Solution to project 1

Stephen Lepp

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First we set up conservation of energy to get

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mv_0^2 - \frac{GMm}{R_E}\,.$$

where G is the gravitational constant, M is mass of sun, m in mass of object and R_E is radius of earths orbit. We solve for v to get

$$v(r) = \pm\sqrt{v_0^2 + 2GM(1/R - 1/R_E)}\,.$$

The $\pm$ comes from the two possibilities for the square root, we clearly want the - solution as we are falling toward the sun. Which gives

$$\frac{dt}{dr} = 1/v(r)$$

or

$$t = \int_{R_E}^{r_S} \frac{-1}{\sqrt{v_0^2 + 2GM(1/R - 1/R_E)d}} dR$$

If we concentrate on the initial case first, $v_0 = 0$, then our expression is

$$t = \int_{R_E}^{r_S} \frac{-1}{\sqrt{2GM(1/R - 1/R_E)}} dR$$

which simplifies to (with the substitution $x = R/R_E$ and $\epsilon = r_S/R_E$)

$$t = \frac{R_E^3}{2GM} \int_1^{\epsilon} \frac{-1}{\sqrt{(1/x - 1)}} dx$$

or

$$t = \frac{R_E^3}{2GM}\int_\epsilon^1 \sqrt{\frac{x}{(1-x)}}dx$$

The integral has a closed from solution of

$$\sqrt{\epsilon-\epsilon^2} + \arccos(\sqrt{\epsilon})$$

but since the purpose of this project is numerical integration we will ignore that for now and proceed to evaluate it numerically.

The first problem with numerical evaluation of this integral is the integral has a singularity at the endpoint $x = 1$. This is a problem for two reasons. First, we cannot evaluate the endpoint. Second, it will reduce our accuracy of the integral, because this is none polynomial behavior. The first problem goes away for the other two starting velocities, but the second remains. The best way to handle the singularity is to remove it. So we subtract off the singularity, which behaves as $1/sqrt(1-x)$ so now we numerically evaluate

$$\int_\epsilon^1 \sqrt{\frac{x}{(1-x)}} - \sqrt{\frac{1}{(1-x)}}dx$$

and then add to it the results of

$$\int_\epsilon^1 \sqrt{\frac{1}{(1-x)}} = 2sqrt(1-e)$$

This makes our integral much easier to evaluate.

The new function is very well behaved, and with 1000 pts by midpoint method gives 7 places of accuracy compared with 2 for the original function. Romberg will converge even faster and it is now easy to write a program to do the problem.

A simple program that calculates the time is given in listing 1. In calculates the integral in its simplified form and adds the analytical result from the removed singularity.

We are now ready to proceed with the case where there is a starting velocity. Note there is still a nearby singularity, which will keep us from getting a accurate numerical integral unless we remove it. The singularity is no longer at RE, but will have moved to a location further from the Sun. We can solve for this new radius of zero kinetic energy by our energy conservation formula.

$$-GMm/R_z = \frac{1}{2}mv_0^2 - GMm/R_E$$

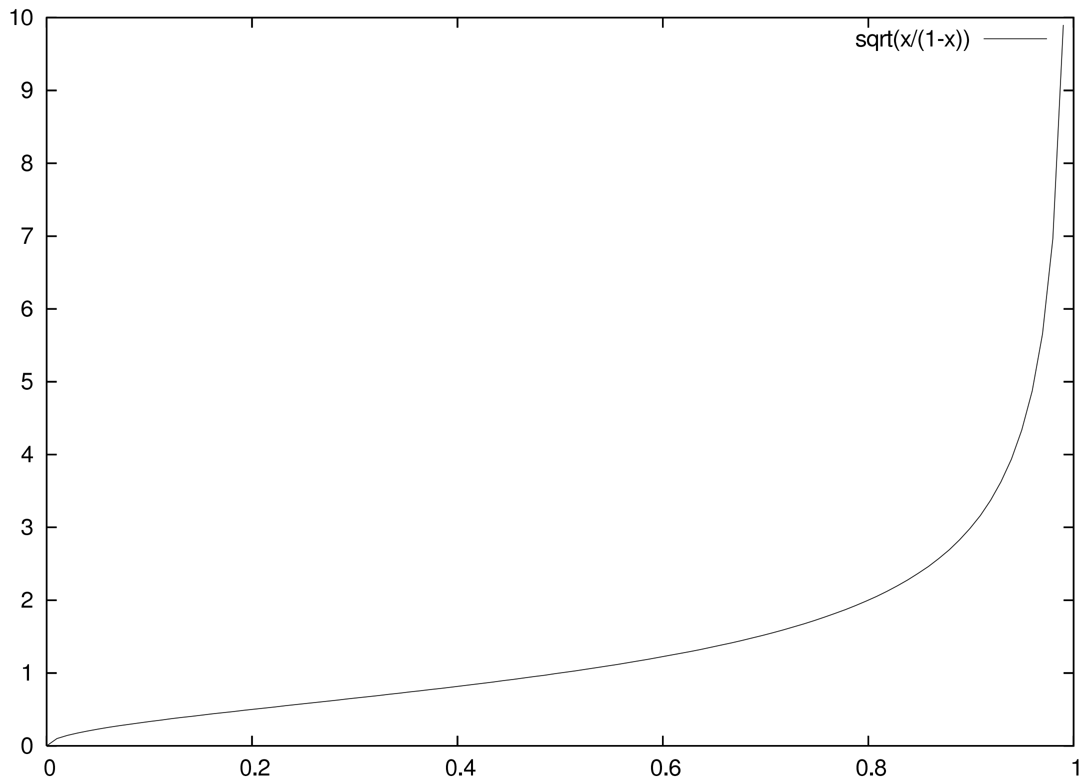

Figure 1: Original function $\sqrt{\frac{x}{1-x}}$

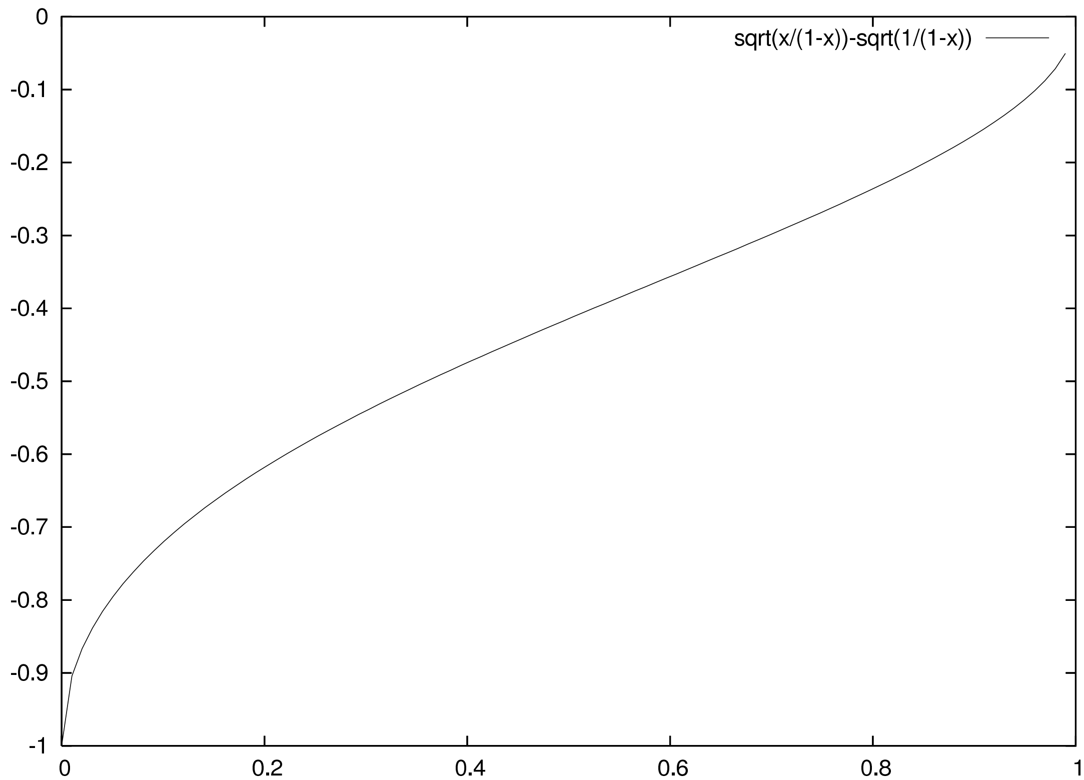

Figure 2: Modified function with singularity removed $\sqrt{\frac{x}{1-x}} - \sqrt{\frac{1}{1-x}}$

v_0 (m/s)	time (s)
0	5586052
10	5584366
1000	5421958

which solving for Rz gives

$$\frac{R_z = R_e}{1 - \frac{R_E v_0^2}{2GM}}$$

We now have the problem in the same form as for $v_0 = 0$, the only difference is we will now integrate from r_S/R_E to R_E/R_Z and use $\sqrt{\frac{R_Z^3}{2GM}}$ as our scaling factor. Making these changes to our program we have the version shown in listing 2, which can be used for all three initial cases.

As a final check, we can integrate all the way to the center of the sun, and compare that to the analytic result for the freefall time of an object the mass of the sun and the radius of the earth. We get 5586809.3716 for the integration compared with $\sqrt{\frac{\pi^2 R_E^3}{2^3 GM}}$ which is rounds to 5586809.3717.

Listing 1

```
import math
import Integrate

G=6.67e-11
M=2e30
rs=7e8
RE=1.5e11


def f1(x):
  if x==1 :return 0 #limit as x->1
  return math.sqrt(x/(1-x)) - math.sqrt(1.0/(1.-x))


eps=rs/RE
result= (Integrate.romberg(f1,eps,1.0,1e-10)+2*math.sqrt(1-eps) )
time=result*math.sqrt(RE**3/(2*G*M))
print "the time to fall is ",time
```

Listing 2

```
import math
import Integrate

G=6.67e-11
M=2e30
rs=7e8
RE=1.5e11
v0=input("v0 = ")
RZ=RE/(1.0-RE*v0**2/(2.0*G*M))



def f1(x):
  if x==1 :return 0 #limit as x->1
  return math.sqrt(x/(1-x)) - math.sqrt(1.0/(1.-x))


eps=rs/RE
result= (Integrate.romberg(f1,eps,1.0,1e-10)+
     2*(-math.sqrt(1-RE/RZ)+math.sqrt(1-eps)))
time=result*math.sqrt(RZ**3/(2*G*M))
print "the time to fall is ",time
```

Questions

What trouble do you run into in trying to evaluate this integral for $v_0 = 0$?

There is a singularity at R_E. It was removed as described in text.

Our formulation of the problem assumed v_0 was directed toward the Sun, how would it change if it was directed away?

It would be necessary to compute how far it went out, which is R_Z in this text and then the total time would be twice the integral from R_Z to R_E plus the integral from R_E to r_S.

Using Newton's Laws, set up a differential equation for r(t) directly?

$$m\frac{d^2\vec{R}}{dt^2} = -\frac{GMm}{R^2}$$