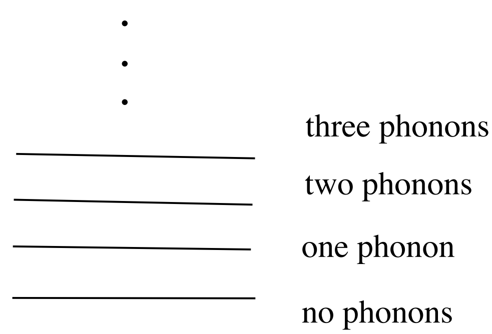
Cirac - Zoller Scheme

Suppose we have cooled the ions situated along the z-axis so that their xy motion is minimized via Laser cooling. However they are still free to move along the z-axis. Image the set of ions all moving in unison so that the center of mass undergoes harmonic motion. In the quantum realm the excitations of this motion are called phonons. For vibrations with lowest energy is called the ground state and no phonons are present, the next highest energy state has one phonon, and so on. We can describe this by a ladder like stucture of energy levels

Each phonon has an energy $h\,\nu$



The ions are sufficently apart from each other so that they can be addressed by a laser individually. The laser can excite, or de-excite the ion from its internal ground state $|g\rangle$ to its excited internal state $|e\rangle$. A laser can also be used to add or emit a phonon. Then the internal ground state of an ion can have several energy states

$|g\rangle|0\rangle_\nu$ Ground internal state+no phonons Energy = E_g

$|g\rangle|1\rangle_\nu$ Ground internal state+1 phonon Energy = E_g + hν

$|g\rangle|2\rangle_\nu$ Ground internal state+2 phonons Energy = E_g + 2 hν

⋮

$|e\rangle|0\rangle_\nu$ Excited internal state+no phonons Energy = $E_g + \hbar\,\omega$

$|e\rangle|1\rangle_\nu$ Excited internal state+1 phonon Energy = $E_g + \hbar\,\omega +$ hν

$|e\rangle|2\rangle_\nu$ Excited internal state+2 phonons Energy = $E_g + \hbar\,\omega$ + 2 hν

⋮

Note we assume that the internal state energy difference $\hbar\,\omega >>$ hν

For a given ion *m* we let $|g\rangle$ represent the Qbit $|0\rangle$ and $|e\rangle$ the Qbit $|1\rangle$
so if we have a group of four ions and the first two are in the ground internal state

and the the last two are in an internal excited state they would represent the computational basis state

$|0011\rangle$

For a string of ions lets consider the m'th ion and label its internal states as $|g\rangle_m$ $|e\rangle_m$ ($|0\rangle_m$ $|1\rangle_m$). Likewise for the n'th ion we use teh notation $|g\rangle_n$ $|e\rangle_n$.

Imagine the system starting out as

$|g\rangle_m|g\rangle_n |0\rangle_\nu$ total energy $2 E_g$

Let' shine a laser on ion m with a frequency of ω -hν. This frequency is not sufficient to excite the ion into its excited state $|e\rangle_m$ because you need at least an energy $\hbar\omega$ to that so the effect of this laser pulse, which we call shorthand U_m on this state is

$U_m |g\rangle_m|g\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m|g\rangle_n |0\rangle_\nu$

Likewise

$U_m |g\rangle_m|e\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m|e\rangle_n |0\rangle_\nu$

However

$U_m |e\rangle_m|g\rangle_n |0\rangle_\nu \longrightarrow -i\ |g\rangle_m|g\rangle_n |1\rangle_\nu$

That is the if U_m acts on state $|e\rangle_m|g\rangle_n |0\rangle_\nu$ it changes to a new state where the m'th Qubit changes its value form 1 to 0 while a phonon is created. The energy of the initial configuration $|e\rangle_m|g\rangle_n |0\rangle_\nu$ is $\hbar\omega + 2E_g$ whereas the energy of $|g\rangle_m|g\rangle_n |1\rangle_\nu$ is $2E_g$ + hν and so the difference in energy is $\hbar\omega$-hν which is exactly the energy of the laser pulse photon applied on ion m. Thus by energetics alone we see why $U_m |e\rangle_m|g\rangle_n |0\rangle_\nu \longrightarrow -i\ |g\rangle_m|g\rangle_n |1\rangle_\nu$ is possible. But what about the -i factor, where does that come from ? It turns out we can adjust the polarization properties of the laser in order the -i phase factor appears (Deeper discussion of this requires a better understanding of quantum optics, which is beyond the scope of this course).
Finally we also have

$U_m |e\rangle_m|e\rangle_n |0\rangle_\nu \longrightarrow -i\ |g\rangle_m|e\rangle_n |1\rangle_\nu$

In summary, after the first pulse

$U_m |g\rangle_m|g\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m|g\rangle_n |0\rangle_\nu$
$U_m |g\rangle_m|e\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m|e\rangle_n |0\rangle_\nu$
$U_m |e\rangle_m|g\rangle_n |0\rangle_\nu \longrightarrow -i\ |g\rangle_m|g\rangle_n |1\rangle_\nu$
$U_m |e\rangle_m|e\rangle_n |0\rangle_\nu \longrightarrow -i\ |g\rangle_m|e\rangle_n |1\rangle_\nu$

We now conisder a similar laser pulse, which we call V_n, on ion n (this will have diffrenet polarization properties than the aforementioned pulse). We then have

$V_n |g\rangle_m |g\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m |g\rangle_n |0\rangle_\nu$
$V_n |g\rangle_m |e\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m |e\rangle_n |0\rangle_\nu$
$V_n |g\rangle_m |g\rangle_n |1\rangle_\nu \longrightarrow - |g\rangle_m |g\rangle_n |1\rangle_\nu$
$V_n |g\rangle_m |e\rangle_n |1\rangle_\nu \longrightarrow |g\rangle_m |e\rangle_n |1\rangle_\nu$

For each state in the first collumn the photon does not have enough energy to change the state of Qbit n.

Now we repeat pulse U_m once again so that

$U_m |g\rangle_m |g\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m |g\rangle_n |0\rangle_\nu$
$U_m |g\rangle_m |e\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m |e\rangle_n |0\rangle_\nu$
$U_m |g\rangle_m |g\rangle_n |1\rangle_\nu \longrightarrow -i |e\rangle_m |g\rangle_n |0\rangle_\nu$
$U_m |g\rangle_m |e\rangle_n |1\rangle_\nu \longrightarrow i |e\rangle_m |e\rangle_n |0\rangle_\nu$

The three-pulses in this sequence W = $U_m V_n U_m$ then lead to the following transformations

W $|g\rangle_m |g\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m |g\rangle_n |0\rangle_\nu$
W $|g\rangle_m |e\rangle_n |0\rangle_\nu \longrightarrow |g\rangle_m |e\rangle_n |0\rangle_\nu$
W $|e\rangle_m |g\rangle_n |0\rangle_\nu \longrightarrow |e\rangle_m |g\rangle_n |0\rangle_\nu$
W $|e\rangle_m |g\rangle_n |0\rangle_\nu \longrightarrow - |e\rangle_m |e\rangle_n |0\rangle_\nu$

Notice that all states return to their original configuartions, except the last state which picks up a negative sign.

Consider the superposition

$$\frac{1}{\sqrt{2}}(|g\rangle_n \pm |e\rangle_n)$$

According to the above table

$$W \ |g\rangle_m \frac{1}{\sqrt{2}}(|g\rangle_n \pm |e\rangle_n) | \ |0\rangle_\nu = |g\rangle_m \frac{1}{\sqrt{2}}(|g\rangle_n \pm |e\rangle_n) | \ |0\rangle_\nu$$

$$W \ |e\rangle_m \frac{1}{\sqrt{2}}(|g\rangle_n \pm |e\rangle_n) | \ |0\rangle_\nu = |e\rangle_m \frac{1}{\sqrt{2}}(|g\rangle_n \mp |e\rangle_n) | \ |0\rangle_\nu$$

So lets define the gate

H_n W H_n here H_n is the Hadamard gate acting on Qbit n. Therefore

H_n W H_n $|g\rangle_m$ $|g\rangle_n$ $|0\rangle_\nu$ = H_n W $|g\rangle_m$ $\frac{1}{\sqrt{2}}$($|g\rangle_n$ + $|e\rangle_n$) $|0\rangle_\nu$ =

H_n $|g\rangle_m$ $\frac{1}{\sqrt{2}}$($|g\rangle_n$ + $|e\rangle_n$) $|0\rangle_\nu$ = $|g\rangle_m|g\rangle_n$ $|0\rangle_\nu$

H_n W H_n $|g\rangle_m$ $|e\rangle_n$ $|0\rangle_\nu$ = H_n W $|g\rangle_m$ $\frac{1}{\sqrt{2}}$($|g\rangle_n$ - $|e\rangle_n$) $|0\rangle_\nu$ =

H_n $|g\rangle_m$ $\frac{1}{\sqrt{2}}$($|g\rangle_n$ - $|e\rangle_n$) $|0\rangle_\nu$ = $|g\rangle_m|$ e $\rangle_n$ $|0\rangle_\nu$

H_n W H_n $|e\rangle_m$ $|g\rangle_n$ $|0\rangle_\nu$ = H_n W $|e\rangle_m$ $\frac{1}{\sqrt{2}}$($|g\rangle_n$ + $|e\rangle_n$) $|0\rangle_\nu$ =

H_n $|e\rangle_m$ $\frac{1}{\sqrt{2}}$($|g\rangle_n$ - $|e\rangle_n$) $|0\rangle_\nu$ = $|e\rangle_m|$ e $\rangle_n$ $|0\rangle_\nu$

H_n W H_n $|e\rangle_m$ $|e\rangle_n$ $|0\rangle_\nu$ = H_n W $|e\rangle_m$ $\frac{1}{\sqrt{2}}$($|g\rangle_n$ - $|e\rangle_n$) $|0\rangle_\nu$ =

H_n $|e\rangle_m$ $\frac{1}{\sqrt{2}}$($|g\rangle_n$ + $|e\rangle_n$) $|0\rangle_\nu$ = $|e\rangle_m|$ g $\rangle_n$ $|0\rangle_\nu$

In summary

H_n W H_n $|g\rangle_m$ $|g\rangle_n$ $|0\rangle_\nu$ = $|g\rangle_m|g\rangle_n$ $|0\rangle_\nu$
H_n W H_n $|g\rangle_m$ $|e\rangle_n$ $|0\rangle_\nu$ = $|g\rangle_m|$ e $\rangle_n$ $|0\rangle_\nu$
H_n W H_n $|e\rangle_m$ $|g\rangle_n$ $|0\rangle_\nu$ = $|e\rangle_m|$ e $\rangle_n$ $|0\rangle_\nu$
H_n W H_n $|e\rangle_m$ $|e\rangle_n$ $|0\rangle_\nu$ = $|e\rangle_m|$ g $\rangle_n$ $|0\rangle_\nu$

Thus we notice that the sequence of operations
H_n W H_n is equivalent to a CNOT gate i.e. Qbit n is flipped if Qbit m is in state $|e\rangle$
but is left alone if Qbit m is in state $|g\rangle$.

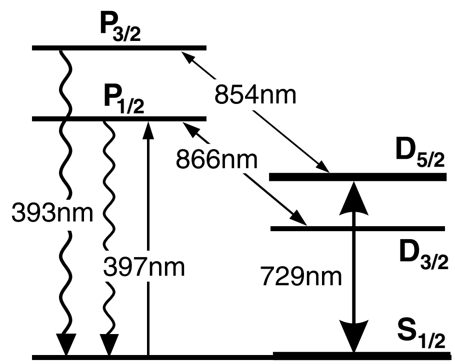


Fig. 1. Level scheme of $^{40}Ca^{+}$. The qubit is implemented using the narrow quadrupole transition. All states split up into the respective Zeeman sublevels.

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