Chapter 4 – Kinematics in Two Dimensions

This chapter combines motion in the x-direction with motion in y-direction. It also introduces relative motion in two dimensions and rotation about an axis. We will see that rotation is analogous to one-dimensional motion with s, the position, replaced by θ, the angle in radians; velocity in m/s becomes ω (omega) angular velocity measured in radians/s; and acceleration in m/s^2 becomes α (alpha) angular acceleration measured in radians/s^2.

Conceptually, two-dimensional motion combines the material in chapters 2 and 3. The position of an object is denoted by a position vector drawn from an origin that is selected by you to the object of interest,

$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$.

If $\mathbf{r}(t)$ is known, then it is easy to find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ by differentiating $\mathbf{r}(t)$ once for $\mathbf{v}(t)$ and twice for $\mathbf{a}(t)$. The vector velocity and the vector acceleration of the object will normally have components in the x and y-directions.

$\mathbf{v}(t) = d\mathbf{r}(t)/dt = dx/dt\,\mathbf{i} + dy/dt\,\mathbf{j} = v_x(t)\,\mathbf{i} + v_y(t)\,\mathbf{j}$, and

$\mathbf{a}(t) = d\mathbf{v}(t)/dt = dv_x/dt\,\mathbf{i} + dv_y/dt\,\mathbf{j} = a_x(t)\,\mathbf{i} + a_y(t)\,\mathbf{j}$.

On the other hand, if $\mathbf{a}(t)$ is given, we use $a_x(t)$ and $a_y(t)$ to find $v_x(t)$ and $v_y(t)$ Once $v_x(t)$ and $v_y(t)$ are known, they are used to find x(t) and y(t) just like we did in chapter 2. The equations from chapter 2 are re-written in terms of x and y, namely,

$v_x(t) = v_x(0) + \int_0^t a_x(t)\,dt$ and $v_y(t) = v_y(0) + \int_0^t a_y(t)\,dt$, and

$x(t) = x(0) + \int_0^t v_x(t)\,dt$ and $y(t) = y(0) + \int_0^t v_y(t)\,dt$,

where inside the integrals $a_x(t)$, $a_y(t)$, $v_x(t)$, and $v_y(t)$ are the components of the vector time-dependent velocity and acceleration in the x and y directions respectively.

If a_x and/or a_y are constant, we can use the simplified equations for constant acceleration to represent the position and velocity of the object in the direction or directions which have constant acceleration,

$v(t) = v(0) + a\,t$ and $s(t) = s(0) + v(0)\,t + \frac{1}{2}\,a\,t^2$.

The projectile motion problems in this chapter give you practice using vectors and the kinematic equations listed above. Notice that for projectile motion, $a_x = 0$ and $a_y = -9.8$ m/s^2 which can be approximated by -10 m/s^2 and the simplified equations describing motion with constant acceleration apply. Make sure you fully appreciate how the projectile problems can be used to reinforce your understanding of material already covered!

Relative motion is an example of vector addition. Suppose Alice and Bob are in Africa on safari. Alice is standing on the ground while Bob is riding in a jeep when a cheetah suddenly starts chasing a young wildebeest. Alice locates the position of the cheetah with a vector $\mathbf{r}_{Ann}(t)$ which stretches from her to the cheetah while Bob keeps track of the cheetah with a vector $\mathbf{r}_{Bob}(t)$ which stretches from his location in the jeep to the cheetah. For convenience assume that at t = 0 Bob and Alice were at the same spot and that Alice's x and y axes are parallel to Bob's x′ and y′ axes. From Alice's perspective Bob is located at $\mathbf{R}(t)$, the vector that goes from Alice to Bob in the jeep. Note that with the choices made earlier $\mathbf{R}(0) = 0$ since Alice and Bob are in the same spot at t = 0.

Draw a picture to convince yourself that the following vector equation describes the above scenario:

$\mathbf{r}_{Ann}(t) = \mathbf{R}(t) + \mathbf{r}_{Bob}(t)$.

It is easy find the velocity of the cheetah according to Alice, $\mathbf{v}_{Ann}(t) = d\mathbf{r}_{Ann}/dt$, or the velocity of the cheetah according to Bob, $\mathbf{v}_{Bob}(t) = d\mathbf{r}_{Bob}/dt$ by differentiating the above equation.

$\mathbf{v}_{Ann}(t) = \mathbf{V}(t) + \mathbf{v}_{Bob}(t)$ where $\mathbf{V}(t)$ is the velocity of the jeep according to Alice, or the relative velocity of Bob with respect to Alice. Relative motion problems all reduce to adding up vectors much like the ones introduced above.

When an object is moving in a circular path, its direction of motion is continuously changing. Since velocity is a vector, circular motion implies a continuously changing velocity vector. If the velocity vector is changing direction, the object is accelerating even when its speed is constant. For an object moving with constant speed v in a circle of radius r, the magnitude of this radial acceleration, which is called centripetal acceleration, is given by v^2/r. Also note that for uniform circular motion the total acceleration points toward the center of the circle. If the object's speed is changing while moving in a circle, the object has both radial (centripetal) and tangential acceleration and the net acceleration does not point toward the center of the circle but is instead the vector sum of the radial and tangential components.

To help picture rotational motion, imagine a nail stuck in a bicycle tire. If the bike is turned upside down and the wheel spun, the nail travels in circles about the wheel's axle. For concreteness assume the nail was located at the highest point of the tire at $t = 0$, then the distance traveled by the nail increases by $2\pi r$ every time the nail makes a complete rotation and returns to its starting point. If we let $s(t)$ be the distance traveled by the nail, the tangential velocity and the tangential acceleration of the nail are given by exactly the same equations introduced in chapter 2, namely, $v(t) = ds/dt$ and $a(t) = dv/dt$.

On the other hand, if $a(t)$, the tangential acceleration of the nail, is known, $v(t)$ and $s(t)$ can be found by integrating just like we did in chapter 2. For example, if the acceleration is constant,

$v(t) = v(0) + a\,t$ and $s(t) = s(0) + v(0)\,t + \frac{1}{2}\,a\,t^2$.

The only other thing we need to know to get the equations for angular motion is that the definition of an angle in radians is the ratio of the arc length along the rim of a circle divided by the radius, $\theta = s/r$. Divide the above equation for $v = ds/dt$ by r to get $v/r = d(s/r)/dt = d\theta/dt$, the angular velocity measured in radians/s. The Greek letter omega, ω, is typically used to denote the angular velocity ($\omega = d\theta/dt$). Now divide the defining equation for acceleration by r to get $a/r = d(v/r)/dt = d\omega/dt$, the angular acceleration. Since ω is the first derivative of θ with respect to time, the angular acceleration is the second derivative of θ with respect to time, $d^2\theta/dt^2$, measured in radians/s^2. The angular acceleration is typically represented by the Greek letter alpha, α.

Now use the same trick to turn the kinematic equations describing linear motion with constant acceleration into equations describing angular motion. That is, divide the equations for $v(t)$ and $s(t)$ by r to get,

$v(t)/r \rightarrow \underline{\omega(t) = \omega(0) + \alpha\,t}$, and $s(t)/r \rightarrow \underline{\theta(t) = \theta(0) + \omega(0)\,t + \frac{1}{2}\,\alpha\,t^2}$.

Again it is important and useful to understand that the equations governing angular motion are just another manifestation of one-dimensional motion!