Gravitational Effect of Spherical Mass

\[ R = \text{radius of earth} \]
\[ M = \text{mass of earth} \]
\[ m = \text{mass of physics student} \]

\[ F_G = \frac{GM_1M_2}{d^2} \]

The gravitational force between two point masses.

What is the gravitational force acting on a physics student standing on Earth?

Each little piece of earth attracts the student.

The net force on the student equals the sum of all the little vector forces pulling the student.

Newton showed the the net force equals

\[ F_{\text{net}} = \frac{GMmM}{R^2} \]
1. Assume the density of earth is only a function of \( r \), the distance from the center. \( \rho = \rho(r) \)

The mass of a thin shell is

\[
\frac{4\pi r^2}{\text{Surface Area}} \cdot \rho(r) \cdot \Delta r = dm
\]

Volume of shell

\[
M = \int 4\pi r^2 \rho(r) \, dr = \text{Total Mass}
\]

2. First consider the force \( F_{\text{shell}} \) produced by one thin shell.

3. We do this by imagining the shell made up of rings.
Rings are good because a piece of the ring is the same distance from the mass m.

As the contributions from the ring are added together only the component of the force in the z-direction remains. The horizontal components cancel.

\[ dF_{\text{ring}} = \left( \text{mass of ring} \right) \frac{GM}{x^2} \]

4. Mass of ring = \( g(r) \) rd\( \phi \) dr \( r \sin \theta \) \( \frac{2\pi}{\text{circumference}} \)

\[ \text{width} \quad \text{thickness} \]

Volume of ring

Using the law of cosines:

\[ x^2 = r^2 + z^2 - 2rz \cos \phi \]

\( r \) and \( z \) are constant for the rings that make up the shell

* \( \phi \) changes from 0 to \( \pi \) as the contribution of each ring is summed.
Use the law of cosines again:

\[ r^2 = x^2 + z^2 - 2xz \cos \theta \]

\[ \cos \theta = \frac{x^2 + z^2 - r^2}{2xz} \]

5. 

\[ dF_{\text{ring}} = \frac{r^2}{2\pi} Gm \frac{dr}{r}\left(\frac{1}{x^2} + \frac{x^2 - r^2}{2xz}\right)\sin^2 \frac{(x^2 + z^2 - r^2)}{2xz}d\phi \]

\[ F_{\text{shell}} = \int_{0}^{2\pi} dF_{\text{ring}} \]

Replace \( x \) with \( \sqrt{r^2 + z^2 - 2rz \cos \phi} \):

\[ F_{\text{shell}} = 2\pi r^2 Gm \frac{dr}{r}\int_{0}^{\pi} \sin^2 \left(\frac{1}{x} + \frac{x^2 - r^2}{2xz}\right)d\phi \]

6. Two integrals:

\[ I_1 = \int_{0}^{\pi} \frac{\sin^2 \phi}{\sqrt{r^2 + z^2 - 2rz \cos \phi}} \sin \phi d\phi \]

\[ I_2 = \int_{0}^{\pi} (r^2 + z^2 - 2rz \cos \phi) \sin \phi d\phi \]

\[ dU = 2rz \sin \phi d\phi \]
\[ I_1 = \int \sin \phi \left( \frac{du}{2rz \sin \phi} \right) \frac{du}{u^{1/2}} = \frac{1}{2rz} \int \frac{du}{u^{1/2}} \]

\[ = \frac{1}{2rz} \left( +2u^{1/2} \right) \]

\[ = \frac{1}{rz} \left[ \sqrt{r^2 + z^2 - 2rz \cos \phi} \right]_0^\pi \]

\[ = \frac{1}{rz} \left[ \frac{\sqrt{r^2 + z^2 + 2rz} - \sqrt{r^2 + z^2 - 2rz}}{\sqrt{(z+r)^2} - \sqrt{(z-r)^2}} \right] \]

Remember \( z > r \)

\[ = \frac{1}{rz} \left[ z+r - (z-r) \right] = \frac{2r}{r^2} = \frac{2}{z} \]

\[ I_2 = (z^2 - r^2) \left( \frac{\sin \phi \cdot 8u}{(r^2 + z^2 - 2rz)^{3/2}} \right) \]

Same substitution for \( u \)

\[ I_2 = (z^2 - r^2) \left( \frac{\sin \phi \cdot d\phi}{2r z \sin \phi} \right) \frac{du}{u^{3/2}} \]

\[ = \frac{(z^2 - r^2)}{2rz} \left( \frac{du}{u^{3/2}} \right) \]
\[ I_2 = \frac{z^2 - l^2}{2r^2} \left[ -2 \theta \right] \]

\[ = -\frac{(z^2 - l^2)}{r^2} \left[ \frac{1}{\sqrt{z^2 + l^2 - 2zl \cos \theta}} \right]_{0}^{\pi} \]

\[ = -\frac{(z^2 - l^2)}{r^2} \left[ \frac{1}{\sqrt{(z+l)^2}} - \frac{1}{\sqrt{(z-l)^2}} \right] \]

\[ = -\frac{(z-l)(z+l)}{r^2} \left[ \frac{1}{z+l} - \frac{1}{z-l} \right] \]

\[ = -\frac{1}{r^2} \left[ z - l - (z+l) \right] = \frac{2}{r} \]

\[ I_1 + I_2 = \frac{4}{r} \]

\[ F_{\text{sheel}} = 4\pi r^2 g(r) G M dr \]

\[ F_{\text{sphere}} = \int_{0}^{R} F_{\text{sheel}} = \frac{GM}{Z^2} \left( 4\pi r^2 g(r) dr \right) \]

The Force produced by the sphere = \( \frac{GM}{Z^2} \)