Problems (2)

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Problem 4. In an astronomical source, electrons are radiating in a random magnetic field via synchrotron radiation. The electron distribution is non-thermal power law: $N(\gamma)d\gamma \propto \gamma^{-p}d\gamma$, with the spectral index $p \sim 2.2$ and the minimum electron Lorentz factor $\gamma_m \sim 100$. The average strength of the magnetic field is about $B \sim 1$ mG (10^{-3}Gauss, Gauss is the unit of magnetic fields in cgs units). What is the synchrotron frequency of the electrons with minimum energy? Neglecting the self-absorption effect, what is the observed emission spectrum below and above this frequency? The observed power law spectrum cuts off above the frequency $\sim 10^{12}$ Hz. Assuming that this break is caused by synchrotron cooling, estimate the age of the source.

Solution:
The synchrotron frequency is

$$\nu_m = \omega_m/2\pi =\sim (3/4\pi)\gamma_m^2(eB/mc) = 4.1 \times 10^7 \text{ Hz} . \quad (1)$$

For the convention $F_\nu \propto \nu^s$, the spectral index is $s = -(p-1)/2 = -0.6$ for $\nu > \nu_m$, and $s = 1/3$ for $\nu < \nu_m$. The emission power of an electron with Lorentz factor γ is

$$P(\gamma) = (4/3)\sigma_T c\gamma^2\beta^2(U_B) \simeq 1.05 \times 10^{-21}\gamma^2 \text{ erg s}^{-1} , \quad (2)$$

where $U_B = B^2/8\pi$. The cooling time scale for an electron with Lorentz factor γ is

$$t_c = \gamma mc^2/P(\gamma) = 7.7 \times 10^{14} \ s\gamma^{-1} \quad (3)$$

From the cooling spectral break at 10^{12} Hz, one can estimate the characteristic "cooling" Lorentz factor of the electrons γ_c, above which electrons cool. From $\nu_c = (3/4\pi)\gamma_c^2(eB/mc)$, one gets

$$\gamma_c = [(4\pi/3)\nu_c(mc/eB)]^{1/2} \sim 1.5 \times 10^4 \quad (4)$$

The lifetime (age) of the source is therefore the cooling time for this Lorentz factor, i.e.

$$\tau = t_c(\gamma = \gamma_c) \sim 5.1 \times 10^{10} \text{ s} \sim 1.6 \times 10^3\text{yr} . \quad (5)$$

Problem 5. This is the famous "compactness" problem of gamma-ray bursts (GRBs). Astronomers usually use the time variability scale δt to estimate the size of the emission region $l = c\delta t$. GRBs are known to happen at cosmological distances (say, $d \sim 10^{28}$ cm). They have a typical time variability scale of $\delta t = 10$ ms. The gamma-ray fluence detected on earth is $\sim 10^{-6}$ erg cm^{-2}. What is the total energy of the burst assuming isotropic emission? The observed typical photon energy is roughly the energy of the rest electron energy $m_e c^2$ (by coincidence). What is the photon number density? At this energy photons interact with each other to produce electron-positron pairs. The cross section is roughly the Thomson cross section σ_T. What is the optical depth of this interaction? (The answer is a huge number.) Is it optically thin or thick? (The answer is definitely optically-thick.) The observed spectrum, however, is an optically-thin power law spectrum. This is the compactness problem. In order to solve the problem, GRBs must be relativistic and beam towards us (you don't need to prove this, but you can have a try).

Solution:
The total isotropic emission energy can be estimated as

$$E = 4\pi d^2 F \sim 1.3 \times 10^{51} \text{ ergs} . \tag{6}$$

The length scale of the emission region may be estimated as

$$l \sim c\delta t \sim 3 \times 10^8 \text{ cm} . \tag{7}$$

the volumn is roughly

$$V \sim l^3 \sim 2.7 \times 10^{25} \text{ cm}^3 \tag{8}$$

The photon number density would be

$$n_{ph} \sim \frac{E}{mc^2 V} \sim 5.9 \times 10^{31} \text{ cm}^{-3} . \tag{9}$$

The optical depth is

$$\tau \sim n_{ph} \sigma_T l \sim 1.2 \times 10^{16} . \tag{10}$$

This huge optical depth suggests a thermal blackbody spectrum, which is not what is observed (optically-thin non-thermal spectrum).

In order to solve the problem, one needs to assume that GRBs are moving towards us with a very large Lorentz factor γ. The ~ 10 ms variability time scale is the "observed" time scale. The emission time scale is longer by a factor of $\sim \gamma^2$. The length scale of the region is therefore larger by a factor of $\sim \gamma^2$, and the volumn is larger by a factor of $\sim \gamma^6$. This greatly reduces the photon number density. This effect, together with Doppler effect, (the observed gamma-ray energy is actually X-rays in the comoving frame), could reduce the optical depth to be lower that unity as long as γ is large enough. This is the argument that GRBs are relativistic outflows beaming towards the Earth.

Problem 6. Pulsars are neutron stars that emit broad-band spectrum of radiation, including radio waves, X-rays and gamma-rays. Near the magnetic polar cap, the magnetic field strength is $B \sim 10^{12}$ G. The field line is curved with a curvature radius of about $\rho \sim 10^7$ cm. Two populations of relativistic electrons are generated near the polar cap: The "primary" particles have a typical Lorentz factor of $\gamma_1 \sim 10^7$ (beaming along the magnetic field line); while the "secondary" particles have a typical Lorentz factor of $\gamma_2 \sim 100$ (with both the parallel and the perpendicular components with respect to the magnetic field line). The polar cap is hot with a temperature of about $T \sim 10^6$ K, and it is likely that there is an oscillating inner accelerator that excites electro-magnetic low frequency waves with frequency $\sim 10^6$ Hz near the polar cap region. Try to find out as many as possible mechanisms to power the observed radio emission in the $\sim 10^8$ - 10^{11} Hz band, and the observed gamma-ray emission in the 1 MeV - 10 GeV band.

Solution:
There are in principle two mechanisms to produce radio waves.
(1) The curvature emission of the secondary particles, with the typical frequency

$$\nu_{cr,2} = \frac{3}{4\pi} \gamma_2^3 \frac{c}{\rho} \sim 7.2 \times 10^8 \text{ Hz} . \tag{11}$$

(2) The inverse Compton scattering of the low frequency wave by the secondary particles, with the typical frequency

$$\nu_{ics,2} \sim \gamma_2^2 \nu_{lfw} \sim 10^{10} \text{ Hz} . \tag{12}$$

There are in principle three mechanisms to produce gamma-ray emission.
(1) The curvature emission of the primary particles, with the typical frequency

$$\nu_{cr,1} = \frac{3}{4\pi}\gamma_1^3\frac{c}{\rho} \sim 7.2 \times 10^{23} \text{ Hz}, \quad h\nu \sim 3.0\text{GeV} . \tag{13}$$

(2) The synchrotron emission of the secondary particles

$$\nu_{sr,2} = \frac{3}{4\pi}\gamma_2^2(eB/mc) \sim 4.1 \times 10^{22} \text{ Hz}, \quad h\nu \sim 170\text{MeV} . \tag{14}$$

(3) The inverse Compton scattering of the thermal X-ray photons by the primary particles

$$\nu_{ics,1} \sim \gamma_1^2 2.8kT \sim 5.8 \times 10^{20} \text{ Hz}, \quad h\nu \sim 2.4 \text{ MeV} . \tag{15}$$