Problems (1)

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Problem 1. Imagine in a nearby solar system like our own (distance 10 kpc, i.e. 10 kilo-parsec) two planets collide. The first planet resembles Jupiter, with mass 1.0×10^{30} g and radius 7.0×10^{9} cm. The second planet resembles Earth, with mass 6×10^{27}g and radius 6.4×10^{8} cm. Estimate the total energy released during the collision. What is the maximum luminosity of the collision? How long does it take to release the total energy with the maximum luminosity? After the collision the Jupiter may temporarily possess a hot spot of the size of Earth. Assuming that the emission from the hot spot has a blackbody spectrum and the maximum luminosity, what is the temperature of the hot spot? At which frequency is the emission peaked? Assuming isotropic emission, what is the specific flux (in unit of ergs $\mathrm{cm^{-2}\ s^{-1}\ Hz^{-1}}$) detected by the Earth astronomers? One parsec is 3.09×10^{18} cm.

Solution:

Since $M_j >> M_e$ (where the subscripts j and e denote Jupiter and Earth, respectively), the gravity center is essentially the center of Jupiter. The collision energy is essentially the gravitational potential energy release of Earth at the radius of R_J, i.e.

$$E = \frac{GM_j M_e}{R_j} = 5.7 \times 10^{40} \text{ ergs} . \tag{1}$$

The maximum luminosity of the collision is the Eddington luminosity

$$L_{max} = L_{Edd} = 1.3 \times 10^{38} \text{ erg s}^{-1} (M_j/M_\odot) = 6.3 \times 10^{34} \text{ erg s}^{-1} . \tag{2}$$

The time to release the total energy E with the luminosity L_{max} is

$$t_{cool} = E/L_{max} = 9.1 \times 10^{5} \text{ s} \sim 10 \text{ days} . \tag{3}$$

The hot spot blackbody emission should satisfy $L/A = \sigma T^4$, so that

$$T = \left(\frac{L_{max}}{\sigma 4\pi R_e^2} \right)^{1/4} \sim 1.2 \times 10^{5} \text{ K} . \tag{4}$$

The peak frequency is

$$\nu_{peak} = 2.8kT/h \sim 7.0 \times 10^{15} \text{ Hz} . \tag{5}$$

This is the extreme ultraviolet band. The distance is $d = 3.09 \times 10^{22}$ cm. Assuming isotropic emission, the specific flux detected on the Earth is

$$F_\nu = \frac{L}{4\pi d^2 \nu_{peak}} = 7.4 \times 10^{-28} \text{ erg s}^{-1}\text{cm}^{-2}\text{Hz}^{-1} . \tag{6}$$

Refernce: Zhang & Sigurdsson, 2003, ApJ, 596, L95

Problem 2. At present time the universe is filled with an isotropic blackbody emission with temperature 2.7 Kelvin. What is the peak frequency and the peak wavelength of this emission? What is the photon energy density? Let's assume that the intergalactic medium number density is $\sim 10^{-6}$ protons per cm^3. Assuming that matter is uniformly distributed in space (in reality matter is clustered in galaxies, and dark matter has more mass than normal matter), what is the typical baryon energy density? Comparing the baryon energy density with the photon energy density, is the current universe matter dominated or radiation dominated?

Solution:
The peak frequency is

$$\nu_{peak} = 2.82kT/h \sim 1.6 \times 10^{11} \text{ Hz} . \tag{7}$$

The peak wavelength is

$$\lambda_{peak} = 0.29/T \sim 0.1 \text{ cm} . \tag{8}$$

The blackbody photon energy density is

$$U_{ph} = aT^4 = 4.0 \times 10^{-13} \text{ erg cm}^{-3} . \tag{9}$$

The majority of the baryons in the universe is Hydrogen. So the baryon energy density could be estimated as

$$U_b = n_p m_p c^2 = 10^{-6} \times 1.67 \times 10^{-24} \times (3 \times 10^{10})^2 \simeq 1.5 \times 10^{-9} \text{ erg cm}^{-3} . \tag{10}$$

Since $U_b \gg U_{ph}$, the universe currently is matter dominated.

Problem 3. An equation of state can be generally expressed as $P = P(\rho, T, \text{composition})$. Under certain conditions, it can be expressed as $P \propto \rho^{\Gamma}$. Derive the equation of state for the following three cases. What are the Γ values for each case? (1) an isothermal ideal gas; (2) an adiabatic ideal gas both for the non-relativistic and relativistic regimes; (3) an degenerate gas both for the non-relativistic and relativistic regimes.

Solution:
A general equation of state of an ideal, non-degenerate gas can be written

$$P = nkT = \frac{N}{V}kT, \tag{11}$$

where P, n, and T are pressure, particle number density, and temperature, respectively, and N is the total number of the particles, which is constant.

(1) For an isothermal gas, one has $T = \text{const}$, so

$$P \propto n \propto \rho\ . \tag{12}$$

So one has $\Gamma = 1$.

(2) According to the 1st law of thermodynamics, one has $dU = dQ - PdV$, where dU is the change of the internal energy, dQ is the change of heat, and PdV is the work done to the environment by the system. For an adiabatic equation of state, one has $dQ = 0$, so that

$$dU + PdV = 0 \tag{13}$$

For a non-relativistic gas, one has

$$U = \frac{3}{2}NkT. \tag{14}$$

So $dU = (3/2)NkdT$. On the other hand, differentiating Eq.(11) gives
$dP = (N/V)kdT - (NKT/V^2)dV = (N/V)dT - (P/V)dV$, or

$$NkdT = VdP + PdV\ . \tag{15}$$

Submitting dU and Eq.(15) into Eq.(13), one gets

$$\frac{dP}{P} = -\frac{5}{3}\frac{dV}{V}\ , \tag{16}$$

or

$$P \propto V^{-5/3} \propto \rho^{5/3}\ . \tag{17}$$

So one has $\Gamma = 5/3$.

For a relativistic gas, since $P = e/3$, one has

$$U = 3NkT. \tag{18}$$

So $dU = 3NkdT$. Submitting dU and Eq.(15) into Eq.(13), one gets

$$\frac{dP}{P} = -\frac{4}{3}\frac{dV}{V}\ , \tag{19}$$

or

$$P \propto V^{-4/3} \propto \rho^{4/3}\ . \tag{20}$$

So one has $\Gamma = 4/3$.

(3) The uncertainty principle reads

$$\Delta x \Delta p \sim \hbar \tag{21}$$

For a degenerate gas with N Fermions in a volumn of V, one should satisfy

$$\frac{V \cdot p_F^3}{\hbar^3} \sim N \ , \tag{22}$$

or

$$n \propto p_F^3 \ , \tag{23}$$

where p_F is the Fermi momentum. This gives $n(p) \propto p^2$.

In general, the gas pressure can be calculated as

$$P = \frac{1}{3} \int_0^\infty n(p) p v dp \ . \tag{24}$$

For a non-relativistic degenerate gas, $v = p/m$, one gets

$$P \propto \int n(p) p^2 dp \propto p^5 \propto n^{5/3} \propto \rho^{5/3} \ . \tag{25}$$

So one has $\Gamma = 5/3$.

For a relativistic degenerate gas, $v \sim c$, one gets

$$P \propto \int n(p) p dp \propto p^4 \propto n^{4/3} \propto \rho^{4/3} \ . \tag{26}$$

So one has $\Gamma = 4/3$.