Continuum Mechanics II

Volume and Surface Forces

We are now ready to tackle three dimensional forces as we did with one dimension in terms of the wire. We could do any shape, but for simplicity let's consider a small rectangular block as shown in Figure 1. The volume element has 6 surfaces and we will define the normal to a surface as the outward unit vector from that surface. The volume is $dV$ and the mass $\sigma dV$ where $\sigma$ is the mass per unit volume.

Most forces acting on this volume element are one of two main types, either volume or surface forces. Volume forces are something that acts over the whole volume like gravity or electro-magnetic forces. Surface forces act over the surface. Volume forces would usually be proportional to $dV$, whereas surface forces are usually proportional to $dA$, the area of the surface they act upon. Surface forces come in three main types. Pressure, force per area, which pushes in on the volume, Tension type forces, again a force per area, which is pulling on an area of the volume (this would perhaps come about if we modeled a thicker wire the tension would be treated this way), and finally shearing forces, forces which are along the surface rather then perpendicular to it.

When is pressure isotropic? Now we will prove that so long as a fluid has no shear forces then pressure will be isotropic.\(^2\)

To prove this we shall imagine a setup as shown in Figure 3. We have three square surfaces set up so that an edge on view makes an

\(^2\) A fluid that has no shear forces is an inviscid or ideal fluid. The viscosity of a real fluid characterizes the shear forces.
equilateral triangle. The end triangular pieces we will ignore and instead we will calculate the pressure forces on the three rectangular pieces, which must be normal to the surfaces and have a value directed inward of $F = pdA$ for each. We may also have a volume force and so the equation of motion for the element is

$$F_1 + F_2 + F_3 + F_V = ma$$

or

$$F_1 + F_2 + F_3 = ma - F_V.$$  

Now consider shrinking the whole setup by a factor $\lambda$, the terms on the left all scale with the area of a side shrink by $\lambda^2$, whereas the terms on the right all go with the volume of the element and shrink by $\lambda^3$. SO this equation becomes

$$\lambda^2(F_1 + F_2 + F_3) = \lambda^3(ma - F_V).$$

The only way this can possibly hold for all values of $\lambda$ is if the expressions in the parenthesis are both equal to zero. And so we have

$$F_1 + F_2 + F_3 = 0$$

These are each vectors and since the area of each face is the same the only way the components perpendicular to $\hat{n}_3$ of $F_1$ and $F_2$ can cancel is if the pressure is also the same. Since the direction is arbitrary the pressure must be independent of direction, that is isotropic.

**Stress, Strain and Elastic Moduli**

The surface forces inside a continuous body (solid, liquid or gas) can be expressed in terms of a three-dimensional tensor called stress. The displacements of a body caused by this stress can be expressed in terms of a second tensor called strain. The relation between these is the Elastic Moduli. The simplest application of this we have already seen is Hooke’s Law, $F = -kx$, which we have used to relate the tension (proportional to stress) in a spring to its displacement (proportional to strain).

Stress is defined as $F/A$, where $F$ is the force and $A$ the area over which it acts. We have already seen examples such as the pressure we just discussed.

$$\text{stress} = \frac{F}{A} = \text{pressure} \quad \text{[in a static fluid]}$$

Similarly for our first example in this chapter the stress on the wire would be the tension divided by the cross sectional area

$$\text{stress} = \frac{\text{tension}}{\text{area}} \quad \text{[in a wire]}$$
For a shearing force on a solid we would also get a stress

$$\text{stress} = \frac{\text{shearing force}}{\text{area}} \quad \text{[of a solid]}$$

**Strain** is a measure of the fractional change in dimensions in response to the stress. So for our examples, the pressure might induce a change in volume for the fluid

$$\text{strain} = \frac{dV}{V} \quad \text{[in a static fluid]}$$

Similarly for our stress on the wire could induce a change in length

$$\text{strain} = \frac{dl}{l} \quad \text{[in a wire]}$$

For a shearing force on a solid the shearing stress could induce a displacement of $dy$ over the length at which it occurs $dx$

$$\text{stress} = \frac{dy}{dx} \quad \text{[of a solid]}$$

**Elastic Moduli** are a way of relating the stress to the strain. If the stress are not too large then the strain in response should be linear. In this regime we will define a moduli. stress $= \text{modulus} \times \text{strain}$ For the fluid it is the Bulk Modulus, $BM$

$$dp = BM \frac{dV}{V}$$

For the wire it is Young’s Modulus, $YM$

$$\frac{dF}{A} = YM \frac{dl}{l}$$

This is our Hooke’s Law. Finally, for our shear we have shear modulus $SM$

$$\frac{F}{A} = SM \frac{dy}{dx}$$

In general

$$\text{stress} = \frac{\text{force}}{\text{Area}}$$

and

$$\text{strain} = \text{fractional deformation}$$

and

$$\text{elastic modulus} = \frac{\text{stress}}{\text{strain}}$$

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3 we could rewrite this $dF = kdl$ where $k = \left( \frac{YM}{A} \right)$ to make this clear.
Stress Tensor

Let us finish this lecture by making a general form for the stress. To do this we will first define a vector which represents our area

\[ d\mathbf{A} = \hat{n}dA \]

where \( \hat{n} \) is the unit vector perpendicular to the area \( d\mathbf{A} \). Then we can extend our argument above for pressure to show that all the surface forces are linear on these infinitesimal areas. This means we can relate the force vector and the area vector with a tensor

\[ \mathbf{F} = \Sigma d\mathbf{A} \]

where the tensor \( \Sigma \), the stress tensor, can be represented by a 3x3 matrix with components \( \sigma_{ij} \). In component form we have

\[ F_i = \sum_{j=1}^{3} \sigma_{ij}dA_j \]

For simplicity of discussion I shall call \( \hat{n}_1, \hat{n}_2, \) and \( \hat{n}_3 \) directions x, y and z. Then the \( \sigma_{11} \) gives us the force per area in the x direction for a \( dA \) whose normal is in the x direction. Similar for y and z on the diagonal. The \( \sigma_{12} \) gives a force per area in the x direction for a \( dA \) whose normal points in the y direction. The units are force per area.

The elements are not independent and again using an argument similar to our isotropic pressure proof above we can prove that

\[ \sigma_{ij} = \sigma_{ji} \]

And so the six off diagonal elements are represent only three independent elements.

The stress tensor for an ideal fluid must be only pressure forces, with no shear and thus no off diagonal elements and can be found from

\[ \mathbf{F}(d\mathbf{A}) = -p d\mathbf{A} \]

and so

\[ \Sigma = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \]

or just \( -p \) times the identity matrix.

As a Numerical Example we will consider a case where we are given the stress tensor

\[ \Sigma = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
with of course some relevant units of force/area. And we know our object has a surface in the plane defined by \( x + y + z = 0 \), then what is the force on the object given a \( dA \) in this plane. First, we need to find a normal to the plane and then multiply by \( \sigma \) to get the force.

The normal is \((1, 1, 1)\)\(^4\) We need to normalize this to get a unit vector of

\[
\hat{n} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

and now just multiply this by our \( \Sigma \) above and we get

\[
F(dA) = \frac{\sqrt{3}}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\]

Notice, the force is not in the normal direction, with shearing forces this is to be expected. By taking the dot product of the two vectors we can find the angle is a bit over 35°.

**problems**

Do chapter 16 problems 2, 4, 6, 16, 20.

**Support**

The website for the class is located at [http://physics.unlv.edu/~lepp/web/class/phy424](http://physics.unlv.edu/~lepp/web/class/phy424). On our website, you’ll find the syllabus for the course and problem assignments. I will be posting lectures, notes and assignments both there and on webcampus.

**References**


\(^4\) An easy way to find the normal is to take the cross product of two vectors in the plane. Since the origin, \((0,0,0)\), is clearly in the plane as is \((1,-1,0)\) and \((1,0,-1)\) we can take the two vectors as \((1,-1,0)\) and \((1,0,-1)\), the cross product gives us \((1, 1, 1)\).