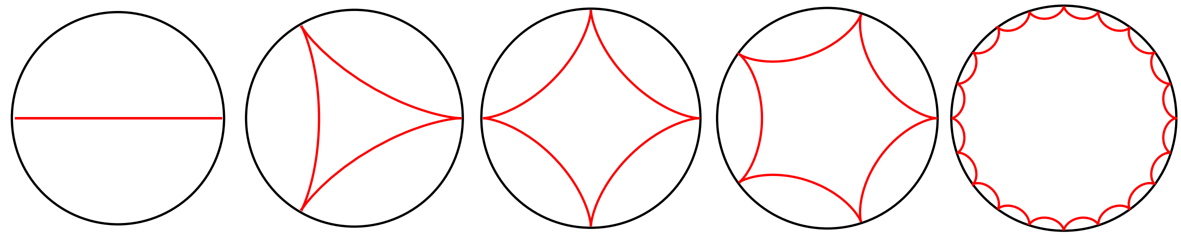

Figure 3: Hypocycloid for integer values of $a/b = 2$, 3, 4, 5, and 20.

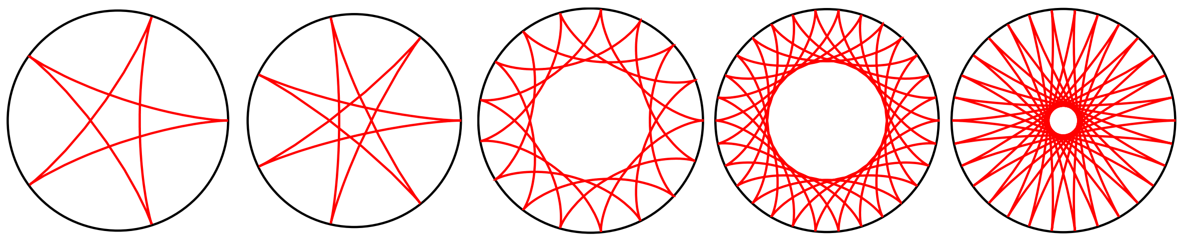

Figure 4: Hypocycloid for rational values of $a/b = \frac{5}{3}, \frac{7}{3}, \frac{17}{4}, \frac{30}{7}$, and $\frac{30}{13}$.

Various curves are plotted in figure 3 for integer values of a/b, from these figure you can see that the number of cusps the curve will have is

$$N(\theta) = \frac{a\theta}{2\pi b} \tag{41}$$

the number of times the curve hits the outer circle is

$$N(\theta) = Floor[\frac{a\theta}{2\pi b}] \tag{42}$$

For integer numbers of a/b, the curve will close after one turn around the big circle ($\theta = 2\pi$). For rational values of a/b, the curve will close, but after a cusps have been drawn, see figure 4.

References

[1] Goldstein, Poole and Safko. *Classical Mechanics*, 3rd edition, 2001.

[2] Smith, Donald R., *Variational Methods in Optimization*, 1974. Section 4.2.

[3] "Brachistochrone Inside the Earth"
http://www.phys.ufl.edu/ dorsey/phy4222/notes/brach_inside.pdf

[4] Cooper, Paul. Through the Earth in Forty Minutes. American Journal of Physics, Volume 34, Issue 1, pp. 68-70 (1966).

[5] Weisstein, Eric W. "Hypocycloid." From MathWorld–A Wolfram Web Resource.
http://mathworld.wolfram.com/Hypocycloid.html