Specific intensity and related quantities (e.g., energy density per unit wavelength) are conventionally given in three representations: photon energy representation $I_E$, frequency representation $I_\nu$, and wavelength representation $I_\lambda$. These representations are related by differential expression

$$I_E \, dE = I_\nu \, d\nu = I_\lambda \, (-d\lambda),$$

where the minus sign is occasionally omitted if one knows what one means—which is that a differential increase in photon energy/frequency corresponds to a differential decrease in wavelength.

b) Suggest two or three reasons why people might want to use the hybrid representation for graphing.

c) Planck’s law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2\hbar \nu^3}{c^2} \frac{1}{e^{x} - 1}, \quad \text{where} \quad x = \frac{\hbar \nu}{kT} = \frac{hc}{kT \lambda}. $$

Derive the energy representation $B_E$, wavelength representation $B_\lambda$, and the hybrid representation $EB_E = \nu B_\nu = \lambda B_\lambda$ in $E$, $\nu$ and $\lambda$ forms.

d) Derive the Rayleigh-Jeans law (small $x$, small $E$, small $\nu$, large $\lambda$ approximation) and the Wien approximation (large $x$, large $E$, large $\nu$, small $\lambda$ approximation) for $B_E$, $B_\nu$, and $B_\lambda$. Hint: This pretty easy albeit tedious.

**SUGGESTED ANSWER:**

a) First,

$$I_E \, dE = I_\nu \, d\nu = I_\lambda \, (-d\lambda)$$

$$EI_E \, d[\ln(E)] = \nu I_\nu \, d[\ln(\nu)] = \lambda I_\lambda \{ -d[\ln(\lambda)] \}.$$

Second,

$$E = \hbar \nu = hc/\lambda$$

$$\ln(E) = \ln(h\nu) = \ln(hc/\lambda)$$

$$d[\ln(E)] = d[\ln(\nu)] = -d[\ln(\lambda)].$$

Dividing the first result by the second gives the required result:

$$EI_E = \nu I_\nu = \lambda I_\lambda \quad \text{QED.}$$

b) First, since $EI_E = \nu I_\nu = \lambda I_\lambda$, there is no wondering about how the values would differ if you graphed the one instead of the other since they are all the same. They hybrid representation is neutral. Second, if you use a logarithmic horizontal axis (which is often convenient for large energy/frequency/wavelength bands), you can integrate up energy by eye which is useful for quick estimates. Third, for the energy and frequency representations, there is often an exponential decline as you go beyond the peak. Among other things, this is due to the inverse exponential behavior of the Planck spectrum beyond the peak: so thermal or semi-thermal will exhibit a rapid decline beyond the peak. If there is a rapid decline beyond the peak, using
the hybrid representation can flatten the spectrum and save you from needing an ugly large
total vertical range to see the whole spectrum.

c) Behold:

\[
B_E = B_\nu \frac{d\nu}{dE} = \frac{2E^3}{h^3c^2} \frac{1}{e^\frac{x}{c^2} - 1}
\quad \text{and} \quad
B_\lambda = -B_\nu \frac{d\nu}{d\lambda} = \frac{2hc^2}{\lambda^5} \frac{1}{e^\frac{x}{c^2} - 1},
\]

and so

\[
EB_E = \frac{2E^4}{h^3c^2} \frac{1}{e^\frac{x}{c^2} - 1} = \nu B_\nu = \frac{2hv^4}{c^2} \frac{1}{e^\frac{x}{c^2} - 1} = \lambda B_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^\frac{x}{c^2} - 1}.
\]

d) Behold:

\[
B_E = \begin{cases} 
\frac{2E^3}{h^3c^2} \frac{1}{e^\frac{x}{c^2} - 1} & \text{in general;} \\
\frac{2E^3}{h^3c^2} \frac{1}{e^\frac{x}{c^2} - 1} & \text{for } x << 1: \text{Rayleigh-Jeans law;} \\
\frac{2E^3}{h^3c^2} \frac{1}{e^\frac{x}{c^2} - 1} & \text{for } x >> 1: \text{Wien approximation;}
\end{cases}
\]

\[
B_\nu = \begin{cases} 
\frac{2hv^3}{c^2} \frac{1}{e^\frac{x}{c^2} - 1} & \text{in general;} \\
\frac{2hv^3}{c^2} \frac{1}{e^\frac{x}{c^2} - 1} & \text{for } x << 1: \text{Rayleigh-Jeans law;} \\
\frac{2hv^3}{c^2} \frac{1}{e^\frac{x}{c^2} - 1} & \text{for } x >> 1: \text{Wien approximation;}
\end{cases}
\]

\[
B_\lambda = \begin{cases} 
\frac{2hc^2}{\lambda^5} \frac{1}{e^\frac{x}{c^2} - 1} & \text{in general;} \\
\frac{2hc^2}{\lambda^5} \frac{1}{e^\frac{x}{c^2} - 1} & \text{for } x << 1: \text{Rayleigh-Jeans law;} \\
\frac{2hc^2}{\lambda^5} \frac{1}{e^\frac{x}{c^2} - 1} & \text{for } x >> 1: \text{Wien approximation;}
\end{cases}
\]

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002 qfull 00510 1 3 0 easy math: Debye function and blackbody radiation results

2. The total Debye function (i.e., the sum of the first and second Debye functions) is

\[
D_z = \int_0^\infty \frac{x^z}{e^x - 1} dx = z! \zeta(z + 1),
\]

(e.g., Wolfram Mathworld: Debye functions; Wikipedia: Debye function) where the factorial function

\[
z! = \begin{cases} 
\int_0^\infty x^ze^{-x} dx = z(z - 1)! & \text{for } z \text{ not a negative integer and also not 0 for the second form;} \\
n! \sqrt{\pi} & \text{for integer } n \geq 0; \\
(2z)!! \sqrt{\pi} & \text{for } z=-1/2; \\
\frac{(2z)!!}{2^{z+1/2}} \sqrt{\pi} & \text{for half-integer } z \geq 1/2;
\end{cases}
\]
and Riemann zeta function (without analytic continuation considered)

\[
\zeta(s) = \begin{cases}
\sum_{\ell=1}^{\infty} \frac{1}{\ell^s} & \text{if } s > 1, \\
\frac{1}{2} + \frac{1}{3} + \ldots & \text{the divergent harmonic series (Ar-279);}
\end{cases}
\]

\[
\zeta(1) = \sum_{\ell=1}^{\infty} \frac{1}{\ell} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots
\]

\[
\zeta(2) = \frac{\pi^2}{6} = \frac{\pi^2}{2 \cdot 3} = 1.644934066848226436472415166646 \ldots
\]

\[
\zeta(3) = 1.2020569031595942853997381615114 \ldots
\]

\[
\zeta(4) = \frac{\pi^4}{90} = \frac{\pi^4}{2 \cdot 3^2 \cdot 5} = 1.0823233711138191516003696541 \ldots
\]

\[
\zeta(5) = 1.03692775514336992633136548645 \ldots
\]

\[
\zeta(6) = \frac{\pi^6}{945} = \frac{\pi^6}{3^3 \cdot 5 \cdot 7} = 1.017343061984491397145179297909 \ldots
\]

\[
\zeta(7) = 1.008349277381922826839797549849 \ldots
\]

\[
\zeta(8) = \frac{\pi^8}{9450} = \frac{\pi^8}{2 \cdot 3^3 \cdot 5^2 \cdot 7} = 1.0040773561979443393786528508 \ldots
\]

\[
\zeta(9) = 1.002008392826082214417852769232 \ldots
\]

\[
\approx 1 + \int_{3/2}^{\infty} x^{-s} dx = 1 + \left(\frac{2/3}{s-1}\right)^{s-1}
\]

\[
1 + \frac{1}{2s}
\]

(e.g., Wikipedia: Riemann zeta function; OEIS: Riemann zeta function).

a) Prove \( D_z = z \zeta(z + 1) \).

b) Determine the general moment formula \( M_n \) (where \( n \) is the moment power) for the distribution \( f(x) = Ax^z/(e^x - 1) \), where \( A \) is the normalization constant which you must determine too. Specialize for \( n = 0 \) (the normalization), \( n = 1 \) (the mean), and \( n = 2 \). Determine the general formula for the variance \( \sigma^2 \).

c) From the Planck’s law specific intensity,

\[
B_\nu = \frac{2h\nu^3}{c^2} \left(\frac{1}{e^x - 1}\right), \quad \text{where} \quad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda},
\]

show the total energy density of a blackbody radiation field is

\[
\epsilon = aT^4,
\]

where

\[
a = \frac{8\pi^3 k^4}{15h^3 c^3} = (7.5657332500339 \ldots) \times 10^{-16} \text{J/m}^3/\text{K}^4 = 1 \text{J/m}^3 \times \left(\frac{1}{6029.6164961230 \text{K}}\right)^4
\]

is the radiation density constant. Remember to change an isotropic specific intensity into a density you must multiply by \( 4\pi/c \).

d) Show that the mean photon energy of blackbody radiation field is

\[
E = \frac{\zeta(4)}{\zeta(3)} (3kT) = (2.70117803291906 \ldots) \times kT
\]

\[
= 2.32769513 \times 10^{-4} \text{eV} \times T = 1 \text{eV} \times \left(\frac{T}{4296.09525 \ldots \text{K}}\right),
\]
where $k = (0.8617333262\ldots) \times 10^4 \text{eV/K}$.

e) It is quite possible to have a radiation field with a Planck’s law spectrum, but not blackbody radiation field energy density. For example, say you have blackbody radiator sphere of radius $R$ and you are a distance $r \geq R$ from the center. The energy density from the sphere is $W = \Omega/(4\pi)$ times that of blackbody radiation field where $\Omega$ is the solid angle subtended by the sphere. The effect is called geometrical dilution and, of course, is approximately true of stars. Show that the geometrical dilution factor

$$W = \frac{\Omega}{4\pi} = \frac{1}{2} \left[ 1 - \sqrt{1 - \left(\frac{r}{R}\right)^2} \right]$$

(Mi-120). **Hint:** Drawing a diagram may help.

**SUGGESTED ANSWER:**

a) Behold:

$$D_z = \int_0^\infty \frac{x^2}{e^x - 1} \, dx = \int_0^\infty x^2 e^{-x} \left( \sum_{\ell=0}^\infty e^{-\ell x} \right) \, dx = \sum_{\ell=0}^\infty \int_0^\infty x^2 e^{-(\ell+1)x} \, dx$$

$$= \sum_{\ell=0}^\infty \frac{1}{(\ell + 1)^2 + 1} \int_0^\infty t^\ell e^{-t} \, dt = z! \sum_{\ell=1}^\infty \frac{1}{\ell^2 + 1} = z! \zeta(z + 1) \text{ QED.}$$

b) Behold:

$$M_n = A \int_0^\infty \frac{x^{z+n}}{e^x - 1} \, dx = \begin{cases} \frac{(z+n)! \zeta(z+n+1)}{z! \zeta(z+1)} & \text{in general;} \\ \frac{1}{(z+1)^2 + 1} \zeta(z + 2) & \text{for normalization } n = 0; \\ \frac{\zeta(z+3)}{\zeta(z+1)} & \text{for the mean } n = 1; \\ \frac{(z+2)(z+1) \zeta(z+3)}{\zeta(z+1)} & \text{for } n = 2. \end{cases}$$

For the variance,

$$\sigma^2 = M_2 - M_1^2 = \left[ \frac{(z+1)! \zeta(z+1)^2}{z! \zeta(z+1)} \right]^2 \left[ \frac{(z+2)}{z+1} \zeta(z+3) \zeta(z+1) - \zeta(z+2)^2 \right].$$

c) Behold:

$$\epsilon = \frac{4\pi}{c} \int_0^\infty B_\nu \, d\nu = \frac{4\pi}{c} \int_0^\infty \frac{2h \nu^3}{e^\nu - 1} \, d\nu$$

$$= \frac{4\pi}{c} \frac{2h}{c^2} \left( \frac{kT}{h} \right)^4 \int_0^\infty x^3 e^{-x} \, dx = \frac{4\pi}{c} \frac{2h}{c^2} \left( \frac{kT}{h} \right)^4 \left[ \zeta(4) \right] = \frac{4\pi}{c} \frac{2h}{c^2} \left( \frac{kT}{h} \right)^4 \frac{\pi^4}{15}$$

$$= 8\pi^5 k^4 \frac{1}{15h^3 c^3} \text{ QED.}$$

d) Behold:

$$E = \frac{\int_0^\infty B_\nu \, d\nu}{\int_0^\infty B_\nu \, d\nu(h}\nu) \, d\nu = kT \frac{3\zeta(4)}{2\zeta(3)} = \left( \frac{\zeta(4)}{\zeta(3)} \right) (3kT) = (2.70117803291906\ldots) \times kT$$

$$= (2.32769513\ldots) \times 10^{-4} \text{eV} \times T = 1 \text{eV} \times \left( \frac{T}{4296.09525\ldots \text{K}} \right) \text{ QED.}$$

e) Let $\mu = \cos \theta$, where $\theta$ is the angle measured from the radial direction from the sphere. Note that

$$d\Omega = \sin \theta \, d\theta \, d\phi = -d\mu \, d\phi$$
and the cosine of the angle from the radial direction to the limb of the sphere is

$$\mu = \cos \theta = \frac{\sqrt{r^2 - R^2}}{r} = \sqrt{1 - \left(\frac{r}{R}\right)^2}.$$ 

Behold:

$$W = \frac{\Omega}{4\pi} = \frac{1}{4\pi} \int_{-\mu}^{1} \int_{0}^{2\pi} d\mu d\phi = \frac{1}{2} (1 - \mu) = \frac{1}{2} \left[ 1 - \sqrt{1 - \left(\frac{r}{R}\right)^2} \right] \quad \text{QED.}$$

Fortran-95 Code

```fortran
! 7.56573250003392847185E-0016 6029.6164961230119483!
! 1 23456789a123456 1234 56789a123456
! 2.7011780329190638961 2.32769513096571802645E-0004!
! 1 23456789a123456 1 234 5b49a1 1234 56789a1
```
c) Planck’s law (AKA the blackbody specific intensity spectrum) in the frequency representation is
\[ B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{hv}{kT}} - 1}, \quad \text{where} \quad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda}. \]

Show that the CBR obeys this law as the observable universe evolves provided it obeys it at the fiducial time and we define temperature evolution to obey the rule found in part (b). **Hint:** The photons in a frequency bin stay in that frequency bin as the universe evolves, and so obey the same energy scaling as the overall CBR. Thus at a general time, we have
\[ I_\nu d\nu = \left( \frac{a_0}{a} \right)^4 B_{\nu_0} d\nu_0, \]
where we have indeed assumed the fiducial time has a Planck-law spectrum. The proof requires showing that \( I_\nu d\nu = B_\nu d\nu \) with the temperature evolution obeying the rule found in part (b).

**SUGGESTED ANSWER:**

a) Since photon number is conserved, photon density \( n \) goes as \( 1/a^3 \). Now since photon energy \( E \) goes as \( 1/a \), we must have
\[ \epsilon \propto nE \propto \frac{1}{a^3} \frac{1}{a} = \frac{1}{a^4}, \quad \text{and so} \quad \epsilon = \epsilon_0 \left( \frac{a_0}{a} \right)^4 \quad \text{QED}. \]

b) We have,
\[ \epsilon = aHT^4 \quad aHT^4 = \epsilon_0 \left( \frac{a_0}{a} \right)^4 \quad T \propto \frac{1}{a} \quad T = T_0 \left( \frac{a_0}{a} \right) \quad \text{QED}. \]

c) Behold:
\[ I_\nu d\nu = \left( \frac{a_0}{a} \right)^4 B_{\nu_0} d\nu_0 = \left( \frac{a_0}{a} \right)^4 \frac{2h\nu_0^3}{c^2} \frac{1}{e^{\frac{h\nu_0}{kT}} - 1} d\nu_0 \]
\[ = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} d\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^x - 1} d\nu \]
\[ = B_\nu d\nu, \]
provided we define \( T \) by
\[ x_0 = \frac{h\nu_0}{kT_0} = \frac{h\nu(a/a_0)}{kT_0} = \frac{h\nu}{kT_0(a_0/a)} = \frac{h\nu}{kT} \quad \text{or} \quad T = T_0 \left( \frac{a_0}{a} \right) \]
which is just the rule we found in part (b): QED.
Actually, the idea is that the steady-state solution is really a quasistatic process: “a thermodynamic process that happens slowly enough for the system to remain in internal equilibrium.” We are crudely/vaguely attempting to understand recombination in this question. But we don’t get too far.

b) Find the limiting forms of solution $X$ for $R \to 0$ (to 1st order in small $R$), $R = 1$, and $R \to \infty$ to first order in small $1/R$. What is special about $X(R = 1)$ from a number point of view?

c) For the nonce, let’s define the recombination temperature of the cosmic radiation field by $R(T) = 1$. Let $N$ be the photon density, we have

$$1 = R = \frac{N_I}{N_H} = \frac{N_I/N}{N_H/N} = \frac{1}{\eta_f} = \frac{1}{\eta} \frac{D^{(2)}(x)}{D_n} \approx \frac{1}{\eta} \frac{e^{-x^2}}{2\zeta(3)} ,$$

where we have approximated the second Debye function by leading term which is valid for $x >> 1$ and where $x = E_R/(kT)$ where $E = 13.605693009(84)$ eV is the Rydberg energy (i.e., the ionization energy of hydrogen) and $T$ is the recombination temperature that we are solving for. The baryon-to-photon ratio $\eta = 6 \times 10^{-10}$ for a fiducial value, $\zeta(3) = 1.2020569031594285395781615114\ldots$, and $k = 8.6173303 \times 10^{-4}$ eV.

Solve for $x$ by iteration and then determine $T$. Remember a iteration formula tends to converge/diverge when its slope is low/high relative to 1. You could write a small computer program to do the solution. Hint: In a test mise en scène, just do the zeroth order solution: i.e., no iteration.

**SUGGESTED ANSWER:**

a) Behold:

$$0 = -CN_e^2 + CN_I(N_H - N_e) \quad 0 = X^2 + R(1 - X) \quad X = \frac{R \pm \sqrt{R^2 + 4R}}{2} \quad X = \frac{\sqrt{R^2 + 4R} - R}{2} ,$$

where only the positive solution is physically relevant.

First, since the differential equation is a 1st order one with no special features that would give a stationary point at a finite time, it can only have stationary points at $t = \pm \infty$: i.e., asymptotic solutions. Second, we note that if right-hand side is positive/negative, then $N_e$ will increase/decrease with time but then the right-hand side will go to zero and the $N_e$ will have a stationary point. But by the first point, this can only be at $t = \infty$. So our solution is the asymptotic solution: i.e., the steady-state solution.

b) Behold:

$$X = \begin{cases} \frac{\sqrt{R^2 + 4R} - R}{2} & \text{in general;} \\ \frac{R}{2} & \text{for } R << 1, \text{ and note} \\ \frac{\sqrt{5} - 1}{2} = 0.618033988749894820458634365638117720\ldots & \text{which is the} \\ 1 - \frac{1}{R} & \text{golden ratio minus 1;} \\ & \text{for } R \to \infty. \end{cases}$$

(c) A good guess at convergent iteration formula is

$$x = -\ln[2\zeta(3)\eta] + 2 \ln(x) \quad \text{since} \quad \frac{dx}{dx} = \frac{2}{x} << 1$$

for $x$ rather large which it probably is. The computer program is displayed below. The values obtained are:

$$x = 26.9444659685049923755 \quad \text{and} \quad T = 5859.74 \text{ K} .$$
The recombination temperature (as we’ve defined it for the nonce) is not decoupling temperature, but should be of the same order of magnitude. The fiducial decoupling temperature is 3000 K. So our result is, indeed, not so far off.

Fortran-95 Code

```fortran
print*  
zeta3=1.2020569031595942853997381615114_np !
http://oeis.org/wiki/Riemann_zeta_function
eryd=13.605693009_np ! (84)
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
boltev=0.86173303e-4_np ! (50)
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
eta=6e-10_np ! fiducial baryon-to-photon ratio
!
https://en.wikipedia.org/wiki/Big_Bang_nucleosynthesis#Characteristics
x0=-log(2.0_np*zeta3*eta)
x=x0
i=0
do
  i=i+1
  xold=x
  x=x0+2.0_np*log(x)
  if(abs(xold-x)/x .le. 1.e-12_np) exit
  print*,i,xold,x
end do

trec=eryd/(boltev*x)
print*,x0,x,trec
print*,x0,x,trec
! 20.3569101047609651092 26.944659685049923755
5859.74001252925990268

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