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But what of other kinds of single-particle eigenstates.

- Well if they exist exactly then the exclusion Principle applies exactly.
- if they exist as an approximation, then the exclusion principle is right for them as far as they go.
- What if single-particle eigenstates are NOT even a good approximation?

Well maybe some other way to use the exclusion principle to characterize the state (but this is apparently unspeakable).

Or maybe in some ~~any~~ kind of observable (H, P etc.) ~~allows~~ single-particle states in principle.

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Examples of Spin 1/2 states

Example 1

~~Two~~ The spin 1/2 particles
(e.g., electrons)
~~and two states eigenstates.~~

$\cancel{\psi_a(\underset{\sim}{r}_1)\chi_{+1}}$,

and two single-particle stationary states

They are orthonormal

$\psi_a(\underset{\sim}{r})\chi_+$ and $\psi_b(\underset{\sim}{r})\chi_+$

both up states for simplicity.

Two product states can be constructed

$$\psi_a(\underset{\sim}{r}_1)\chi_{+1}\,\psi_b(\underset{\sim}{r}_2)\chi_{+2}$$

and $$\psi_a(\underset{\sim}{r}_2)\chi_{+2}\,\psi_b(\underset{\sim}{r}_1)\chi_{+1}$$

Question: Construct a symmetrized state.

Ans:

$$\Psi(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1)\right]$$

$$* \ \chi_{+1}\chi_{+2}$$

$$\Psi(2,1) = -\Psi(1,2)$$

and so antisymmetric as ~~antisymmetrized~~ required.

Normalization? $|\Psi(1,2)|^2 = 1$?

— The cross terms vanish since $\int \psi_a^*(r)\psi_b(r)\,dr = 0$

etc.

$$|\Psi(12)|^2 = \frac{1}{2}\left[1\cdot 1 + 1\cdot 1\right]\chi_{+1}^\dagger \chi_{+1}$$

$$* \ \chi_{+2}^\dagger \chi_{+2}$$

$$= 1$$

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~~Example~~

Note if $x = x_1 = x_2$

then $\psi(12) \rightarrow 0$

Not a real force — a result ~~result~~ of ~~symmetrization~~ symmetrization

The exchange force repulsion.

If $a = b$, $\psi(12) = 0$ everywhere for any x_1, x_2 value

We couldn't construct a 2-particle state in this case.

Both examples of the Pauli ~~exchange p~~ exclusion principle.

What happens if $\psi_a \rightarrow \psi_b$ as one ~~adjusts the~~ system in the right way?

Does Pauli exclusion principle for states just suddenly turn on?

Maybe can't happen for quantized single-particle states.

(8)

~~Example~~ Slater Determinant developed for Fermions

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What if we had

N identical fermions

and N single-particle eigenstates

(usually thinking of energy eigenstates AKA stationary states)

$$\left.\begin{array}{l} \psi_1(\underline{r})\chi_1 \\ \psi_2(\underline{r})\chi_2 \\ \vdots \\ \psi_N(\underline{r})\chi_N \end{array}\right\}$$

e.g., $\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$\chi_2 = \begin{pmatrix}0\\1\end{pmatrix}$

for example, for spin 1/2 particles

We label the particles

$1', 2', \ldots, N'$

— primes are so we can keep

particles and states separate

Well a product state that solves the sch. eq. is

Heck of an indexing problem with many particle and single-particle states.

$$\Psi_{product} = \psi_1(1')\psi_2(2')\psi_3(3')\ldots\psi_N(N')$$

a short form for $\psi_2(\mathbf{r}_{2'})\chi_{22'}\ \psi_3(\mathbf{r}_{3'})\chi_{33'}$

But the product state is ~~Not~~ symmetrized (unless all single-particle states are actually the same state, In which case the product state is symmetric and that is ruled out for fermions)

~~If any two single particle states~~

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Can we create a fully <u>antisymmetric state</u> by linear combination of the product state with permuted particles (or permuted indices if you prefer)?

Yes.

$$\Psi = \frac{1}{\sqrt{N!}} \sum_{P} (-1)^{p} P \, \chi_1(1')\chi_2(2')\ldots\chi_N(N')$$

p is the parity of the permutation

Normalization

~~We~~ We sum over all permutations P of particles (or particle indexes)

How many unique permutations are there?

Ans. N! of course.

5-198

Each permutation has a unique parity for creating it from 2-particle exchanges

$$P_p = \begin{cases} 0 & \text{for even permutation} \\ 1 & \text{for odd permutation.} \end{cases}$$

Proving this point is a bit tedious, but we've got to bite the bullet sometimes.

Proof in steps (Maybe there's an easier proof, but this is all I can think of)

a) First going from the fiducial or original ~~order~~ order (permutation) to any other order can always be done by a series of 2-particle exchanges.

Which could be any permutation you like.

from $1' \; 2' \; 3' \ldots N'$ Call this permutation X

any ~~per~~mutation. Call this permutation Y

~~First~~ 1) exchange $1'$ (from wherever it is) and whatever to where you want $1'$.

2) exchange $2'$ (from wherever it is) and whatever to where you want $2'$.

and so on

until you have the <u>permutation</u> you want.

b) Every 2-particle exchange can be built from a <u>series of 2-adjacent-particle</u> exchanges.

Say exchange i and n

~~2 3 4~~

$i \; j \; k \; l \; m \; n$

$\xrightarrow{1} \; \xrightarrow{2} \; \xrightarrow{3} \; \xrightarrow{4} \; \xrightarrow{5}$

$j \; k \; l \; m \; n \; i$

$\xleftarrow{4} \; \xleftarrow{3} \; \xleftarrow{2} \; \xleftarrow{1}$

These <u>series</u> are always <u>odd</u> in <u>number</u>.

If particle 2 is t steps from particle 1

particle 1 takes t steps to reach particle 2's position

and particle 2 takes t − 1 steps to reach particle 1 old position.

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So $2t-1$ steps in all.

c) Say you started from the ~~fiducial~~ permutation X

and went on series of 2-adjacent particle exchange ~~for all~~ ~~pair~~ any ~~the~~ set of particles you like

and returned to the ~~original~~ permutation X

particles → . . . g'

position → . . . . g

To get particle g' back to position g

always takes an even ~~set~~ number of 2-adjacent particle exchanges.

There is no other way.

— all steps to the right must equal in number all steps to the left if you return to the same place.

d) Now say you went by ~~2 parti~~ 2-adjacent particle exchanges from the ~~fiducial~~ permutation X to ~~any~~ permutation Y.

— Say it took an even/odd number of 2-adjacent particle exchanges by some definite sequence

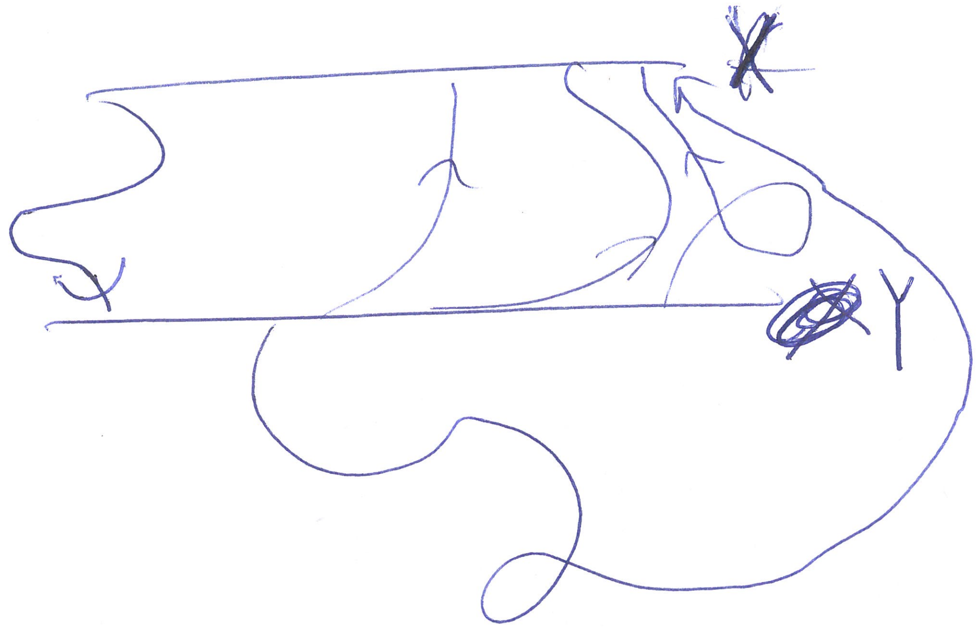
~~Then it~~

—— ~~fiducial~~ X

—— Y

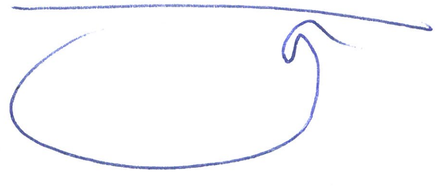
5-202

Now by ~~any~~ all possible (of which there are actually infinitely many) sequence at all come back from Y to ~~fiducial~~ X



All the back sequences must amount to an even/odd number of exchanges

or else (c) is violated which cannot be.

Since  ~~fiducial~~ X to X ~~to fiducial~~ is an even number ~~of 2 particle~~

of 2-adjacent-particle exchanges.

∴ Every permutation Y

is a definite even or odd number of 2-adjacent particle exchanges from permutation ~~[illegible]~~ X

Since permutations X and Y are general every permutation is a definate even or odd number of 2-adjacent-particle exchanges from any other.

If X is the original or ~~[illegible]~~ fiducial permutation.

5-204

~~all a~~ assigned

to be an even permutation

by definition

every other permutation

is definitely

an even

or odd permutation

(has definite parity)

Say Y and Z ~~even/odd~~

general ~~even~~ permutations

going from Y to Z

would take even (if both were even or both odd

odd (if one even + one odd)

and ~~even/odd~~ number

of 2-adjacent particle

exchanges.

X odd (even)

odd (even) Y Z

even

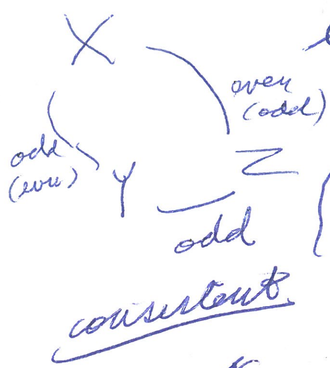
consistent

~~If it took an odd/even number~~

~~then there'd be a~~

even + odd + even = odd

or odd + even + odd = odd



set of 2-adjacent-particle exchanges from X to ~~o~~ X ~~which~~ would take an odd number of 2 adjacent particle exchanges again violates (c)

If these ~~...~~ results violates

A definite ~~perm~~ parity ~~is that~~ assigned to every permutation is then fully consistent.

By symmetry, there must

$\frac{N!}{2}$ even parity permutations

$\frac{N!}{2}$ odd parity permutations

~~(Note N!)~~ unless $N = 1$

in which case there is only 1 even permutation by definition.

Another argument is say you worked your way systematically ~~through~~ all permutations ~~The sequence~~ without repeats — then you go even, odd, even, ..., odd.

by 2-~~particle~~ adjacent particle exchanges

and if ever ran out of evens
~~at~~ with more than one odd
to go you couldn't
do the ~~system~~ repeatless run through

And if you ran out of odds
with ~~any evens~~ more than one even left
you couldn't ~~do~~
the the repeatless
run through.

Ah but can you do a repeatless
run through?
I think so, but can't see a simple
proof at the moment.

I give up on this for now

Picking up the thread from
p. 5-198.
Is our sum of product
states fully antisymmetric?

e) Now we've established ~~that~~ ~~2-particle~~ 2-adjacent particle exchanges ~~transitions~~ always change the parity of the permutation from even/odd to odd/even.
(see p. 5-201)

What of general 2 particle exchanges?

Well as established on p. 5-199, every general 2-particle exchange is equivalent ~~is~~ to an odd number of 2 adjacent particle exchanges.

∴ Every 2-particle exchange changes the permutation parity.

Now to resume ~~from~~ from p. 5-198.

Is our sum of products states fully antisymmetric

$$\Psi = \frac{1}{\sqrt{N!}} \sum_{\wp} (-1)^{P_\wp} \wp \psi_1(1')\psi_2(2')\ldots\psi_N(N')$$

$\wp$ $\wp$ is a particular permutation operator

with the fiducial $\psi_1(1')\ldots\psi_N(N')$ product being defined even

$P_\wp$ is the parity of the permutation

$$P_\wp = \begin{cases} 0 & \text{for even} \\ 1 & \text{for odd.} \end{cases}$$

Consider a particular term

~~$\psi_1(\;\;)\psi_2(\;\;)\ldots\psi_9($~~

$(-1)^{P_\wp} \ldots \psi_i(l') \ldots \psi_j(k') \ldots$

if we exchange these particles

→ we get another

permutation, ~~and~~
~~the on l~~

One which a different
parity.

If we exchange particles l' and k'
in all terms,
we've created
an overall sign change.

Examples might help here.

<u>Ex 1</u> 2 particles, 2 single particle ~~state~~

$$\Psi = \frac{1}{\sqrt{2}}\left[\psi_1(1')\psi_2(2') - \psi_1(2')\psi(1')\right]$$

exchange

$$\Psi_{ex} = \frac{1}{\sqrt{2}}\left[\psi_1(2')\psi(1') - \psi_2(1')\psi(2')\right]$$

$$= -\Psi$$

J-210

Ex 2 3-particles and 3-states

This is tedious

fiducial product state

$$\Psi = \frac{1}{\sqrt{3!}}\Big[\psi_1(1')\psi_2(2')\psi_3(3')$$
$$-\psi_1(2')\psi_2(1')\psi_3(3')$$
$$+\psi_1(2')\psi_2(3')\psi_3(1')$$
$$-\psi_1(3')\psi_2(2')\psi_3(1')$$
$$+\psi_1(3')\psi_2(1')\psi_3(2')$$
$$-\psi_1(1')\psi_2(3')\psi_3(2')\Big]$$

+ve parity ones are cyclic

−ve ones are anticyclic

Say we exchange $1' \longleftrightarrow 2'$

$$\Psi_{ex} = \frac{1}{\sqrt{3!}}\Big[\psi_1(2')\psi_2(1')\psi_3(3')$$
$$-\psi_1(1')\psi_2(2')\psi_3(3')$$
$$+\psi_1(1')\psi_2(3')\psi_3(1')$$
$$-\psi_1(3')\psi_2(1')\psi_3(2')$$
$$+\psi_1(3')\psi_2(2')\psi_3(1')$$
$$-\psi_1(2')\psi_2(3')\psi_3(1')\Big]$$

$$= -\Psi$$

So yes.

We can believe in general

$$\Psi = \sum_{\mathcal{P}} (-1)^{p_{\mathcal{P}}} \mathcal{P}\, \psi_1(1') \ldots \psi_N(N')$$

has the right exchange properties. (i.e., it is antisymmetric)

Does it obey the Pauli exchange Principle?

~~Say $i' \neq j'$~~

In some sense we see that it should already maybe

~~Say coordinate $i' \neq j'$~~

~~$\psi_i(i') \ldots \psi_j(j') \ldots$~~

~~In the $N!$ permutations there must pairs of terms which differ only in having i' and j' exchanged~~

~~$\ldots \psi_k(i') \ldots \psi_l(j') \ldots$~~

5-212

Say we ~~exchange~~ exchange $i' \leftrightarrow j'$

$$\psi_{ex} = -\psi$$

but say $i' = j'$

then $\psi_{ex} = \psi$.

Only consistent when $\psi = 0$

To be more concrete, say we pick ~~term~~ a general term (every term must have i' and j' somewhere)

$$(-1)^{p_{p_a}} \ldots \psi_k(i') \ldots \psi_l(j') \ldots$$

k and l ~~are~~ are general states.

Out of $N_i!-1$ possibilities, there must be a term exactly alike but with i' and j' exchange

$$(-1)^{p_{p_b}} \ldots \psi_k(j') \ldots \psi_l(i') \ldots$$

Must have opposite signs

5-213

If $i' = j'$, these terms differ only by sign, and so cancel.

So all terms cancel <u>pairwise</u>

So the <u>Pauli exclusion principle</u> in the position-spin sense is satisfied.

= exchange boson or antibonding case.

What about in the single-particle state sense (other than space-spin)?

Well say two states included in the original <u>product</u> <u>state</u> were identical.

Any term in sum must have ψ_k & ψ_l in it.

$$(-1)^{p_{p_q}} \ldots \psi_k(\cancel{i}') \ldots \psi_l(\cancel{j}') \ldots$$

$k = l$ actually.

If we picked any term at random ~~the~~ with general i', j', then out of the $N! - 1$ other terms there is one with

$$(-1)^{P_b} \ldots \psi_k(j') \ldots \psi_l(i') \ldots$$

this has a different sign,

but since $k = l$,

the two terms cancel pairwise.

So all terms cancel pairwise.

Our antisymmetric state

is ~~exactly zero~~ identically zero,

and so not a physical state.

So the Pauli Exclusion principle

is satisfied in Both senses.

At this point you

may ask how we could

systematically create
a symmetrized state.

Actually the process is exactly
that of finding the determinant
of matrix.

$$\det \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{3} \end{vmatrix}$$

$$\det(A^T) = \det(A) \quad (\text{Wik})$$

In our case particle label
can be take as column index
~~label~~
and single-particle state
label as
row index.

Or vice versa since

5-216

$$\Psi = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(1') & \psi_2(1') & \psi_3(1') & \cdots & \psi_N(1') \\ \psi_1(2') & \psi_2(2') & \psi_3(2') & \cdots & \psi_N(2') \\ \psi_1(3') & \psi_2(3') & \psi_3(3') & & \vdots \\ \psi_1(4') & \cdots & & & \vdots \\ \vdots & & & & \\ \psi_1(N') & \cdots & & & \psi_N(N') \end{vmatrix}$$

(CT-1390)

One can evaluate systematically by Laplace expansion (Wik) or some other means.

~~What of bosons?~~

~~If all the single particle states are same~~

Normalization & orthogonality.

$$|\Psi|^2 = \frac{1}{N!}\sum_{P}\sum_{P'}$$

a double sum which awfully complex. in general.

~~$\int |\Psi|^2 d$~~ and has terms like

$$(-1)^{p_a}(-1)^{p_b}\Psi_1(i')^* \Psi_2(j')^* \ldots \Psi_N(k')^* \Psi_1(l') \Psi_2(m') \ldots \Psi_N(n')$$

Now normalization.

$$1 = \int |\Psi|^2 d1' d2' d3' \ldots dN'$$

stand for space-spin integration ~~normalization~~ normalize

spinor formalism takes care of sum over spin states.

5-218

Well consider

$$(-1)^{P_k P_l} \psi_1^*(i') \psi_2^*(j') \cdots \psi_N(k')^* \psi_1(l') \psi_2(m') \cdots \psi_N(n')$$

$\chi_f(t')$

t' in the first factor meets up with a t' in the second factor

$$\delta_{fg} = \int \psi_f^*(j') \psi_g(j') dj$$

since we assume the single-particle states are all orthonormal.

Any term in the double sum which is not a direct term vanishes because at least two single-particle states

{ i.e., a cross term

are orthogonal in the integration for normalization.

So only the direct terms contribute

$$\underbrace{(-1)^{\#p}\,\varphi_1(i')^*\varphi_2(j')\ldots\varphi_N(k')^*\,\varphi_1(i')\,\varphi_2(j')\ldots\varphi_N(k')}_{\text{in } 1}$$

$$\int \qquad di'\,dj'\,dk'$$

$$= 1 * 1 * \ldots * 1 = 1$$

$$\int |\Psi|^2 = \frac{1}{N!}\sum_p 1 = \frac{N!}{N!} = 1$$

So $\Psi = \frac{1}{\sqrt{N!}}\sum_p (-1)^{\#p}\varphi_1(1')\ldots\varphi_N(N')$

is correctly Normalized

by $\frac{1}{\sqrt{N!}}$

5-220

What if had two symmetrized states for ~~that~~ N particles that differed by ~~a single~~ one single-particle state

$\Psi_2 =$

$$\Psi_1 = \frac{1}{\sqrt{N!}} \sum_p (-1)^{p_p} \varphi_1(1') \ldots \varphi_l(l') \ldots \varphi_N(N')$$

$$\Psi_2 = \frac{1}{\sqrt{N!}} \sum_p (-1)^{p_p} \varphi_1(1') \ldots \varphi_{\hat{l}}(l') \ldots \varphi_N(N')$$

$$\varphi_l \neq \varphi_{\hat{l}}$$

but they are both orthonormal.

$$\langle \Psi_1 | \Psi_2 \rangle = \int \Psi_1^* \Psi_2 \, d1' \ldots dN'$$

In ~~any~~ every terms in this product we'll have ~~a~~ ~~where~~

$$\int \varphi_l(i')^* \varphi_m(i') \, di' = 0$$