Quantum Mechanics **NAME:**

Homework 4: Quantum Mechanics in Three Dimensions: Homeworks are not handed in or marked. But you get a mark for reporting that you have done them. Once you've reported completion, you may look at the already posted supposedly super-perfect solutions.

Answer Table

Name:

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	31.	O	O	O	O	O
2.	O	O	O	O	O	32.	O	O	O	O	O
3.	O	O	O	O	O	33.	O	O	O	O	O
4.	O	O	O	O	O	34.	O	O	O	O	O
5.	O	O	O	O	O	35.	O	O	O	O	O
6.	O	O	O	O	O	36.	O	O	O	O	O
7.	O	O	O	O	O	37.	O	O	O	O	O
8.	O	O	O	O	O	38.	O	O	O	O	O
9.	O	O	O	O	O	39.	O	O	O	O	O
10.	O	O	O	O	O	40.	O	O	O	O	O
11.	O	O	O	O	O	41.	O	O	O	O	O
12.	O	O	O	O	O	42.	O	O	O	O	O
13.	O	O	O	O	O	43.	O	O	O	O	O
14.	O	O	O	O	O	44.	O	O	O	O	O
15.	O	O	O	O	O	45.	O	O	O	O	O
16.	O	O	O	O	O	46.	O	O	O	O	O
17.	O	O	O	O	O	47.	O	O	O	O	O
18.	O	O	O	O	O	48.	O	O	O	O	O
19.	O	O	O	O	O	49.	O	O	O	O	O
20.	O	O	O	O	O	50.	O	O	O	O	O
21.	O	O	O	O	O	51.	O	O	O	O	O
22.	O	O	O	O	O	52.	O	O	O	O	O
23.	O	O	O	O	O	53.	O	O	O	O	O
24.	O	O	O	O	O	54.	O	O	O	O	O
25.	O	O	O	O	O	55.	O	O	O	O	O
26.	O	O	O	O	O	56.	O	O	O	O	O
27.	O	O	O	O	O	57.	O	O	O	O	O
28.	O	O	O	O	O	58.	O	O	O	O	O
29.	O	O	O	O	O	59.	O	O	O	O	O
30.	O	O	O	O	O	60.	O	O	O	O	O

011 qmult 00100 1 4 3 easy deducto-memory: central-force

1. In a central-force problem, the magnitude of central force depends only on:

   a) the angle of the particle.
   b) the vector $\vec{r}$ from the center to the particle.
   c) the radial distance r from the center to the particle.
   d) the magnetic quantum number of the particle.
   e) the uncertainty principle.

   SUGGESTED ANSWER: (c)

   Wrong Answers:
   a) Nah.
   b) Exactly wrong.

   Redaction: Jeffery, 2001jan01

011 qmult 00200 1 1 2 easy memory: separation of variables

2. The usual approach to getting the eigenfunctions of the Hamiltonian in multi-dimensions is:

   a) non-separation of variables. b) separation of variables. c) separation of invariables.
   d) non-separation of invariables. e) non-separation of variables/invariables.

   SUGGESTED ANSWER: (b) Yes separation of variables is the conventional name. See Ar-86.

   Wrong Answers:
   e) A nonsense answer

   Redaction: Jeffery, 2001jan01

011 qmult 00210 1 1 3 easy memory: separation of variables

3. Say you have a differential equation of two independent variables x and y and you want to look for solutions that can be factorized thusly $f(x, y) = g(x)h(y)$. Say then it is possible to reorder equation into the form

$$\text{LHS}(x) = \text{RHS}(y) ,$$

   where LHS stands for left-hand side and RHS for right-hand side. Well LHS is explicitly independent of y and implicitly independent of x:

$$\frac{\partial \text{LHS}}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \text{LHS}}{\partial x} = \frac{\partial \text{RHS}}{\partial x} = 0 .$$

   Thus, LHS is equal to a constant C and necessarily RHS is equal to the same constant C which is called the constant of separation (e.g., Arf-383). The solutions for $g(x)$ and $h(y)$ can be found separately and are related to each other through C. The solutions for $f(x, y)$ that cannot be factorized are not obtained, of course, by the described procedured. However, if one obtains complete sets of $g(x)$ and $h(y)$ solutions for the x-y region of interest, then any solution $f(x, y)$ can be constructed at least to within some approximation (Arf-443). Thus, the generalization of the described procedure is very general and powerful. It is called:

   a) separation of the left- and right-hand sides. b) partitioning.
   c) separation of the variables. d) solution factorization. e) the King Lear method.

   SUGGESTED ANSWER: (c)

   Wrong answers:
   d) Seems reasonable.
   e) Metaphorical names due turn up in physics like the Monte Carlo method (named after a famous casino in Monaco) and the Urca process (named after a casino in Rio de Janeiro). One sometimes gets the feeling that theoretical physicists spend a lot of time in casinos. I used to wander through them all the time in my Vegas years.

   Redaction: Jeffery, 2008jan01

011 qmult 00400 1 4 2 easy deducto memory: spherical harmonics 1

4. The eigensolutions of the angular part of the Hamiltonian for the central force problem are the:

   a) linear harmonics. b) spherical harmonics. c) square harmonics.
   d) Pythagorean harmonics. e) Galilean harmonics.

   SUGGESTED ANSWER: (b)

   Wrong Answers:

   d) Legend has it that Pythagoras discovered the harmonic properties of strings.
   e) Vincenzo Galileo, father of the other Galileo, was a scientist too and studied music scientifically.

   Redaction: Jeffery, 2001jan01

---

011 qmult 00420 1 4 3 easy deducto memory: spherical harmonic Y00

5. Just about the only spherical harmonic that people remember—and they really should remember it too—is $Y_{00} =$:

   a) $e^{im\phi}$. b) r^2. c) $\dfrac{1}{\sqrt{4\pi}}$. d) θ^2. e) $2a^{-3/2}e^{-r/a}$.

   SUGGESTED ANSWER: (c)

   Wrong Answers:

   a) This is the general azimuthal component of the spherical harmonics: $m = 0, \pm 1, \pm 2, \ldots$
   b) This is radial and it's not normalizable.
   d) Except for Y_{00} itself, the spherical harmonics are all combinations of sinusoidal functions of the θ and ϕ.
   e) This is the R_{10} hydrogenic radial wave function where a is the scale radius

   $$a = a_0 \frac{m_e}{m} \frac{1}{Z} ,$$

   where m_e is the electron mass, m is the reduced mass, Z is the number of unit charges of the central particle, and a_0 is the Bohr radius (Gr2005-137). The Bohr radius in MKS units is given by

   $$a_0 = \frac{\hbar^2}{m_e[e^2/(4\pi\varepsilon_0)]} = \frac{\lambda_{\rm C}}{2\pi} \frac{1}{\alpha} = 0.52917720859(36)\,\text{Å} ,$$

   where e is the elementary charge, $\lambda_C = \hbar/(m_e c)$ is the Compton wavelength, and $\alpha \approx /137$ is the fine structure constant.

   Redaction: Jeffery, 2001jan01

---

011 qfull 00200 2 3 0 moderate math: central-force azimuthal component solution

Extra keywords: solving the azimuthal component of the central force problem

6. In the central force problem, the separated azimuthal part of the Schrödinger equation is:

   $$\frac{d^2\Phi}{d\phi^2} = -m_\ell^2 \Phi ,$$

   where $-m_\ell^2$ is the constant of separation for the azimuthal part. The constant has been parameterized in terms of m_ℓ (which is not mass) since it turns out that for normalizable (and therefore physically allowed) solutions that m must be an integer. The m_ℓ quantity is the z-component angular momentum quantum number or magnetic quantum number (MEL-59; ER-240). The latter name arises since the z-components of the angular momentum manifest themselves most noticeably in magnetic field phenomena.

   a) Since the differential equation is second order, there should should be two independent solutions for each value of m_ℓ^2. Solve for the general solution Φ for each m_ℓ^2: i.e., the solution that is a linear combination of the two independent solutions with undetermined coefficients. Note that writing the separation constant as m_ℓ^2 is so far just a parameterization and nothing yet demands that m_ℓ^2

be greater than zero or pure real. **HINT:** Use an exponential trial function with exponent $\pm(a+ib)$ with a and b real. Also remember the special case of $m_\ell^2 = 0$.

b) The solutions are continuous and so that quantum mechanical requirement is met. But another one must be imposed for the azimuthal coordinate: the single-valuedness condition. Since we have no interpretation for multi-valuedness, we micropostulate that it doesn't happen. Impose the single-valuedness condition on the generl solution

$$\Phi = Ae^{(a+ib)\phi} + Be^{-(a+ib)\phi} ,$$

and show that $a = 0$ and m_ℓ must be an integer. Remember to consider the special case where $m_\ell = 0$?

c) What are the eigenfunction solutions for the z-component of the angular momentum operator

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} .$$

What are the eigenvalues that satisfy single-valuedness and continuity? What is the relationship between these eigenfunction solutions and the azimuthal angle part of the hydrogenic atom wave functions?

d) Normalize the allowed eigensolutions of L_z Note these solutions are, in fact, conventionally left unnormalized: i.e., the coefficient of the special function that is the solution is left as just 1. Normalization is conventionally imposed on the total orbital angular momentum solutions, spherical harmonics.

SUGGESTED ANSWER:

a) The trial solution

$$\Phi = e^{\pm(a+ib)\phi}$$

obviously satisfies the differential equation for

$$(a+ib)^2 = -m_\ell^2 \qquad \text{or} \qquad (a+ib) = i\sqrt{m_\ell^2} .$$

The general solution for each m_ℓ is then

$$\Phi = Ae^{(a+ib)\phi} + Be^{-(a+ib)\phi} ,$$

where A and B are undetermined constants.

In the special case of $m_\ell^2 = 0$, we have

$$\Phi = A\phi + B ,$$

where A and B are undetermined constants.

Note that a 2nd order linear differential equation has only two independent solutions. So we have found all the linearly independent solutions.

b) Let's consider the exponential solutions first: i.e., those with $m_\ell^2 \neq 0$.

Rather obviously, for the solution to be single-valued the a parameter must be zero. If it wasn't, then as you go around the axis in either direction the function would grow exponentially. The function couldn't be single-valued.

Now if we just had one term, then it would be pretty obvious that b would have to be an integer to give single-valuedness. But with two terms, we actually do need a proof. We demand that $\Phi(\phi + 2\pi) = \Phi(\phi)$ for all ϕ. Thus we must have

$$Ae^{ib(\phi+2\pi)} + Be^{-ib(\phi+2\pi)} = Ae^{ib\phi} + Be^{-ib\phi} .$$

Since symmetry might help, let's rearrange the last expression to get

$$Ae^{ib(\phi+\pi)}\left[e^{ib\pi} - e^{-ib\pi}\right] = Be^{-ib(\phi+\pi)}\left[e^{ib\pi} - e^{-ib\pi}\right] .$$

If

$$e^{ib\pi} - e^{-ib\pi} \neq 0 \ ,$$

then we can cancel out that factor and obtain

$$Ae^{2ib(\phi+\pi)} = B$$

which can only be true for general ϕ if $b = 0$ which implies $e^{ib\pi} - e^{-ib\pi} = 0$ which contradicts our assumption or if $A = B = 0$ which does not give a normalizable solution. The only case allowed then is when

$$e^{ib\pi} - e^{-ib\pi} = 0$$

which implies

$$e^{ib(2\pi)} = 1$$

which in turn imples that b is an integer.

Thus, for the exponential solution case, we conclude that the only allowed m_ℓ values are given by

$$m_\ell = 0, \pm 1, \pm 2, \pm 3, \ldots$$

and the general exponential solution is

$$\Phi = Ae^{i|m_\ell|\phi} + Be^{-i|m_\ell|\phi} \ .$$

For the linear solution

$$\Phi = A\phi + B \ ,$$

single-valuedness requires that $A = 0$. The constant solution is just the $m_\ell = 0$ solution all over again.

c) Say we parameterize the eigenvalues as $m_\ell\hbar$. Thus, the eigenproblem is

$$L_z\Phi = m\hbar\Phi \ .$$

The solutions that satisfy single-valuedness based on parts (a) and (b) are obviously

$$\Phi = e^{im_\ell\phi}$$

where

$$m_\ell = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The azimuthal angle parts of the hydrogenic atom wave function can be constructed from the eigenstates of the L_z operator.

d) By inspection, all the allowed normalized solutions are given by

$$\Phi = \frac{1}{\sqrt{2\pi}} e^{im\phi} \ .$$

In actual fact, one seldom normalizes the azimuthal solutions when they stand alone. One normalizes the total angular solutions which are the spherical harmonics.

Redaction: Jeffery, 2001jan01

---

012 qmult 00050 1 1 1 easy memory: hydrogen atom, 2-body

7. The hydrogen atom is the simplest of all neutral atoms because:

a) it is a 2-body system. b) it is a 3-body system. c) it has no electrons.
d) it has many electrons. e) hydrogen is the most abundant element in the universe.

SUGGESTED ANSWER: (a)

Wrong answers:

e) It is the most abundant element in the universe. But this doesn't make it the simplest element. In fact, perhaps it is the other way: because it is the simplest element, it is most abundant.

However, even this is not necessarily so. The abundances of the elements depend on how things were cooked up in the beginning. A different set of initial conditions would lead to different universal abundances.

Redaction: Jeffery, 2001jan01

---

012 qmult 00100 1 1 3 easy memory: radial wave function requirements

8. What basic requirements must the radial part of hydrogenic atom wave function meet in order to be a physical radial wave function?

   a) Satisfy the radial part of the Schrödinger equation and grow exponentially as $r \to \infty$.
   b) Not satisfy the radial part of the Schrödinger equation and grow exponentially as $r \to \infty$.
   c) Satisfy the radial part of the Schrödinger equation and be normalizable.
   d) Not satisfy the radial part of the Schrödinger equation and be normalizable.
   e) None at all.

   SUGGESTED ANSWER: (c) The Schrödinger equation is our basics physics in nonrelativistic quantum mechanics. It must be satisfied. And of course a radial function must also be normalizable (i.e., be square-integrable).

   Wrong Answers:
   b) Everything is wrong.
   e) Oh c'mon.

   Redaction: Jeffery, 2001jan01

---

012 qmult 00190 1 1 2 easy memory: hydrogen wave functions

9. The hydrogenic atom eigenstate wave functions contain a factor that causes them to:

   a) increase exponentially with radius. b) decrease exponentially with radius.
   c) increase logarithmically with radius. d) increase quadratically with radius.
   e) increase linearly with wavelength.

   SUGGESTED ANSWER: (b) The wave function must decrease rapidly with radius in order for it to be normalizable.

   Wrong answers:
   a) Exactly wrong.

   Redaction: Jeffery, 2001jan01

---

012 qmult 00200 1 4 1 easy deducto-memory: associated Laguerre polyn.

10. What special functions are factors in the radial part of the of the hydrogenic atom eigenstate wave functions?

   a) The associated Laguerre polynomials. b) The unassociated Laguerre polynomials.
   c) The associated Jaguar polynomials. d) The unassociated jaguar polynomials.
   e) The Hermite polynomials.

   SUGGESTED ANSWER: (a)

   Wrong Answers:
   e) These are factors in the simple harmonic oscillator wave functions.

   Redaction: Jeffery, 2001jan01

---

012 qmult 01000 1 4 1 easy deducto-memory: atomic spectroscopy

11. Almost all would agree that the most important empirical means for learning about atomic energy eigenstates is:

   a) spectroscopy. b) microscopy. c) telescopy. d) pathology. e) astrology.

   SUGGESTED ANSWER: (a)

   Wrong Answers:
   e) It doesn't even pretend to reveal atomic energy eigenstates.

Redaction: Jeffery, 2001jan01

---

012 qfull 01200 2 3 0 moderate math: s electron in nucleus
Extra keywords: (Gr-142:4.14)

12. Let us consider the probability that the electron of a hydrogenic atom in the ground state will be in the nucleus. Recall the wave function for ground state is

$$\Psi_{100}(\vec{r}) = R_{10}(r)Y_{00}(\theta,\phi) = 2a^{-3/2}e^{-r/a} \times \frac{1}{\sqrt{4\pi}}$$

(Gr2005-154), where $a = a_{\rm Bohr}[m_e/(mZ_{\rm N})]$: $a_{\rm Bohr} \approx 0.529$Å is the Bohr radius, $Z_{\rm N}$ is the nuclear charge, m_e is the electron mass, and m is the reduced mass of the actual hydrogenic atom.

a) First assume that the wave function is accurate down to $r = 0$. It actually can't be, of course. The wave function was derived assuming a point nucleus and the nucleus is, in fact, extended. However, the extension of the nucleus is of order 10^5 times smaller than the Bohr radius, and so the effect of a finite nucleus is a small perturbation. Given that the nuclear radius is b, calculate the probability of finding the electron in the nucleus. Use $\epsilon = b/(a/2) = 2b/a$ to simplify the formula. **HINT:** The formula

$$g(n,x) = \int_0^x e^{-t}t^n\,dt = n!\left(1 - e^{-x}\sum_{\ell=0}^{n}\frac{x^\ell}{\ell!}\right)$$

could be of use.

b) Expand the part (a) answer in ϵ power series and show to lowest non-zero order that

$$P(r < b, \epsilon << 1) = \frac{1}{6}\epsilon^3 = \frac{4}{3}\left(\frac{b}{a}\right)^3 .$$

c) An alternate approach to find the probability of the electron being in the nucleus is assume $\Psi(\vec{r})$ can be approximated by $\Psi(0)$ over nucleus. Thus

$$P(r < b) \approx \left(\frac{4\pi}{3}\right) b^3|\Psi(0)|^2 .$$

Is this result consistent with the part (b) answer?

d) Assume $b \approx 10^{-15}$ m and $a = 0.5 \times 10^{-10}$ m. What is the approximate numerical value for finding the electron in the nucleus? You can't interpret this result as "the fraction of the time the electron spends in the nucleus". Nothing in quantum mechanics tells us that the electron spends time definitely anywhere. One should simply stop with what quantum mechanics gives: the result is the probability of finding the electron in nucleus.

SUGGESTED ANSWER:

a) Behold:

$$\begin{aligned} P(r < b) &= \int_0^b\int_0^\pi\int_0^{2\pi} |\Psi_{100}|^2 r^2 \sin\theta\,dr\,d\theta\,d\phi = \int_0^b |R_{10}|^2 r^2\,dr \\ &= 4a^{-3}\left(\frac{a}{2}\right)^3\int_0^\epsilon x^2e^{-x}\,dx = \frac{1}{2}\int_0^\epsilon x^2e^{-x}\,dx \\ &= 1 - e^{-\epsilon}\left(1 + \epsilon + \frac{1}{2}\epsilon^2\right) , \end{aligned}$$

where we have transformed the integration variable to $x = r/(a/2)$ and have we have defined

$$\epsilon = \frac{b}{(a/2)} = \frac{2b}{a} .$$

b) Behold:

$$P(r<b,\epsilon<<1)=1-e^{-\epsilon}\left(1+\epsilon+\frac{1}{2}\epsilon^2\right)$$
$$\approx 1-e^{-\epsilon}\left(e^{\epsilon}-\frac{1}{6}\epsilon^3\right)\approx\frac{1}{6}\epsilon^3e^{-\epsilon}\approx\frac{1}{6}\epsilon^3$$
$$=\frac{4}{3}\left(\frac{b}{a}\right)^3 .$$

Note that each approximate equality is an equality to 3rd order in ϵ.

It is a trick, of course, to recognize that

$$1+\epsilon+\frac{1}{2}\epsilon^2 \qquad \text{and} \qquad e^{\epsilon}-\frac{1}{6}\epsilon^3$$

are equal to 3rd order in ϵ. If you didn't know the trick, then one would have to tediously expand, multiply, collect, and cancel:

$$e^{-\epsilon}\left(1+\epsilon+\frac{1}{2}\epsilon^2\right)=\left(1-\epsilon+\frac{1}{2}\epsilon^2-\frac{1}{6}\epsilon^3\right)\left(1+\epsilon+\frac{1}{2}\epsilon^2\right)$$
$$=1+\epsilon(1-1)+\epsilon^2\left(-1+\frac{1}{2}+\frac{1}{2}\right)+\epsilon^3\left(-\frac{1}{2}+\frac{1}{2}-\frac{1}{6}\right)+\ldots$$
$$=1-\frac{1}{6}\epsilon^3+\ldots .$$

c) Behold:

$$P(r<b)\approx\left(\frac{4\pi}{3}\right)b^3|\Psi(0)|^2=\left(\frac{4\pi}{3}\right)b^3\times 4a^{-3}\frac{1}{4\pi}=\frac{4}{3}\left(\frac{b}{a}\right)^3 .$$

This result is exactly the part (b) answer. So we have exact consistency.

d) Behold:

$$P(r<b)\approx\frac{4}{3}\left(\frac{b}{a}\right)^3\approx\frac{4}{3}\left(2\times10^{-5}\right)^3=\frac{32}{3}\times10^{-15}\approx10^{-14} .$$

This is a very small probability. But the probability of finding the electron merely close to the nucleus could be much higher depending on how you define close.

Redaction: Jeffery, 2001jan01

---

013 qmult 00100 1 1 4 easy memory: ang. mom. commutation relations

13. The fundamental angular momentum commutation relation and a key corollary are, respectively:

a) $[J_i,J_j]=0$ and $[J^2,J_i]=J_i$. b) $[J_i,J_j]=J_k$ and $[J^2,J_i]=0$.
c) $[J_i,J_j]=0$ and $[J^2,J_i]=0$. d) $[J_i,J_j]=i\hbar\varepsilon_{ijk}J_k$ and $[J^2,J_i]=0$.
e) $[x_i,p_j]=i\hbar\delta_{ij}$, $[x_i,x_j]=0$, and $[p_i,p_j]=0$.

SUGGESTED ANSWER: (d)

Wrong Answers:

a) Both are wrong.
b) The first one is wrong.
c) The first one is dead wrong.
e) These are often called the canonical commutation relations (see, e.g., CT-150). They are true relations, but not a right answer.

Redaction: Jeffery, 2001jan01

---

013 qmult 00910 1 1 3 easy memory: vector model

14. In the vector model for angular momentum of a quantum system with the standard axis for the eigenstates being the z axis, the particles in the eigenstates are thought of as having definite z-components of angular momentum $m_j\hbar$ and definite total angular momenta of magnitude $\sqrt{j(j+1)}\hbar$,

where j can stand for orbital, spin, or total angular momentum quantum number and m_j is the z-component quantum number. Recall j can be only be integer or half-integer and there are $2j+1$ possible values of m_j given by $-j, -j+1, \ldots, j-1, j$. The x-y component of the angular momementum has magnitude $\sqrt{j(j+1)-m_j^2}\hbar$, but it has no definite direction. Rather this component can be thought of as pointing all x-y directions in simultaneous: i.e., it is in a superposition state of all direction states. Diagramatically, the momentum vectors can be represented by

a) cones with axis aligned with the x-axis.
b) cones with axis aligned with the y-axis.
c) cones with axis aligned with the z-axis.
d) cones with axis aligned with the x-y-axis.
e) the cones of silence.

SUGGESTED ANSWER: (c)

Wrong answers:

e) "I demand the cones of silence Chief."
"But Max you know those never work."
"I insist Chief."
Whirr.
"What did you say?"

Redaction: Jeffery, 2008jan01

---

013 qmult 02000 1 1 1 easy memory: added ang. mom. operators

15. Does the fundamental commutation relation for angular momentum operators (i.e., $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$) apply to angular momentum operators formed by summation from angular momentum operators applying to individual particles or to spatial and spin degrees of freedom? The answer is:

a) Yes. b) No. c) Maybe. d) All of the above. e) None of the above.

SUGGESTED ANSWER: (a)

The result actually follows from the way one constructs these operators. If one has particles 1 and 2, then

$$J_i = J_{1i} + J_{2i} \ .$$

The operators for different particles commute by micropostulate. Now one finds

$$\begin{aligned}[J_i, J_j] &= [J_{1i} + J_{2i}, J_{1j} + 2j] = [J_{1i}, J_{1j}] + [J_{1i}, J_{2j}] + [J_{2i}, J_{1j}] + [J_{2i}, J_{2j}] \\ &= i\hbar\varepsilon_{ijk}J_{1k} + 0 + 0 + i\hbar\varepsilon_{ijk}J_{2k} = i\hbar\varepsilon_{ijk}J_k \ .\end{aligned}$$

Wrong Answers:

b) Exactly wrong.

Redaction: Jeffery, 2001jan01

---

013 qmult 02100 1 4 5 easy deducto-memory: Clebsch-Gordan coefficients

16. "Let's play *Jeopardy*! For \$100, the answer is: The name for the coefficients used in the expansion of a total angular momentum state for 2 angular momentum degrees of freedom in terms of products of individual angular momemtum states."

What are the ____________, Alex?

a) Racah W coefficients b) Wigner $6j$ symbols c) Buck-Rogers coefficients
d) Flash-Gordon coefficients e) Clebsch-Gordan coefficients

SUGGESTED ANSWER: (e)

Wrong answers:

a) Nah. Those are for summing 3 angular momentum degrees of freedom (Ba-346).
b) Same thing as (a) by a different name. Even Baym confesses they are exceedingly unpleasant to calculate (Ba-346)—but that was before Mathematica.

Redaction: Jeffery, 2001jan01

---

013 qmult 02200 1 4 5 easy deducto-memory: Clebsch-Gordan m rule

17. "Let's play *Jeopardy*! For \$100, the answer is: In constructing a set of $|j_1 j_2 jm\rangle$ states from a set of $|j_1 j_2 m_1 m_2\rangle$ states using Clebsch-Gordan coefficients, this is a strict constraint on the non-zero coefficients."

    What is the rule ____________, Alex?

    a) of complete overtures b) of incomplete overtures c) $m = m_1^2 + m_2^2$ d) $m = m_1 - m_2$
    e) $m = m_1 + m_2$

    SUGGESTED ANSWER: (e)

    Wrong answers:

    a) Huh?

    Redaction: Jeffery, 2001jan01

---

013 qfull 00100 2 5 0 moderate thinking: angular momentum operator identities

18. Prove the following angular momentum operator identities. **HINT:** Recall the fundamental angular momentum commutator identity,

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k \ , \qquad \text{and the definition} \qquad J_\pm \equiv J_x \pm iJ_y \ .$$

a) $[J_i, J^2] = 0$.

b) $[J^2, J_\pm] = 0$.

c) $[J_z, J_\pm] = \pm\hbar J_\pm$.

d) $J_\pm^\dagger J_\pm = J_\mp J_\pm = J^2 - J_z(J_z \pm \hbar)$.

e)

$$J_x = \frac{1}{2}(J_+ + J_-) \qquad \text{and} \qquad J_y = \frac{1}{2i}(J_+ - J_-) \ .$$

f) $[J_+, J_-] = 2\hbar J_z$.

g)

$$J^2_{\left\{\substack{x \\ y}\right\}} = \pm\frac{1}{4}\left(J_+^2 + J_-^2 \pm \{J_+, J_-\}\right) \ ,$$

where the upper case is for J_x^2 and the lower case for J_y^2 and where recall that $\{A, B\} = AB + BA$ is the anticommutator of A and B.

h)

$$J^2 = \frac{1}{2}\{J_+, J_-\} + J_z^2 \ .$$

SUGGESTED ANSWER:

a) Behold:

$$\begin{aligned}
[J_i, J^2] &= [J_i, J_jJ_j] = J_iJ_jJ_j - J_jJ_jJ_i \\
&= J_iJ_jJ_j - J_jJ_iJ_j + J_jJ_iJ_j - J_jJ_jJ_i \\
&= [J_i, J_j]J_j + J_j[J_i, J_j] = i\hbar\varepsilon_{ijk}J_kJ_j + i\hbar\varepsilon_{ijk}J_jJ_k \\
&= i\hbar\varepsilon_{ijk}J_kJ_j + i\hbar\varepsilon_{ikj}J_kJ_j = i\hbar\varepsilon_{ijk}J_kJ_j - i\hbar\varepsilon_{ijk}J_kJ_j \\
&= 0 \ ,
\end{aligned}$$

where we have used relabeling of the dummy indices.

b) Behold:

$$[J^2, J_\pm] = [J^2, J_x] \pm i[J^2, J_y] = 0 \ .$$

c) Behold:

$$[J_z, J_\pm] = [J_z, J_x] \pm i[J_z, J_y] = i\hbar J_y \pm i^2\hbar(-J_x) = \pm\hbar J_\pm \ .$$

d) Behold:

$$J_{\pm}^{\dagger}J_{\pm} = J_{\mp}J_{\pm} = J_x^2 + J_y^2 \pm iJ_xJ_y \mp iJ_yJ_x = J^2 - J_z^2 \pm i[J_x, J_y]$$
$$= J^2 - J_z^2 \mp \hbar J_z$$
$$= J^2 - J_z(J_z \pm \hbar) \ .$$

e) By inspection.

f) Behold:

$$[J_+, J_-] = -i[J_x, J_y] + i[J_y, J_x] = 2\hbar J_z \ .$$

The $[J_x, J_x]$ and $[J_y, J_y]$ terms are zero of course.

g) Behold:

$$J^2_{\left\{ {x \atop y} \right\}} = \pm\frac{1}{4}\left[J_+^2 + J_-^2 \pm (J_+J_- + J_-J_+)\right] = \pm\frac{1}{4}\left(J_+^2 + J_-^2 \pm \{J_+, J_-\}_+\right) \ .$$

h) Behold:

$$J^2 = J_x^2 + J_y^2 + J_z^2 = \frac{1}{2}\{J_+, J_-\} + J_z^2 \ .$$

Redaction: Jeffery, 2001jan01

---

013 qfull 00200 2 3 0 mod math: diagonalization of J_x for 3-d

Extra keywords: diagonalization of the J-x angular momentum matrix for 3-d

19. The x-component angular momentum operator matrix in a three-dimensional angular momentum space expressed in terms of the z-component orthonormal basis (i.e., the standard basis with eigenvectors $|1\rangle$, $|0\rangle$, and $|-1\rangle$) is:

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(Co-659) and yes the $1/\sqrt{2}$ factor is correct. Is this matrix Hermitian? Diagonalize this matrix: i.e., solve for its eigenvalues and normalized eigenvectors (written in terms of the standard basis ket eigenvectors) or, if you prefer in column vector form. Note the solution is somewhat simpler if you solve the reduced eigen problem. Just divide both sides of the eigen equation by $\hbar/\sqrt{2}$ and solve for the reduced eigenvalues. The physical eigenvalues are the reduced ones times $\hbar/\sqrt{2}$. Verify that the eigenvectors are orthonormal.

NOTE: Albeit some consider it a sloppy notation since kets and bras are abstract vectors and columns vectors are from a concrete representation, its concretely useful to equate them at times. In the present case, the kets equate like so

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ , \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ , \qquad |-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ ,$$

and the bras, like so

$$\langle 1| = (1,0,0)^* \ , \qquad \langle 0| = (0,1,0)^* \ , \qquad \langle -1| = (0,0,1)^* \ .$$

SUGGESTED ANSWER:

The x-component matrix is obviously Hermitian. Complex conjugating and transposing (which amounts to Hermitian conjugating) leaves the matrix unchanged.

The eigenproblem for the x-component matrix in reduced dimensionless form (i.e., sans the physical constant coefficients) is:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_0 \\ C_{-1} \end{pmatrix} = \lambda_{\rm red}\begin{pmatrix} C_1 \\ C_0 \\ C_{-1} \end{pmatrix} \ ,$$

where the eigenvalues λ_{red} and eigenvectors $\vec{C}$ are unknown. We subtract the right-hand side from the left to create a homogeneous matrix equation which only has non-trivial solutions (i.e., solutions without all vectors being zero vectors) if the determinant of the matrix vanishes. The determinant equation is

$$\det \begin{vmatrix} -\lambda_{\text{red}} & 1 & 0 \\ 1 & -\lambda_{\text{red}} & 1 \\ 0 & 1 & -\lambda_{\text{red}} \end{vmatrix} = 0 \ .$$

The determinant equation allows us to determine the eigenvalues from eigenvalue or characteristic equation. For the 3×3 matrix the characteristic equation is easily found to be

$$(-\lambda_{\text{red}})^3 + 2\lambda_{\text{red}} = 0 \ .$$

One solution is obviously $\lambda_{\text{red}} = 0$ and the other two are $\lambda_{\text{red}} = \pm\sqrt{2}$. These values are real as they should be for a Hermitian matrix. Restoring the physical constants the eigenvalues are

$$\lambda = 0 \qquad \text{and} \qquad \lambda = \pm\hbar \ .$$

The eigenvector components can be obtained by substituting the eigenvalues back into the original eigenproblem equation. There are three equations for the coefficients, but one is redundant. This leaves the absolute scale of the vectors undetermined: this lack of absolute scale is a characteristic of homogenous matrix problems. Normalization fixes the scale in quantum mechanics.

For $\lambda_{\text{red}} = 0$, we find by inspection that

$$\vec{C}_{x,0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

or in ket form

$$|0\rangle_x = \frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle) \ .$$

For $\lambda_{\text{red}} = \pm\sqrt{2}$, we have

$$C_0 = \pm\sqrt{2}C_1 \qquad \text{and} \qquad C_0 = \pm\sqrt{2}C_{-1} \ , \qquad \text{and so} \qquad C_{-1} = C_1 \ .$$

Normalizing, we have

$$\vec{C}_{x,\pm 1} = \frac{1}{2} \begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix}$$

or in ket form

$$|\pm 1\rangle_x = \frac{1}{2}(|1\rangle \pm \sqrt{2}|0\rangle + |-1\rangle) \ .$$

By inspection the eigenvectors are orthonormal.

Recall eigenvectors are unique only to within a global phase factor. Caveat markor.

Redaction: Jeffery, 2001jan01

---

014 qmult 00100 1 4 5 easy deducto-memory: spin defined

Extra keywords: mathematical physics

20. "Let's play *Jeopardy*! For \$100, the answer is: It is the intrinsic angular momentum of a fundamental (or fundamental-for-most-purposes) particle. It is invariant and its quantum number s is always an integer or half-integer.

What is ____________, Alex?

a) rotation b) quantum number c) magnetic moment d) orbital angular momentum e) spin

SUGGESTED ANSWER: (e)

Wrong answers:

a) Well no, but not a bad guess.

Redaction: Jeffery, 2008jan01

---

014 qmult 00110 1 4 1 easy deducto-memory: Goudsmit and Uhlenbeck, spin

Extra keywords: Don't abbreviate: it ruins the joke

21. "Let's play *Jeopardy*! For \$100, the answer is: Goudsmit and Ulhenbeck."

a) Who are the original proposers of electron spin in 1925, Alex?
b) Who performed the Stern-Gerlach experiment, Alex?
c) Who are Wolfgang Pauli's evil triplet brothers, Alex?
d) What are two delightful Dutch cheeses, Alex?
e) What were Rosencrantz and Gildenstern's first names, Alex?

SUGGESTED ANSWER: (a) See Le-185 and ER-276. Actually Compton hinted at the idea in 1921, but didn't follow up on it.

Wrong Answers:

b) Now wouldn't you think Stern & Gerlach performed the Stern-Gerlach experiment? CT-897 calls it the Stern-Gerlach experiment.
e) Rosencrantz and Gildenstern were real people: members of the Danish embassy to England in 1592. Frederick (Fred) Rosenkrantz later met up with Johannes Kepler and thus provides the missing link between Kepler and Shakespeare. Rosenkrantz died tragically in a duel—trying to stop it, not fight it—but Shakespeare and Stoppard have made him immortal.

Redaction: Jeffery, 2001jan01

---

014 qmult 00120 1 1 1 easy memory: spin magnitude

22. A spin s particle's angular momentum vector magnitude (in the vector model picture) is

a) $\sqrt{s(s+1)}\hbar$. b) $s\hbar$ c) $\sqrt{s(s-1)}\hbar$ d) $-s\hbar$ e) $s(s+1)\hbar^2$

SUGGESTED ANSWER: (a)

Wrong answers:

e) This is the magnitude squared or the eigenvalue of the $S_{\rm op}^2$ spin operator.

Redaction: Jeffery, 2008jan01

---

014 qmult 00130 1 1 5 easy memory: eigenvalues of spin 1/2 particle

23. The eigenvalues of a **COMPONENT** of the spin of a spin 1/2 particle are always:

a) $\pm\hbar$. b) $\pm\frac{\hbar}{3}$. c) $\pm\frac{\hbar}{4}$. d) $\pm\frac{\hbar}{5}$. e) $\pm\frac{\hbar}{2}$.

SUGGESTED ANSWER: (e)

Wrong Answers:

Redaction: Jeffery, 2001jan01

---

014 qmult 00130 1 1 2 easy memory: eigenvalues of spin s particle

24. The quantum numbers for the component of the spin of a spin s particle are always:

a) ± 1. b) $s, s-1, s-2, \ldots, -s+1, -s$. c) $\pm\frac{1}{2}$. d) ± 2. e) $\pm\frac{1}{4}$.

SUGGESTED ANSWER: (b)

Wrong Answers:

c) This is only correct for electrons.

Redaction: Jeffery, 2001jan01

---

014 qmult 00140 1 4 2 easy deducto-memory: spin and environment

25. Is the spin (not spin component) of an electron dependent on the electron's environment?

a) Always.

b) No. Spin is an intrinsic, unchanging property of a particle.
c) In atomic systems, no, but when free, yes.
d) Both yes and no.
e) It depends on a recount in Palm Beach.

SUGGESTED ANSWER: (b)

Wrong Answers:

e) Only for those who recall the American presidential election of 2000.

Redaction: Jeffery, 2001jan01

---

014 qmult 00400 1 4 5 easy deducto-memory: spin commutation relation

26. "Let's play *Jeopardy*! For \$100, the answer is:

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k \text{ .''}$$

What is ____________, Alex?

a) the spin anticommutator relation b) an implicit equation for ε_{ijk} c) an impostulate
d) an inobservable e) the fundamental spin commutation relation

SUGGESTED ANSWER: (e)

Wrong answers:

a) Spin 1/2 has the anticommutator relation $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}$, where the sigmas are the Pauli spin matrices and $\mathbf{1}$ is the unit matrix.
b) Not the best the answer in this context.
c) Arguably untrue.

Redaction: Jeffery, 2001jan01

---

014 qmult 00500 1 4 2 easy deducto-memory: Pauli spin matrices

27. "Let's play *Jeopardy*! For \$100, the answer is:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ .''}$$

What are ____________, Alex?

a) dimensioned spin 1/2 matrices b) the Pauli spin matrices
c) the Pauli principle matrices d) non-Hermitian matrices
e) matrix-look-alikes, not matrices

SUGGESTED ANSWER: (b)

Wrong answers:

a) The Pauli spin matrices are the dimensionless spin 1/2 matrices.
d) Wrong.

Redaction: Jeffery, 2001jan01

---

014 qmult 00600 1 1 1 easy memory: spin anticommutator relation

28. The expression

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}$$

is

a) the Pauli spin matrix anticommutator relation. b) the Pauli spin matrix commutator relation.
c) the fundamental spin commutator relation. d) the covariance of two standard deviations.
e) an oddish relation.

SUGGESTED ANSWER: (a)

Wrong Answers:

d) It sort of looks like that doesn't it.

Redaction: Jeffery, 2001jan01

---

014 qmult 00900 1 1 3 easy memory: space and spin operators commute

29. A spatial operator and a spin operator commute:

a) never. b) sometimes. c) always. d) always and never. e) to the office.

SUGGESTED ANSWER: (c)

Wrong Answers:

e) I don't think this could reasonably be interpreted as a right answer.

Redaction: Jeffery, 2001jan01

---

014 qmult 01100 1 4 5 easy deducto-memory: spin-magnetic interaction

30. "Let's play *Jeopardy*! For \$100, the answer is:

$$\vec{\mu} = g\frac{q}{2m}\vec{J} \ , \qquad \vec{F} = \nabla(\vec{\mu}\cdot\vec{B}) \ , \qquad \vec{\tau} = \vec{\mu}\times\vec{B} \ , \qquad H = -\vec{\mu}\cdot\vec{B} \ ."$$

a) What are Maxwell's equations, Alex?
b) What are incorrect formulae, Alex?
c) What are classical formulae sans any quantum mechanical analogs, Alex?
d) What are quantum mechanical formulae sans any classical analogs, Alex?
e) What are formulae needed to treat the interaction of angular momentum of a particle and magnetic field in classical and quantum mechanics, Alex?

SUGGESTED ANSWER: (e)

Wrong answers:

a) Nyet.

Redaction: Jeffery, 2001jan01

---

014 qmult 01200 1 1 2 easy memory: Bohr magneton

31. What is

$$\mu_{\rm B} = \frac{e\hbar}{2m_e} = 9.27400915(23)\times 10^{-24}\,{\rm J/T} = 5.7883817555(79)\times 10^{-5}\,{\rm eV/T} \ ?$$

a) The nuclear magneton, the characteristic magnetic moment of nuclear systems.
b) The Bohr magneton, the characteristic magnetic moment of electronic systems.
c) The intrinsic magnetic dipole moment of an electron.
d) The coefficient of sliding friction.
e) The zero-point energy of an electron.

SUGGESTED ANSWER: (b)

Wrong Answers:

a) The subscript "b" and the values should tell you this must be wrong.
c) No. For magnitude of the intrinsic magnetic dipole moment is $g\sqrt{3}/2$ times the Bohr magneton (ER-274).

Redaction: Jeffery, 2001jan01

---

014 qmult 01210 1 1 3 easy memory: g factor g-factor

32. The g factor in quantum mechanics is the dimensionless factor for some system that multiplied by the appropriate magneton (e.g., Bohr magneton for electron systems) times the angular momentum of the system divided by $\hbar$ gives the magnetic moment of the system. Sometimes the sign of the magnetic moment is included in the g factor and sometimes it is just shown explicitly. The modern way seems to be to include it, but yours truly finds that awkward and so for now yours truly doesn't do it. For the electron, the intrinsic magnetic moment operator associated with intrinsic spin is given by

$$\vec{\mu}_{\rm op} = -g\mu_{\rm B}\frac{\vec{S}_{\rm op}}{\hbar} \ ,$$

where $\mu_{\rm B}$ is the Bohr magneton and $S_{\rm op}$ is the spin vector operator. What is g for the intrinsic magnetic moment operator of an electron to modern accuracy?

a) 1. b) 2. c) 2.0023193043622(15). d) 1/137. e) 137.

SUGGESTED ANSWER: (c)

Wrong answers:

b) This is what Dirac's relativistic quantum theory gives. Modern quantum electrodynamics gives to g to about 1 in 10^{12} (Wikipedia: 2008apr30: Precision tests of QED).

Redaction: Jeffery, 2008jan01

---

014 qmult 01210 1 1 4 easy memory: magnetic moment precession

Extra keywords: The precession is also called Larmor precession (En-114)

33. An object in a uniform magnetic field with magnetic moment due to the object's angular momentum will both classically and quantum mechanically:

a) Lancy progress. b) Lorenzo regress. c) London recess. d) Larmor precess. e) Lamermoor transgress.

SUGGESTED ANSWER: (d)

Wrong Answers:

e) That too.

Redaction: Jeffery, 2001jan01

---

014 qfull 00100 2 5 0 moderate thinking: Pauli matrices in detail

34. The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \text{and} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

a) Are the Pauli matrices Hermitian?

b) What is the result when Pauli matrices act on general vector

$$\begin{pmatrix} a \\ b \end{pmatrix} ?$$

c) Diagonalize the Pauli matrices: i.e., solve for their eigenvalues and **NORMALIZED** eigenvectors. **NOTE:** The verb 'diagonalize' takes its name from the fact that a matrix transformed to the representation of its own eigenvectors is diagonal with the eigenvalues being the diagonal elements. One often doesn't actually write the diagonal matrix explicitly.

d) Prove that

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i \varepsilon_{ijk} \sigma_k ,$$

where i, j, and k stand for any of x, y, and z, $\mathbf{1}$ is the unit matrix (which can often be left as understood), δ_{ij} is the Kronecker delta, ε_{ijk} is the Levi-Civita symbol, and Einstein summation is used. **HINT:** I rather think by exhaustion is the only way: i.e., extreme tiredness.

e) Prove

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \qquad \text{and} \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} ,$$

where $\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i$ is the anticommutator of Pauli matrices. **HINT:** You should make use of the part (d) expression.

f) Show that a general 2×2 matrix can be expanded in the Pauli spin matrices plus the unit matrix: i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \mathbf{1} + \vec{\beta} \cdot \vec{\sigma} ,$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the Pauli matrices. **HINT:** Find expressions for the expansion coefficients α, β_x, β_y, and β_z.

g) Let $\vec{A}$ and $\vec{B}$ be vectors of operators in general and let the components of $\vec{B}$ commute with the Pauli matrices. Prove

$$(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma} \ .$$

HINT: Make use of the part (d) expression.

SUGGESTED ANSWER:

a) By inspection $\sigma_i^\dagger = (\sigma_i^T)^* = \sigma_i$ for all i. Thus all the Pauli matrices are Hermitian.

b) Behold:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \ , \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ ia \end{pmatrix} \ ,$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} \ .$$

c) By inspection the eigen equation for all the Pauli matrices is $\lambda^2 - 1 = 0$ with eigenvalues $\lambda = \pm 1$. The eigen vectors for the first two matrices are solved for from the following equations

$$c_2 = \pm c_1 \qquad \text{and} \qquad -ic_2 = \pm c_1 \ .$$

Thus solutions for the eigenvectors—the third set follow by inspection—are

$$|x\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \ , \qquad |y\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} \ ,$$

and

$$|z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ .$$

Note that eigenvectors are unique only to within a global phase factor. Thus eigenvectors that look quite different than those above can also be solutions. Caveat markor. But I think I've chosen the simplest look set of eigenvectors.

d) First, using the results of the part (b) answer or by any other means, we find

$$\sigma_x^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \ , \qquad \sigma_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \ , \qquad \text{and} \qquad \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \ .$$

Second, using the results of the part (b) answer, $(AB)^\dagger = B^\dagger A^\dagger$, the Hermiticity of the Pauli matrices, and the fact that $i^\dagger = -i$, we find

$$\sigma_x\sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z \ ,$$
$$\sigma_y\sigma_x = -i\sigma_z \ ,$$
$$\sigma_y\sigma_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_x \ ,$$
$$\sigma_z\sigma_y = -i\sigma_x \ ,$$
$$\sigma_z\sigma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y \ ,$$

and

$$\sigma_x\sigma_y = -i\sigma_y \ .$$

The above results exhaust the possibilies for a product of two Pauli spin matrices. Collectively they prove the desired result (which, in fact, summarizes them):

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k \ .$$

e) Making use of the part (d) expression and the properties of the Levi-Civita symbol, we find

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = i\varepsilon_{ijk}\sigma_k - i\varepsilon_{jik}\sigma_k = 2i\varepsilon_{ijk}\sigma_k$$

and

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} + i\varepsilon_{ijk}\sigma_k + i\varepsilon_{jik}\sigma_k = 2\delta_{ij} \ .$$

f) Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\mathbf{1} + \vec{\beta}\cdot\vec{\sigma} = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \beta_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \beta_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

It follows that

$$a = \alpha + \beta_z \ , \qquad d = \alpha - \beta_z \ , \qquad b = \beta_x - i\beta_y \ , \qquad \text{and} \qquad c = \beta_x + i\beta_y \ .$$

The solutions for the expansion coefficients follow at once:

$$\alpha = \frac{1}{2}(a+d) \ , \qquad \beta_z = \frac{1}{2}(a-d) \ , \qquad \beta_x = \frac{1}{2}(b+c) \ , \qquad \text{and} \qquad \beta_y = \frac{i}{2}(b-c) \ .$$

Since we have found explicit expressions for the coefficients of a general 2×2 matrix expansion, we have proven that a general 2×2 matrix can be so expanded: QED.

g) Behold:

$$\begin{aligned}(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) &= A_i\sigma_i B_j\sigma_j = A_iB_j\sigma_i\sigma_j = A_iB_j(\delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k) = A_iB_i + i\varepsilon_{kij}A_iB_j\sigma_k \\ &= \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma} \ .\end{aligned}$$

Redaction: Jeffery, 2001jan01

---

014 qfull 00400 1 5 0 easy thinking: electron spin in B-field Hamiltonian

Extra keywords: electron spin in magnetic field Hamiltonian

35. What is the Hamiltonian fragment (piece, part) that describes the energy of an electron spin magnetic moment in a magnetic field? This fragment in a Schrödinger equation can sometimes be separated from the rest of the equation and solved as separate eigenvalue problem. Solve this separated problem. The intrinsic angular momentum operator is $\vec{S}$ and assume the magnetic field points in the z direction. **HINTS:** Think of the classical energy of a magnetic dipole in a magnetic field and use the correspondence principle. This is not a long question.

SUGGESTED ANSWER:

This is easy. There is no translational kinetic term because we assume motionlessness (or in another view have separated off the translational kinetic terms of the Hamiltonian) and no intrinsic kinetic term needs to be considered (for some darn good reason such as the mythical rotational kinetic energy is invariant). Thus no kinetic term is needed at all. Classically the potential energy of a magnetic dipole in space is:

$$U_{\rm cl} = -\vec{\mu}_{\rm cl}\cdot\vec{B} \ .$$

Quantization is easy since $\vec{B}$ doesn't need to be quantized and the spin operator commutes with $\vec{B}$. Admittedly both these notions are justified by experimental results ultimately, but they are plausible a priori. Thus

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} = -\gamma B_z S_z \ ,$$

where γ (which is called the gyromagnetic ratio) is given by

$$\gamma = -g\frac{e\hbar}{2m} = -g\frac{\mu_{\rm B}}{\hbar} \ .$$

The g-factor for the electron spin is given by

$$g = 2\left(1 + \frac{\alpha}{2\pi} + \ldots\right) = 2.0023193043622(15)\ldots \ ,$$

where α is the fine structure constant and where we note that some people include the negative sign for the electron charge in the g-factor. The $\mu_{\rm B}$ is the Bohr magneton: i.e., the characteristic magnetic moment of electronic systems. Well the eigenvalues of the eigen problem are just

$$E_{\pm} = \mp\frac{g}{2}\mu_{\rm B}B_z$$

and the eigenstates are just the z-basis eigenstates

$$|z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ .$$

Redaction: Jeffery, 2001jan01

---

014 qfull 00500 2 5 0 moderate thinking: classical Larmor precession

36. Let's tackle the classical Larmor precession.

a) What is Newton's 2nd law in rotational form?

b) What is the torque on a magnetic dipole moment $\vec{\mu}$ in a magnetic field $\vec{B}$? **HINT:** Any first-year text will tell you.

c) Say that the magnetic moment of a system is given by $\vec{\mu} = \gamma\vec{L}$, where γ is the gyromagnetic ratio and $\vec{L}$ is the system's angular momemtum. Say also that there is a magnetic field $\vec{B} = (0, 0, B_z)$. Solve for the time evolution of $\vec{L}$ using Newton's 2nd law in rotational form assuming the **INITIAL CONDITION** $\vec{L}(t = 0) = (L_{x,0}, 0, L_{z,0})$. **HINTS:** You should get coupled differential equations for two components of $\vec{L}$. They are not so hard to solve. For niceness you should define an appropriate Larmor frequency ω.

SUGGESTED ANSWER:

a) Naturlich

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\rm net} \ .$$

b) Auch naturlich

$$\vec{\tau} = \mu \times \vec{B} \ .$$

c) First we need to evaluate $\mu \times \vec{B}$. Using the determinant rule,

$$\mu \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \mu_x & \mu_y & \mu_z \\ 0 & 0 & B_z \end{vmatrix} = (\mu_y B_z, -\mu_x B_z, 0) = (\gamma L_y B_z, -\gamma L_x B_z, 0) \ .$$

Clearly,

$$L_z' = 0 \ ,$$

and so L_z is a constant given by

$$L_z = L_{z,0} \ .$$

For the other components we have the coupled differential equations

$$L_x' = \omega L_y \qquad \text{and} \qquad L_y' = -\omega L_x \ ,$$

where we have defined

$$\omega = \gamma B_z \ .$$

With minimal hesitation we see that

$$L_x'' = \omega L_y' = -\omega^2 L_x$$

which is just the simple harmonic oscillator equation: "it's deja vue all over again" (Yogi Berra). Given the initial conditions

$$L_x = L_{x,0}\cos(\omega t)$$

and, with a mental bound,

$$L_y = \frac{1}{\omega}L'_x = -L_{x,0}\sin(\omega t) \ .$$

Thus we have the complete solution

$$\vec{L} = [L_{x,0}\cos(\omega t), -L_{x,0}\sin(\omega t), L_{z,0}] \ .$$

The behavior is Larmor precession with angular frequency $\omega = \gamma B_z$ and clockwise direction (assuming $\omega > 0$) when viewing down from positive z-axis. One can also call the fiducial positive direction as the right-hand rule direction for the z-axis. In this usage, $\omega > 0$ gives a negative precession.

Note that $\sqrt{L_x^2 + L_y^2 + L_z^2} = \sqrt{L_{x,0}^2 + L_{z,0}^2}$ is constant. Thus the magnitude of $\vec{L}$ is constant and only the direction changes in time. We can get this result in another way. Take the inner product of the vector differential equation for $\vec{L}$

$$\frac{d\vec{L}}{dt} = \gamma\vec{L}\times\vec{B}$$

with $\vec{L}$ and you find

$$\vec{L}\cdot\frac{d\vec{L}}{dt} = \frac{1}{2}\frac{dL^2}{dt} = \gamma\vec{L}\cdot(\vec{L}\times\vec{B}) = \gamma L_i\varepsilon_{ijk}L_jB_k = \gamma B_k\varepsilon_{kij}L_iL_j = -\gamma B_k\varepsilon_{kij}L_iL_j \ ,$$

where we relabeled the dummy indices to make the last equality. If any number $a = -a$, then $a = 0$. Thus $dL^2/dt = 0$, and L^2 is a constant. Thus again we find that the magnitude of $\vec{L}$ is constant. This result is true even if $\vec{B}$ varies in time and space provided our differential equation holds. I think it always does—hm.

Redaction: Jeffery, 2001jan01

Appendix 2 Quantum Mechanics Equation Sheet

Note: This equation sheet is intended for students writing tests or reviewing material. Therefore it neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things.

1 Constants not to High Accuracy

Constant Name	Symbol	Derived from CODATA 1998
Bohr radius	$a_{\mathrm{Bohr}} = \dfrac{\lambda_{\mathrm{Compton}}}{2\pi\alpha}$	$= 0.529\,\text{Å}$
Boltzmann's constant	k	$= 0.8617 \times 10^{-6}\,\mathrm{eV\,K^{-1}}$
		$= 1.381 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$
Compton wavelength	$\lambda_{\mathrm{Compton}} = \dfrac{h}{m_e c}$	$= 0.0246\,\text{Å}$
Electron rest energy	$m_e c^2$	$= 5.11 \times 10^5\,\mathrm{eV}$
Elementary charge squared	e^2	$= 14.40\,\mathrm{eV}\,\text{Å}$
Fine Structure constant	$\alpha = \dfrac{e^2}{\hbar c}$	$= 1/137.036$
Kinetic energy coefficient	$\dfrac{\hbar^2}{2m_e}$	$= 3.81\,\mathrm{eV}\,\text{Å}^2$
	$\dfrac{\hbar^2}{m_e}$	$= 7.62\,\mathrm{eV}\,\text{Å}^2$
Planck's constant	h	$= 4.15 \times 10^{-15}\,\mathrm{eV}$
Planck's h-bar	$\hbar$	$= 6.58 \times 10^{-16}\,\mathrm{eV}$
	hc	$= 12398.42\,\mathrm{eV}\,\text{Å}$
	$\hbar c$	$= 1973.27\,\mathrm{eV}\,\text{Å}$
Rydberg Energy	$E_{\mathrm{Ryd}} = \dfrac{1}{2} m_e c^2 \alpha^2$	$= 13.606\,\mathrm{eV}$

2 Some Useful Formulae

Leibniz's formula
$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$$

Normalized Gaussian
$$P = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - \langle x \rangle)^2}{2\sigma^2}\right]$$

3 Schrödinger's Equation

$$H\Psi(x,t) = \left[\frac{p^2}{2m} + V(x)\right]\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

$$H\psi(x) = \left[\frac{p^2}{2m} + V(x)\right]\psi(x) = E\psi(x)$$

$$H\Psi(\vec{r},t) = \left[\frac{p^2}{2m} + V(\vec{r})\right]\Psi(\vec{r},t) = i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} \qquad H|\Psi\rangle = i\hbar\frac{\partial}{\partial t}|\Psi\rangle$$

$$H\psi(\vec{r}) = \left[\frac{p^2}{2m} + V(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r}) \qquad H|\psi\rangle = E|\psi\rangle$$

4 Some Operators

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$p = \frac{\hbar}{i}\nabla \qquad p^2 = -\hbar^2 \nabla^2$$

$$H = \frac{p^2}{2m} + V(\vec{r}\,) = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}\,)$$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

---

5 Kronecker Delta and Levi-Civita Symbol

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & \text{otherwise} \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} 1, & ijk \text{ cyclic;} \\ -1, & ijk \text{ anticyclic;} \\ 0, & \text{if two indices the same.} \end{cases}$$

$$\varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell} \qquad \text{(Einstein summation on } i\text{)}$$

---

6 Time Evolution Formulae

$$\text{General} \qquad \frac{d\langle A\rangle}{dt} = \left\langle \frac{\partial A}{\partial t}\right\rangle + \frac{1}{\hbar}\langle i[H(t), A]\rangle$$

$$\text{Ehrenfest's Theorem} \qquad \frac{d\langle \vec{r}\,\rangle}{dt} = \frac{1}{m}\langle \vec{p}\,\rangle \qquad \text{and} \qquad \frac{d\langle \vec{p}\,\rangle}{dt} = -\langle \nabla V(\vec{r}\,)\rangle$$

$$|\Psi(t)\rangle = \sum_j c_j(0)e^{-iE_jt/\hbar}|\phi_j\rangle$$

---

7 Simple Harmonic Oscillator (SHO) Formulae

$$V(x) = \frac{1}{2}m\omega^2x^2 \qquad \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2\right)\psi = E\psi$$

$$\beta = \sqrt{\frac{m\omega}{\hbar}} \qquad \psi_n(x) = \frac{\beta^{1/2}}{\pi^{1/4}}\frac{1}{\sqrt{2^n n!}}H_n(\beta x)e^{-\beta^2x^2/2} \qquad E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$H_0(\beta x) = H_0(\xi) = 1 \qquad H_1(\beta x) = H_1(\xi) = 2\xi$$

$$H_2(\beta x) = H_2(\xi) = 4\xi^2 - 2 \qquad H_3(\beta x) = H_3(\xi) = 8\xi^3 - 12\xi$$

8 Position, Momentum, and Wavenumber Representations

$$p = \hbar k \qquad E_{\text{kinetic}} = E_T = \frac{\hbar^2 k^2}{2m}$$

$$|\Psi(p,t)|^2\,dp = |\Psi(k,t)|^2\,dk \qquad \Psi(p,t) = \frac{\Psi(k,t)}{\sqrt{\hbar}}$$

$$x_{\text{op}} = x \qquad p_{\text{op}} = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, t\right) \qquad \text{position representation}$$

$$x_{\text{op}} = -\frac{\hbar}{i}\frac{\partial}{\partial p} \qquad p_{\text{op}} = p \qquad Q\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}, p, t\right) \qquad \text{momentum representation}$$

$$\delta(x) = \int_{-\infty}^{\infty} \frac{e^{ipx/\hbar}}{2\pi\hbar}\,dp \qquad \delta(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{2\pi}\,dk$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \Psi(p,t)\frac{e^{ipx/\hbar}}{(2\pi\hbar)^{1/2}}\,dp \qquad \Psi(x,t) = \int_{-\infty}^{\infty} \Psi(k,t)\frac{e^{ikx}}{(2\pi)^{1/2}}\,dk$$

$$\Psi(p,t) = \int_{-\infty}^{\infty} \Psi(x,t)\frac{e^{-ipx/\hbar}}{(2\pi\hbar)^{1/2}}\,dx \qquad \Psi(k,t) = \int_{-\infty}^{\infty} \Psi(x,t)\frac{e^{-ikx}}{(2\pi)^{1/2}}\,dx$$

$$\Psi(\vec{r},t) = \int_{\text{all space}} \Psi(\vec{p},t)\frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}\,d^3p \qquad \Psi(\vec{r},t) = \int_{\text{all space}} \Psi(\vec{k},t)\frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}\,d^3k$$

$$\Psi(\vec{p},t) = \int_{\text{all space}} \Psi(\vec{r},t)\frac{e^{-i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}\,d^3r \qquad \Psi(\vec{k},t) = \int_{\text{all space}} \Psi(\vec{r},t)\frac{e^{-i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}\,d^3r$$

9 Commutator Formulae

$$[A,BC] = [A,B]C + B[A,C] \qquad \left[\sum_i a_i A_i, \sum_j b_j B_j\right] = \sum_{i,j} a_i b_j [A_i, b_j]$$

$$\text{if} \qquad [B,[A,B]] = 0 \qquad \text{then} \qquad [A,F(B)] = [A,B]F'(B)$$

$$[x,p] = i\hbar \qquad [x,f(p)] = i\hbar f'(p) \qquad [p,g(x)] = -i\hbar g'(x)$$

$$[a,a^\dagger] = 1 \qquad [N,a] = -a \qquad [N,a^\dagger] = a^\dagger$$

10 Uncertainty Relations and Inequalities

$$\sigma_x \sigma_p = \Delta x \Delta p \geq \frac{\hbar}{2} \qquad \sigma_Q \sigma_Q = \Delta Q \Delta R \geq \frac{1}{2} \left| \langle i[Q, R] \rangle \right|$$

$$\sigma_H \Delta t_{\text{scale time}} = \Delta E \Delta t_{\text{scale time}} \geq \frac{\hbar}{2}$$

---

11 **Probability Amplitudes and Probabilities**

$$\Psi(x,t) = \langle x | \Psi(t) \rangle \qquad P(dx) = |\Psi(x,t)|^2 \, dx \qquad c_i(t) = \langle \phi_i | \Psi(t) \rangle \qquad P(i) = |c_i(t)|^2$$

---

12 **Spherical Harmonics**

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}} \qquad Y_{1,0} = \left(\frac{3}{4\pi} \right)^{1/2} \cos(\theta) \qquad Y_{1,\pm 1} = \mp \left(\frac{3}{8\pi} \right)^{1/2} \sin(\theta) e^{\pm i\phi}$$

$$L^2 Y_{\ell m} = \ell(\ell+1) \hbar^2 Y_{\ell m} \qquad L_z Y_{\ell m} = m\hbar Y_{\ell m} \qquad |m| \leq \ell \qquad m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

0	1	2	3	4	5	6	...
s	p	d	f	g	h	i	...

---

13 **Hydrogenic Atom**

$$\psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}(\theta, \phi) \qquad \ell \leq n-1 \qquad \ell = 0, 1, 2, \ldots, n-1$$

$$a_z = \frac{a}{Z} \left(\frac{m_e}{m_{\text{reduced}}} \right) \qquad a_0 = \frac{\hbar}{m_e c \alpha} = \frac{\lambda_{\text{C}}}{2\pi\alpha} \qquad \alpha = \frac{e^2}{\hbar c}$$

$$R_{10} = 2 a_Z^{-3/2} e^{-r/a_Z} \qquad R_{20} = \frac{1}{\sqrt{2}} a_Z^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a_Z} \right) e^{-r/(2a_Z)}$$

$$R_{21} = \frac{1}{\sqrt{24}} a_Z^{-3/2} \frac{r}{a_Z} e^{-r/(2a_Z)}$$

$$R_{n\ell} = -\left\{ \left(\frac{2}{n a_Z} \right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3} \right\}^{1/2} e^{-\rho/2} \rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \qquad \rho = \frac{2r}{n r_Z}$$

$$L_q(x) = e^x \left(\frac{d}{dx} \right)^q \left(e^{-x} x^q \right) \quad \text{Rodrigues's formula for the Laguerre polynomials}$$

$$L_q^j(x) = \left(\frac{d}{dx} \right)^j L_q(x) \qquad \text{Associated Laguerre polynomials}$$

$$\langle r \rangle_{n\ell m} = \frac{a_Z}{2} \left[3n^2 - \ell(\ell+1) \right]$$

$$\text{Nodes} = (n-1) - \ell \qquad \text{not counting zero or infinity}$$

$$E_n = -\frac{1}{2} m_e c^2 \alpha^2 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -E_{\text{Ryd}} \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -13.606 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \text{ eV}$$

14 General Angular Momentum Formulae

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k \qquad \text{(Einstein summation on } k\text{)} \qquad [J^2, \vec{J}] = 0$$

$$J^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle \qquad J_z |jm\rangle = m\hbar |jm\rangle$$

$$J_\pm = J_x \pm i J_y \qquad J_\pm |jm\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |jm \pm 1\rangle$$

$$J_{\left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right\}} = \left\{ \begin{matrix} \frac{1}{2} \\ \frac{1}{2i} \end{matrix} \right\} (J_+ \pm J_-) \qquad J_\pm^\dagger J_\pm = J_\mp J_\pm = J^2 - J_z (J_z \pm \hbar)$$

$$[J_{fi}, J_{gj}] = \delta_{fg} i\hbar \varepsilon_{ijk} J_k \qquad \vec{J} = \vec{J}_1 + \vec{J}_2 \qquad J^2 = J_1^2 + J_2^2 + J_{1+} J_{2-} + J_{1-} J_{2+} + 2 J_{1z} J_{2z}$$

$$J_\pm = J_{1\pm} + J_{2\pm} \qquad |j_1 j_2 j m\rangle = \sum_{m_1 m_2, m = m_1 + m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle j_1 j_2 j m\rangle$$

$$|j_1 - j_2| \leq j \leq j_1 + j_2 \qquad \sum_{|j_1 - j_2|}^{j_1 + j_2} (2j+1) = (2j_1 + 1)(2j_2 + 1)$$

15 Spin 1/2 Formulae

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\pm\rangle_x = \frac{1}{\sqrt{2}} (|+\rangle \pm |-\rangle) \qquad |\pm\rangle_y = \frac{1}{\sqrt{2}} (|+\rangle \pm i|-\rangle) \qquad |\pm\rangle_z = |\pm\rangle$$

$$|++\rangle = |1,+\rangle|2,+\rangle \qquad |+-\rangle = \frac{1}{\sqrt{2}} (|1,+\rangle|2,-\rangle \pm |1,-\rangle|2,+\rangle) \qquad |--\rangle = |1,-\rangle|2,-\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk} \sigma_k \qquad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk} \sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = \vec{A} \cdot \vec{B} + i(\vec{A} \times \vec{B}) \cdot \vec{\sigma}$$

$$\frac{d(\vec{S} \cdot \hat{n})}{d\alpha} = -\frac{i}{\hbar} [\vec{S} \cdot \hat{\alpha}, \vec{S} \cdot \hat{n}] \qquad \vec{S} \cdot \hat{n} = e^{-i\vec{S} \cdot \vec{\alpha}} \vec{S} \cdot \hat{n}_0 e^{i\vec{S} \cdot \vec{\alpha}} \qquad |\hat{n}_\pm\rangle = e^{-i\vec{S} \cdot \vec{\alpha}} |\hat{z}_\pm\rangle$$

$$e^{ixA} = \mathbf{1}\cos(x) + iA\sin(x) \quad \text{if } A^2 = \mathbf{1} \qquad\qquad e^{-i\vec{\sigma}\cdot\vec{\alpha}/2} = \mathbf{1}\cos(x) - i\vec{\sigma}\cdot\hat{\alpha}\sin(x)$$

$$\sigma_i f(\sigma_j) = f(\sigma_j)\sigma_i\delta_{ij} + f(-\sigma_j)\sigma_i(1-\delta_{ij})$$

$$\mu_{\text{Bohr}} = \frac{e\hbar}{2m} = 0.927400915(23)\times 10^{-24}\,\text{J/T} = 5.7883817555(79)\times 10^{-5}\,\text{eV/T}$$

$$g = 2\left(1 + \frac{\alpha}{2\pi} + \ldots\right) = 2.0023193043622(15)$$

$$\vec{\mu}_{\text{orbital}} = -\mu_{\text{Bohr}}\frac{\vec{L}}{\hbar} \qquad \vec{\mu}_{\text{spin}} = -g\mu_{\text{Bohr}}\frac{\vec{S}}{\hbar} \qquad \vec{\mu}_{\text{total}} = \vec{\mu}_{\text{orbital}} + \vec{\mu}_{\text{spin}} = -\mu_{\text{Bohr}}\frac{(\vec{L}+g\vec{S})}{\hbar}$$

$$H_\mu = -\vec{\mu}\cdot\vec{B} \qquad H_\mu = \mu_{\text{Bohr}}B_z\frac{(L_z+gS_z)}{\hbar}$$

16 **Time-Independent Approximation Methods**

$$H = H^{(0)} + \lambda H^{(1)} \qquad |\psi\rangle = N(\lambda)\sum_{k=0}^{\infty}\lambda^k|\psi_n^{(k)}\rangle$$

$$H^{(1)}|\psi_n^{(m-1)}\rangle(1-\delta_{m,0}) + H^{(0)}|\psi_n^{(m)}\rangle = \sum_{\ell=0}^{m}E^{(m-\ell)}|\psi_n^{(\ell)}\rangle \qquad |\psi_n^{(\ell>0)}\rangle = \sum_{m=0,\ m\neq n}^{\infty}a_{nm}|\psi_n^{(0)}\rangle$$

$$|\psi_n^{\text{1st}}\rangle = |\psi_n^{(0)}\rangle + \lambda\sum_{\text{all } k,\ k\neq n}\frac{\left\langle\psi_k^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle}{E_n^{(0)}-E_k^{(0)}}|\psi_k^{(0)}\rangle$$

$$E_n^{\text{1st}} = E_n^{(0)} + \lambda\left\langle\psi_n^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle$$

$$E_n^{\text{2nd}} = E_n^{(0)} + \lambda\left\langle\psi_n^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle + \lambda^2\sum_{\text{all } k,\ k\neq n}\frac{\left|\left\langle\psi_k^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle\right|^2}{E_n^{(0)}-E_k^{(0)}}$$

$$E(\phi) = \frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle} \qquad \delta E(\phi) = 0$$

$$H_{kj} = \langle\phi_k|H|\phi_j\rangle \qquad H\vec{c} = E\vec{c}$$

17 **Time-Dependent Perturbation Theory**

$$\pi = \int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}\,dx$$

$$\Gamma_{0\to n} = \frac{2\pi}{\hbar}|\langle n|H_{\mathrm{perturbation}}|0\rangle|^2\delta(E_n - E_0)$$

18 Interaction of Radiation and Matter

$$\vec{E}_{\mathrm{op}} = -\frac{1}{c}\frac{\partial \vec{A}_{\mathrm{op}}}{\partial t} \qquad \vec{B}_{\mathrm{op}} = \nabla \times \vec{A}_{\mathrm{op}}$$

19 Box Quantization

$$kL = 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \qquad k = \frac{2\pi n}{L} \qquad \Delta k_{\mathrm{cell}} = \frac{2\pi}{L} \qquad \Delta k^3_{\mathrm{cell}} = \frac{(2\pi)^3}{V}$$

$$dN_{\mathrm{states}} = g\frac{k^2\,dk\,d\Omega}{(2\pi)^3/V}$$

20 Identical Particles

$$|a, b\rangle = \frac{1}{\sqrt{2}}\left(|1, a; 2, b\rangle \pm |1, b; 2, a\rangle\right)$$

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right)$$

21 Second Quantization

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = 0 \qquad [a_i^\dagger, a_j^\dagger] = 0 \qquad |N_1, \ldots, N_n\rangle = \frac{(a_n^\dagger)^{N_n}}{\sqrt{N_n!}} \cdots \frac{(a_1^\dagger)^{N_1}}{\sqrt{N_1!}}|0\rangle$$

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad \{a_i, a_j\} = 0 \qquad \{a_i^\dagger, a_j^\dagger\} = 0 \qquad |N_1, \ldots, N_n\rangle = (a_n^\dagger)^{N_n} \cdots (a_1^\dagger)^{N_1}|0\rangle$$

$$\Psi_s(\vec{r})^\dagger = \sum_{\vec{p}} \frac{e^{-i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a^\dagger_{\vec{p}s} \qquad \Psi_s(\vec{r}) = \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a_{\vec{p}s}$$

$$[\Psi_s(\vec{r}), \Psi_{s'}(\vec{r}\,')]_\mp = 0 \qquad [\Psi_s(\vec{r})^\dagger, \Psi_{s'}(\vec{r}\,')^\dagger]_\mp = 0 \qquad [\Psi_s(\vec{r}), \Psi_{s'}(\vec{r}\,')^\dagger]_\mp = \delta(\vec{r} - \vec{r}\,')\delta_{ss'}$$

$$|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle = \frac{1}{\sqrt{n!}}\Psi_{s_n}(\vec{r}_n)^\dagger \cdots \Psi_{s_n}(\vec{r}_n)^\dagger|0\rangle$$

$$\Psi_s(\vec{r})^\dagger|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle\sqrt{n+1}|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n, \vec{r}s\rangle$$

$$|\Phi\rangle = \int d\vec{r}_1 \ldots d\vec{r}_n\, \Phi(\vec{r}_1, \ldots, \vec{r}_n)|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle$$

$$1_n = \sum_{s_1 \ldots s_n} \int d\vec{r}_1 \ldots d\vec{r}_n\, |\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle\langle\vec{r}_1 s_1, \ldots, \vec{r}_n s_n| \qquad 1 = |0\rangle\langle 0| + \sum_{n=1}^{\infty} 1_n$$

$$N = \sum_{\vec{p}s} a^\dagger_{\vec{p}s} a_{\vec{p}s} \qquad T = \sum_{\vec{p}s} \frac{p^2}{2m} a^\dagger_{\vec{p}s} a_{\vec{p}s}$$

$$\rho_s(\vec{r}\,) = \Psi_s(\vec{r}\,)^\dagger \Psi_s(\vec{r}\,) \qquad N = \sum_s \int d\vec{r}\, \rho_s(\vec{r}\,) \qquad T = \frac{1}{2m} \sum_s \int d\vec{r}\, \nabla\Psi_s(\vec{r}\,)^\dagger \cdot \nabla\Psi_s(\vec{r}\,)$$

$$\vec{j}_s(\vec{r}\,) = \frac{1}{2im} \left[\Psi_s(\vec{r}\,)^\dagger \nabla\Psi_s(\vec{r}\,) - \Psi_s(\vec{r}\,)\nabla\Psi_s(\vec{r}\,)^\dagger \right]$$

$$G_s(\vec{r} - \vec{r}\,') = \frac{3n}{2} \frac{\sin(x) - x\cos(x)}{x^3} \qquad g_{ss'}(\vec{r} - \vec{r}\,') = 1 - \delta_{ss'} \frac{G_s(\vec{r} - \vec{r}\,')^2}{(n/2)^2}$$

$$v_{\text{2nd}} = \frac{1}{2} \sum_{ss'} \int d\vec{r} d\vec{r}\,'\, v(\vec{r} - \vec{r}\,') \Psi_s(\vec{r}\,)^\dagger \Psi_{s'}(\vec{r}\,')^\dagger \Psi_{s'}(\vec{r}\,') \Psi_s(\vec{r}\,)$$

$$v_{\text{2nd}} = \frac{1}{2V} \sum_{pp'qq'} \sum_{ss'} v_{\vec{p}-\vec{p}\,'} \delta_{\vec{p}+\vec{q},\vec{p}'+\vec{q}'} a^\dagger_{\vec{p}s} a^\dagger_{\vec{q}s'} a_{\vec{q}\,'s'} a_{\vec{p}\,'s} \qquad v_{\vec{p}-\vec{p}\,'} = \int d\vec{r}\, e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{r}} v(\vec{r}\,)$$

22 Klein-Gordon Equation

$$E = \sqrt{p^2c^2 + m^2c^4} \qquad \frac{1}{c^2} \left(i\hbar \frac{\partial}{\partial t} \right)^2 \Psi(\vec{r}, t) = \left[\left(\frac{\hbar}{i} \nabla \right)^2 + m^2c^2 \right] \Psi(\vec{r}, t)$$

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar} \right)^2 \right] \Psi(\vec{r}, t) = 0$$

$$\rho = \frac{i\hbar}{2mc^2} \left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right) \qquad \vec{j} = \frac{\hbar}{2im} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right)$$

$$\frac{1}{c^2} \left(i\hbar \frac{\partial}{\partial t} - e\Phi \right)^2 \Psi(\vec{r}, t) = \left[\left(\frac{\hbar}{i} \nabla - \frac{e}{c} \vec{A} \right)^2 + m^2c^2 \right] \Psi(\vec{r}, t)$$

$$\Psi_+(\vec{p}, E) = e^{i(\vec{p}\cdot\vec{r} - Et)/\hbar} \qquad \Psi_-(\vec{p}, E) = e^{-i(\vec{p}\cdot\vec{r} - Et)/\hbar}$$