So

$$n = \int_0^\infty \frac{1}{e^{(E-\mu)/kT}+1} \frac{g}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}\, dE$$

If $n$ is held fixed and $T\uparrow$, $\mu \downarrow$ to compensate and it goes negative in MB limit.

$$P = \frac{2}{3}\varepsilon = \frac{2}{3}\int_0^\infty \frac{1}{e^{(E-\mu)/kT}} \frac{g}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right) E^{3/2}\, dE$$

What one would like is a simple analytic formula

For $T=0$ there is a simple formula. See p. 5-596.

$$P = P(n, T)$$
$$P = P(\rho, T)$$

the EOS equation of state — Useful in astro modeling

So to eliminate $\mu$.

But there's no analytic to this.

There are various approximations and there are Tabulations

Actually for White Dwarfs (free electron gas)

5-702)

and neutron stars where (neutron gas) you have Fermi or degenerate gases, One needs to include relativistic effects too since $E_F$ ~~gets to~~ is comparable to or larger than the rest mass electrons or neutrons.

i) Bosons (identical)

From p. 5-629

$$\cancel{W = \prod_i \binom{g_i - 1 + N_i}{N_i}}$$

$$W = \prod_i \binom{g_i - 1 + N_i}{N_i}$$

$$\ln W = \sum_i \ln (g_i - 1 + N_i)!$$

$$- \sum_i \ln (g_i - 1)! - \sum_i \ln N_i!$$

$$h = \ln W + \alpha\left(\sum_i N_i - N\right) + \beta\left(\sum_i N_i E_i - E\right)$$

To find the maximizing $N_i$ again

$$0 = h(N_j) - h(N_j - 1)$$

$$= \ln(g_j - 1 + N_j)$$

$$-(\ln N_j) + \alpha + \beta E_j$$

$$\frac{N_j}{g_j - 1 + N_j} = e^{\alpha + \beta E_j} = e$$

$$N_j(1 - e) = e(g_j - 1)$$

$$N_j = \frac{g_j - 1}{e^{-1} - 1} = \frac{g_j - 1}{e^{-(\alpha + \beta E_j)} - 1}$$

5-~~I~~04

$$N_i = \mathrm{Int}\left(\frac{g_i - 1}{e^{-(\alpha+\beta E_i)} - 1}\right)$$ Restoring the use of index $i$

Using Stirling's approximation

$$0 = \frac{\partial h}{\partial N_j} = \ln(g_j - 1 + N_j) + 1 + 1$$
$$- (\ln N_j + 1 + 1)$$
$$+ \alpha + \beta E_j$$

$$0 = \ln\left(\frac{g_j - 1 + N_j}{N_j}\right) + \alpha + \beta E_j$$

which ~~is the~~ gives some formula before applying the Int ~~expression~~ function, and so is guaranteed to be an overestimate

5-705

If $g_i$ is very large, the $-1$ is negligible."
The Int function is analytically intractable and generally insignificant.
So our final formula is

$$N_i = \frac{g_i}{e^{(E_i-\mu)/kT} - 1}$$

and

$$f = \begin{cases} \dfrac{1}{e^{(E-\mu)/kT} - 1} & \text{is the Bose-Einstein distribution} \\ e^{-(E-\mu)/kT} & \text{if } e^{(E-\mu)/kT} \gg 1 \text{ MB limit} \\ \dfrac{kT}{E-\mu} & \text{if } e^{(E-\mu)/kT} \ll 1 \end{cases}$$

5-706

All the distribution functions are useful when the degeneracy $g$ is a single fraction that separated off and in continuum of states unit

$$f = \begin{cases} e^{-(E-\mu)/kT} & \text{Maxwell-Boltzmann} \\ \dfrac{1}{e^{(E-\mu)/kT}+1} & \text{Fermi-Dirac} \\ \dfrac{1}{e^{(E-\mu)/kT}-1} & \text{Bose Einstien} \end{cases}$$

They all have different behavior as $e^{(E-\mu)/kT}$ gets small, but the same as $e^{(E-\mu)/kT}$ gets big.

Where does the difference and similarities originate?

- Differences in the counting procedures which reflect
  - distinct particles
  - fermions with exclusion principle and identical particles
  - bosons with identical particles and NO exclusion principle.

Recall p. 5-629

$$W = \begin{cases} N! \prod_i \frac{g_i^{N_i}}{N_i!} & \text{distinct particles} \\ \prod_i \binom{g_i}{N_i} = \prod_i \frac{g_i!}{(g_i-N_i)!\,N_i!} & \text{fermions} \\ \prod_i \binom{g_i-1+N_i}{N_i} = \prod_i \frac{(g_i-1+N_i)!}{(g-1)!\,N_i!} & \text{bosons} \end{cases}$$

5 - 70 8

What if $g_i >> N_i$

~~so that the probability of a particle being in a state << 1 and the probability of two particles being in a state is negligible.~~

$$\frac{g_i!}{(g_i - N_i)!\, N_i!} = \frac{g_i(g_i-1)\cdots(g_i - N_i + 1)}{N_i!}$$

$$\le \frac{g_i^N}{N_i!}$$

$$\frac{(g-1+N_i)!}{(g-1)!\, N_i!} = \frac{(g_i-1+N_i)(g_i-2+N_i)\cdots(g_i)}{N_i!}$$

$$\le \frac{g_i^{N_i}}{N_i!}$$

In this case, the fermion and boson W's lead to the same distribution as distinct particle W. The $N!$ constant with the <u>distinct particle W</u> has no effect on the maximizing $N_i$ formula.

What does $g_i >> N_i$ sufficiently mean.

Well in counting the Fermion states, the single-particle state choices decreases because of the Pauli exclusion principle

$$g_i(g_i - 1) \ldots (g_i - N_i + 1)$$

but if $g_i >> N_i$

this $\simeq g_i^{N_i}$

Then the weight for a level $= W_i = \dfrac{g_i^{N_i}}{N_i!}$

5-71 U)

For the Boson case,
if we assume probability
of 2 particles in the
same single particle
states is negligible,
then in counting we
just have $g_i$ choices
for each particle and
get $\frac{g_i^{N_i}}{N_i!}$ as the weight
for a level.

i) ~~Photon Gas~~

~~photons are bosons,~~
~~but they are relativistic~~
~~— extremely so.~~

j) Bose-Einstien Condensate [5-71]

~~Ideal Boson gas~~

$$f = \frac{1}{e^{(E-\mu)/kT} - 1} \geq 0$$

$$\therefore \quad e^{(E-\mu)/kT} \geq 1$$

$$\frac{E-\mu}{kT} \geq 0$$

$$E \geq \mu \quad \text{for } T > 0$$

Now if for our continum approximation free gas $E = 0$ is the lowest energy state.

$$\therefore \quad \mu \leq 0$$

$$n = \int_0^\infty \frac{1}{e^{(E-\mu)/kT} - 1} \frac{g}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} \, dE$$

5-712

As $T \uparrow$, $\mu \downarrow$ to compensate.

As $T \downarrow$, $\mu$ $\uparrow$ and approach 0

But say $\mu = 0$ for a finite T.

Then
$$n = \int_0^\infty \frac{1}{e^{E/kT}-1} \frac{g}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}\, dE$$

defines a critical temperature.

$$n = \frac{g}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(kT)^{3/2}\int_0^\infty \frac{x^{1/2}}{e^x-1}\,dx$$

$$\int_0^\infty \frac{x^{1/2}}{e^x-1}\,dx = \int_0^\infty e^{-x}x^{1/2}\sum_{\ell=0}^{\infty} e^{-\ell x}\,dx$$

using the geometric series $\frac{1}{1-r} = \sum_{\ell=0}^{\infty} r^\ell$
(Art-23.8)

which converges for $|v|<1$

$v = e^{-x}$ in our case

and we integrate ~~to~~ $x = 0$

when $v = 1$

but that's a <u>zero area</u>

point and we assume

causes no problem.

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1}\,dx = \sum_{\ell=0}^\infty \int x^{1/2} e^{-(\ell+1)x}\,dx$$

$$= \sum_{\ell=1}^\infty \int x^{1/2} e^{-\ell x}\,dx$$

$$= \sum_{\ell=1}^\infty \frac{1}{\ell^{3/2}} \underbrace{\int_0^\infty z^{1/2} e^{-z}\,dz}_{\begin{array}{c} z! = (\frac{1}{2})! \\ = \frac{1}{2}\sqrt{\pi} \\ (\text{Arf} - 493) \end{array}}$$

$$= \frac{\sqrt{\pi}}{2} \sum_{\ell=1}^\infty \frac{1}{\ell^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2}\,\zeta(3/2)$$

This is the Riemann zeta function. Arf - 282

5-714

$$\zeta(3/2) \cong 2.612 \quad \text{(Wik)}$$

$$T_{critical} = \frac{1}{k}\left[\frac{n}{\frac{g}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\sqrt{\pi}}{2}\,\zeta(3/2)}\right]^{2/3}$$

$$= \frac{1}{k}\left(\frac{\hbar^2}{2m}\right)\left[\frac{n}{\frac{g}{2^3\pi^{3/2}}\,\zeta(3/2)}\right]^{2/3}$$

$$= \frac{4\pi}{k}\left(\frac{\hbar^2}{2m}\right)\left[\frac{n}{g\,\zeta(3/2)}\right]^{2/3}$$

(Gr-Solutions - 148)

$$n = \frac{\rho}{A\,m_{amu}}$$ where $A$ is atomic weight and $m_{amu}$ is the AMU.

$$T_{cr} = \frac{4\pi}{k}\left(\frac{\hbar^2}{\underbrace{2m}_{m = A m_{amu}}}\right)\left[\frac{1}{g\,\zeta(3/2)\,m_{amu}}\right]^{2/3}\left(\frac{\rho}{A}\right)^{2/3}$$

5 - 71 5

$$T_{cr} = \frac{4\pi}{k}\left(\frac{\hbar^2}{2m_{amu}}\right)\left[\frac{1}{g\,\zeta(3/2)\,m_{amu}}\right]^{2/3}\frac{\rho^{2/3}}{A^{5/3}}$$

$$= \underbrace{114.6\,\mathrm{K}}_{\text{Kelvins}}\,\frac{\rho^{2/3}}{g^{2/3}A^{5/3}}$$

$= 3.135$ K for He-4

(Gr-Solutions - 148)

$\rho = .145$ g/cm at melting

$A = 4.002602$

As temperature gets very low near $T_c$, our continuum density of states approximation breaksdown.

— ~~The~~ discrete ~~sum of~~ set and a sum is

5-716

is needed to find the occupation numbers and total n.

What happens is the bosons can all pack into the lowest energy states.

— A <u>Bose Einstein condensate</u> is when all or a large fraction are in the ground state (Wik)

Now liquid He-4 below 2.17 K is a <u>superfluid</u>.

which is a ~~non-simple~~ Bose-Einstein Condensate-like

~~State~~

x the He4-Atoms interact ~~strongly~~

and so it's not a simple BOSE-Einstein condensate.

Superconductivity is a related phenomena to Bose Einstein Condensates.

But again not simple

Simple Bose-Einstein condensates have been created in cooled potential traps using suitable atoms (Wik)

— but let's not go into that.

5-718

k) Photons

Photons are extremely relativistic, and so are ~~B~~ out of scope NR QM formally.

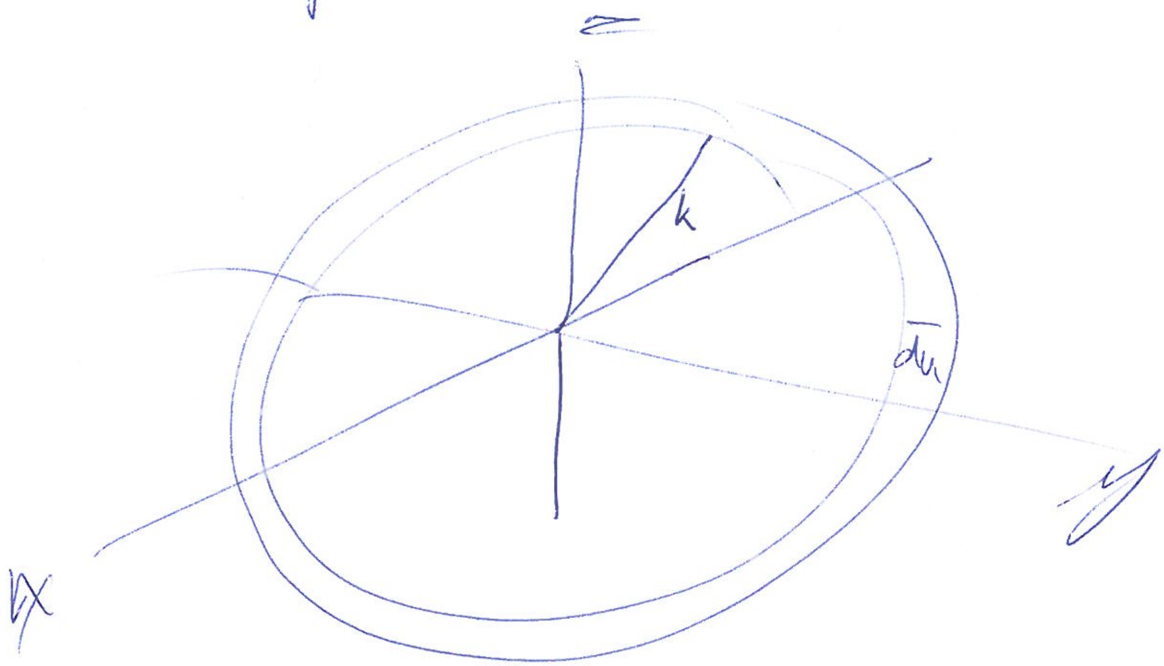
But if we ansatz that they have periodic BC single particle states of the form $\psi = \frac{e^{ik \cdot x}}{\sqrt{V}}$

$$k_i l_i = 2\pi n_i \quad n = 0, \pm 1, \pm 2, \ldots$$

Then $\Delta k_i l = 2\pi$

$$\rho_{k,V} = \frac{1}{(2\pi)^3} \text{ is the density}$$

of states in k-space
per unit volume




$$\therefore \quad \rho_k \, dk = \frac{g}{(2\pi)^3} \, 4\pi k^2 \, dk$$

density of states
per unit k
per unit volume.

g is internal degeneracy.

photons are bosons
of spin $s = 1$

but $m = -1$ and $1$.

The state $m = 0$ is excluded.

— for handwaving relativistic reasons.

↳ Maybe just so. So $g = 2$

Since photons ~~are relat~~ have no rest mass

~~And no~~ so

$E = pc = \hbar kc$ assuming the same connection between wave number and momentum as for non-relativistic particles. Which is plausible since Einstein & de Broglie started from that idea.

$$= h\nu$$

$\nu$ is frequency

$$k = \frac{2\pi}{\lambda}$$

$$kc = 2\pi\nu$$

$$\hbar kc = h\nu$$

For photons, it is conventional to use $\nu$ or $\lambda$ rather than $E$ or $k$

$$\nu = \frac{kc}{2\pi}$$

$$\rho_k \, dk = \rho_\nu \, d\nu$$

$$k = \frac{2\pi\nu}{c}$$

$$\rho_\nu = \rho_k \frac{dk}{d\nu}$$

$$= \frac{g}{(2\pi)^3} 4\pi k^2 \frac{2\pi}{c} d\nu$$

$$= \frac{g}{(2\pi)^3} 4\pi \frac{(2\pi)^2}{c^2} \nu^2 \frac{2\pi}{c} d\nu$$

$$= \frac{4\pi g}{c^3} \nu^2 d\nu$$

$$n(\nu) = \frac{4\pi g}{c^3} \nu^2 \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

5-722

What happened to $\alpha = \frac{\mu}{kT}$?

Well photons in a container in equilibrium are NOT conserved.

They are created and destroyed by interactions with the walls — whose existence we otherwise don't seem to need to invoke.

So we should maximize the entropy leaving number of photons unconstrained. Somehow "processes" will find the right number to maximize entropy ($\ln W$).

So we never impose the particle constraint which is the same as setting $\alpha = 0$

in our derivations.

Now usually we don't ask for the number of photons ~~in a dν or~~ per $\nu$ per volume but the photon energy per $\nu$ per volume

and

in astro context

we want what we call the Planck function $B_\nu$ Mihalas - 7 which is energy flux per unit solid angle.

or Blackbody spectrum.

$$B_\nu = \frac{c}{4\pi} h\nu \, n_\nu$$

$$= \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

G - 724

setting $g = 2$

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

(Mihalas-7)

which is a
very familiar function
to me —— the thermodynamic
equilibrium specific intensity

If we want the energy density

$$\mathcal{E} = \frac{4\pi}{c} \int_0^\infty B_\nu \, d\nu = \frac{4\pi}{c} B$$

integrate over angle
and convert back to a density

~~[illegible]~~

5-725

$$B = \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \, d\nu$$

$$= \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1} \, dx$$

$$= \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty e^{-x} x^3 \sum_{\ell=0}^{\infty} e^{-\ell x} \, dx$$

$$\sum_{\ell=1}^{\infty} \int_0^\infty x^3 e^{-\ell x} \, dx$$

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell^4} \underbrace{\int_0^\infty z^3 e^{-z} \, dz}_{3! = 6}$$

(Ar-453)

$$6 \, \zeta(4)$$

$$6 \cdot \frac{\pi^4}{90}$$

(Arf-285)

5-726

$$B = \frac{2\pi^4 k^4}{15 c^2 h^3} T^4$$

$$= \frac{\sigma T^4}{\pi}$$

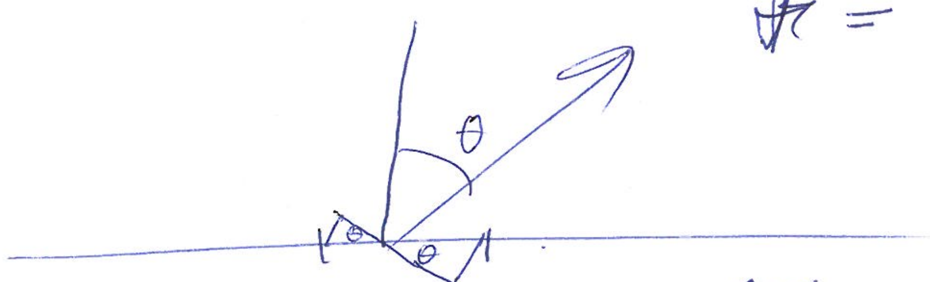
where $\sigma \equiv \frac{2\pi^5 k^4}{15 c^2 h^3} = 5.67.. \times 10^{-5}$ cgs $= 5.67.. \times 10^{-8}$ mks

is the Stefan-Boltzmann (Wik) constant.

If you have a surface radiating exactly like a Blackbody

$$F = 2\pi \int_0^1 B \mu \, d\mu$$

$$\mu = \cos\theta$$



$$F = 2\pi B \left.\frac{\mu^2}{2}\right|_0^1$$

$$= \pi B$$

$$= \sigma T^4$$

The energy per unit time per unit area emitted by a Blackbody radiator.

$$\mathcal{E} = \frac{4\pi}{c} B$$

$$= \frac{4\pi}{c} \frac{\sigma T^4}{\pi}$$

$$= \frac{4\sigma}{c} T^4$$

$$= aT^4$$

$a = \frac{4\sigma}{c}$ is the radiation constant (Wiki: Stefan-Boltzmann constant)

Where does $B_\nu$ reach a maximum as a function of $\nu$

Let's consider the general class functions of form

$$f(x) = \begin{cases} \dfrac{x^p}{e^x - 1} & \\ 0 & x \to \infty \\ x^{p-1} & x \to 0 \end{cases}$$

$\left\{ \cancel{\;p \neq 0\;} \right.$ ~~which is a special case of~~

5-728

Note $p = 0$ handled to ... we require $0 \cdot x^{-1} = 0$

$$\frac{\partial f}{\partial x} = \frac{p x^{p-1}}{e^x - 1} - \frac{x^p}{(e^x - 1)^2} e^x$$

$$= \frac{x^{p-1}}{(e^x - 1)^2}\left[p(e^x - 1) - x\right]$$

If $p = 0$ $f$ decreases from $x = 0$ to infinity

$$\frac{\partial f}{\partial x} = -\frac{e^x}{(e^x - 1)^2}$$

$\rightarrow -\frac{(1+x)}{x^2}$ $x$ small

$\rightarrow -\infty$ at $x = 0$

$0$ at $x = \infty$

only stationary point.

If $p > 0$,

$\cancel{p(e^x - 1)}$

Let's only consider the $p > 1$ cases where the function is zero at $x = 0$ and $x = \infty$

$$\frac{\partial f}{\partial x} = \frac{p x^{p-1}}{e^x - 1} - \frac{x^p e^x}{(e^x - 1)^2}$$

$$= \begin{cases} \dfrac{x^{p-1} e^x}{(e^x - 1)^2}\left[p(1 - e^{-x}) - x\right] \\[2ex] 0 \qquad\qquad x = \infty \quad \text{a stationary point} \\[2ex] \dfrac{x^{p-1}}{x^2}(px - x) = x^{p-2}(p-1), \quad x \rightarrow 0 \end{cases}$$

a stationary point for $p > 2$ and $p = 1$

Anyway, for $p > 1$,

the function is $0$ at $x = 0$ and $x = \infty$

And $p(1 - e^{-x}) - x = 0$

gives only one stationary point. $\longrightarrow$ It must be a maximum.

$$x = f(x) = p(1 - e^{-x})$$

is a good iteration function

for $\frac{df}{dx} = p e^{-x} < 1$

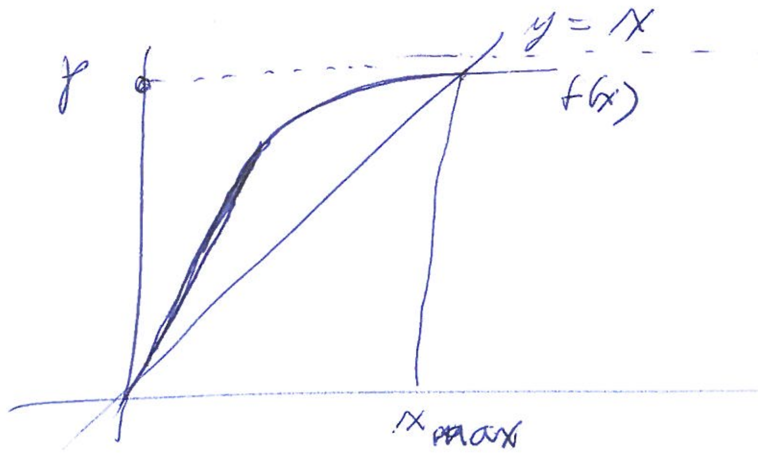
which is likely. For

$x \ll 1$

$f(x) = px$

$x \gg 1$

$f(x) = p$




A good initial guess for $p \geq 2$

is $p$

$x_0 = p$

5-130

then

$$x_1 = P(1-e^{-P})$$

and iterate

So where does

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$$

have a maximum

Well $x_{max} \triangleq 3$, $x_{max} = 2.8214393722122\ldots$

$$\nu_{max} = \frac{kT}{h}x_{max}$$

Now often we want the wavelength representation

$$B_\lambda = B_\nu\frac{d\nu}{d\lambda} = B_\nu\left(\frac{-c}{\lambda^2}\right)$$

but the minus is just for the flipped limits.

$$B_\lambda = \frac{2h\nu^3}{c^2}\frac{c}{\lambda^2}\frac{1}{e^{\frac{hc}{kT\lambda}}-1}$$

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{kT\lambda}}-1}$$

(Mihalas-7)

$$\frac{2hc^2}{\lambda^5}\frac{kT\lambda}{hc} \qquad x << 1$$

$$= \frac{2kTc}{\lambda^4}$$

The Rayleigh-Jeans has the ultraviolet catastrophe.

$$\int_0^\infty B_{\lambda\,RJ}\,d\lambda$$

$$\propto -\lambda^{-3}\Big|_0^\infty = \infty$$

which is the <u>Rayleigh-Jeans law</u> 1900 - 1905 (wiki) by which time it was already out of date because Planck had his law in 1900

~~5~~ T-732

Where does $B_\lambda$ reach a max?

Well $x = 5(1 - e^{-x})$

and so $x_{max} \doteq 5$

$x_{max} = 4.96511423174...$

$$\therefore \lambda = \frac{hc}{kT x_{max}}$$

$$= \frac{.2897768 \text{ cm-K}}{T}$$

$$= \frac{2897.768 \text{ nm-K}}{T}$$

Wein's law.